UNIFORM ESTIMATES ON THE NUMBER OF COLLISIONS IN SEMI-DISPERSING BILLIARDS.

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ABSTRACT. We use the methods of non-standard Riemannian geometry to obtain complete solutions of two classic billiard problems:

1. We establish local uniform estimates on the number of collisions in non-degenerate semi-dispersing billiards (moreover, our results apply to billiards on arbitrary manifolds).

2. We find an explicit uniform estimate on the maximal possible number of collisions, in the infinite period of time, in an arbitrary system of hard spheres in empty space, in terms of the number of spheres, their masses and radii.

Our methods are based on the construction of a certain Alexandrov space of curvature bounded from above which geodesics naturally correspond to the trajectories of the billiard flow.

1. INTRODUCTION.

Estimating the number of collisions in a neighborhood of a given point is a central problem in the theory of billiard systems. Moreover, the existence of a uniform estimate on the number of collisions is related to various important properties of a billiard system. For example, the Sinai-Chernov formulas for the metric entropy of billiards are proved under the assumption that such an estimate exists ([2], [9]).

It is quite obvious that if the boundary of the billiard has any concave parts then a trajectory can have arbitrarily many collisions in a neighborhood of a "point of concavity," and sometimes even infinitely many collisions (see, for example, [12]). Therefore, the class of billiards for which uniform estimates are possible are the semi-dispersing billiards, which have been studied extensively in the numerous works (see the review in [7]). The most important examples of semi-dispersing billiards are the billiards that correspond to hard balls gas models.

Vaserstein (1979, [12]) and Galperin (1981, [4]) proved that in semi-dispersing billiards any trajectory has only finitely many collisions in any finite period of time.

Sinai (1978, [8]) proved the existence of a uniform estimate for billiards inside polyhedral angles, and pointed out that his results should also hold for semi-dispersing billiards in a neighborhood of a point x with linearly

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independent normals to the "walls" of the billiard at x (see the remark at the end of [8]).

In this paper we deal with semi-dispersing billiards on arbitrary manifolds. All our results apply to the usual billiards in $\mathbb{R}^k$ and $\mathbb{T}^k$.

First we prove that for semi-dispersing billiards on arbitrary manifolds any trajectory has only a finite number of collisions in a finite period of time.

Then we establish local uniform estimates for such billiards under a mild non-degeneracy condition. We also give effective estimates on the maximal number of collisions and the size of the neighborhood in which they may occur in terms of the non-degeneracy condition and some geometric properties of the manifold.

Some non-degeneracy condition is necessary to prohibit obvious counterexamples. We believe that our condition is not only sufficient but also necessary for the existence of a uniform estimate, and we plan to prove it in a forthcoming paper. In particular, our condition is always satisfied for a system of hard balls in empty space for which other more "natural" conditions, as, for example, the condition on the normals to the "walls" to be in the general position, are known to fail. For a system of balls in a jar with concave walls our non-degeneracy condition is satisfied except for some special sets of radii, when it is possible to "squeeze the balls tightly between the walls." Actually, it is known that in those situations the system may have arbitrarily many collisions locally.

As an application, we consider the systems of hard balls in empty space. We obtain a global uniform estimate on the maximal possible number of collisions, in the infinite time interval $(-\infty, \infty)$, for hard balls systems in empty space, in terms of masses, radii and the number of balls. In spite of the fact that the finiteness of the number of collisions in such a system has been known for a long time (due to Vaserstein (1979, [12]) and Galperin (1981, [4]) and also, much later, an alternative proof by Illner (1988, [5])), uniform estimates on the number of collisions were obtained only for the system of three balls of equal radii (by Thurston and Sandri (1964, [11])) and for systems of particles (balls of zero radius) on a line (by Galperin (1981, [4])). [1]

Although this paper deals with "dynamical" and "billiard" types of problems, our methods are purely geometrical. The gist of the geometrical counterpart of our arguments is the following. Consider a collection of Riemannian triangles with concave sides and curvature bounded from above. Assume that one glues them along isometric sides so that, at every vertex,

[1]Notice that a system of particles corresponds to a billiard system only if the particles are on a line. In a higher dimensional space its behavior is very different from the balls of non-zero radii (for example, the future of the system is not uniquely determined by the initial conditions). Recently, however, Semenovich and Vaserstein [10] obtained explicit estimates on the number of collisions and some other important results for systems of particles in $\mathbb{R}^k$ for arbitrary k.

the shortest homotopically non-trivial loop in the link is no shorter than 2π (in angular metric). Then it is known that the resulting intrinsic metric space is also (locally) an Alexandrov space of curvature bounded from above. In this paper we prepare all necessary tools for generalizing this statement to higher dimensions. In a forthcoming paper [1] we plan to address this geometric problem, as well as to show some other dynamical applications of our methods. We glue together finitely many copies of a given billiard and get a certain natural space which in a sense eliminates the discontinuity of the billiard flow. We obtain some estimates for the topological entropy of a non-degenerate semi-dispersing billiard B. In particular, we show the finiteness of the topological entropy for the time-one map. For the first-return map to the boundary we prove the finiteness of the topological entropy for non-degenerate semi-dispersing billiards with a "finite horizon" (i.e. such billiards that the length of geodesics in B is bounded). Note that for billiards with "infinite horizon" the first-return map may have infinite topological entropy (for example, the "Lorentz gas," [3]).

2. Summary of Results.

Let M be an arbitrary C^2 Riemannian manifold without boundary, with bounded sectional curvature and with the injectivity radius $r > 0$. Consider a collection of n geodesically convex subsets (**walls**) $B_i \subset M, i = 1, \dots, n$ of M, such that their boundaries are C^1 submanifolds of codimension one. Let $B = M \backslash (\bigcup_{i=1}^{n} Int(B_i))$, where $Int(B_i)$ denotes the interior of the set B_i. A semi-dispersing billiard flow (or a semi-dispersing billiard system) acts on a certain subset of the unit tangent bundle to B (see, for example, [7] for the rigorous definitions). The projections of the orbits of that flow to B are called the billiard trajectories and correspond to free motions of particles inside B. Namely, the particle moves inside the set B with unit speed along a geodesic until it reaches one of the sets B_i (**collision**) where it reflects according to the law "the angle of incidence is equal to the angle of reflection." If it reaches one of the sets $B_i \bigcap B_j, i \neq j$, the trajectory is not defined after that moment. For any two points X, Y of a trajectory T we will denote the piece of the trajectory between X and Y by $T(X,Y)$ and its length by $|T(X,Y)|$.

The following result is a variable curvature counter-part of the results of Vaserstein [12] and Galperin [4] and is an easy corollary of our Comparison Lemma.

Theorem 1. *For any trajectory the number of collisions during any finite time interval is finite.*

An immediate corollary of Theorem 1 is that a trajectory of any point can be defined for all $t \in (-\infty, \infty)$ as long as it does not intersect $B_i \bigcap B_j, i \neq j$.

Now we introduce a non-degeneracy condition under which we establish the existence of uniform local estimates. This condition is weaker than other non-degeneracy conditions that were considered in the local estimate problem. Moreover, it has the important advantage of not being intrinsically local which, in particular, allows us to solve the global problem for hard balls (Section 7).

The non-degeneracy of B in a neighborhood of $x \in B$ follows, in particular, from the following local condition: for any $I \subset \{1, \ldots, n\}$ the intersection of the tangent cones to $B_i \bigcap B$, $i \in I$ at x is equal to the tangent cone to $\bigcap_I (B_i \bigcap B)$.

Definition 2.1. *A billiard table B is* **non-degenerate** *in a subset $U \subset M$ (with constant $C > 0$), if for any $I \subset \{1, \ldots, n\}$ and for any $y \in (U \bigcap B) \backslash (\bigcap_{j \in I} B_j)$,*

$$\max_{k \in I} \frac{dist(y, B_k)}{dist(y, \bigcap_{j \in I} B_j)} \geq C,$$

whenever $\bigcap_{j \in I} B_j$ is non-empty.

Roughly speaking, it means that if a point is d-close to all the walls from I then it is Cd-close to their intersection.

We will say that B is **non-degenerate** if there exist $\delta > 0$ and $C > 0$ such that B is non-degenerate, with constant C, in any δ-ball.

¿From here on out we denote by K, r, n and C the upper bound on the curvature of M, its injectivity radius, the number of walls B_i and the constant of non-degeneracy, respectively.

We will prove the following

Theorem 2. *Let a semi-dispersing billiard B be non-degenerate in an open neighborhood $U \subset M$ of $x \in U$. Then there exist a neighborhood U_x of x and a number P_x such that every billiard trajectory entering U_x leaves it after making no more than P_x collisions with the walls.*

As an immediate application of Theorem 2 (or, to be more rigorous, of Proposition 6.1) we obtain the following global linear estimate:

Corollary 2.1. *For any non-degenerate semi-dispersing billiard there exists a constant P such that, for every t, every trajectory of the billiard flow makes no more than $P(t+1)$ collisions with the boundary in the time interval $[0, t]$.*

Let $(T_1)^m$ be the m-th return map of the billiard flow to the boundary N of the phase space of the billiard flow. To be more precise, N consists of those points $X = (x, v)$ in the unit tangent bundle to B that satisfy the following conditions

1. X belongs to the domain of definition of the billiard flow T^t;
2. x belongs to the boundary of B in M.

Then, obviously, T^t may be represented as a special flow over $(T_1)^m$ under a function $\tau_m : N \to \mathbb{R}$. To be more precise, $\tau_m(X)$ is equal to the time that passes before the orbit of T^t that starts at X returns to N for the $m-th$ time.

A simple application of Theorem 2 is the following

Corollary 2.2. *For any non-degenerate semi-dispersing billiard there exist $m \in \mathbb{N}$ and $\epsilon > 0$ such that T^t is a special flow over $(T_1)^m$ under a function τ_m, with $\tau_m > \epsilon$.*

Now, consider a system of N hard elastic balls in $\mathbb{R}^k$ that move freely and collide elastically. This means that if two balls collide they reflect (that is their speed vector change) according to the following laws: In a (moving) coordinate system chosen so that the impulse of the pair of colliding balls with respect to this coordinate system is equal to zero

1. for each ball, the component of the speed vector parallel to the common tangent plane to the balls at the point of collision remains unchanged;
2. for each ball, the component of the speed vector perpendicular to the common tangent plane to the balls at the point of collision changes sign.

(As usually, we assume that no more then two balls collide with each other at any time.)

Effective estimates on the number of collisions and the size of the neighborhood allow us to prove the following global estimate:

Theorem 3. *In the system of N hard elastic balls in $\mathbb{R}^k$ such that the maximal mass and radius of the balls in the system are M and R and the minimal ones are m and r, there can be no more than*

$$\left(32\sqrt{\frac{M}{m}}\frac{R}{r}N^{\frac{3}{2}}\right)^{N^2}$$

collisions in the infinite period of time $(-\infty, \infty)$.

3. Plan of the Proof of Theorem 2.

The proof is based on two geometric observations: the Comparison Lemma, which is proved in Section 4 and the Shortening Procedure, which is explained in detail in Section 6.

By a **combinatorial type** of a trajectory $T(X,Y)$ we mean the sequence $B_{i_1}, \dots, B_{i_k}$ of the walls which it encounters. We say that a curve $\sigma(t), \sigma(0) = X, \sigma(1) = Y$ has the same combinatorial type as $T(X,Y)$ if it visits the same bodies B_{i_j} as does $T(X,Y)$ in the same order, i.e., there are $0 \le t_1 \le t_2 \cdots \le t_k \le 1$ with $\sigma(t_j) \in B_{i_j}$. Note that the inequalities are not strict, so rather than visiting B_i and then B_j the curve $\sigma(t)$ may visit their intersection $B_i \cap B_j$ instead. For example, any curve between X and Y visiting the intersection of all B_i has the same combinatorial type as any trajectory between X and Y.

Comparison Lemma. *Let $\Delta = \min(r, \frac{1}{\sqrt{\max\{K,0\}}})$. Then every trajectory $T(X,Y)$ which lies inside some Δ-ball is shorter than every other curve of the same combinatorial type, provided $|T(X,Y)| < \Delta$.*

Though this statement may seem rather unexpected, it becomes very transparent when we construct a metric space where $T(X, Y)$ is a geodesic (Section 4).

Since Theorem 2 is a local existence result we can assume in its proof that $Q = \bigcap_1^n B_i$ is non-empty. (Notice that we do not require this in the Comparison Lemma.)

Theorem 2 follows immediately from the following statement:

There exist $R(M, C)$ and $M(C, n)$ such that every trajectory inside a ball of radius R on M, which makes more than $M(C, n)$ collisions, can be shortened in its combinatorial class.

Let us outline the construction of a shorter curve of the same combinatorial class (the detailed argument can be found in Section 6). Let $C > 0$ be as in the non-degeneracy condition. We define a **special triple** to be any three points X_1, Y and X_2 on T such that

1. $X_1, X_2 \in B_i$, where $i \in \{1, \dots, n\}$ is such that $\frac{dist(Y, B_i)}{dist(Y, Q)} \geq C$;
2. X_1 precedes Y on T and X_2 follows Y on T.

For each special triple we consider two possible modifications of the trajectory:

1. join X_1 and X_2 by a shortest arc in B_i, decreasing the length by some quantity S;
2. join X_1 with a point $q \in Q$ and then join q and X_2.

The second modification may lead to some increase of the length, which, however, can be estimated by $SK(C)$, where $K(C)$ is a certain function of C (Lemma 5.1). Hence, as soon as the number of such special triples exceeds $K(C) + 1$, we choose one with the smallest value of S and use modification (2) for it and modification (1) for others. This way we obtain a curve which is shorter than the original trajectory $T(X, Y)$, connects X and Y and visits Q. Hence it has the same combinatorial type. Contradiction.

To apply our results to the hard balls gas, we notice that, for $M = \mathbb{R}^k$ and $\bigcap_i B_i$ non-empty, one can choose $R = \infty$. Then we estimate C (Section 7) and thus obtain an estimate on $M(C, n)$.

4. Proof of the Comparison Lemma and of Theorem 1.

Given $T(X = X_0, Y = X_{j+1})$ with collision points $X_1, \dots, X_j$ we construct a generalized Riemannian space (or Alexandrov's space, see [6]) $\tilde{M}$ in the following way: take $j+1$ isometric copies M_i, $i = 0, \dots, j$ of M and, for all $i = 0, \dots, j-1$, glue together M_i and M_{i+1} by the set B_k, which contains X_{i+1}. Notice that by construction, for each i, there is a canonical isometric embedding $E_i : M \to \tilde{M}$, which is an isometry between M and M_j and maps the subsets B_i in M into the subsets B_i in M_j.

The curve $G(T(X, Y)) = \bigcup_{i=0}^{j} E_i(X_i X_{i+1}) \in \tilde{M}$ is a geodesic in $\tilde{M}$ corresponding to the trajectory $T(X, Y)$ in M and it has the same length in $\tilde{M}$ as $T(X, Y)$ in M.

¿From here on out we will denote the shortest geodesic in M connecting points X and Y by XY (when it is unique), and its length by $|XY|$. Denote by $B_\delta(x)$ a ball of radius δ centered at $x \in M$.

Let Δ be as in the Comparison Lemma. Then $B_\Delta(x)$ is a simply-connected P_K-domain (for a background information on non-regular Riemannian geometry see [6]). Then, applying j times Reshetnyak's theorem [6], we see that the Δ-ball $B_\Delta^{\tilde{M}}(x)$ around x in $\tilde{M}$ is also a simply connected P_K-domain with a unique geodesic joining any two points. (Recall that Reshetnyak's theorem asserts that if we glue two P_K-domains by isometric convex sets then we obtain another P_K-domain.) The geodesic $G(T(X,Y))$ lies inside $B_\Delta^{\tilde{M}}(x)$ and thus is the shortest curve inside $\tilde{M}$ joining its end points $E_0(X)$ and $E_j(Y)$ in $\tilde{M}$. Consider a curve $\sigma(t)$ between X and Y of the same combinatorial type as $T(X,Y)$. Then the curve $G(\sigma) = \bigcup_{i=0}^{j} E_i(\sigma[t_i, t_{i+1}]) \in \tilde{M}$ has the same length as $\sigma(t)$ in M and joins $E_0(X)$ and $E_j(Y)$ (here $t_0 = 0, t_{j+1} = 1$). Now, since $G(T(X,Y))$ is shorter then $G(r)$ we see that $T(X,Y)$ is shorter than $\sigma(t)$.

Remark. *Using the same arguments we see that for any $\delta < \Delta$ any piece of a trajectory inside $B_\delta(x)$, $x \in Q$, has length less than 2δ.*

Proof of Theorem 1. Assume that a finite length trajectory T has infinitely many collision points that accumulate to a point q. Without loss of generality we may assume that $q \in Q$ (otherwise we consider a billiard system outside of a smaller number of bodies) and $T \subset B_\Delta(q)$. Let $x_n, n = 1, 2, \dots$ be the points of collision that converge to q. Clearly $|x_1 q| < |T|$, therefore we can find k such that $|x_1 q| + |q x_k| < |T(x_1, x_k)|$ which contradicts the Comparison Lemma since $T(x_1, x_k)$ has only a finite number of collisions.

Now we present an elementary proof of a weaker version of the Comparison Lemma for $M = \mathbb{R}^k$, which however can be used to complete the proofs of all our results for the case $M = \mathbb{R}^k$. Namely, we show that if $Q = \bigcap B_i$ is non-empty, then $|Xq| + |qY| > |T(X,Y)|$, for every $q \in Q$.

Fix an arbitrary point $q \in Q$ and a billiard trajectory $T(X = x_0, Y = x_{j+1})$ in B with the collision points $x_i \in \partial B, i = 1, \dots, j$. Consider the triangulated surface which is formed by the triangles $q x_i x_{i+1}, i = 0, \dots, j$. Let $O X_0 \dots X_{j+1} \in \mathbb{R}^2$ be the *development* of this surface.

The following simple lemma first appeared in the work of Vaserstein [12].

Lemma. *The curve $\gamma = X_0 \dots X_{j+1}$ is convex in the following sense: For any $i \in \{0, \dots, j\}$, the whole γ lies on the opposite side of $X_i X_{i+1}$ from O.*

Proof of the lemma is very simple and is based only on "the angle of intersection is equal to the angle of reflection" property.

Now all we have to do is to notice that it follows from the convexity of γ that

$$|Xq| + |qY| = |OX_0| + |OX_{j+1}| \geq \sum_{i=0}^{j} |X_i X_{i+1}| = \sum_{i=0}^{j} |x_i x_{i+1}| = |T(X,Y)|.$$

We remark here that although the above lemma has been known for over 15 years we noticed the elementary proof of the Comparison Lemma for $\mathbb{R}^k$ only after we proved the general case using the methods of non-regular Riemannian geometry and then tried to get an elementary proof by adjusting the proofs of the results about gluing P_K domains to $\mathbb{R}^k$. However, our future dynamical applications [1] essentially rely on the non-elementary construction even in the case of billiards in $\mathbb{R}^k$.

5. Auxiliary Lemma

Before we proceed with the proofs of our main results we need to prove the following auxiliary lemma.

Lemma 5.1. *There exists a constant $R(M, c)$ such that for any A, B, C, E in a ball U of radius $R(M, c)$ the inequality $\frac{dist(C,AB)}{|CE|} \geq c$ implies*

$$\frac{|AE| + |EB| - |AC| - |CB|}{|AC| + |CB| - |AB|} \leq K(c) = 32(1/c + 1)^2.$$

In the case $M = \mathbb{R}^k$ one can take $R(M, c) = \infty$, and $K(c) = 16(1/c + 1)^2$

Proof. Let $D \in AB$ be the point closest to C. We have $\frac{|ED|}{|CD|} \leq \frac{|CD|+|CE|}{|CD|} \leq c_1 = 1 + 1/c$. Denote $|CD| = d$.

First we will prove the lemma for $M = \mathbb{R}^k$.

We will consider two cases. The first case: $|AD| > 2c_1 d$ and $|BD| > 2c_1 d$. Then D belongs to the interior of AB and thus $CD \perp AB$.

Note that in every right triangle with sides $y \leq x$ and hypotenuse z one has $\frac{y^2}{4x} \leq z - x \leq \frac{y^2}{2x}$.

Let D_1 be such that $D_1 \in (AB), ED_1 \perp AB$. Then $D_1 \in AB$ and $|AD_1| \geq |AD| - |ED| \geq \frac{1}{2}|AD|$, $|BD_1| \geq \frac{1}{2}|BD|$. Thus,

$$|AE| + |EB| - |AB| = |AE| - |AD_1| + |EB| - |BD_1| \leq \frac{|ED_1|^2}{2|AD_1|} +$$

$$+ \frac{|ED_1|^2}{2|BD_1|} \leq \frac{|ED|^2}{2}\left(\frac{1}{|AD_1|} + \frac{1}{|BD_1|}\right) \leq$$

$$\leq \frac{c_1^2}{2}d^2\left(\frac{1}{|AD_1|} + \frac{1}{|BD_1|}\right) \leq c_1^2 d^2\left(\frac{1}{|AD|} + \frac{1}{|BD|}\right),$$

$$|AC| + |CB| - |AB| = |AC| - |AD| + |BC| - |BD| \geq \frac{d^2}{4|AD|} + \frac{d^2}{4|BD|} =$$

$$= \frac{d^2}{4}\left(\frac{1}{|AD|} + \frac{1}{|BD|}\right).$$

Therefore,

$$\frac{|AE| + |EB| - |AC| - |CB|}{|AC| + |CB| - |AB|} = \frac{(|AE| + |EB| - |AB|) - (|AC| + |CB| - |AB|)}{|AC| + |CB| - |AB|} \leq$$

$$\leq 4c_1^2 - 1 = 3 + \frac{8}{c} + \frac{1}{c^2} \leq 16(1/c+1)^2.$$

Now we consider the second case: $|AD| \leq 2c_1 d$ or $|BD| \leq 2c_1 d$. Then $|AE| + |EB| - |AB| \leq 2|ED| \leq 2c_1 d$,

$$|AC| + |CB| - |AB| \geq \sqrt{d^2 + |AD|^2} + \sqrt{d^2 + |BD|^2} - |AD| - |BD| \geq$$

$$\geq d\left(\sqrt{1 + \left(\frac{\min(AD,BD)}{|CD|}^2\right)} - \frac{\min(AD,BD)}{|CD|}\right) \geq$$

$$\geq d(\sqrt{1+4c_1^2} - 2c_1) \geq \frac{d}{8c_1},$$

since $c_1 > 1$. Therefore

$$\frac{|AE| + |EB| - |AC| - |CB|}{|AC| + |CB| - |AB|} \leq 16\frac{c_1}{c} = 16/c + 16/c^2 \leq 16(1/c+1)^2$$

Lemma 5.1 is proved for $M = \mathbb{R}^N$. Moreover, assuming that $R(M,c)$, and thus d and $|AB|$, are small enough we see that

$$|AC| + |CB| - |AB| \geq d^2. \tag{1}$$

Let us now prove the Lemma in the general case.

Assume that $R(M,c)$ is so small that for all $A, B \in U$ and $D \in AB$ it is possible to introduce normal coordinates along the geodesic (AB) defined on the whole U and with the origin at D.

Let $f : U \to \mathbb{R}^m$, where m is the dimension of M, be the coordinate map. Then there exists a constant L such that for any $z \in U$,

$$1 - L(dist(z,AB))^2 \leq |D(f)(z)| \leq 1 + L(dist(z,AB))^2.$$

Let $A_1, B_1, C_1, D_1, E_1 \in \mathbb{R}^m$ be the images of A, B, C, D, E under f.

Then, from equation (1), it easily follows that

$$\frac{|AE| + |EB| - |AC| - |CB|}{|AC| + |CB| - |AB|} \leq$$

$$\leq \frac{|A_1E_1| + |E_1B_1| - |A_1C_1| - |C_1B_1| + 4Ld^2R(M,c)}{|A_1C_1| + |C_1B_1| - |A_1B_1| - 3Ld^2R(M,c)} \leq$$

$$\leq 2\frac{|A_1E_1| + |E_1B_1| - |A_1C_1| - |C_1B_1|}{|A_1C_1| + |C_1B_1| - |A_1B_1|} \leq 32(1/c+1)^2,$$

provided that $R(M,c)$ is small enough compared to c and L. This finishes the proof of Lemma 5.1. ☐

6. Shortening Procedure and Proof of Theorem 2.

In proving Theorem 2 we can and will assume that $Q = \bigcap_1^n B_i$ is non-empty and that $x \in Q$. Assume that B is non-degenerate in some neighborhood U of x with constant C. We will show that the conclusion of Theorem 2 is true for $U_x = U \bigcap B_{\Delta/2}(x) \bigcap B_{R(M,C)}(x)$.

First of all notice that from the Remark that follows the proof of Comparison Lemma we immediately see that Comparison Lemma is applicable to every trajectory inside $\bigcap B_{\Delta/2}(x)$ and, thus, to every trajectory inside U_x.

Consider a trajectory T inside U_x. Recall that a triple $P = \{X_1, Y, X_2\}$ is called special if

1. $X_1, X_2 \in B_i$, where $i \in \{1, \ldots, n\}$ is such that $\frac{dist(Y,B_i)}{dist(Y,Q)} \geq C$;
2. X_1 precedes Y on T and X_2 follows Y on T.

For a special triple $P = \{X_1, Y, X_2\}$ we introduce the following operations with T :

1. Replace $T(X_1, X_2)$ by $X_1E \bigcup EX_2$, where $E \in Q$ is the point closest to Y in Q. In this case we will say that we stretched T with respect to P. We denote $|X_1E| + |EX_2| - |X_1Y| - |YX_2|$ by $S(P)$.
2. Replace $T(X_1, X_2)$ by X_1X_2. In this case we will say that we contracted T with respect to P. We denote $|X_1Y| + |YX_2| - |X_1X_2|$ by $C(P)$.

Then by Lemma 5.1 we have

$$S(P) \leq K(C)C(P).$$

Call a piece $T(a, b)$ of trajectory T a **full cycle** if T intersects all bodies B_i between the points a and b.

Two full cycles will be called **non-intersecting** if they have no common points except maybe for the end points of the cycles.

Let $L = [K(C)] + 2 < 32(1/C + 1)^2 + 2 < 32(1/C + 2)^2$.

Lemma 6.1. *A trajectory inside U_x can not have more than $2L$ pairwise non-intersecting full cycles.*

Proof. Assume that there are pairwise non-intersecting full cycles T_j, $j = \{1, \ldots, 2L\}$, numbered according to their order on T. Let $T(a_1, b_1)$ and $T(a_2, b_2)$ be two consecutive full cycles. Set $Y = a_2$. Let i be such that $\frac{dist(Y,B_i)}{dist(Y,Q)} \geq C$. Then there are points $X_1 \in T(a_1, b_1)$ and $X_2 \in T(a_2, b_2)$ such that $X_1, X_2 \in B_i$. Hence each two consecutive full cycles contain a special triple.

Therefore, inside T we have at least L special triples $P_q = \{X_{1,q}, Y_q, X_{2,q}\}$, $q = \{1, \ldots, L\}$. Moreover, $P_q \subset T_{2q-1} \bigcup T_{2q}$.

Consider, $S(P_q)$ and $C(P_q)$. Let k be such that $S(P_k) = \min_q S(P_q)$.

Let us stretch T with respect to P_k and contract it with respect to P_q for all $q \neq k$. Denote the resulting curve by T_1. Then $|T| - |T_1| \geq (\sum_{q \neq k} C(P_q)) - S(P_k) \geq (\sum_{q \neq k} \frac{1}{K(C)} S(P_q)) - S(P_k) \geq \frac{L-1}{K(C)} S(P_k) - S(P_k) > 0$. This contradicts the Comparison Lemma. $\square$

Proposition 6.1. *Any piece of a trajectory that lies inside U_x has no more than*

$$M_n = (16(1/C + 2))^{2(n-1)}$$

collisions (where n is the number of bodies B_i).

Proof. We prove Proposition 6.1 by induction on the number of bodies B_i. Assume that for all $k < n$ any trajectory inside U_x, which visits no more than k bodies B_i, has no more than M_k collisions in U_x. By the inductive assumption every piece of trajectory with more than M_{n-1} collisions contains a full cycle. Therefore Lemma 6.1 implies that any trajectory inside U_x has no more than $M_n = 2L(M_{n-1}+1) \leq 4LM_{n-1}$ collisions in U_x. Since, clearly $M_1 = 1$, we see that $M_n \leq (4L)^{n-1} < (16(1/C+2))^{2(n-1)}$. $\square$

Theorem 2 immediately follows from Proposition 6.1.

Remark. *For billiards in $\mathbb{R}^k$ many things simplify significantly. In particular, $\Delta = \infty$ and $R(M, C) = \infty$ and therefore $U_x = U$. Thus, the estimate for M_n becomes "global" in the sense that it depends only on the neighborhood U in which B is non-degenerate. (Also the estimate improves a little: $M_n = (8(1/C+2))^{2n-2}$.)*

7. Application to the Systems of Hard Balls. Proof of Theorem 3.

Consider a system of N balls V_i, $i = 1, \dots, N$, in $\mathbb{R}^k$ of radii r_i, $i = 1, \dots, N$ and masses m_i, $i = 1, \dots, N$, respectively, which move in $\mathbb{R}^k$ freely with arbitrary speeds and collide with each other elastically. Without loss of generality we may assume that $\min_i r_i = 1$, $\min_i m_i = 1$, $\max_i r_i = R$ and $\max_i m_i = M$.

Under the condition that the kinetic energy of the system is equal to some constant E the evolution of such a system in time corresponds to the movement of a particle in a certain billiard B in $\mathbb{R}^{Nk}$. To be more precise, a configuration x of balls V_i, $i = 1, \dots, N$ in $\mathbb{R}^k$ with centers $c_i = \{x_{i,j}, j = 1, \dots, k\} \in \mathbb{R}^k$, $i = 1, \dots, N$, is represented by the point $X = F(c_i) = \{x_{i,j}, j = 1, \dots, k, i = 1, \dots, N\} \in \mathbb{R}^{Nk}$. Equip $\mathbb{R}^{Nk}$ with the Euclidean metric

$$||\{x_{i,j}\}||^2 = \frac{1}{2E}\sum_i\sum_j m_i x_{i,j}^2 = \frac{1}{2E}\sum_i m_i |c_i|^2.$$

Then the corresponding billiard system is defined in the complement B of the $\frac{N(N-1)}{2}$ bodies $B_{m,l}$, which correspond to the pairs of balls. Namely for $m = 1, \dots, N$; $l = 1, \dots, N$; $m \neq l$ and

$$B_{m,l} = \{\{x_{i,j}\} \in \mathbb{R}^{Nn} | |c_m - c_l|^2 = \sum_j (x_{m,j} - x_{l,j})^2 \leq (r_m + r_l)^2\}.$$

Now we will check the non-degeneracy condition for B. The intuitive meaning of the following rather tedious argument is the following: For a fixed set I of pairs of balls and every configuration of balls in $\mathbb{R}^k$ we move the balls from I in the direction of each other until all the pairs from I intersect. We show how to move them as to ensure that the total displacement of the corresponding point in the configuration space of B is bounded by a constant times the largest distance between the balls from I in the original configuration. In order to do so we divide the collection of balls into several groups corresponding to the connected components of the relation I. Then we contract each group homothetically with a center in one of the balls of the group.

Fix a subset I of bodies $B_{m,l}$. Let us estimate $C_{I,\mathbb{R}^{Nk}}(X)$ for some $X \in B \backslash (\bigcap_{B_{m,l} \in I} B_{m,l})$.

Let m and l such that $B_{m,l} \in I$, and let $d_{m,l}$ be the distance between c_m and c_l in $\mathbb{R}^k$. Let $e_{m,l} = d_{m,l} - r_m - r_l \geq 0$, since $X \in B$. Let $e = e_{m_1,l_1} = \max_{B_{m,l} \in I} e_{m,l}$. Since $X \in B \backslash (\bigcap_{B_{m,l} \in I} B_{m,l})$, we have that $e > 0$.

Let us estimate $\max_{B_{m,l} \in I} dist(X, B_{m,l})$. The distance from X to any point $X' = F(c_i')$ of B_{m_1,l_1} is no less than

$$\sqrt{\frac{1}{2E}(m_{m_1}|c_{m_1} - c'_{m_1}|^2 + m_{l_1}|c_{l_1} - c'_{l_1}|^2)} \geq \sqrt{\frac{1}{2E}e^2/4} \geq \frac{e}{2\sqrt{2E}}.$$

Therefore,

$$dist(X, B_{m_1,l_1}) \geq \frac{e}{2\sqrt{2E}}. \tag{2}$$

Let us estimate $dist(X, \bigcap_{B_{m,l} \in I} B_{m,l})$. Let $\lambda = \frac{2}{2+e}$. Then, for all $B_{m,l} \in I$

$$\lambda d_{m,l} \leq r_m + r_l. \tag{3}$$

Construct a graph G in the following way:

1. G has N vertices marked by the integers $i = 1, \dots, N$;
2. Vertices m and l are connected by an edge if and only if $B_{m,l} \in I$.

Let $G = \bigcup_j G_j$, where G_j are the connected components of G. Fix arbitrary vertices $v_j \in G_j$ (one in each connected component).

Consider dilations H_j in $\mathbb{R}^k$ with the centers at c_{v_j} and contraction coefficient λ. Let $c_i' = H_j(c_i)$, where j is such that $i \in G_j$. Then for all $i = 1, \dots, N$ the distance between c_i' and c_i is not greater than

$$(N \max_{B_{m,l} \in I} d_{m,l})(1 - \lambda) \leq \frac{(2R + e)eN}{2 + e}.$$

Let $X' = F(c_i')$. Then equation (2) implies that $X' \in \bigcap_{B_{m,l} \in I} B_{m,l}$. And

$$||X - X'||^2 = \frac{1}{2E}\sum_i m_i|c_i - c_i'|^2 \leq \frac{1}{2E}NM\frac{(2R+e)^2e^2N^2}{(2+e)^2} \leq \frac{1}{2E}N^3MR^2e^2.$$

Therefore,

$$dist(X, \bigcap_{B_{m,l} \in I} B_{m,l}) \leq \frac{NRe\sqrt{MN}}{\sqrt{2E}}. \tag{4}$$

¿From equations (2) and (4) we have

$$C_{I,\mathbb{R}^{Nk}}(X) \geq \frac{1}{2RN\sqrt{MN}}$$

for any I. Therefore $C = C_{I,\mathbb{R}^{Nk}}(X) \geq \frac{1}{2RN\sqrt{MN}}$ for all $X \in \mathbb{R}^{kN}$.

Due to this fact and the remark after Proposition 6.1 we see that the maximal number of collisions is no greater than

$$(8(2RN\sqrt{MN}))^{N(N-1)-2} < (32RN\sqrt{MN})^{N^2}$$

which proves Theorem 3.

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