Bosonization of Massive Fermions

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Abstract

We study the Euclidean sine-Gordon field theory on the plane with $\beta < 16\pi/3$ and an interaction density confined to a finite square. For $\beta = 4\pi$ we construct correlation functions for the field $: \sin\phi :$ and show that they are equal to the pseudoscalar $\overline{\psi}\Gamma\psi$ correlation functions for a free fermion theory with mass term confined to the finite square.

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Contents

1 Introduction

On a two dimensional space time certain boson and fermion quantum field theories are equivalent, a phenomenon discovered by Coleman [10]. This equivalence has found applications to statistical mechanics and string theory.

The simplest case is the equivalence of massless free fermions and massless free bosons on the plane with a Euclidean metric. The massless free fermion theory is given formally by a (fermionic) functional measure

$$\exp\left(\int(\bar{\psi}i\partial\!\!\!/\psi)(x)dx\right)d\overline{\psi}d\psi \tag{1}$$

The massless free boson theory is given formally by the functional measure

$$\exp\left(-\frac{1}{2\beta}\int|\partial\phi(x)|^2dx\right)d\phi \tag{2}$$

The equivalence is the statement that all correlation functions (moments) are equal if $\beta=4\pi$ and we make the correspondance

$$\begin{array}{rcl} c:\cos(\phi(x)): & \Longleftrightarrow & (\overline{\psi}\psi)(x) \\ ic:\sin(\phi(x)): & \Longleftrightarrow & (\overline{\psi}\Gamma\psi)(x) \end{array} \tag{3}$$

where $\Gamma = \gamma_5 = i\gamma_0\gamma_1$ and c is a positive constant.

This massless equivalence is now well-understood. There is a large literature, but for our purposes the best references are [15], [13]. Equivalence has been found on other two dimensional manifolds as well, see for example [1]. The correspondance becomes increasingly complicated with the topology of the manifold. Already on the torus it seems that one must take a circle valued field to make the correspondance.

In any case the massless equivalence suggests the equivalence of the massive free fermions with mass μ and measure

$$\exp\left(\int((\bar{\psi}i\partial\!\!\!/\psi)(x) + \mu(\bar{\psi}\psi)(x))dx\right) d\overline{\psi}d\psi \tag{4}$$

and sine-Gordon bosons at $\beta = 4\pi$ with coupling constant ζ and measure

$$\exp\left(\int(-\frac{1}{2\beta}|\partial\phi(x)|^2 + \zeta : \cos(\phi(x)) :)dx\right) d\phi \tag{5}$$

provided $\zeta = \mu c$. Indeed the two theories are formally equivalent order by order in perturbation theory. The result is remarkable because it is an equivalence between a linear and a non-linear field theory. The sine-Gordon theory is of particular interest because it describes the classical statistical mechanics of a Coulomb gas at temperature β^{-1} and activity $\zeta/2$.

The purpose of this paper is to give a rigorous proof of this widely conjectured equivalence. What has been missing till now is a proof that perturbation theory for the sine-Gordon model converges to the actual model at $\beta = 4\pi$. This involves showing that the model exists and is analytic for ζ in a neighborhood of the origin and explicitly computing the derivatives at $\zeta = 0$. Both steps are non-trivial due to the fact that at this value of β the model requires an infinite energy renormalization.

The sine-Gordon model on the torus for $\beta < 8\pi$ was studied by Dimock and Hurd [12]. (See however the remarks in section 3.1). Here we give a similar, but simpler, treatment on the plane. Working with cutoffs we study correlation functions and prove analyticity in ζ and bounds uniform in the cutoffs for $\beta < 16\pi/3$. Then for $\beta = 4\pi$ we prove existence and analyticity of the theory with no cutoffs, and compute the derivatives at $\zeta = 0$. A parallel and much more elementary analysis is given for fermions. Comparing derivatives at $\zeta = \mu = 0$ we conclude that correlation functions at non-coinciding points are the same in the two models.

The following qualifying remarks need to be made. First, although the free action is taken on the whole plane, the interaction/mass term is confined to a finite volume. Taking the infinite volume limit becomes a separate issue which is not addressed here. Secondly we only show equality between correlation functions for $: \sin\phi :$ and $\overline{\psi}\Gamma\psi$. The ordinary correlation functions for $: \cos\phi :$ and $\overline{\psi}\psi$ do not exist. (Truncated functions are a different story).

An earlier result is due to Fröhlich and Seiler [15] who consider the sine-Gordon theory for $\beta < 4\pi$. In this case no energy renormalization is required and the convergence of perturbation theory is simpler. They prove equivalence in perturbation theory with a more complicated Thirring-Schwinger model.

Our use of the free massless action for the whole plane introduces some special infrared divergences not present or not important for compact manifolds like the torus, and also not present for the massive sine-Gordon model (Yukawa gas). The massive sine-Gordon model is studied in [14],[3], [2], [18], [9]. We particularly note the paper of Brydges and Kennedy [9], where the analyticity of the renormalized partition function is established for $\beta < 16\pi/3$ by a precursor of the method used in the present paper.

The organization of the paper is as follows. In section 2.1 we review some material about exponential functions of the free boson field, and prove a new result about truncated correlation functions. In section 2.2 we prove several forms of the equivalence with the free fermi field. In section 2.3 we define the sine-Gordon theory in a box, quote basic existence and analyticity results for $\beta < 16\pi/3$ and then prove the equivalence at $\beta = 4\pi$ with massive free fermions in a box. In section 3 we use a renormalization group technique to prove the facts about sine-Gordon needed in section 2.

2 Boson-fermion equivalence

2.1 Exponentials of the free boson field

Our treatment follows Fröhlich-Seiler [15] , and Fröhlich- Marchetti [13], but with some innovations (lemmas 2, 5).

The massless boson field theory on $\mathbf{R}^2$ is defined as the family of Gaussian random variables $\phi(x)$ with covariance which is the inverse Laplacian. We start with a regularized version of this adding a mass r to remove the infrared difficulties and an ultraviolet cutoff at L^N (for fixed large L) to get a more regular theory. Thus consider the operator $v_{r,N}$ whose kernel is

$$v_{r,N}(x-y) = (2\pi)^{-2} \int \frac{e^{ip(x-y)}}{(p^2+r^2)} e^{-p^2 L^{-2N}} dp \tag{6}$$

and consider the Gaussian random variables $\phi(x)$ with covariance $\beta v_{r,N}$. We can realize it concretely by taking the Gaussian measure $\mu_{\beta,r,N}$ on $\mathcal{S}'(\mathbf{R^2})$ with covariance $\beta v_{r,N}$ Because of the smoothness of the kernel the measure is actually supported on smooth functions and we take $\phi(x)$ to be evaluation at x. We have the expectation:

$$\begin{aligned} < \exp(i \sum_i s_i \phi(x_i)) >_{\beta,r,N} &= \int \exp(i \sum_i s_i \phi(x_i)) d\mu_{\beta,r,N}(\phi) \\ &= \exp\left(-\frac{1}{2} \sum_{i,j} s_i s_j \beta v_{r,N}(x_i - x_j) \right) \end{aligned} \tag{7}$$

We are interested in taking the limits $N \to \infty$ and $r \to 0$. To get something non-trivial we need to make some adjustments. First define Wick-ordered exponentials by

$$: e^{i\phi(x)} :_{r,N} = \exp(\beta v_{r,N}(0)/2)\ e^{i\phi(x)} \tag{8}$$

(β dependence is supressed from the notation here). These normalize the exponential so that

$$<: e^{i\phi(x)} :_{r,N} >_{\beta,r,N} = 1 \tag{9}$$

We consider the case where there is a missmatch between the Wick ordering, taken to be $: e^{i\phi(x)} :_{1,N}$, and the measure $\mu_{\beta,r,N}$.

Lemma 1 *For non-coinciding points $x_1, ..., x_n, y_1, ..., y_m$ the limit*

$$< \prod_{i=1}^{n} : e^{i\phi(x_i)} : \prod_{j=1}^{m} : e^{-i\phi(y_j)} :>_\beta$$

$$= \lim_{r\to 0} \lim_{N\to\infty} < \prod_{i=1}^{n} : e^{i\phi(x_i)} :_{1,N} \prod_{j=1}^{m} : e^{i\phi(y_j)} :_{1,N} >_{\beta,r,N} \tag{10}$$

exists. The limit is zero if $n \neq m$, and if $n = m$ is given by

$$c_0^{-n-m} \left(\frac{\prod_{1\le i<i'\le n} |x_i - x_{i'}| \prod_{1\le j<j'\le m} |y_j - y_{j'}|}{\prod_{i=1}^{n} \prod_{j=1}^{m} |x_i - y_j|} \right)^{\beta/2\pi} \tag{11}$$

for some positive constant c_0.

Proof. The expectation with $r > 0, N < \infty$ is evaluated as

$$\exp\left(\frac{\beta}{2}\left[v_{1,N}(0)(n+m) - \sum_{i,i'} v_{r,N}(x_i - x_i') - \sum_{j,j'} v_{r,N}(y_j - y_j') + 2\sum_{i,j} v_{r,N}(x_i - y_j)\right]\right) \tag{12}$$

The terms $i = i'$ and $j = j'$ contribute $-v_{r,N}(0)(n+m)$ to the bracket. combining this with the first term and using

$$\lim_{N\to\infty} (v_{1,N}(0) - v_{r,N}(0)) = (2\pi)^{-1} \log r \tag{13}$$

we get a contribution $(r^{\beta/4\pi})^{(n+m)}$ as $N \to \infty$. We also have for $x \neq 0$ the existence of

$$v_r(x) = \lim_{N\to\infty} v_{r,N}(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{-\sqrt{p^2+r^2}|x|}}{2\sqrt{p^2+r^2}} dp \tag{14}$$

Thus we find that the $N \to \infty$ limit of the expectation is

$$(r^{\beta/4\pi})^{(n+m)} \exp\left(\beta\left[-\sum_{i<i'} v_r(x_i - x_i') - \sum_{j<j'} v_r(y_j - y_j') + \sum_{i,j} v_r(x_i - y_j)\right]\right) \tag{15}$$

Now as $r \to 0$ we have

$$v_r(x) = v_1(rx) = -(2\pi)^{-1}\log(r|x|) + \gamma + h(rx) \tag{16}$$

where $h(x)$ is bounded on compact subsets of $\mathbf{R}^2$ and goes to 0 as $x \to 0$. With $c_0 = e^{-\gamma}$ we find that our expression can be written

$$(r^{\beta/4\pi})^{(n-m)^2} c_0^{-n-m+(n-m)^2} \left(\frac{\prod_{1\leq i<i'\leq n} |x_i - x_{i'}| \prod_{1\leq j<j'\leq m} |y_j - y_{j'}|}{\prod_{i=1}^n \prod_{j=1}^m |x_i - y_j|} \right)^{\beta/2\pi} H(rx, ry) \tag{17}$$

where $H(x,y)$ is bounded on compact subsets of $\mathbf{R}^{2(n+m)}$ and goes to one as $(x,y) \to 0$. The $r \to 0$ limit gives 0 if $n \neq m$ and the stated result if $n = m$.

Remark. It is useful to introduce complex variables $z_i = x_i^1 + ix_i^0$ and $w_j = y_j^1 + iy_j^0$. Then $|x_i - y_j| = |z_i - w_j|$ and so forth. Then use the Cauchy lemma (see for example [16]) to obtain

$$\frac{\prod_{1\leq i<i'\leq n}(z_i - z_j) \prod_{1\leq j<j'\leq n}(w_i - w_j)}{\prod_{i=1}^n \prod_{j=1}^n (z_i - w_j)} = \det\{(z_i - w_j)^{-1}\} \tag{18}$$

Now we can write our result for $n = m$ as

$$c_0^{2n} < \prod_{i=1}^n : e^{i\phi(x_i)} : \prod_{j=1}^n : e^{-i\phi(y_j)} :>_\beta = |\det\{(z_i - w_j)^{-1}\}|^{\beta/2\pi} \tag{19}$$

This is useful for estimates and essential for the identification with fermions.

More generally we have the following result which is valid for $n \neq m$. Note that when $n = 0$ it is the standard expansion of the Vandermode determinant.

Lemma 2 *For $n \leq m$*

$$\frac{\prod_{1\leq i<i'\leq n}(z_i - z_{i'}) \prod_{1\leq j<j'\leq m}(w_j - w_{j'})}{\prod_{i=1}^n \prod_{j=1}^m (z_i - w_j)} = \det\{a_{ij}\} \tag{20}$$

where $\{a_{ij}\}$ is the $m \times m$ matrix with entries

$$a_{ij} = \begin{cases} (z_i - w_j)^{-1} & 1 \leq i \leq n \\ (w_j)^{m-i} & n < i \leq m \end{cases} \tag{21}$$

Proof. The proof is by induction on n. If $n = m$ it is the Cauchy formula. We assume it is true for n and prove it for $n-1$. On the left side of the identity for n the factor z_n occurs $n-1$ times in the numerator and m times in the denominator. If we let $k = m - n + 1$ multiply by z_n^k and then let $z_n \to \infty$ we get the left side of the identity for $n-1$. We have to check that the same operation applied to the right side for n gives the right side for $n-1$.

The first $n-1$ rows of the determinant are independent of z_n and expanding out these rows we see that the problem is reduced to an evaluation of the limit of certain determinants of $k \times k$ matrices. Relabeling the w_j's the required limit is

$$\lim_{z\to\infty} z^k \det \begin{pmatrix} (z-w_1)^{-1} & ... & (z-w_k)^{-1} \\ w_1^{k-2} & ... & w_k^{k-2} \\ ... & ... & ... \\ w_1 & ... & w_k \\ 1 & ... & 1 \end{pmatrix} = \det \begin{pmatrix} w_1^{k-1} & ... & w_k^{k-1} \\ ... & ... & ... \\ w_1 & ... & w_k \\ 1 & ... & 1 \end{pmatrix} \tag{22}$$

But the first row on the left can be expanded as

$$z^{-1}(1,...,1) + z^{-2}(w_1,...,w_k) + ... + z^{-k}(w_1^{k-1},...,w_k^{k-1}) + \mathcal{O}(z^{-k-1}) \tag{23}$$

The first $k-1$ terms do not contribute to the determinant, the k^{th} term gives the answer, and the remainder goes to zero.

Next we investigate the integrability of these correlation functions.

Lemma 3 *For $\beta < 4\pi$ the expectation*

$$< \prod_{i=1}^{n} : e^{i\phi(x_i)} : \prod_{j=1}^{m} : e^{-i\phi(y_j)} :>_\beta$$

is locally integrable.

Proof. First consider the case $n = m$ and $2\pi \leq \beta \leq 4\pi$. Then

$$\begin{aligned} |\det\{(z_i - w_j)^{-1}\}|^{\beta/2\pi} &\leq (\sum_\pi \prod_{i=1}^{n} |z_i - w_{\pi(i)}|^{-1})^{\beta/2\pi} \\ &\leq \text{const} \sum_\pi \prod_{i=1}^{n} |z_i - w_{\pi(i)}|^{-\beta/2\pi} \end{aligned} \tag{24}$$

where the sum is over permutations π of $(1,...,n)$ and in the last step we have used the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ valid for $p \geq 1$. This shows the local integrability. If $\beta < 2\pi$ take out the region where the determinant is less than one and then increase β to 2π to get an integrable function. If $n \neq m$ we use (2) to get the result by a similar argument.

Now define

$$: e^{i\phi} :_{1,N} (f) = \int : e^{i\phi(x)} :_{1,N} f(x)dx \tag{25}$$

where the test functions f are taken to be bounded measurable functions of compact support.

Lemma 4 *For $\beta < 4\pi$ and test functions f_i, g_j the limit*

$$< \prod_{i=1}^{n} : e^{i\phi} : (f_i) \ \prod_{j=1}^{m} : e^{-i\phi} : (g_j) >_{\beta}$$

$$= \lim_{r\to 0} \lim_{N\to\infty} < \prod_{i=1}^{n} : e^{i\phi} :_{1,N} (f_i) \ \prod_{j=1}^{m} : e^{-i\phi} :_{1,N} (g_j) >_{\beta,r,N} \tag{26}$$

exists and is equal to

$$= \int < \prod_{i=1}^{n} : e^{i\phi(x_i)} : \ \prod_{j=1}^{m} : e^{-i\phi(y_j)} :>_{\beta} \prod_i f_i(x_i) \prod_j g_j(y_j) dx dy \tag{27}$$

Proof. With the cutoffs in place it is easy to change the order of intergration and obtain

$$< \prod_{i=1}^{n} : e^{i\phi} :_{1,N} (f_i) \ \prod_{j=1}^{m} : e^{-i\phi} :_{1,N} (g_j) >_{\beta,r,N}$$

$$= \int < \prod_{i=1}^{n} : e^{i\phi(x_i)} :_{1,N} \ \prod_{j=1}^{m} : e^{-i\phi(y_j)} :_{1,N}>_{\beta,r,N} \prod_i f_i(x_i) \prod_j g_j(y_j) dx dy \tag{28}$$

Thus to prove the result one has to show that the right side of this equation converges to the same thing without cutoffs. However it is difficult to find a dominating function. since one does not exactly have a determinant representation for $N < \infty$. The actual proof for the $N \to \infty$ limit requires some trickery, see Fröhlich [14], Theorem 3.7., for the details. The proof is given for $n = m$ and does use the determinant representation (18) at $N = \infty$. It also works for $n \neq m$ if we use the representation (2) at $N = \infty$.

The $r \to 0$ limit is easier. One uses (17), (18) and dominated convergence .

We also need to discuss truncated correlation functions . For these we will be able to integrate over coinciding points even at $\beta = 4\pi$.

Truncated correlations are defined for any random variables $\{\chi_i\}$ and any expectation $< ... >$ (including our singular limits) inductively by

$$< \prod_{i=1}^{n} \chi_i >^T = < \prod_{i=1}^{n} \chi_i > - \sum_{\Pi} \prod_j < \prod_{i\in\Pi_j} \chi_i >^T \tag{29}$$

where the sum is over the proper partitions $\Pi = \{\Pi_j\}$ of $(1, ..., n)$. The truncated functions are sums of products of ordinary correlation functions.

For $\beta < 4\pi$ we have by the previous lemma that

$$< \prod_{i=1}^{n} : e^{i\phi} : (f_i) \ \prod_{j=1}^{m} : e^{-i\phi} : (g_j) >_{\beta}^T$$

$$= \int < \prod_{i=1}^{n} : e^{i\phi(x_i)} : \ \prod_{j=1}^{m} : e^{-i\phi(y_j)} :>_{\beta}^T \prod_i f_i(x_i) \prod_j g_j(y_j) dx dy \tag{30}$$

Note that these expectations are zero unless $n = m$. We want to take the limit $\beta \nearrow 4\pi$

Lemma 5 *Except for $n = m = 1$*

$$< \prod_{i=1}^{n} : e^{i\phi} : (f_i) \prod_{j=1}^{m} : e^{-i\phi} : (g_j) >^T_{4\pi} = \lim_{\beta \nearrow 4\pi} < \prod_{i=1}^{n} : e^{i\phi} : (f_i) \prod_{j=1}^{m} : e^{-i\phi} : (g_j) >^T_{\beta} \quad (31)$$

exists and is equal to

$$\int < \prod_{i=1}^{n} : e^{i\phi(x_i)} : \prod_{j=1}^{m} : e^{-i\phi(y_j)} :>^T_{4\pi} \prod_i f_i(x_i) \prod_j g_j(y_j) dx dy \quad (32)$$

Proof. It suffices to take $n = m \geq 2$. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ and define

$$\mathcal{G}_\beta(x, y) = < \prod_{i=1}^{n} : e^{i\phi(x_i)} : \prod_{j=1}^{n} : e^{-i\phi(y_j)} :>_\beta \quad (33)$$

and let $\mathcal{G}^T_\beta(x, y)$ be the truncated function. We claim that for each n there is a locally integrable function $h(x, y)$ such that

$$|\mathcal{G}^T_\beta(x, y)| \leq h(x, y) \quad (34)$$

for all $\beta \leq 4\pi$.

Then the integral in (32) exists. Furthermore we can take the limit $\beta \nearrow 4\pi$ in (30) by the dominated convergence theorem and obtain (31).

The proof of (34) is by induction on n. We suppose the result is true for all smaller n (and make no assumption in case n=2). It suffices to find a dominating function in the sector

$$|x_1 - y_1| \leq |x_i - y_j| \quad (35)$$

We write $x = (x_1, \tilde{x})$ and have

$$\mathcal{G}^T_\beta(x, y) = \mathcal{G}_\beta(x, y) - \mathcal{G}_\beta(x_1, y_1)\mathcal{G}_\beta(\tilde{x}, \tilde{y}) - \sum_{\{(\Pi_j, \Pi_{j'})\}}{}' \prod_i \mathcal{G}^T_\beta(x_{\Pi_j}, y_{\Pi_{j'}}) \quad (36)$$

where the sum is over partitions $\{(\Pi_j, \Pi_{j'})\}$ of two copies of $(1, 2, ...n)$ except that partitions containing $(1, 1)$ are excluded.

The first two terms in (36) can be written as a constant times

$$|\det(z_i - w_j)^{-1}|^{\beta/2\pi} - |(z_1 - w_1)^{-1} \det(\tilde{z}_i - \tilde{w}_j)^{-1}|^{\beta/2\pi} \quad (37)$$

We use the identity for $\alpha \geq 1$

$$||a|^\alpha - |b|^\alpha| \leq |a - b|\alpha(|a|^{\alpha-1} + |b|^{\alpha-1}) \quad (38)$$

to bound this by a constant times:

$$\begin{aligned} & |\det(z_i - w_j)^{-1} - (z_1 - w_1)^{-1} \det(\tilde{z}_i - \tilde{w}_j)^{-1}| \\ \times \quad & (|\det(z_i - w_j)^{-1}|^{\beta/2\pi-1} + |(z_1 - w_1)^{-1} \det(\tilde{z}_i - \tilde{w}_j)^{-1}|^{\beta/2\pi}) \end{aligned} \quad (39)$$

which in turn is bounded by

$$\left(\sum_{\pi:\pi(1)\neq 1}\prod_i |z_i - w_{\pi(i)}|^{-1}\right)\left(2+|\det(z_i-w_j)^{-1}|+|(z_1-w_1)^{-1}\det(\tilde{z}_i-\tilde{w}_j)^{-1}|\right) \quad (40)$$

This function is locally integrable since the second factor is locally in $L^{2-\epsilon}$ for any $\epsilon > 0$ as in (24) and the first factor is locally $L^{2+\epsilon}$ for ϵ sufficiently small. To see this last statement let B_{2n} be a ball in $\mathbf{R}^{2n}$ and let B'_{2n} be the intersection with the sector (35). For a permutation π with $\pi(1) \neq 1$ the corresponding term in the first factor has an $L^{2+\epsilon}$ norm on B'_{2n} which is

$$\begin{aligned} & \int_{B'_{2n}} |\prod_i |x_i - y_{\pi(i)}|^{-2-\epsilon} dx dy \\ \leq \quad & \text{const} \int_{B_n} |x_1 - y_1|^{-n\epsilon} dx_1 dy_1 ... (dy_{\pi(1)})^{\wedge} ... dy_n \end{aligned} \quad (41)$$

and this is finite if $n\epsilon < 2$

Now consider the last term in (36). Each term in the sum will be a product of some two-point truncated functions (n=m=1), and some truncated functions which are dominated by integrable functions by the inductive hypothesis. The truncated two point functions equal the ordinary two point functions which are a constant times $|x-y|^{-\beta/2\pi}$. Thus we have to dominate a terms of the form

$$\left(\prod_{i\in I} |x_i - y_{\pi(i)}|\right)^{-\beta/2\pi} \quad (42)$$

where I is a subset of $(1, ..., n)$ and π is a permutation with $\pi(1) \neq 1$. This is dominated by one on the set where the product is greater than one and by its value at $\beta = 4\pi$ on the complement. The latter is locally integrable as in (41).

2.2 Massless fermions and equivalence

Now we turn to massless fermions, mostly following [13]. The massless free fermion field theory on the plane is defined by the two point function which is a fundamental solution $S(x-y)$ for the Dirac operator $i\partial\!\!\!/$. Here $\partial\!\!\!/ = \gamma_\mu \partial_\mu$ where γ_μ are the gamma matrices satisfying $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$. We choose a skew adjoint representation in which $\gamma_0 = -i\sigma_2$, $\gamma_1 = -i\sigma_1$ and $\Gamma = i\gamma_0\gamma_1 = \sigma_3$ where the σ_i are the Pauli matrices.

One can write down $S(x-y)$ directly, but we prefer to obtain it as a limit of regularized quantities. We define

$$S_{MN}(x) = (2\pi)^{-2} \int_{-M}^{M} dp_0 \int_{-N}^{N} dp_1 e^{ipx} \frac{p\!\!\!/}{p^2} \quad (43)$$

In the limit $M \to \infty$ one can do the p_0 integral by closing the contour. For $x_0 > 0$ one finds

$$S_N(x) = (2\pi)^{-1} \int_{-N}^{N} e^{(-|p_1|x_0 + ip_1x_1)} \frac{(i\gamma_0|p_1| + \gamma_1 p_1)}{2|p_1|} dp_1 \tag{44}$$

which is a bounded function.

Lemma 6

1. *For $x \neq 0$, $S(x) = \lim_{M,N\to\infty} S_{MN}(x)$ exists, is continuous, and is a fundamental solution for the Dirac operator.*
2. *If $z = ix_0 + x_1$ then for $x \neq 0$*

$$S(x) = \lim_{N\to\infty} S_N(x) = (2\pi)^{-1} \begin{pmatrix} 0 & \bar{z}^{-1} \\ z^{-1} & 0 \end{pmatrix} \tag{45}$$

3. *Uniformly in N:*

$$|S_N(x)| \leq \mathcal{O}(1)|x|^{-1} \tag{46}$$

Proof. The first part follows by integrating by parts twice. This argument also shows that one can take the limit in either order. Thus for $x_0 > 0$

$$S(x) = \lim_{N\to\infty} S_N(x) = (2\pi)^{-1} \int \frac{e^{(-|p_1|x_0 + ip_1x_1)}}{2|p_1|} (i\gamma_0|p_1| + \gamma_1 p_1) dp_1 \tag{47}$$

Then one can compute matrix elements

$$\begin{aligned} S_{21}(x) &= -i(2\pi)^{-1} \int \frac{e^{(-|p_1|x_0 + ip_1x_1)}}{2|p_1|} (|p_1| + p_1) dp_1 \\ &= -i(2\pi)^{-1} \int_0^\infty e^{ip_1 z} dp_1 \\ &= (2\pi z)^{-1} \end{aligned} \tag{48}$$

Similarly $S_{12}(x) = (2\pi\bar{z})^{-1}, S_{11}(x) = 0, S_{22}(x) = 0$ The same formulas hold for $x_0 < 0$, and by continuity for all $x \neq 0$

With a cutoff we have instead

$$(S_N)_{21}(x) = -i(2\pi)^{-1} \int_0^N e^{ip_1 z} dp_1 = \mathcal{O}(|z|^{-1}) \tag{49}$$

This completes the proof.

The regularized theory is defined by a fermionic functional integral with respect to the Gaussian measure with covariance S_N. We have for correlation functions

$$\begin{aligned} < \prod_{i=1}^n \psi_{\alpha_i}(x_i) \overline{\psi}_{\beta_i}(y_i) >_N &= \int \prod_{i=1}^n \psi_{\alpha_i}(x_i) \overline{\psi}_{\beta_i}(y_i) d\mu_{S_N} \\ &= \det\{(S_N)_{\alpha_i\beta_i}(x_i - y_j)\} \end{aligned} \tag{50}$$

with all other correlation functions equal to zero. For non-coinciding points we can take the limit $N \to \infty$ and obtain

$$< \prod_{i=1}^{n} \psi_{\alpha_i}(x_i)\overline{\psi}_{\beta_i}(y_i) >= \det\{S_{\alpha_i\beta_i}(x_i - y_j)\} \tag{51}$$

We introduce chiral fermions b, c by

$$\begin{aligned} b &= (1+\Gamma)\psi/2 = \psi_1 \\ \bar{c} &= (1-\Gamma)\psi/2 = \psi_2 \end{aligned} \tag{52}$$

Then our results so far can be characterized by the statement that we have a free fermion field theory with two point functions

$$\begin{aligned} < \bar{b}(x)\bar{c}(y) > &= (2\pi\bar{z})^{-1} \\ < c(x)b(y) > &= (2\pi z)^{-1} \end{aligned} \tag{53}$$

and all others equal to zero.

Lemma 7 *For non-coinciding points*

$$< \prod_{i=1}^{n} \bar{b}b(x_i) \ \prod_{j=1}^{m} c\bar{c}(y_j) >= \left\{ \begin{array}{cc} 0 & n \neq m \\ (2\pi)^{-2n}|\det\{\frac{1}{z_i - w_j}\}|^2 & n = m \end{array} \right\} \tag{54}$$

Remark. Since $< \bar{b}(x)b(x) >_N = 0$ we can allow $\bar{b}, b$ to be taken at the same point and still have an $N \to \infty$ limit. No Wick ordering is required.

Proof. The expectation can be explicitly computed by Wicks theorem. We have two independent field theories (namely (b, c) and $(\bar{b}, \bar{c})$) and they give complex conjugate expectations whence

$$< \prod_{i=1}^{n} \bar{b}b(x_i) \ \prod_{j=1}^{m} c\bar{c}(y_j) >= | < \prod_{i=1}^{n} b(x_i) \ \prod_{j=1}^{m} c(y_j) > |^2 \tag{55}$$

If $n \neq m$ the expectation is zero, while if $n = m$ we have

$$< \prod_{i=1}^{n} b(x_i) \ \prod_{j=1}^{n} c(y_j) >= (2\pi)^{-n} \det\{\frac{1}{z_i - w_j}\} \tag{56}$$

This completes the proof

Now by comparing (19) and (54) we have:

Theorem 1 *(massless equivalence). For non coinciding points and $\beta = 4\pi$*

$$\begin{aligned}
& c_0^{n+m} < \prod_{i=1}^{n} : \exp(i\phi(x_i)) : \prod_{j=1}^{m} : \exp(-i\phi(y_j)) :>_\beta \\
= \quad & (2\pi)^{n+m} < \prod_{i=1}^{n} : \bar{b}b : (x_i) \prod_{j=1}^{m} : c\bar{c} : (y_j) > \qquad (57)
\end{aligned}$$

This shows that the massless free theories are equivalent if we take $c = c_0/\pi$ and make the identification

$$\begin{aligned}
\frac{c}{2} : \exp(i\phi) : \quad & \Longleftrightarrow \quad \bar{b}b \\
\frac{c}{2} : \exp(-i\phi) : \quad & \Longleftrightarrow \quad c\bar{c} \qquad (58)
\end{aligned}$$

We can also write this as

$$\begin{aligned}
c : \cos(\phi) : \quad & \Longleftrightarrow \quad \overline{\psi}\psi = \bar{b}b + c\bar{c} \\
ic : \sin(\phi) : \quad & \Longleftrightarrow \quad \overline{\psi}\Gamma\psi = \bar{b}b - c\bar{c} \qquad (59)
\end{aligned}$$

Next we smear with test functions, still taken to be bounded measurable functions with compact support. We study scalars formally given by

$$(\overline{\psi}\psi)(f) = \int (\overline{\psi}\psi)(x) f(x) dx \qquad (60)$$

These are not defined directly but by their expectations given by

$$< (\overline{\psi}\psi)(f_1)...(\overline{\psi}\psi)(f_n) >_N = \int < (\overline{\psi}\psi)(x_1)...(\overline{\psi}\psi)(x_n) >_N f(x_1)...f(x_n) dx \qquad (61)$$

Similarly we define expectations with pseudo-scalars $\overline{\psi}\Gamma\psi$

We cannot take the limit $N \to \infty$ here unless the test functions have non-overlapping supports. However for truncated correlation functions like

$$< (\overline{\psi}\psi)(f_1)...(\overline{\psi}\psi)(f_n) >_N^T = \int < (\overline{\psi}\psi)(x_1)...(\overline{\psi}\psi)(x_n) >_N^T f(x_1)...f(x_n) dx \qquad (62)$$

(with $\overline{\psi}\psi$ treated as a unit) we can take the limit.

Lemma 8 *For $n + m \geq 3$*

$$< \prod_{i=1}^{n} (\overline{\psi}\psi)(x_i) \prod_{j=1}^{m} (\overline{\psi}\Gamma\psi)(y_j) >^T \qquad (63)$$

is locally integrable and

$$<\prod_{i=1}^{n}(\overline{\psi}\psi)(f_i)\prod_{j=1}^{m}(\overline{\psi}\Gamma\psi)(g_j)>^T=\lim_{N\to\infty}<\prod_{i=1}^{n}(\overline{\psi}\psi)(f_i)\prod_{j=1}^{m}\overline{\psi}\Gamma\psi(g_j)>_N^T \tag{64}$$

exists and equals

$$\int<\prod_{i=1}^{n}(\overline{\psi}\psi)(x_i)\prod_{j=1}^{m}(\overline{\psi}\Gamma\psi)(y_j)>^T\prod_{i=1}^{n}f_i(x_i)\prod_{j=1}^{m}g_j(y_j)dxdy \tag{65}$$

Proof. Suppose that $m=0$ and $n\geq 3$. The general case would just involve inserting $\Gamma's$ at appropriate places. It suffices to bound $<\overline{\psi}\psi(x_1)...\overline{\psi}\psi(x_n)>_N^T$ by a locally integrable function independent of N. The the integrability follows and one can use dominated convergence to take the limit in (62).

Truncated correlations for free field theories are given as sums over connected Feymann diagrams. In this case the only connected diagram is a circle and we have

$$<(\overline{\psi}\psi)(x_1)...(\overline{\psi}\psi)(x_n)>_N^T=\sum_{\pi}tr(S_N(x_1-x_{\pi(1)})...S_N(x_{\pi^n(1)}-x_1)) \tag{66}$$

where the sum is over the cyclic permutations of $(1,2,...,n)$. Each term has the form

$$tr(S_N(x_1-x_2)S_N(x_2-x_3)...S_N(x_n-x_1)) \tag{67}$$

On any ball B use lemma 6 to bound this by

$$h(x_1,...x_n)=\mathcal{O}(1)g(x_1-x_2)g(x_2-x_3)...g(x_n-x_1) \tag{68}$$

where

$$g(x)=\chi_{2B}(x)|x|^{-1} \tag{69}$$

Then by Young's inequality

$$\begin{aligned}\int_B h &\leq |B|\|g*g*...*g\|_\infty\\ &\leq |B|\|g\|_{3/2}\|g*...*g\|_3\\ &\leq |B|\|g\|_{3/2}^2\|g*...*g\|_{3/2}\\ &\leq |B|\|g\|_{3/2}^3\|g*...*g\|_1\\ &\leq |B|\|g\|_{3/2}^3\|g\|_1^{n-3}\end{aligned} \tag{70}$$

and this is finite since $g\in L_p$ for $p<2$.

Remark. Note that since the number of permutations in bounded by $n!$, it follows from our proof that for any f there are constants A,B such that

$$<(\overline{\psi}\psi)(f)^n>_N^T\leq AB^n n! \tag{71}$$

for all N.

Theorem 2 *For $n+m \geq 3$*

$$< \prod_{i=1}^{n} (\overline{\psi}\psi)(f_i) \prod_{j=1}^{m} (\overline{\psi}\Gamma\psi)(g_j) >^T = (c)^n (ic)^m < \prod_{i=1}^{n} :\cos\phi: (f_i) \prod_{j=1}^{m} :\sin\phi: (g_j) >^T_{4\pi} \tag{72}$$

Proof. The pointwise identity follows from theorem 1. The correlations smeared with test functions are given as integrals over the pointwise correlation functions by lemma 5 and lemma 8. Hence the result.

2.3 The main result

2.3.1 The sine-Gordon model

We consider the the sine-Gordon model on the plane with an interaction density supported on a square Λ. The model is defined by the interaction

$$\zeta :\cos\phi:_{1,N} (\chi_\Lambda) = \zeta \int_\Lambda :\cos\phi(x):_{1,N} dx \tag{73}$$

By scaling we can always arrange that $\Lambda = \Delta =$ a unit square, and we make this choice for the rest of the paper.

Define normalized expectations by

$$< ... >_{\zeta,\beta,r,N} = \frac{\int [...] \exp(\zeta :\cos\phi:_{1,N} (\chi_\Delta)) d\mu_{\beta,r,N}(\phi)}{\int \exp(\zeta :\cos\phi:_{1,N} (\chi_\Delta)) d\mu_{\beta,r,N}(\phi)} \tag{74}$$

We study expectations of products of the fields $\sin\phi$.

The main technical result is:

Theorem 3 *Let $f_1, ..., f_n$ be test functions in $\mathcal{C}_0^\infty(\mathbf{R}^2)$ with disjoint supports in Δ. Then for all r, N and for β in a compact subset of $(0, 16\pi/3)$ the functions*

$$\mathcal{W}_{\beta,r,N}(\zeta; f_1, ..., f_n) = \zeta^n < \prod_{i=1}^{n} :\sin\phi:_{1,N} (f_i) >_{\zeta,\beta,r,N} \tag{75}$$

are analytic in ζ and uniformly bounded on a fixed a neighborhood of the origin. The same is true for the truncated functions $\mathcal{W}^T_{\beta,r,N}(\zeta; f_1, ..., f_n)$.

The proof is given in subsequent sections and involves taking some shortcuts. This accounts for the presence of the ζ^n which is not really necessary. Assuming the theorem we can prove:

Lemma 9 *Under the hypotheses of the theorem,*

$$\mathcal{W}(\zeta; f_1, ..., f_n) = \zeta^n < \prod_{i=1}^{n} : \sin\phi : (f_i) :>_{\zeta,4\pi} \tag{76}$$

defined by

$$\mathcal{W}(\zeta; f_1, ..., f_n) = \lim_{\beta\nearrow 4\pi} \lim_{r\to 0} \lim_{N\to\infty} \mathcal{W}_{\beta,r,N}(\zeta; f_1, ..., f_n) \tag{77}$$

exists and is analytic in a neighborhood of the origin. The same is true for truncated functions

$$\mathcal{W}^T(\zeta; f_1, ..., f_n) = \zeta^n < \prod_{i=1}^{n} : \sin\phi : (f_i) :>^T_{\zeta,4\pi} \tag{78}$$

and for these we have the derivatives at zero

$$\begin{aligned}&(\mathcal{W}^T)^{(k)}(0; f_1, ..., f_n) = \\ &\begin{cases} 0 & k < n \\ \frac{k!}{(k-n)!} < \prod_{i=1}^{n} : \sin\phi : (f_i)(: \cos\phi : (\chi_\Delta))^{k-n} >^T_{4\pi} & k \geq n \end{cases}\end{aligned} \tag{79}$$

Remark. We are only asserting that the $N \to \infty, r \to 0$ limits exist for $\beta < 4\pi$ and then defining the theory for $\beta = 4\pi$ as the limit from below. In fact the $N \to \infty, r \to 0$ limits no doubt exist for all $\beta < 16\pi/3$ and give the same answer at $\beta = 4\pi$. This should follow from our subsequent analysis as in [12]

Proof. It suffices to prove that the limit exists for the truncated functions. By a standard computation we have the derivatives

$$\begin{aligned}&(\mathcal{W}^T)^{(k)}_{\beta,r,N}(0; f_1, ..., f_n) = \\ &\begin{cases} 0 & k < n \\ \frac{k!}{(k-n)!} < \prod_{i=1}^{n} : \sin\phi :_{1,N} (f_i)(: \cos\phi :_{1,N} (\chi_\Delta))^{k-n} >^T_{\beta,r,N} & k \geq n \end{cases}\end{aligned} \tag{80}$$

By the theorem we have Cauchy bounds $|(\mathcal{W}^T)^{(k)}_{\beta,r,N}(0)| \leq AB^k k!$ uniformly in β, r, N.

The derivatives also have a limits as $N \to \infty$ given by (79). For $n + k \geq 3$ this follows by lemma 4 and and lemma 5. For $n = 1, k - n = 0$ and $n = 1, k - n = 1$ the limit vanishes, in the latter case by $\phi \to -\phi$ symmetry. For $n = 2, k - n = 0$ the limit exists by the pointwise limit and the assumption of non-overlapping supports.

Thus in in the Taylor series expansion

$$(\mathcal{W}^T)_{\beta,r,N}(\zeta; f_1, ..., f_n) = \sum_{k=0}^{\infty} (\mathcal{W}^T)^{(k)}_{\beta,r,N}(0; f_1, ..., f_n)\zeta^k/k! \tag{81}$$

we can take the various limits and all the claimed results follow.

2.3.2 Massive fermions

Now consider massive fermions. We again start with a regularized theory defining

$$< ... >_{\mu,N} = \frac{< [...] e^{\mu(\overline{\psi}\psi)(\chi_\Lambda)} >_N}{< e^{\mu(\overline{\psi}\psi)(\chi_\Lambda)} >_N} \tag{82}$$

Again by scaling it suffices to consider $\Lambda = \Delta =$ a unit square.

We study particularly expectations of products of $\overline{\psi}\psi$ and $\psi\Gamma\psi$. With the cutoff in place an expression like

$$< \prod_{i=1}^{n} (\overline{\psi}\psi)(f_i) e^{\mu(\overline{\psi}\psi)(\chi_\Delta)} >_N$$

can be defined by a power series in μ. The coefficient of $\mu^k/k!$ is an $(n+k)\times(n+k)$ determinant with bounded entries. For fixed n it can be estimated by $\mathcal{O}(1)^k k!$ and hence the series converges for $|\mu|$ small. (Actually it converges for all μ). These considerations show that $< \prod_{i=1}^{n} (\overline{\psi}\psi)(f_i) >_{\mu,N}$ is well-defined and analytic for for $|\mu|$ small. The same is true with the fields $\overline{\psi}\Gamma\psi$.

Lemma 10 *For test functions with disjoint supports the limit*

$$< \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_\mu = \lim_{N\to\infty} < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_{\mu,N} \tag{83}$$

exists and is analytic in μ in a neighborhood of the origin. The same is true for the truncated functions and these satisfy:

$$\frac{d^k}{d\mu^k} < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_\mu^T |_{\mu=0} = < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i)((\overline{\psi}\psi)(\chi_\Delta))^k :>^T \tag{84}$$

Proof. It suffices to consider the truncated functions. The derivatives at zero can be computed as

$$\frac{d^k}{d\mu^k} < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_{\mu,N}^T |_{\mu=0} = < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i)((\overline{\psi}\psi)(\chi_\Delta))^k >_N^T \tag{85}$$

The derivatives have bounds of the form $AB^k k!$ uniformly in N by estimates like (71).

The derivatives have a limits as $N\to\infty$ given by (84). For $n+k\geq 3$ this follows by lemma 8. For $n=1, k=0$ and $n=1, k=1$ the expression vanishes, in the latter case by reflection invariance. For $n=2, k=0$ the limit exists by the assumption of non-overlapping supports.

Now in in the Taylor series expansion for $< \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_\mu$ we can take the limit $N\to\infty$ and all the results follow.

2.3.3 Massive equivalence

Our main result is the following

Theorem 4 *Let $f_1, ...f_n$ be test functions with disjoint support in Δ. If $\zeta = \mu c$ is sufficiently small*

$$(ic)^n < \prod_{i=1}^{n} : \sin\phi : (f_i) :>_{\zeta,4\pi} = < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >_{\mu} \tag{86}$$

Proof. It suffices to show that the truncated functions are equal. If $\zeta = \mu c = 0$ the result follow from theorem 2. If $\zeta = \mu c \neq 0$ it suffices to establish:

$$(i\zeta)^n < \prod_{i=1}^{n} : \sin\phi : (f_i) :>^T_{\zeta,4\pi} = \mu^n < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i) >^T_{\mu} \tag{87}$$

Both sides are now analytic functions of μ on a neighborhood of the origin. To establish the identity it suffices to show that the derivatives at zero agree. The derivatives are computed in lemma 9 and lemma 10. To prove the identity we need that

$$(ic)^n c^k < \prod_{i=1}^{n} : \sin\phi : (f_i)(: \cos\phi : (\chi_\Delta))^k >^T_{4\pi} = < \prod_{i=1}^{n} (\overline{\psi}\Gamma\psi)(f_i)((\overline{\psi}\psi)(\chi_\Delta))^k >^T \tag{88}$$

But this follows from theorem 2. This completes the proof of equivalence.

Remark. If we try to include terms like $: \cos\phi :$ and $\overline{\psi}\psi$ in the statement of the theorem we run into the difficulty that divergent terms like $< (\overline{\psi}\psi)(f)(\overline{\psi}\psi)(\chi_\Delta) >_N$ occur in the perturbation expansion.

3 The sine-Gordon model

3.1 Overview

We now study the sine-Gordon model on the plane with an interaction density supported on a square. The analysis uses the renormalization group, and is based on an earlier treatment of the sine-Gordon model due to Dimock and Hurd [11], [12]. Unfortunately at the time of this writing there are some problems with the proofs in [11], [12], and it is not yet clear whether they can be resolved. Thus we do not follow the proof too closely. Instead we adopt a simpler strategy in which the leading contributions in perturbation theory play a prominent role. This makes it possible to treat remainder terms more crudely. In particular we do not make any extractions from the remainders, which is problematic for dimensionless fields. The closer one is to 8π the more orders of pertubation theory one has to take to render the remainder irrelevant. For this paper we only go to second order perturbation theory , and the proof works for $\beta < 16\pi/3$ which is sufficient for our purposes. Possibly something similar will work all the way up to $\beta = 8\pi$.

The present model also differs from from [12] in that the base manifold is the plane and not the torus, and because there are now genuine infrared divergences in the problem (the $r \to 0$ limit).

Our version of the renormalization group follows the general framework of Brydges and Yau [8]. We also adopt some recent innovations in this technique due to Brydges, Dimock, and Hurd [5], [6], [7]. In particular we use open polymers and background potentials.

3.2 The Renormalization Group

The model is defined in section 2.3.1. Consider expressions of the form

$$\begin{aligned} \Xi_{r,N}(\zeta,\phi) &= \mu_{\beta,r,N} * e^{\tilde{V}_N(\zeta,\Lambda)}(\phi) = \int e^{\tilde{V}_N(\zeta,\Lambda,\phi+\phi')} d\mu_{\beta,r,N}(\phi') \\ \tilde{V}_N(\zeta,\Lambda,\phi) &= \zeta \int_\Lambda :\cos(\phi(x)):_{1,N} dx \end{aligned} \tag{89}$$

and $\mu_{\beta,r,N}$ has covariance $\beta v_{r,N}$ defined in (6).

Then $\Xi_{r,N}(\zeta,0)$ is the partition function, and derivatives of $\Xi_{r,N}(\zeta,\phi)$ yield correlations functions. We want to obtain control over $\Xi_{r,N}$ uniformly r, N as $N \to \infty$ and $r \to 0$.

Scaling up by a factor L^N we find

$$\begin{aligned} \Xi_{r,N}(\zeta,\phi) &= (\mu_{\beta,r_N,0} * e^{V(\zeta_N,\Lambda_N)})(\phi(\cdot/L^N)) \\ V(\zeta,X,\phi) &= \zeta \int_X \cos\phi(x) dx \end{aligned} \tag{90}$$

where

$$\begin{aligned} \Lambda_N &= L^N \Lambda \\ \zeta_N &= L^{-2N} e^{\beta v_{1,N}(0)/2} \zeta \\ r_N &= L^{-N} r \end{aligned} \tag{91}$$

Note that the Wick ordering constant $\exp(\beta v_{1,N}(0)/2)$ has been absorbed into the coupling constant.

The measure $\mu_{\beta,r_N,0}$ has the covariance $\beta v_{r_N,0}$ and it is convenient to write this as $\beta v_{r_N,0} = \beta' \hat{v}_N$ where

$$\begin{aligned} \beta' &= e^{r_N^2} \beta \\ \hat{v}_N(x) &= e^{-r_N^2} v_{r_N,0}(x) \\ &= (2\pi)^{-2} \int \frac{e^{ipx}}{p^2 + r_N^2} e^{-(p^2+r_N^2)} dp \end{aligned} \tag{92}$$

We seek a sequence of densities $\mathcal{Z}_i^{r,N}(\phi)$ for $i = N, N-1, ..., 1, 0$ localized in Λ_i such that

$$\Xi_{r,N}(\phi) = (\mu_{\beta'\hat{v}_i} * \mathcal{Z}_i^{r,N})(\phi(\cdot/L^i)) \tag{93}$$

Initially this holds with $\mathcal{Z}_N^{r,N} = e^{V(\zeta_N,\Lambda_N)}$. Given such a density for i we define the density for $i-1$ as follows. First define C_i by

$$C_i(x) = \hat{v}_i(x) - \hat{v}_{i-1}(x/L) \tag{94}$$

whence

$$\mu_{\beta'\hat{v}_i} = \mu_{\beta'\hat{v}_{i-1}(\cdot/L)} * \mu_{\beta' C_i} \tag{95}$$

Then scaling down by L changes the first factor to $\mu_{\beta'\hat{v}_{i-1}}$ and gives the expression for $i-1$ with

$$(\mathcal{Z}_{i-1}^{r,N})(\phi) = (\mu_{\beta' C_i} * \mathcal{Z}_i^{r,N})(\phi(\cdot/L)) \tag{96}$$

This is the basic renormalization group transformation. Since C_i is only weakly dependent on i the study of $\mathcal{Z}_i^{r,N}$ is essentially the study of iterations of this mapping. Following the flow down to $i = 0$ gives $\mathcal{Z}_0^{r,N}$, and for this functional it will be possible to analyze the r, N dependence.

3.3 Polymer expansions

Control over the densities $\mathcal{Z}_i^{r,N}$ depends on keeping careful track of the localization of the functional. This is accomplished by polymer expansions. We only sketch the formalism here, referring to [8], [4], [5] for more details

A (open) polymer X is a union of open unit squares centered on points of the integer lattice in $\mathbf{R}^2$. A polymer activity is a function $K(X,\phi)$ depending on polymers X and fields ϕ with the property that the dependence on ϕ is localized in X. One can define a product on polymer activities and also an exponential function $\mathcal{E}xp$. If $\Box$ is the characteristic function of unit blocks, then we have for an additive polymer activity V and a general polymer activity K

$$\mathcal{E}xp(\Box e^V + K)(X,\phi) = \sum_{\{X_i\}} \prod_i K(X_i,\phi) e^{V(X \setminus \cup_i X_i, \phi)} \tag{97}$$

where the sum is over collections of disjoint polymers $\{X_i\}$ in X. The densities will all be represented in this form. In particular the intial density has this form with $K = 0$:

$$\mathcal{Z}_N^{r,N}(\phi) = e^{V(\zeta_N,\Lambda_N,\phi)} = \mathcal{E}xp(\Box e^{V(\zeta_N)})(\Lambda_N,\phi) \tag{98}$$

It is useful to keep track of the dependence on fields and their derivatives separately. Thus we consider modified polymer activities $K(X,\psi)$ defined on pairs $\psi = (\psi_0,\psi_1)$ consisting of a continuous function ψ_0 and a continuous vector field ψ_1. A basic polymer activity $K^0(X,\phi)$ is represented as a modified polymer activity $K(X,\psi)$ if $K(X,\psi_\phi) =$

$K^0(X,\phi)$ where $\psi_\phi = (\phi, \partial\phi)$. Modified polymer activities K, K' are equivalent, written $K \approx K'$, if $K(X,\psi_\phi) = K'(X,\psi_\phi)$.

Norms are defined on the modified polymer activities. Let $K_n(X,\psi)$ be the n^{th} derivative of $K(X,\psi)$ with respect to ψ. This is a multilinear functional on pairs $f = (f_0, f_1)$ of continuous functions, and we assume that restricted to components it is given by by a measure of bounded variation supported on X. We let $\|K_n(X,\psi)\|$ denote total variation norm (with a sum over components) and let $\|K_n(X,\psi)\|_{\mathbf{\Delta}}$ denote this norm restricted to a product of unit blocks $\mathbf{\Delta} = \Delta_1 \times \ldots \times \Delta_n$.

Next we define a large field regulator

$$G(\kappa, X, \phi) = \exp\left(\kappa \sum_{1 \leq |\alpha| \leq 3} \int_X |\partial^\alpha \phi|^2\right) \tag{99}$$

with constant $0 \leq \kappa \leq 1$. Then define

$$\|K_n(X)\|_G = \sum_{\mathbf{\Delta}} \sup_{\phi \in \mathcal{C}^3} \|K_n(X,\psi_\phi)\|_{\mathbf{\Delta}} G(X,\phi)^{-1}. \tag{100}$$

Dependence on the set X is controlled by a large set regulator $\Gamma(X)$ which is such that $\Gamma(X)^{-1}$ has polynomial tree decay in X. We define

$$\|K_n\|_{G,\Gamma} = \sup_{\Delta} \sum_{X \supset \Delta} \Gamma(X) \|K_n(X)\|_G \tag{101}$$

Finally, for $h > 0$ we define

$$\|K\|_{G,\Gamma,h} = \sum_n (h^n/n!)\, \|K_n\|_{G,\Gamma} \tag{102}$$

Our polymer expansions will always involve modified polymer activities $K(X,\psi)$ for which this norm is finite. Such activities are analytic in ψ in a strip of width h.

Sometimes we do that last two sums in the opposite order and call the norm $\|K\|_{G,h,\Gamma}$. If K is translation invariant then one can drop the supremum over Δ in (101) and so $\|K\|_{G,h,\Gamma} = \|K\|_{G,\Gamma,h}$. Our K 's are generally translation invariant, although they appear in expressions like $\mathcal{E}xp(\Box e^V + K)(\Lambda, \psi_\phi)$ which are not.

Another variation is that sometimes we replace Γ by

$$\Gamma_n(X) = 2^{n|X|}\Gamma(X) \tag{103}$$

3.4 The RG flow to second order

As a preliminary we study some aspects of the RG flow to second order in ζ. Suppose that after some number of steps we have we have basic polymer activities of the form $K = Qe^V + \mathcal{O}(\zeta^3)$ where $V = \mathcal{O}(\zeta), Q = \mathcal{O}(\zeta^2)$. Then our density has the form

$$\mathcal{E}xp((\Box + Q)e^V)(\Lambda, \phi) + \mathcal{O}(\zeta^3) \tag{104}$$

We want to find new densities Q_1, V_1 and more generally continuous functions $Q_t, V_t, t \in [0,1]$ such that $Q_0 = Q, V_0 = V$ and such that for $\beta = \beta', C = C_i$

$$\mu_{t\beta C} * \mathcal{E}xp((\square + Q)e^{V}) = \mathcal{E}xp((\square + Q_t)e^{V_t}) + \mathcal{O}(\zeta^3) \tag{105}$$

In general if one defines A_t by

$$\mu_{t\beta C} * \mathcal{E}xp(A) = \mathcal{E}xp(A_t) \tag{106}$$

then one finds the A_t satisfies the differential equation

$$E(A_t) \equiv (\frac{\partial}{\partial t} - \Delta_{\beta C})A_t - \mathcal{B}_{\beta C}(A_t, A_t) = 0 \tag{107}$$

where

$$\begin{aligned} \Delta_C A(\phi) &= \frac{1}{2}\int d\mu_C(\zeta)\ A_2(\phi;\zeta,\zeta) \\ \mathcal{B}_C(A,B)(\phi) &= \frac{1}{2}\int d\mu_C(\zeta)\ A_1(\phi;\zeta)\circ B_1(\phi;\zeta) \end{aligned} \tag{108}$$

and where $A_n(\phi)$ is the n^{th} derivative of $A(\phi)$. Futhermore approximate solutions to (107) give approximate solutions to (106). This is the approach we take. Starting with $A = (\square + Q)e^{V}$ we ask for $A_t = (\square + Q_t)e^{V_t}$ so that $E(A_t) = \mathcal{O}(\zeta^3)$.

Associated with $V(\zeta) = V(\zeta, X, \phi)$ defined in (90), we define polymer activities $\mathcal{Q}^0(\zeta, W)$ for $W = (W_1, W_2)$ and $|X| \leq 2$ by

$$\begin{aligned} \mathcal{Q}^0(\zeta, W; X, \phi) &= \frac{\zeta^2}{4}\int_{\tilde{X}} \cos(\phi(x)+\phi(y))W_1(x-y)dxdy \\ &+ \frac{\zeta^2}{4}\int_{\tilde{X}} \cos(\phi(x)-\phi(y))W_2(x-y)dxdy \end{aligned} \tag{109}$$

Here for $X = \Delta \cup \Delta'$ we define

$$\tilde{X} = \Delta \times \Delta' \cup \Delta' \times \Delta \tag{110}$$

If $|X| \geq 3$ then $\mathcal{Q}^0(\zeta, W; X, \phi) = 0$

Lemma 11 *Let $V_t = V(\zeta_t)$ and $Q_t^0 = \mathcal{Q}^0(\zeta_t, W_t)$. Then $A_t = (\square + Q_t^0)e^{V_t}$ satisfies $E(A_t) = \mathcal{O}(\zeta^3)$ provided*

$$\begin{aligned} \zeta_t &= e^{-t\beta C(0)/2}\zeta \\ W_{1,t}(x) + 1 &= e^{-t\beta C(x)}(W_{1,0}(x) + 1) \\ W_{2,t}(x) + 1 &= e^{+t\beta C(x)}(W_{2,0}(x) + 1) \end{aligned} \tag{111}$$

Proof. For the first order terms in E to vanish we need

$$\frac{\partial V_t}{\partial t} - \Delta_{\beta C} V_t = 0 \tag{112}$$

This follows by our choice of ζ_t. For the second order terms in E to vanish it turns out we need:

$$(\frac{\partial}{\partial t} - \Delta_{\beta C}) Q_t^0 - J_t = 0 \tag{113}$$

where for $|X| \leq 2$

$$\begin{aligned} J_t(X,\phi) &= \frac{1}{2}\int_{\tilde{X}} \beta C(x-y)(V_t)_1(\phi,x)(V_t)_1(\phi,y)dxdy \\ &= -\frac{\zeta_t^2}{8}\sum_{p,q=\pm 1}\int_{\tilde{X}} e^{ip\phi(x)} pq\beta C(x-y) e^{iq\phi(y)}dxdy \end{aligned} \tag{114}$$

If Q_t^0 has the general form

$$Q_t^0(X,\phi) = \frac{\zeta_t^2}{8}\sum_{p,q=\pm 1}\int_{\tilde{X}} e^{ip\phi(x)} W_{pq,t}(x-y) e^{iq\phi(y)}dxdy \tag{115}$$

Then for (113) to vanish the kernel must satisfy

$$(\frac{\partial}{\partial t} + pq\beta C(x))W_{pq,t}(x) = -pq\beta C(x) \tag{116}$$

which can also be written

$$(\frac{\partial}{\partial t} + pq\beta C(x))(W_{pq,t}(x) + 1) = 0 \tag{117}$$

This has the solution

$$W_{pq,t}(x) + 1 = e^{-tpq\beta C(x)}(W_{pq,0}(x) + 1) \tag{118}$$

The theorem is the special case $W_{1,1} = W_{-1,-1}$ and $W_{1,-1} = W_{-1,1}$. If it is true at $t=0$ then it is true for all t. Denoting the first pair by W_1 and the second pair by W_2 we have $Q_t^0 = \mathcal{Q}^0(\zeta_t, W_t)$ and (118) becomes (111).

Now we modify our definition of $\mathcal{Q}$ by defining $\mathcal{Q}(\zeta, W; X, \psi)$. We say that a polymer with $|X| \leq 2$ is a small set if $\bar{X}$ is connected. Otherwise it is a large set. We define

$$\mathcal{Q}(\zeta, W; X, \psi) = \mathcal{Q}_1(\zeta, W_1; X, \psi) + \mathcal{Q}_2(\zeta, W_2; X, \psi) \tag{119}$$

where

$$\begin{aligned} \mathcal{Q}_1(\zeta, W; X, \psi) &= \frac{\zeta^2}{4}\int_{\tilde{X}} \cos\left(\psi_0(x) + \psi_0(y)\right) W(x-y)dxdy \\ \mathcal{Q}_2(\zeta, W; X, \psi) &= \begin{cases} \frac{\zeta^2}{4}\int_{\tilde{X}} \cos\left(\psi_0(x) - \psi_0(y)\right) W(x-y)dxdy & \text{X large} \\ \frac{\zeta^2}{4}\int_{\tilde{X}} \cos\left(\int_y^x \psi_1\right) W(x-y)dxdy & \text{X small} \end{cases} \end{aligned} \tag{120}$$

and the integral $\int_y^x$ is over the shortest path in $\bar{X}$. We recover our earlier definition when we restrict:

$$\mathcal{Q}(\zeta, W; X, \psi_\phi) = \mathcal{Q}^0(\zeta, W; X, \phi) \tag{121}$$

Now define $Q_t(X, \psi) = \mathcal{Q}(\zeta_t, W_t; X, \psi)$. We do not have (113) for all ψ, but (113) does imply

$$\left((\frac{\partial}{\partial t} - \Delta_{\beta C}) Q_t - J_t \right) (X, \psi_\phi) = 0 \tag{122}$$

The result follows from the identity $(\Delta_{\beta C} Q_t)(X, \psi_\phi) = (\Delta_{\beta C} Q_t^0)(X, \phi)$ where $\Delta_{\beta C}$ is the Laplacian in ψ; see the Appendix.

3.5 Estimates

In estimates we make the convention that constants denoted $\mathcal{O}(1)$ are independent of L, while constants denoted C may depend on L. The value of C may vary from line to line.

We start with some estimates on $C_i(x)$ which is given by

$$\begin{aligned} C_i(x) &= (2\pi)^{-2} \int \frac{e^{ipx}}{p^2 + r_i^2} (e^{-(p^2+r_i^2)} - e^{-L^2(p^2+r_i^2)}) dp \\ &= (2\pi)^{-2} \int e^{ipx} \int_1^{L^2} e^{-\alpha(p^2+r_i^2)} d\alpha dp \\ &= (4\pi)^{-1} \int_1^{L^2} \alpha^{-1} e^{-x^2/4\alpha} e^{-\alpha r_i^2} d\alpha \end{aligned} \tag{123}$$

Then we have the following uniform bounds:

Lemma 12 *For any multi-index $\beta \neq 0$*

$$\begin{aligned} 0 < C_i(x) &\leq \mathcal{O}(\log L) e^{-|x|/L} \\ |\partial^\beta C_i(x)| &\leq \mathcal{O}(1) e^{-|x|/2L} \end{aligned} \tag{124}$$

Proof. For $\alpha \leq L^2$ we have

$$\frac{|x|^2}{4\alpha} \geq \frac{|x|}{\sqrt{\alpha}} - 1 \geq \frac{|x|}{L} - 1 \tag{125}$$

This gives the first bound. The second bound is similar. We always get at least one extra power of α^{-1} so the α integral has a bound independent of L.

Next we study the evolution of the kernel $W = (W_1, W_2)$ under the transformation (111) at $t = 1$, with $\beta = \beta'$, followed by a scaling operation. Starting with $W_N = (W_{1,N}, W_{2,N})$ define inductively

$$\begin{aligned} W_{1,i-1}(x) + 1 &= e^{-\beta' C_i(Lx)} (W_{1,i}(Lx) + 1) \\ W_{2,i-1}(x) + 1 &= e^{+\beta' C_i(Lx)} (W_{2,i}(Lx) + 1) \end{aligned} \tag{126}$$

This is equivalent to

$$\begin{aligned} W_{1,i}(x)+1 &= e^{-\beta' w_i(x)} \\ W_{2,i}(x)+1 &= e^{+\beta' w_i(x)} \end{aligned} \tag{127}$$

where

$$\begin{aligned} w_i(x) &= \sum_{k=i+1}^{N} C_k(L^{k-i}x) \\ &= (2\pi)^{-2}\int \frac{e^{ipx}}{p^2+r_i^2}\left(e^{-(p^2+r_i^2)L^{-2(N-i)}}-e^{-(p^2+r_i^2)}\right)dp \\ &= (4\pi)^{-1}\int_{L^{-2(N-i)}}^{1} \alpha^{-1}e^{-x^2/4\alpha}e^{-\alpha r_i^2}d\alpha \end{aligned} \tag{128}$$

Lemma 13

1. *for* $|x|\leq 1$

$$\begin{aligned} 0<w_i(x) &\leq \frac{1}{2\pi}\log|x|^{-1}+\mathcal{O}(1) \\ |W_{1,i}(x)|,\ |x|^{\beta'/2\pi}|W_{2,i}(x)| &\leq \mathcal{O}(1) \end{aligned} \tag{129}$$

2. *for* $|x|\geq 1$

$$\begin{aligned} 0<w_i(x) &\leq \mathcal{O}(1)e^{-|x|/2} \\ |W_{1,i}(x)|,\ |W_{2,i}(x)| &\leq \mathcal{O}(1)e^{-|x|/2} \end{aligned} \tag{130}$$

Proof. For $|x|\leq 1$

$$\begin{aligned} w_i(x) &\leq (4\pi)^{-1}\int_0^1 \alpha^{-1}e^{-x^2/4\alpha}d\alpha \\ &= (4\pi)^{-1}\int_0^{|x|^{-2}} \alpha^{-1}e^{-1/4\alpha}d\alpha \\ &\leq \mathcal{O}(1)+(4\pi)^{-1}\int_1^{|x|^{-2}}\alpha^{-1}d\alpha \\ &= \mathcal{O}(1)+\frac{1}{2\pi}\log|x|^{-1} \end{aligned} \tag{131}$$

This gives the first bound on w_i and the bounds on $W_{1,i}, W_{2,i}$ follow.

For $|x|\geq 1$ and $\alpha\leq 1$ we have

$$\frac{|x|^2}{4\alpha}\geq\frac{|x|}{\sqrt{\alpha}}-1\geq\frac{|x|}{2}+\frac{1}{2\sqrt{\alpha}}-1 \tag{132}$$

This gives the second bound on w_i and the bounds on $W_{1,i}, W_{2,i}$ follow.

Now we turn to some bounds on polymer activities. . We start with bounds on $V(X)=V(\zeta,X,\phi)$ defined in (90). The following hold for any $G=G(\kappa,X,\phi)$.

Lemma 14 *For a unit block Δ:*

$$\|V(\Delta)\|_{G,h} \leq |\zeta| e^h \tag{133}$$

Proof. The n^{th} derivative of $V(\Delta, \phi)$ with respect to ϕ is the multilinear function on continuous $f_j = (f_j, 0)$ given by

$$V_n(\Delta, \phi; f_1, ...f_n) = \zeta \int_\Delta \cos^{(n)}(\phi(x)) f_1(x)...f_n(x) dx \tag{134}$$

Then $V_n(\Delta, \phi)$ is given by a measure satisfying

$$\|V_n(\Delta, \phi)\| \leq |\zeta| \tag{135}$$

Similarly $\|V_n(\zeta, \Delta)\|_G \leq |\zeta|$. Multiplying by $h^n/n!$ and summing over n gives the result.

Lemma 15 *For $|\zeta| e^{4h}$ sufficiently small*

$$\begin{aligned} \|e^{V(\Delta)}\|_{G,h} &\leq 2 \\ \|e^{V(\Delta)} - 1\|_{G,h} &\leq 2|\zeta| e^h \\ \|e^{V(X)}\|_{G,h} &\leq 2^{|X|} \end{aligned} \tag{136}$$

Proof. First suppose that $X = \Delta$. Then after some combinatorics (see [5]) one finds that

$$\frac{h^n}{n!} \|(e^{V(\Delta,\phi)})_n\| \leq \exp(\sum_{n=0}^{\infty} \frac{h^n}{n!} \|V_n(\Delta, \phi)\|) \leq \exp(|\zeta| e^h) \tag{137}$$

Similarly

$$\frac{h^n}{n!} \|(e^{V(\Delta)})_n\|_G \leq \exp(|\zeta| e^h) \tag{138}$$

Now replace h by $4h$, multiply by 4^{-n}, and sum over n to obtain the first bound for $|\zeta| e^{4h}$ small. The second bound follows from the first and the representation

$$e^{V(\Delta)} - 1 = \int_0^1 V(\Delta) e^{sV(\Delta)} ds \tag{139}$$

The last bound follows by

$$\|e^{V(X)}\|_{G,h} \leq \prod_{\Delta \subset X} \|e^{V(\Delta)}\|_{G,h} \tag{140}$$

Lemma 16 *Let $|\zeta| e^{4h}$ be sufficiently small, and let Q be a translation invariant polymer activity. Then*

$$\|Q e^V\|_{G,\Gamma,h} \leq \|Q\|_{G,\Gamma_1,h} \tag{141}$$

Proof. Using lemma 15 we have

$$\|Q(X)e^{V(X)}\|_{G,h} \leq \|Q(X)\|_{G,h}\|e^{V(X)}\|_{1,h} \leq 2^{|X|}\|Q(X)\|_{G,h} \tag{142}$$

Then multiplying by $\Gamma(X)$ and summing gives the result with norm $\|\cdot\|_{G,h,\Gamma} = \|\cdot\|_{G,\Gamma,h}$

Next we give some bounds on $\mathcal{Q}(\zeta, W)$. For good bounds one would like W_1, W_2 to be locally integrable and exponentially decaying. Refering to lemma 13 we see that this is true for W_1, but not for W_2 if $\beta' \geq 4\pi$. Thus we only assume the weaker bounds

$$\begin{aligned}
\int_{\Delta\times\Delta'} |W_1(x-y)|dxdy &\leq Ce^{-d(\Delta,\Delta')/2L} \\
\int_{\Delta\times\Delta'} |x-y||W_2(x-y)|dxdy &\leq Ce^{-d(\Delta,\Delta')/2L}
\end{aligned} \tag{143}$$

Note that $W_{1,i}, W_{2,i}$ satisfy this, even with stronger decay $e^{-d(\Delta,\Delta')/2}$. For $W_{2,i}$ we need that $|x|^{1-\beta'/2\pi}$ is locally integrable and this is true for $\beta' < 6\pi$.

For a good uniform bound on $\mathcal{Q}_2(\zeta, W_2)$ we subtract the most singular part which is the constant

$$\begin{aligned}
\mathcal{Q}_{sing}(\zeta, W_2; X, \psi) &= \mathcal{Q}_2(\zeta, W_2; X, 0) \\
&= \frac{\zeta^2}{4}\int_{\tilde{X}} W_2(x-y)dxdy
\end{aligned} \tag{144}$$

For the difference the bound (143) will be adequate.

Lemma 17 *Let W_1, W_2 satisfy (143). Then for some constant C:*

$$\begin{aligned}
\|\mathcal{Q}_1(\zeta, W_1)\|_{G,\Gamma,h} &\leq C|\zeta|^2 e^{\mathcal{O}(1)h} \\
\|\mathcal{Q}_2(\zeta, W_2) - \mathcal{Q}_{sing}(\zeta, W_2)\|_{G,\Gamma,h} &\leq C|\zeta|^2 e^{\mathcal{O}(1)h}\kappa^{-1/2} \\
\|\mathcal{Q}(\zeta, W) - \mathcal{Q}_{sing}(\zeta, W_2)\|_{G,\Gamma,h} &\leq C|\zeta|^2 e^{\mathcal{O}(1)h}\kappa^{-1/2}
\end{aligned} \tag{145}$$

Proof. The last follows from the first two. For the first bound we compute for $f_j = (f_j, 0)$

$$\begin{aligned}
&(\mathcal{Q}_1)_n(X, \psi_\phi; f_1, ...f_n) \\
&= \frac{\zeta^2}{4}\int_{\tilde{X}} \cos^{(n)}(\phi(x)+\phi(y))W_1(x-y)\prod_{j=1}^n (f_j(x)+f_j(y))dxdy
\end{aligned} \tag{146}$$

This leads to

$$\begin{aligned}
\|(\mathcal{Q}_1)_n(\Delta\cup\Delta', \psi_\phi)\| &\leq \mathcal{O}(1)^n|\zeta|^2\int_{\Delta\times\Delta'} |W_1(x-y)|dxdy \\
&\leq C\mathcal{O}(1)^n|\zeta|^2 e^{-d(\Delta,\Delta')/2L}
\end{aligned} \tag{147}$$

The norm restricted to $\mathbf{\Delta}$ has the same bound, and since there are 2^n terms in the sum over $\mathbf{\Delta}$ it follows that

$$\|(\mathcal{Q}_1)_n(\Delta \cup \Delta')\|_G \leq C\mathcal{O}(1)^n|\zeta|^2 e^{-d(\Delta,\Delta')/2L} \tag{148}$$

Now multiply by $\Gamma(X)$ and sum over X, then multiply by $h^n/n!$ and sum over n, and get the result.

The second bound proceeds in just the same way for large sets. We avoid the singularity at $x = y$ by the large set hypothesis and the subtraction has the same bound as the function.

For small sets X we compute for continuous vector-valued $f_j = (0, f_j)$

$$\begin{aligned}
&(\mathcal{Q}_2 - \mathcal{Q}_{sing})_n(X, \psi_\phi; f_1, ...f_n) \\
&= \frac{\zeta^2}{4}\int_{\tilde{X}}(\cos^{(n)}(\int_y^x \partial\phi) - \delta_{n,0})W_2(x-y)\prod_{j=1}^n(\int_y^x f_j)dxdy
\end{aligned} \tag{149}$$

It follows that

$$\begin{aligned}
&\|(\mathcal{Q}_2 - \mathcal{Q}_{sing})_n(X, \psi_\phi)\| \\
&\leq \mathcal{O}(1)|\zeta|^2\int_{\tilde{X}}|(\cos^{(n)}(\int_y^x \partial\phi) - \delta_{n,0})||W_2(x-y)||x-y|^n dxdy
\end{aligned} \tag{150}$$

For $n = 0$ a Sobolev inequality gives

$$|\cos(\int_y^x \partial\phi) - 1| \leq |x-y|\sup_{x\in X}|\partial\phi(x)| \leq |x-y|\kappa^{-1/2}G(\kappa, X, \phi) \tag{151}$$

Thus we always have a factor $|x - y|$, and so

$$\begin{aligned}
&\|(\mathcal{Q}_2 - \mathcal{Q}_{sing})_n(\Delta \cup \Delta', \psi_\phi)\| \\
&\leq \mathcal{O}(1)^n|\zeta|^2\kappa^{-1/2}G(\kappa, X, \phi)\int_{\Delta\times\Delta'}|x-y||W_2(x-y)|dxdy \\
&\leq C\mathcal{O}(1)^n|\zeta|^2\kappa^{-1/2}G(\kappa, X, \phi)
\end{aligned} \tag{152}$$

The restriction to $\mathbf{\Delta}$ has the same bound. Now multiply by G^{-1}, taking the supremum over ϕ, and sum over 2^n terms $\mathbf{\Delta}$ to obtain

$$\|(\mathcal{Q}_2 - \mathcal{Q}_{sing})_n(\Delta \cup \Delta')\|_G \leq C\mathcal{O}(1)^n|\zeta|^2\kappa^{-1/2} \tag{153}$$

The result now follows as before.

3.6 The full RG flow

We now study the full RG flow. The building blocks are the polymer activities

$$\begin{aligned}
V_i^{r,N} &= V(\zeta_i^{r,N}) \\
Q_i^{r,N} &= \mathcal{Q}(\zeta_i^{r,N}, W_i) - \mathcal{Q}_{sing}(\zeta_i^{r,N}, W_{2,i})
\end{aligned} \tag{154}$$

where

$$\zeta_i^{r,N} = L^{-2i} \exp(\frac{\beta}{2}(v_{1,N}(0) - v_{r,N}(0)) + \frac{\beta'}{2}\hat{v}_i(0))\zeta \tag{155}$$

Note that initially $\zeta_N^{r,N} = \zeta_N$. Also note using $\beta'\hat{v}_i(0) \leq \beta v_{r_i,0}(0)$ and $v_{r,N}(0) = v_{r_N,0}(0)$ and

$$v_{r,0}(0) = -\log r/2\pi + \mathcal{O}(1) \tag{156}$$

we have that

$$\begin{aligned} |\zeta_i^{r,N}| &\leq \mathcal{O}(1)|\zeta|_i \\ |\zeta|_i &= L^{-(2-\beta/4\pi)i}|\zeta| \end{aligned} \tag{157}$$

We define

$$h_i = 3 - \sum_{j=i}^{\infty} 2^{-j} \tag{158}$$

so that $1 \leq h_i \leq 3$. We also define $G_i(X,\phi) = G(\kappa_i, X, \phi)$ where

$$\kappa_i = \kappa_0 2^{-i} \tag{159}$$

for some constant κ_0.

For the main theorem we have the following restrictions on the parameter values. First fix $\beta_0 < 16\pi/3$ and allow $\beta \leq \beta_0$. We assume r_N is small enough so $\beta' = e^{r_N^2}\beta \leq \beta_1 < 16\pi/3$. Then choose ϵ sufficiently small (depending on β_0), L sufficiently large (depending on β_0, ϵ), κ_0 sufficiently small (depending on β_0, ϵ, L), and finally $|\zeta|$ sufficiently small (depending on $\beta_0, \epsilon, L, \kappa_0$).

By the previous lemmas we have

$$\begin{aligned} \|\Box e^{V_i^{r,N}}\|_{G_i,\Gamma,h_i} &\leq C \\ \|\Box(e^{V_i^{r,N}} - 1)\|_{G_i,\Gamma,h_i} &\leq C|\zeta|_i \leq |\zeta|_i^{1-\epsilon} \\ \|Q_i^{r,N} e^{V_i^{r,N}}\|_{G_i,\Gamma,h_i} &\leq C|\zeta|_i^2 \kappa_i^{-1/2} \leq |\zeta|_i^{2-\epsilon} \end{aligned} \tag{160}$$

Here we have used that that $L^{-\epsilon(2-\beta/4\pi)i}2^{i/2} \leq 1$ for L sufficiently large, and that $C|\zeta|^{\epsilon}\kappa_0^{-1/2} \leq 1$ for $|\zeta|$ sufficiently small.

Theorem 5 *For $\beta < 16\pi/3$ and $|\zeta|$ sufficiently small as above, there exist polymer activities $K_i^{r,N}(X,\psi)$ and constants $\Omega_i^{r,N}$ such that*

$$\mathcal{Z}_i^{r,N}(\phi) = e^{\Omega_i^{r,N}} \mathcal{E}xp(\Box e^{V_i^{r,N}} + K_i^{r,N})(\Lambda_i, \psi_\phi) \tag{161}$$

The $K_i^{r,N}$ are analytic in ζ and translation invariant. They have the form

$$K_i^{r,N} = Q_i^{r,N} e^{V_i^{r,N}} + R_i^{r,N} \tag{162}$$

where for all β, r, N, i:

$$\|R_i^{r,N}\|_{G_i,\Gamma,h_i} \leq |\zeta|_i^{3-\epsilon} \tag{163}$$

and hence

$$\|K_i^{r,N}\|_{G_i,\Gamma,h_i} \leq 2|\zeta|_i^{2-\epsilon} \tag{164}$$

Remark. The term $\Omega_i^{r,N}$ corresponds to the vacuum energy in second order perturbation theory. It does not have a uniform bound for $\beta \geq 4\pi$

Proof. We suppress r, N from the notation. The proof that Ω_i, R_i exist and that R_i satisfies the bounds is by induction on i, starting with $i = N$ and working down to $i = 0$. We assume the result is true for i and prove it for $i-1$. For i=N the result holds with $W_N = 0, Q_N = 0, R_N = 0, \Omega_N = 0$. Each step preserves the analyticity in ζ.

Step 1. For the fluctuation integral we have in general that for any polymer activity M

$$\mu_{tC} * \mathcal{E}xp(\Box + M) = \mathcal{E}xp(\Box + \mathcal{F}_t(M)) \tag{165}$$

where $\mathcal{F}_t(M)$ is a certain analytic function of M. (See the Appendix for more details). For a family $G_t(X, \phi)$, $0 \leq t \leq 1$, of large field regulators satisfying for $t > s$

$$\mu_{(t-s)C} * G_s \leq G_t \tag{166}$$

we have the general bound [8]

$$\|\mathcal{F}_t(M)\|_{G_t,\Gamma,h'} \leq \|M\|_{G_0,\Gamma,h} \tag{167}$$

provided $h' < h$ and

$$\|M\|_{G,\Gamma,h} \leq (h - h')^2/16\|C\|_\theta \tag{168}$$

where $\|C\|_\theta$ is a certain norm on C.

We study

$$\mathcal{Z}_{i-1}(\phi) = e^{\Omega_i}\mu_{\beta' C_i} * \mathcal{E}xp(\Box e^{V_i} + K_i)(\Lambda_i, \psi_{\phi,L^{-1}}) \tag{169}$$

where $\psi_{L^{-1}}(x) = (\psi_0(x/L), L^{-1}\psi_1(x/L))$. Before the scaling the polymer exponential is rewritten

$$\begin{aligned}\mu_{t\beta' C_i} * \mathcal{E}xp(\Box e^{V_i} + K_i)(\Lambda_i, \psi_\phi) &= \mu_{t\beta' C_i} * \mathcal{E}xp(\Box + M_i)(\Lambda_i, \psi_\phi) \\ &= \mathcal{E}xp(\Box + \mathcal{F}_t(M_i))(\Lambda_i, \psi_\phi)\end{aligned} \tag{170}$$

where

$$M_i = \Box(e^{V_i} - 1) + K_i \tag{171}$$

Now define

$$G_t(X, \phi) = (2^{|X|}G_{i-1}(X, \phi))^t G_i(X, \phi)^{1-t} \tag{172}$$

This satisfies (166) as in [6]. We use that $\kappa_{i-1} = 2\kappa_i$ and that κ_0 is sufficiently small in order that $\kappa_i\|\beta' C_i\|$ be sufficiently small for various Sobolev norms. We conclude from (167) that

$$\|\mathcal{F}_t(M_i)\|_{G_t,\Gamma,h_{i-1}} \leq \|M_i\|_{G_i,\Gamma,h_i} \leq 3|\zeta|_i^{1-\epsilon} \tag{173}$$

The condition (168) is satisfied since for L sufficiently large and $|\zeta|$ sufficiently small

$$16\|M_i\|_{G_i,\Gamma,h_i}\|\beta' C_i\|_\theta \leq C|\zeta|_i^{1-\epsilon} \leq 2^{-2i+2} = (h_i - h_{i-1})^2 \tag{174}$$

Next we isolate the second order contribution. Write $M_i = N_i + R_i$ where

$$N_i = \square(e^{V_i} - 1) + Q_i e^{V_i} \tag{175}$$

As the leading term in $\mathcal{F}_t(M_i) = \mathcal{F}_t(N_i + R_i)$ we take

$$N_t = \square(e^{V_t} - 1) + Q_t e^{V_t} \tag{176}$$

where with $\zeta_t = e^{-t\beta' C_i(0)/2}\zeta_i$

$$\begin{aligned} V_t &= V(\zeta_t) \\ Q_t &= \mathcal{Q}(\zeta_t, W_t) - \mathcal{Q}_{sing}(\zeta_i, W_{2,i}) \end{aligned} \tag{177}$$

and W_t is defined by (111) with $\beta = \beta', C = C_i$, and $W_0 = W_i$. This is consistent with the discussion in section 3.4; the subtraction of the constant $\mathcal{Q}_{sing}$ will have no effect to second order.

Now provided that $R_t \approx \mathcal{F}_t(N_i + R_i) - N_t$ we can rewrite (170) as

$$\mathcal{E}xp(\square + R_t + N_t)(\Lambda_i, \psi_\phi) = \mathcal{E}xp(\square e^{V_t} + K_t)(\Lambda_i, \psi_\phi) \tag{178}$$

where $K_t = Q_t e^{V_t} + R_t$

To define R_t we write

$$\begin{aligned} & \mathcal{F}_t(N_i + R_i) - N_t \\ =\ & (\mathcal{F}_t(N_i + R_i) - \mathcal{F}_t(N_i)) + (\mathcal{F}_t(N_i) - N_t) \\ =\ & \int_0^1 (\mathcal{F}_t)_1(N_i + sR_i; R_i)ds - \int_0^t (\mathcal{F}_{t-s})_1 (N_s; E_s)ds \end{aligned} \tag{179}$$

where $(\mathcal{F}_t)_1$ is the derivative of $\mathcal{F}_t$ and $E_t = E(N_t) = E(\square + N_t)$. An explicit computation [5] shows that

$$\begin{aligned} E_t &= \left((\frac{\partial}{\partial t} - \Delta_{\beta' C_i})Q_t - J_t \right) e^{V_t} + \beta' C_i \left((V_t)_1, (V_t)_1 \right) Q_t e^{V_t} \\ &- \beta' C_i \left((Q_t)_1, (V_t)_1 \right) e^{V_t} - 2\mathcal{B}_{\beta' C_i} \left(\square e^{V_t}, Q_t e^{V_t} \right) - \mathcal{B}_{\beta' C_i} \left(Q_t e^{V_t}, Q_t e^{V_t} \right) \end{aligned} \tag{180}$$

In the first term $\mathcal{Q}_{sing}$ does not contribute and so the term is equivalent to zero by (122). Define E'_t to be equal to E_t but with the first term removed so $E'_t \approx E_t$. Now define

$$R_t = \int_0^1 (\mathcal{F}_t)_1(N_i + sR_i; R_i)ds - \int_0^t (\mathcal{F}_{t-s})_1 (N_s; E'_s)ds \tag{181}$$

and we have $R_t \approx \mathcal{F}_t(N_i + R_i) - N_t$ as required (see the Appendix).

Now we give some bounds. First write

$$\begin{aligned} Q_t &= \mathcal{Q}(\zeta_t, W_t) - \mathcal{Q}_{sing}(\zeta_t, W_{2,t}) \\ &+ \mathcal{Q}_{sing}(\zeta_t, W_{2,t}) - \mathcal{Q}_{sing}(\zeta_i, W_{2,i}) \end{aligned} \tag{182}$$

The kernels W_t still satisfy the bounds (143) and we apply lemma 17 with $\kappa = t\kappa_{i-1} + (1-t)\kappa_i$. Just as in the bound on $\mathcal{Q}_i$ we find that

$$\|\mathcal{Q}(\zeta_t, W_t) - \mathcal{Q}_{sing}(\zeta_t, W_{2,t})\|_{G_t,\Gamma,h_i} \leq |\zeta|_i^{2-\epsilon} \tag{183}$$

For the second term we write

$$\begin{aligned} &\mathcal{Q}_{sing}(\zeta_t, W_{2,t}, X) - \mathcal{Q}_{sing}(\zeta_i, W_{2,i}, X) \\ &= \frac{\zeta_i^2}{4}\int_{\tilde{X}} (e^{t\beta'(C_i(x-y)-C_i(0))} - 1)W_{2,i}(x-y)dxdy \\ &+ \frac{\zeta_i^2}{4}\int_{\tilde{X}} e^{-t\beta' C_i(0)}(e^{t\beta' C_i(x-y)} - 1)dxdy \end{aligned} \tag{184}$$

This also satisfies

$$\|\mathcal{Q}_{sing}(\zeta_t, W_{2,t}) - \mathcal{Q}_{sing}(\zeta_i, W_{2,i})\|_{G_t,\Gamma,h_i} \leq |\zeta|_i^{2-\epsilon} \tag{185}$$

Indeed in the first term the $(C_i(x-y) - C_i(0))$ supplies the factor $|x-y|$ needed for a good bound on $W_{2,i}(x-y)$, and in the second term the $C_i(x-y)$ gives the needed exponential decay. We conclude that $\|Q_t\|_{G_t,\Gamma,h_{i-1}} \leq \mathcal{O}(1)|\zeta|_i^{2-\epsilon}$. The same bound holds with Γ_1 so $\|Q_t e^{V_t}\|_{G_t,\Gamma,h_{i-1}} \leq \mathcal{O}(1)|\zeta|_i^{2-\epsilon}$. A similar first order bound holds for N_t.

Estimating $(\mathcal{F}_t)_1$ and $(\mathcal{F}_{t-s})_1$ by Cauchy bounds we have from (181)

$$\|R_t\|_{G_t,\Gamma,h_{i-1}} \leq 2(\|R_i\|_{G,\Gamma,h_i} + \sup_{s\leq t}\|E'_s\|_{G_s,\Gamma,h_i}) \tag{186}$$

All the terms in E'_s are $\mathcal{O}(\zeta^3)$ and by the techniques we have been using one can show that $\|E'_s\|_{G_s,\Gamma,h_i} \leq |\zeta|_i^{3-\epsilon}$. It follows that $\|R_t\|_{G_t,\Gamma,h_{i-1}} \leq \mathcal{O}(1)|\zeta|_i^{3-\epsilon}$

Now let $\zeta^\#, W^\#, V^\#, Q^\#, K^\#, R^\#$ be the values of $\zeta_t, W_t, V_t, Q_t, K_t, R_t$ at $t=1$. Then we have

$$\mathcal{Z}_{i-1}(\phi) = e^{\Omega_i}\mathcal{E}xp(\Box e^{V^\#} + K^\#)(\Lambda_i, \psi_{\phi,L^{-1}}) \tag{187}$$

where $K^\# = Q^\# e^{V^\#} + R^\#$. After switching a factor $2^{-|X|}$ from G_1^{-1} to Γ we have

$$\begin{aligned} \|Q^\# e^{V^\#}\|_{G_{i-1},\Gamma_{-1},h_{i-1}} &\leq \mathcal{O}(1)|\zeta|_i^{2-\epsilon} \\ \|R^\#\|_{G_{i-1},\Gamma_{-1},h_{i-1}} &\leq \mathcal{O}(1)|\zeta|_i^{3-\epsilon} \end{aligned} \tag{188}$$

and hence a bound on $K^\#$.

Step 2. We extract a constant $F_i(X)$ which is the last two terms in $Q^\#$:

$$F_i(X) = \mathcal{Q}_{sing}(\zeta^\#, W^\#, X) - \mathcal{Q}_{sing}(\zeta_i, W_{2,i}, X) \tag{189}$$

¿From (185) we have $\|F_i\|_{G_{i-1},\Gamma_{-1},h_{i-1}} \leq |\zeta|_i^{2-\epsilon}$. The extraction is a rearrangement [5] [1]

$$\mathcal{E}xp(\Box e^{V^\#} + K^\#)(\Lambda_i, \psi) = e^{\delta\Omega_i}\mathcal{E}xp(\Box e^{V^*} + K^*)(\Lambda_i, \psi) \tag{190}$$

[1]In [5], Λ_i was a torus, not a rectangle in the plane. Nevertheless this formula holds with the same $\mathcal{E}$ and the same $\delta\Omega$

where $V^* = V^\#$ and

$$\begin{aligned}
K^* &= \mathcal{E}(K^\#, F_i) \\
&= (K^\# - F_i e^{V^\#}) + \mathcal{E}_{\geq 2}(K^\#, F_i) \\
\delta\Omega_i &= \sum_{X \subset \Lambda_i} F_i(X)
\end{aligned} \tag{191}$$

Now we can write $K^* = Q^* e^{V^*} + R^*$ where

$$\begin{aligned}
Q^* &= \mathcal{Q}(\zeta^\#, W^\#) - \mathcal{Q}_{sing}(\zeta^\#, W_2^\#) \\
R^* &= R^\# + \mathcal{E}_{\geq 2}(K^\#, F_i)
\end{aligned} \tag{192}$$

A general bound on $\mathcal{E}_{\geq 2}$ is

$$\|\mathcal{E}_{\geq 2}(K, F)\|_{G,\Gamma,h} \leq \mathcal{O}(1) \|K\|_{G,\Gamma_1,h} \|F\|_{G,\Gamma,h}) \tag{193}$$

Using this and (188) to bound R^*, and (183) to bound Q^* we have:

$$\begin{aligned}
\|Q^* e^{V^*}\|_{G_{i-1},\Gamma_{-2},h_{i-1}} &\leq \mathcal{O}(1) |\zeta|_i^{2-\epsilon} \\
\|R^*\|_{G_{i-1},\Gamma_{-2},h_{i-1}} &\leq \mathcal{O}(1) |\zeta|_i^{3-\epsilon}
\end{aligned} \tag{194}$$

and hence a bound on K^*.

Now we have

$$\begin{aligned}
\mathcal{Z}_{i-1}(\phi) &= e^{\Omega_{i-1}} \mathcal{E}xp(\Box e^{V^*} + K^*)(\Lambda_i, \psi_{\phi,L^{-1}}) \\
\Omega_{i-1} &= \Omega_i + \delta\Omega_i
\end{aligned} \tag{195}$$

Step 3. Finally we scale down. The scaling transformation is [5]:

$$\mathcal{E}xp(\Box e^{V^*} + K^*))(\Lambda_i, \psi_{\phi,L^{-1}}) = \mathcal{E}xp(\Box e^{V_{i-1}} + K')(\Lambda_{i-1}, \psi_\phi) \tag{196}$$

where $K' = \mathcal{S}(K^*)$ is a certain function of K^*. We have used

$$L^2 \zeta^\# = L^2 e^{-\beta'/2\pi C_i(0)} \zeta_i = \zeta_{i-1} \tag{197}$$

to identify $V^*(LX, \phi(\cdot/L)) = V_{i-1}(X, \phi)$. Now write $\mathcal{S}(K^*) = \mathcal{S}_1(K^*) + \mathcal{S}_{\geq 2}(K^*)$ where $\mathcal{S}_1$ is the linearization. Then we have $K' = \mathcal{S}_1(Q^* e^{V^*}) + R_{i-1}$ where

$$R_{i-1} = \mathcal{S}_1(R^*) + \mathcal{S}_{\geq 2}(K^*) \tag{198}$$

However an explicit calculation using (121), (197), and $W^\#(Lx) = W_{i-1}(x)$ shows that

$$\mathcal{S}_1(Q^* e^{V^*}) \approx Q_{i-1} e^{V_{i-1}} \tag{199}$$

Thus if we define $K_{i-1} = Q_{i-1}e^{V_{i-1}} + R_{i-1}$, then $K_{i-1} \approx K'$, we can replace K' by K_{i-1} in (196), and obtain the required

$$\mathcal{Z}_{i-1}(\phi) = e^{\Omega_{i-1}}\mathcal{E}xp(\Box e^{V_{i-1}} + K_{i-1})(\Lambda_{i-1}, \psi_\phi) \tag{200}$$

A basic $\mathcal{S}$ bound is

$$\|\mathcal{S}(K)\|_{G,\Gamma,h} \leq \mathcal{O}(1)L^2\|K\|_{G,\Gamma_{-2},h} \tag{201}$$

The same bound holds for $\mathcal{S}_1$ and $\mathcal{S}_{\geq 2}$ has a bound proportional to $\|K\|^2$. We conclude for $\beta \leq \beta_0 < 16\pi/3$, ϵ sufficiently small, and L sufficiently large

$$\begin{aligned} \|R_{i-1}\|_{G_{i-1},\Gamma,h_{i-1}} &\leq \mathcal{O}(1)L^2|\zeta|_i^{3-\epsilon} \\ &= \mathcal{O}(1)L^2L^{-(2-\beta/4\pi)(3-\epsilon)}|\zeta|_{i-1}^{3-\epsilon} \\ &\leq |\zeta|_{i-1}^{3-\epsilon} \end{aligned} \tag{202}$$

This completes the proof.

3.7 Correlation functions

We now study the correlation functions, see also Hurd [17]. As in (75) we consider:

$$\mathcal{W}_{\beta,r,N}(\zeta; f_1, ..., f_n) = \zeta^n <: \sin\phi :_{1,N} (f_1) ... : \sin\phi :_{1,N} (f_n) >_{\zeta,\beta,r,N} \tag{203}$$

with test functions $f_i \in \mathcal{C}_0^\infty(\mathbf{R}^2)$ with non-overlapping supports in Δ . These have a nice representation as functional derivatives of $\Xi_{r,N}(\zeta,\phi)$ as defined in (89). Indeed we have

$$(-1)^n\mathcal{W}_{\beta,r,N}(\zeta, f_1, ..., f_n) = \frac{(\Xi_{r,N})_n(\zeta, 0; f_1...., f_n)}{\Xi_{r,N}(\zeta, 0)} \tag{204}$$

Equations (93) and (161) for $i = 0$ give the representation for the $\Xi_{r,N}$. We have

$$\begin{aligned} \Xi_{r,N}(\zeta,\phi) &= e^{\Omega_0^{r,N}}\mu_{\beta'\hat{v}_0} * \mathcal{E}xp(A_0^{r,N})(\Delta, \psi_\phi) \\ &= e^{\Omega_0^{r,N}}\mu_{\beta'\hat{v}_0} * A_0^{r,N}(\Delta, \psi_\phi) \end{aligned} \tag{205}$$

where

$$A_0^{r,N} = \Box e^{V_0^{r,N}} + K_0^{r,N} \tag{206}$$

The divergent constants $\exp(\Omega_0^{r,N})$ cancel in the expression (204) and the functional derivatives can be taken inside the integral to obtain

$$(-1)^n\mathcal{W}_{\beta,r,N}(\zeta, f_1, ..., f_n) = \frac{\int (A_0^{r,N})_n(\Delta, \psi_\phi; \psi_{f_1}, ..., \psi_{f_n})d\mu_{\beta'\hat{v}_0}(\phi)}{\int A_0^{r,N}(\Delta, \psi_\phi)d\mu_{\beta'\hat{v}_0}(\phi)} \tag{207}$$

We need uniform bounds on this function. In prepraration we have

Lemma 18 *For κ_0 sufficiently small $\int G(\kappa_0,\Delta,\phi)d\mu_{\beta'\hat{v}_0}(\phi)$ is bounded by a constant independent of r.*

Proof. Let B be the quadratic form

$$(\phi, B\phi) = \sum_{1\leq|\alpha|\leq 3}\int_\Delta |\partial^\alpha\phi|^2 \tag{208}$$

Then we have

$$\begin{aligned}\int G(\kappa_0,\Delta,\phi)d\mu_{\beta'\hat{v}_0}(\phi) &= \int e^{\kappa_0(\phi,B\phi)}d\mu_{\beta'\hat{v}_0}(\phi)\\ &= (\det(1-\kappa_0\beta'\hat{v}_0^{1/2}B\hat{v}_0^{1/2}))^{-1/2}\end{aligned} \tag{209}$$

provided $\hat{v}_0^{1/2}B\hat{v}_0^{1/2}$ is trace class and κ_0 is sufficiently small. We give a bound on the trace norm which is uniform in r and hence get a uniform bound on the integral for κ_0 sufficiently small. We have $\hat{v}_0^{1/2}B\hat{v}_0^{1/2}=\sum_\alpha A_\alpha^*A_\alpha$ where $A_\alpha=\chi_\Delta\partial^\alpha\hat{v}_0^{1/2}$. We show that A_α is Hilbert-Schmidt with a uniformly bounded Hilbert- Schmidt norm to obtain the result. We have

$$\begin{aligned}\|A_\alpha\|_{HS}^2 &= \int_\Delta dx\int dy|(\partial^\alpha\hat{v}_0^{1/2})(x-y)|^2\\ &= \|\partial^\alpha\hat{v}_0^{1/2}\|_2^2\\ &= \mathcal{O}(1)\int\frac{|p^\alpha|^2}{p^2+r^2}e^{-(p^2+r^2)}dp\end{aligned} \tag{210}$$

and this is bounded uniformly in r since $|\alpha|\geq 1$. This completes the proof.

We paraphrase theorem 3 as:

Theorem 6 *Under the hypothese of theorem 5, $\mathcal{W}_{\beta,r,N}(\zeta,f_1,....,f_n)$ is analytic in ζ in a neighborhood of the origin and is bounded there uniformly in β,r,N.*

Proof. If suffices to assume $\|\psi_{f_i}\|\leq 1$. By the definition of the norm

$$|(A_0^{r,N})_n(\Delta,\psi_\phi;\psi_{f_1},...,\psi_{f_n})|\leq n!h_0^{-n}G(\kappa_0,\Delta,\phi)\Gamma(\Delta)^{-1}\|A_0^{r,N}\|_{G_0,\Gamma,h_0} \tag{211}$$

Using this and the previous lemma we get for each n

$$|\int(A_0^{r,N})_n(\Delta,\psi_\phi;\psi_{f_1},...,\psi_{f_n})d\mu_{\beta'\hat{v}_0}(\phi)|\leq\mathcal{O}(1)\|A_0^{r,N}\|_{G_0,\Gamma,h_0} \tag{212}$$

By theorem 5, $\|A_0^{r,N}\|_{G_0,\Gamma,h_0}\leq C$. Thus the numerator in (207) has a uniform bound and the analyticity follows from the anlayticity of $A_0^{r,N}$.

The same holds for the denominator, but we also have using (160)

$$|1-\int A_0^{r,N}(\Delta,\psi_\phi)\ d\mu_{\beta'\hat{v}_0}(\phi)|\leq\mathcal{O}(1)\|\square-A_0^{r,N}\|_{G_0,\Gamma,h_0}\leq\mathcal{O}(1)|\zeta|^{1-\epsilon} \tag{213}$$

so the denominator is bounded away from zero. This completes the proof.

4 Appendix

We collect some notes about the fluctuation step. First suppose we have basic polymer activities $K = K(X, \phi)$. We have the definitions

$$\begin{aligned}
(\mu_C * K)(\phi) &= \int K(\phi + \zeta) d\mu_C(\zeta) \\
(\Delta_C K)(\phi) &= \frac{1}{2} \int \ K_2(\phi; \zeta, \zeta) \ d\mu_C(\zeta) \\
\mathcal{B}_C(K, K)(\phi) &= \frac{1}{2} \int \ \ K_1(\phi; \zeta) \circ K_1(\phi; \zeta) d\mu_C(\zeta)
\end{aligned} \tag{214}$$

We define $K_t(X, \phi)$ by

$$\mu_{tC} * \mathcal{E}xp(\Box + K) = \mathcal{E}xp(\Box + K_t) \tag{215}$$

This determines K_t since the exponential function $\mathcal{E}xp$ is invertible. Then one finds the K_t satisfy the the differential equation $E(K_t) = E(\Box + K_t) = 0$ or

$$(\frac{\partial}{\partial t} - \Delta_C) K_t - \mathcal{B}_C(K_t, K_t) = 0 \tag{216}$$

with the intial condition $K_0 = K$. This can be integrated to give an integral equation

$$K_t = \mu_{tC} * K + \int_0^t \mu_{(t-s)C} * \mathcal{B}_C(K_s, K_s) ds \tag{217}$$

A finite iteration gives an explicit formula for $K_t(X, \phi)$, also written $(\mathcal{F}_t K)(X, \phi)$.

Now suppose we have modified activities $K = K(X, \psi)$. Now define

$$\begin{aligned}
(\mu_C * K)(\psi) &= \int K(\psi + \psi_\zeta) \ d\mu_C(\zeta) \\
(\Delta_C K)(\psi) &= \frac{1}{2} \int \ K_2(\psi; \psi_\zeta, \psi_\zeta) \ d\mu_C(\zeta) \\
\mathcal{B}_C(K, K)(\psi) &= \frac{1}{2} \int \ K_1(\psi; \psi_\zeta) \circ K_1(\psi; \psi_\zeta) \ d\mu_C(\zeta)
\end{aligned} \tag{218}$$

where we recall $\psi_\zeta = (\zeta, \partial\zeta)$. Again define $K_t(X, \psi) = (\mathcal{F}_t K)(X, \psi)$ by (215), and again the equations (216), (217) hold.

Now consider the relation between the two formulations. Suppose that $K(\psi_\phi) = K^0(\phi)$. Directly from the definition we have $(\mu_C * K)(\psi_\phi) = (\mu_C * K^0)(\phi)$ and it follows that $(\mathcal{F}_t K)(\psi_\phi) = (\mathcal{F}_t K^0)(\phi)$. A corollary is that If $K \approx K'$ then $\mathcal{F}_t(K) \approx \mathcal{F}_t(K')$. If also $A \approx A'$ then $(\mathcal{F}_t)_1(K; A) \approx (\mathcal{F}_t)_1(K'; A')$

We also note explicitly that if $K(\psi_\phi) = K^0(\phi)$ then

$$\begin{aligned}
(\Delta_C K)(\psi_\phi) &= (\Delta_C K^0)(\phi) \\
\mathcal{B}_C(K, K)(\psi_\phi) &= \mathcal{B}_C(K^0, K^0)(\phi)
\end{aligned} \tag{219}$$

This follows from

$$\begin{aligned} K_1(\psi_\phi;\psi_f) &= \frac{d}{ds}K(\psi_\phi + s\psi_f)|_{s=0} \\ &= \frac{d}{ds}K^0(\phi + sf)|_{s=0} \\ &= (K^0)_1(\phi; f) \end{aligned} \tag{220}$$

and similarly for the second derivative.

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References

[1] L. Alvarez-Gaume, J.B. Bost, G. Moore, P. Nelson, C. Vafa. Bosonization on higher genus Riemann surfaces *Commun. Math. Phys.* 112:503-552, 1987.

[2] G. Benfatto, An iterated expansion for the Yukawa gas, *J. Stat. Phys.* **41**, 671, 1985.

[3] G. Benfatto, G. Gallavotti, F. Nicolo: On the massive sine-Gordon equation in the first few regions of collapse. *Commun. Math. Phys.* **83**, 387-410 (1982)

[4] D.C.Brydges, J.Dimock, and T.R. Hurd. Weak perturbations of Gaussian measures. In J.Feldman, R.Froese, and L.Rosen, editors, *Mathematical Quantum Theory I: Field Theory and Many–body Theory.* AMS, 1994.

[5] D.C. Brydges, J.Dimock, and T.R. Hurd. The short distance behavior of ϕ^4_3. *Commun. Math. Phys.*, 172:143–186, 1995.

[6] D.C. Brydges, J.Dimock, and T.R. Hurd. Estimates on renormalization group transformations , preprint

[7] D.C. Brydges, J.Dimock, and T.R. Hurd. A non-Gaussian fixed point for ϕ^4 in $4-\epsilon$ dimensions -I., preprint

[8] D.C. Brydges and H.T. Yau. Grad ϕ perturbations of massless Gaussian fields. *Commun. Math. Phys.*, 129:351–392, 1990.

[9] D.C. Brydges and T. Kennedy, Mayer expansions and the Hamilton-Jacobi equation. *J. Stat. Phys.* 48, 19-49, 1987.

[10] S. Coleman. Quantum sine-Gordon equation as the massive Thirring model. *Phys. Rev. D*, 11:2088, 1975

[11] J.Dimock and T.R. Hurd. A renormalization group analysis of the Kosterlitz-Thouless phase. *Commun. Math. Phys.*, 137:263–287, 1991.

[12] J.Dimock and T.R.Hurd. Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. *Commun. Math. Phys.*, 156:547–580, 1993.

[13] J.Fröhlich and P.Marchetti. Bosonization, topological solitons, and fractional charges in two dimensional quantum field theory. *Commun. Math. Phys.*, 116:127–173, 1988.

[14] J.Fröhlich. Classical and quantum statistical mechanics in one and two dimensions: two component Yukawa and Coulomb systems. *Commun. Math. Phys.*, 47:233–268, 1976.

[15] J.Fröhlich and E.Seiler. The massive Thirring-Schwinger model (qed_2): convergence of perturbation theory and particle structure. *Helv. Phys. Acta*, 49:889–, 1976.

[16] L. Hua. *Harmonic analysis of functions of several complex variables in the classical domains.* AMS, Providence, RI, 1963.

[17] T.R. Hurd. Charge correlations for the two-dimensional Coulomb gas. in V. Rivasseau, editor, *Constructive Physics*, 311-326, Berlin, 1995, Springer.

[18] F. Nicolo, J. Renn, and A. Steinmann: On the massive sine-Gordon equation in all regions of collapse. *Commun. Math. Phys* **105**, 291-326 (1986)