INTERFACE PINNING AND FINITE SIZE EFFECTS IN THE 2D ISING MODEL

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Abstract: We apply new techniques developed in [PV1] to the study of some surface effects in the 2D Ising model. We examine in particular the pinning-depinning transition. The results are valid for all subcritical temperatures. By duality we obtained new finite size effects on the asymptotic behaviour of the two–point correlation function above the critical temperature. The key–point of the analysis is to obtain good concentration properties of the measure defined on the random lines giving the high–temperature representation of the two–point correlation function, as a consequence of the **sharp triangle inequality**: let $\hat{\tau}(x)$ be the surface tension of an interface perpendicular to x; then for any x, y

$$\hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(x+y) \geq \frac{1}{\kappa} \left(\|x\| + \|y\| - \|x+y\| \right) ,$$

where κ is the maximum curvature of the Wulff shape and $\|x\|$ the Euclidean norm of x.

Key Words: Ising model, pinning transition, reentrance, interface, surface tension, positive stiffness, correlation length, two–point function, finite size effects, concentration of measures.

1 Introduction

Consider a 2D Ising model in some rectangular box with boundary conditions implying the presence of a phase separation line crossing the box from one vertical side to the other one. The bottom side of the box, which we call the wall, is subject to a magnetic field h. By varying β and/or h, when we are in the phase coexistence region, we can observe the so-called pinning–depinning transition, which occurs when the + phase, which is above the interface, begins to wet the wall with the result that the equilibrium shape of the interface changes from a straight line crossing the box to a broken line touching a macroscopic part of the wall. This phenomenon has been recently studied by Patrick [Pa] in the SOS model using exact calculations. In the 2D Ising model this phenomenon has a dual interpretation at high temperature in terms of finite size effects on the asymptotic behaviour of two–point function for large distances. These questions can be analyzed in the 2D Ising model by the new non–perturbative results developed in [Pf2] and [PV1] in the context of large deviations and separation of phases. Some parts of the paper , like section 4, are written directly for the two–point function. Pinning–depinning transition and the finite size effects on the two–point correlation function are treated in section 6.

The fundamental thermodynamical function associated with an interface is the surface tension. The interface[1] between the two phases of the model is a non–random object. On the other hand, at the scale of the lattice spacing, we have the random line, which is a geometrical object separating the two phases. The interface is therefore defined at a scale where the fluctuations of the phase separation line become negligible. Its main properties are described by a functional of the surface tension. The observed interface at equilibrium is a minimum of this functional (section 3). The study of fluctuations of the phase separation line is an important and difficult problem; some works in that directions are [Hi], [BLP1][2], [DH].

A key–point of the present analysis is the role of the sharp triangle inequality of

[1] The concept of "interface" as a macroscopic phenomenon is advocated in the recent paper [ABCP]. Moreover, Talagrand in his analysis [T] about the Law of Large Numbers for independent random variables develops similar ideas. See also footnote 2.

[2] The local structure of the phase separation line is studied in [BLP1] at low temperatures for the case of the so–called $\pm$ boundary condition, which corresponds to $a = b = 1/2$ and $h = 1$ of the present paper. The definition of the phase separation line in [BLP1] coincides with the one of Gallavotti in his work [G] about the phase separation in the 2D Ising model; it differs slightly from the one used here, but not in an essential way. (Notice that the terminology "interface" is sometimes used for "phase separation line" in [BLP1].) It is shown that the phase separation line has a well-defined intrinsic width, which is finite at the scale of the lattice spacing, but that its position has fluctuations typically of the order of $O(L^{1/2})$, L being the linear size of the box Λ_L containing the system. Because of these fluctuations the projection of the corresponding limiting Gibbs state, at the middle of the box, when $L \to \infty$, is translation invariant; the magnetization (at the middle of the box) is zero. However, the results of this paper show that, at a suitable mesoscopic scale of order $O(L^\alpha)$, $\alpha > 1/2$, the system has a well–defined non–random horizontal interface. To describe the system at the scale $O(L^\alpha)$ we partition the box Λ_L into square boxes C_i of linear size $O(L^\alpha)$; the state of the system in each of these boxes is specified by the empirical magnetization $|C_i|^{-1} \sum_{t \in C_i} \sigma(t)$ (normalized block-spin). Then we rescale all lengths by $1/L$ in order to get a measure for these normalized block-spins in the fixed (macroscopic box) Q. When $L \to \infty$ these measures converge to a non–random macroscopic configuration with a well-defined horizontal interface separating the two phases of the model, characterized by a value $\pm m^*$ of the normalized block-spins, m^* being the spontaneous magnetization of the model.

the surface tension [I], which, combined with our recent results, leads to good concentration properties for the measure defined on the random lines giving the high–temperature representation of the two–point correlation function (section 4). (These random lines coincide with the phase–separation lines.) Because of its importance, we devote section 7 to a geometrical study of the sharp triangle inequality. This section can be read independently; Proposition 7.1 has its own interest.

The paper is not self–contained, because we use in an essential way results of [PV1], in particular those of section 5. They are carefully stated in Propositions 2.3 and 4.2 and Lemmas 5.1 and 5.2. This has the advantage that we can focus our attention on the essential points of the proofs. Motivated by [Pa] we have chosen pinning–depinning transition to illustrate the technique of [PV1]; we can consider more complicated situations than the ones of this paper.

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2 Definitions and notations

2.1 Phase separation line

We follow [PV1] for the notation and terminology. Throughout the paper $O(x)$ denotes a non–negative function of $x \in \mathbb{R}^+$, such that there exists a constant C with $O(x) \leq Cx$. The function $O(x)$ may be different at different places.

Let Q be the square box in $\mathbb{R}^2$,

$$Q := \{\, x = (x(1), x(2)) \in \mathbb{R}^2 : \, |x(1)| \leq 1/2 \, , \; 0 \leq x(2) \leq 1 \,\} \, , \tag{2.1}$$

and Λ_L be the subset of $\mathbb{Z}^2$ (L an even integer)

$$\Lambda_L := \{\, x = (x(1), x(2)) \in \mathbb{Z}^2 : \, |x(1)| \leq L/2 \, , \; 0 \leq x(2) \leq L \,\} \, . \tag{2.2}$$

The spin variable at $x \in \mathbb{Z}^2$ is the random variable $\sigma(x) = \pm 1$; spin configurations are denoted by $\omega \in \{-1, +1\}^{\mathbb{Z}^2}$, so that $\sigma(t)(\omega) = \pm 1$ if $\omega(t) = \pm 1$. We always suppose that we have for the box Λ_L either the ab boundary condition (ab b.c.) or the $-$ boundary condition ($-$ b.c.). Let $0 \leq a \leq 1$ and $0 \leq b \leq 1$ be given; the ab b.c. specifies the values of the spins outside Λ_L as follows,

$$\forall x \notin \Lambda_L \, , \; \sigma(x) := \begin{cases} -1 & \text{if } x(2) \leq a \cdot L, \, x(1) < 0, \\ -1 & \text{if } x(2) \leq b \cdot L, \, x(1) \geq 0, \\ +1 & \text{otherwise.} \end{cases} \tag{2.3}$$

The $-$ b.c. specifies the values of the spins outside Λ_L as follows,

$$\forall x \notin \Lambda_L \, , \; \sigma(x) := -1 \, . \tag{2.4}$$

In Λ_L we consider the Ising model defined by the Hamiltonian

$$H_{\Lambda_L} = - \sum_{\langle t,t' \rangle \cap \Lambda_L \neq \emptyset} J(t,t') \sigma(t) \sigma(t') \, , \tag{2.5}$$

where $\langle t, t' \rangle$ denotes a pair of nearest neighbours points of the lattice $\mathbb{Z}^2$, or the corresponding edge (considered as a unit–length segment) with end–points t, t'; the coupling constants $J(t, t')$ are given by

$$J(t,t') := \begin{cases} 1 & \text{if } t(2) \geq 0 \text{ and } t'(2) \geq 0, \\ h & \text{otherwise, with } h > 0. \end{cases} \tag{2.6}$$

Let β be the inverse temperature. The Boltzmann factor is $\exp\{-\beta H_{\Lambda_L}\}$ and the Gibbs measures in Λ_L with ab b.c., respectively $-$ b.c., are denoted by

$$\langle \cdot \rangle_L^{ab} = \langle \cdot \rangle_L^{ab}(\beta, h) \quad \text{resp.} \quad \langle \cdot \rangle_L^{-} = \langle \cdot \rangle_L^{-}(\beta, h)\,. \tag{2.7}$$

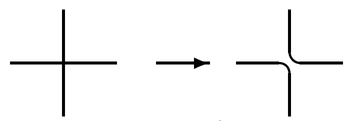
We introduce the dual lattice to $\mathbb{Z}^2$

$$\mathbb{Z}^{2*} := \{\, x = (x(1), x(2)) : \, x + (1/2, 1/2) \in \mathbb{Z}^2 \,\}\,, \tag{2.8}$$

and describe the spin configurations ω by a set of edges $\mathcal{E}^*(\omega)$ of the dual lattice. For each edge e of $\mathbb{Z}^2$ there is a unique edge e^* of $\mathbb{Z}^{2*}$ which crosses e, which is written $e^* \dagger e$. Let $\omega \in \{-1, +1\}^{\mathbb{Z}^2}$ be a spin configuration satisfying the ab b.c.. We set

$$\mathcal{E}^*(\omega) := \{ e^* \subset \Lambda_L^* : \exists \, \langle t, t' \rangle \dagger e^* \text{ with } \sigma(t)(\omega)\sigma(t')(\omega) = -1 \,\}\,. \tag{2.9}$$

We decompose the set $\mathcal{E}^*(\omega)$ into connected components and use rule A defined in the picture below in order to get a set of disjoint simple lines called **contours**.



rule A

Each configuration ω satisfying the ab b.c. is uniquely specified by a family $(\underline{\gamma}(\omega), \lambda(\omega))$ of disjoint contours; all contours of $\underline{\gamma} = \{\gamma_1, \gamma_2, \ldots\}$ are closed[3] and λ is open, with end–points t_l^L and t_r^L. We call λ the **phase separation line**[4]. Conversely, a family of contours $(\underline{\gamma}', \lambda')$ is called ab **compatible**[5] if there exists ω such that ω satisfies the ab b.c. and $\underline{\gamma}(\omega) = \underline{\gamma}'$, $\lambda(\omega) = \lambda'$. In the same way each configuration ω satisfying the $-$ b.c. is uniquely specified by a family $\underline{\gamma}$ of closed contours and we have a notion of $-$ compatibility.

[3]Let A be a set of edges; the boundary δA of A is the set of $x \in \mathbb{Z}^{2*}$ such that there is an odd number of edges of A adjacent to x. A is **closed** if $\delta A = \emptyset$ and **open** if $\delta A \neq \emptyset$.

[4]As already mentioned in the introduction we make a distinction between the concept of "phase separation line", which is defined for each configuration at the scale of the lattice spacing, and the concept of "interface", which is associated with the fact that there is a separation of the two phases in the model due to our choice of boundary condition. The "interface" concept emerges at a scale large enough so that it is a non–random object, whose free energy is given in terms of the surface tension.

[5]To be precise we should say that $(\underline{\gamma}', \lambda')$ is ab compatible in Λ_L. Compare this notion of compatibility with the notion of compatibility used in the high–temperature expansion (see subsection 2.3).

For each contour η, closed or open, we define a set of edges of $\mathbb{Z}^2$,

$$\mathrm{co}(\eta) := \{\, e \in \mathbb{Z}^2 : \exists e^* \in \eta,\ e \dagger e^* \,\}\,. \tag{2.10}$$

The Boltzmann weight of η is

$$w(\eta) := \prod_{\langle t,t'\rangle \in \mathrm{co}(\eta)} \exp(-2\beta J(t,t'))\,. \tag{2.11}$$

Next we define three (normalized) partition functions, $Z^{ab}(\Lambda_L)$, $Z^{ab}(\Lambda_L|\lambda)$ and $Z^-(\Lambda_L)$. By definition

$$Z^{ab}(\Lambda_L) := \sum_{\omega \text{ with } ab \text{ b.c.}} w(\lambda(\omega)) \prod_{\gamma \in \underline{\gamma}(\omega)} w(\gamma)\,; \tag{2.12}$$

$$Z^{ab}(\Lambda_L|\lambda) := \sum_{\substack{\omega \text{ with } ab \text{ b.c.:} \\ \lambda(\omega)=\lambda}} \prod_{\gamma \in \underline{\gamma}(\omega)} w(\gamma)\,; \tag{2.13}$$

$$Z^-(\Lambda_L) := \sum_{\omega \text{ with } - \text{ b.c.}} \prod_{\gamma \in \underline{\gamma}(\omega)} w(\gamma)\,. \tag{2.14}$$

We define a weight $q_L(\lambda) = q_L(\lambda;\beta,h)$ for each phase separation line λ of a configuration ω satisfying the ab b.c.,

$$q_L(\lambda) := \begin{cases} w(\lambda)\dfrac{Z^{ab}(\Lambda_L|\lambda)}{Z^-(\Lambda_L)} & \text{if } \lambda \text{ is } ab \text{ compatible in } \Lambda_L, \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

2.2 Surface tension

Consider the model defined in Λ_L, with coupling constants $J(t,t') \equiv 1$, i.e. $h = 1$ in (2.6). For each ω compatible with the ab b.c. there is a well-defined phase separation line $\lambda(\omega)$ with end–points t_l^L and t_r^L. Let $n = (n(1), n(2))$ be the unit vector in $\mathbb{R}^2$ which is perpendicular to the straight line passing through t_l^L and t_r^L. By definition **the surface tension** $\hat\tau(n;\beta) = \hat\tau(n)$ is

$$\hat\tau(n) = \hat\tau(n(1), n(2)) := -\lim_{L\to\infty} \frac{1}{\|t_l^L - t_r^L\|} \log \frac{Z^{ab}(\Lambda_L)}{Z^-(\Lambda_L)}\,, \tag{2.16}$$

where $\|t_l^L - t_r^L\|$ is the Euclidean distance between t_l^L and t_r^L. By symmetry of the model we have

$$\hat\tau(n(1), n(2)) = \hat\tau(-n(1), -n(2)) = \hat\tau(n(2), -n(1)) = \hat\tau(n(2), n(1))\,. \tag{2.17}$$

Using (2.15) we can write

$$\hat\tau(n) = -\lim_{L\to\infty} \frac{1}{\|t_l^L - t_r^L\|} \log \sum_{\substack{\lambda(\omega): \\ \omega \text{ with } ab-\text{b.c.}}} q_L(\lambda(\omega))\,. \tag{2.18}$$

We extend the definition of the surface tension to $\mathbb{R}^2$ by homogeneity,

$$\hat\tau(x) := \|x\| \hat\tau(x/\|x\|)\,. \tag{2.19}$$

Proposition 2.1 *The surface tension is a uniformly Lipschitz convex function on* $\mathbb{R}^2$*, such that* $\hat{\tau}(x) = \hat{\tau}(-x)$*. It is identically zero above the critical temperature and strictly positive for all* $x \neq 0$ *when the temperature is strictly smaller than the critical temperature.*

The main property of $\hat{\tau}$ *is the sharp triangle inequality. For all* $\beta > \beta_c$ *there exists a strictly positive constant* $\Delta = \Delta(\beta)$ *such that for any* x*,* $y \in \mathbb{R}^2$*, the norm* $\hat{\tau}(\,\cdot\,)$ *satisfies*

$$\hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(x+y) \geq \Delta(\|x\| + \|y\| - \|x+y\|)\,. \tag{2.20}$$

Let $x(\theta) := (\cos\theta, \sin\theta)$ *and* $\hat{\tau}(\theta) := \hat{\tau}(x(\theta))$*. Then the best constant* Δ *is*

$$\Delta := \inf_{\theta} \left(\frac{d^2}{d\theta^2}\hat{\tau}(\theta) + \hat{\tau}(\theta) \right) > 0\,. \tag{2.21}$$

Remark: The first part of the proposition follows from [MW] or [LP]. The second part is proven in section 7. (2.20) was introduced and proven by Ioffe in [I]. The statement of (2.20) is different in [I], but equivalent to the present one (see proof of Proposition 7.1). The constant Δ is not optimal in [I]. (2.21) is called the **positive stiffness property**. Geometrically, (2.21) means that the curvature of the Wulff shape is bounded above by $1/\Delta$.

2.3 Duality

A basic property of the 2D Ising model is self-duality. As a consequence of that property many questions about the model below the critical temperature can be translated into dual questions for the dual model above the critical temperature. For example, questions about the surface tension are translated into questions about the correlation length. We refer to [PV1] for a more complete discussion and recall here the main results, which we shall use in section 4.

Consider the model defined in the box Λ_L with coupling constants given by (2.6). We suppose that we have $-$ b.c.. Let

$$\mathcal{E}_L := \{\, \langle t, t' \rangle : \, t \text{ or } t' \in \Lambda_L \,\}\,, \tag{2.22}$$

be the set of edges in the sum (2.5). The dual set of edges is

$$\mathcal{E}_L^* := \{\, e^* : \, \exists e \in \mathcal{E}_L\,,\, e^* \dagger e \,\}\,, \tag{2.23}$$

and the dual model is defined on the dual box

$$\begin{aligned} \Lambda_L^* &:= \{ x \in \mathbb{Z}^{2*} : \, \exists e^* \in \mathcal{E}_L^*\,,\, x \in e^* \,\} \\ &= \{\, x \in \mathbb{Z}^{2*} : \, |x(1)| \leq (L+1)/2\,,\; -1/2 \leq x(2) \leq L + 1/2 \,\}\,. \end{aligned} \tag{2.24}$$

The dual Hamiltonian is the free boundary condition (free b.c.) Hamiltonian, that is,

$$H_{\Lambda_L^*} = -\sum_{\langle t,t' \rangle \subset \Lambda_L^*} J^*(t,t')\sigma(t)\sigma(t')\,. \tag{2.25}$$

In (2.25) the dual coupling constants are related to the coupling constants (2.6) as follows:

$$J^*(t,t') := \begin{cases} h^* & \text{if } t(2) = t'(2) = -1/2, \\ 1 & \text{otherwise.} \end{cases} \tag{2.26}$$

h^* and the dual temperature β^* are defined by

$$\tanh\beta^* := \exp\{-2\beta\}\,; \tag{2.27}$$
$$\tanh\beta^* h^* := \exp\{-2\beta h\}\,. \tag{2.28}$$

The critical inverse temperature β_c is characterized by

$$\tanh\beta_c := \exp\{-2\beta_c\}\,. \tag{2.29}$$

Notice that when h or β are small, then h^* or β^* are large. The Gibbs measure in Λ_L^* with free boundary condition is denoted by $\langle\,\cdot\,\rangle_{\Lambda_L^*} = \langle\,\cdot\,\rangle_{\Lambda_L^*}(\beta^*, h^*)$. In this paper $\beta > \beta_c$ so that $\beta^* < \beta_c$. For those values of β^* there is a unique Gibbs measure on $\mathbb{Z}^2$, which we denote by $\langle\,\cdot\,\rangle = \langle\,\cdot\,\rangle(\beta^*)$[6]. The most important quantity for the 2D Ising model with free b.c. is the covariance function, or two–point function,

$$\langle\sigma(t)\sigma(t')\rangle(\beta^*)\,. \tag{2.30}$$

The **decay-rate** of the covariance function is defined for all $t, t' \in \mathbb{Z}^{2*}$ as

$$\tau(t-t') = \tau(t-t';\beta^*) := -\lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \frac{1}{n}\log\langle\sigma(nt)\sigma(nt')\rangle(\beta^*)\,. \tag{2.31}$$

Proposition 2.2 *For the 2D Ising model the surface tension $\hat{\tau}(x;\beta)$ and the decay–rate $\tau(x;\beta^*)$ are equal,*

$$\hat{\tau}(x;\beta) = \tau(x;\beta^*) \quad \forall x\,. \tag{2.32}$$

For a proof see [BLP2].

Proposition 2.2 indicates that the decay–rate and surface tension are dual quantities. Moreover, properties of the phase separation line λ at β are related to properties of the covariance function at β^* through the random–line representation of the covariance (see [PV1] for detailed discussion). The random-line representation follows from the high–temperature expansion. The terms of this expansion are indexed by sets of edges, called contours. Throughout the paper we use the following notations: if $A \subset \mathbb{Z}^{2*}$, then $\mathcal{E}^*(A)$ is the set of all edges of $\mathbb{Z}^{2*}$ with both end–points in A. Consider the partition function in Λ_L^* with free b.c., which can be written as

$$\sum_{\sigma(t),\, t\in\Lambda_L^*} \prod_{\langle t,t'\rangle\subset\Lambda_L^*} \cosh J^*(t,t')(1+\sigma(t)\sigma(t')\tanh J^*(t,t'))\,. \tag{2.33}$$

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[6]This measure is the limit of Gibbs measures in finite subsets Λ with free boundary condition, when $\Lambda \uparrow \mathbb{Z}^2$. As long as $\beta^* < \beta_c$, the choice of the boundary conditions does not matter since there is a unique Gibbs state. Notice that $\langle\,\cdot\,\rangle(\beta^*)$ is not the limit of the measures $\langle\,\cdot\,\rangle_{\Lambda_L^*}(\beta^*, h^*)$ when $L \to \infty$, since the subsets Λ_L^* do not converge to $\mathbb{Z}^2$. There is a limiting measure for $\langle\,\cdot\,\rangle_{\Lambda_L^*}$ when $L \to \infty$, which is defined on the semi–infinite lattice $\mathbb{L}^* := \{x \in \mathbb{Z}^{2*} : x(i) \geq -1/2\,, i = 1, 2\}$, and which depends on β^* *and* h^* [FP2].

We expand the product; each term of the expansion is labeled by a set of edges $\langle t, t'\rangle$: we specify the edges corresponding to factors $\tanh J^*(t,t')$. Then we sum over $\sigma(t)$, $t \in \Lambda$; after summation only terms labeled by sets of edges of the dual lattice $\mathbb{Z}^{2*}$ with empty boundary give a non–zero contribution. We decompose this set uniquely into a family of connected closed contours using the rule A. Any such family of contours is called **compatible**[7]. For each (closed) contour γ we set

$$w^*(\gamma) := \prod_{\langle t,t'\rangle \in \gamma} \tanh J^*(t,t')\,, \tag{2.34}$$

and we introduce a normalized partition function

$$Z(\Lambda_L^*) := \sum_{\substack{\underline{\gamma}:\\ \text{comp. in } \Lambda_L^*}} \prod_{\gamma \in \underline{\gamma}} w^*(\gamma)\,. \tag{2.35}$$

We treat the numerator of the two–point function $\langle \sigma(t)\sigma(t')\rangle_{\Lambda_L^*}$ in a similar way. In this case all non–zero terms of the expansion are labeled by compatible families $(\underline{\gamma}, \lambda)$, where all $\gamma \in \underline{\gamma}$ are closed, λ is open with end–points t, t'. Given an open contour λ, we introduce a partition function as in (2.13),

$$Z(\Lambda_L^*|\lambda) := \sum_{\substack{\underline{\gamma}:\\ (\underline{\gamma},\lambda)\,\text{comp.}}} \prod_{\gamma \in \underline{\gamma}} w^*(\gamma)\,. \tag{2.36}$$

The next two formulas are fundamental. For each open contour λ we define the weight of the contour as

$$q_L^*(\lambda) = q_L^*(\lambda;\beta^*,h^*) := \begin{cases} w^*(\lambda)\dfrac{Z(\Lambda_L^*|\lambda)}{Z(\Lambda_L^*)} & \text{if } \lambda \subset \mathcal{E}^*(\Lambda_L^*), \\ 0 & \text{otherwise.} \end{cases} \tag{2.37}$$

Using this weight we get a random–line representation for the two–point function $\langle \sigma(t)\sigma(t')\rangle_{\Lambda_L^*}$ as

$$\langle \sigma(t)\sigma(t')\rangle_{\Lambda_L^*} = \sum_{\lambda: t \to t'} q_L^*(\lambda)\,. \tag{2.38}$$

There are similar representations for $\langle \sigma(t)\sigma(t')\rangle$ and for

$$\lim_{L\to\infty} \langle \sigma(t)\sigma(t')\rangle_{\Lambda_L^*}(\beta^*,h^*) = \langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}^*}(\beta^*,h^*) \tag{2.39}$$

when $\beta^* < \beta_c$.

Let λ be such that $\delta\lambda = \{t,t'\}$; we also write $\lambda : t \to t'$. Given $\lambda : t \to t'$ we can define weights $q^*(\lambda;\beta^*)$ and $q^*_{\mathbb{L}^*}(\lambda;\beta^*,h^*)$ such that

$$\langle \sigma(t)\sigma(t')\rangle(\beta^*) = \sum_{\lambda: t\to t'} q^*(\lambda;\beta^*)\,, \tag{2.40}$$

and

$$\langle \sigma(t)\sigma(t')\rangle_{\mathbb{L}^*}(\beta^*,h^*) = \sum_{\lambda: t\to t'} q^*_{\mathbb{L}^*}(\lambda;\beta^*,h^*)\,. \tag{2.41}$$

[7]To be precise we should say compatible in Λ_L^*, since each contour is a subset $\mathcal{E}^*(\Lambda_L^*)$. A family of closed contours in Λ_L^* is compatible if and only if they are disjoint according to rule A. This is a purely geometrical property, contrary to the definition of $-$ compatibility. A family of closed contours which is $-$ compatible in Λ_L is also compatible in Λ_L^*. Because of our choice of Λ_L the converse is also true, but in general $-$ compatibility does not imply compatibility.

Proposition 2.3 *Let $\beta^* < \beta_c$. Let λ_1 and λ_2 be two open contours such that $\lambda := \lambda_1 \cup \lambda_2$ is an open contour and $\lambda \subset \mathcal{E}(\Lambda_L^*)$. Then*

$$q_L^*(\lambda;\beta^*,h^*) \geq q_{\mathbb{L}^*}^*(\lambda;\beta^*,h^*) \quad \text{and} \quad q_L^*(\lambda) \geq q_L^*(\lambda_1)q_L^*(\lambda_2)\,. \tag{2.42}$$

If $\delta\lambda = \{t_l^L, t_r^L\}$, then $q_L(\lambda;\beta^*,h^*)$ is equal to (2.15), that is,

$$q_L^*(\lambda;\beta^*,h^*) = q_L(\lambda;\beta,h)\,. \tag{2.43}$$

Proposition 2.3[8] is a key–result which is proven in [PV1]. It allows us to study properties of the phase separation line through the two–point correlation function $\langle\sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}(\beta^*,h^*)$. From Proposition 2.3 and GKS inequalities we get the interesting inequalities

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{t,t'\}\\ \lambda\subset\mathcal{E}^*(\Lambda_L^*)}} q_{\mathbb{L}^*}^*(\lambda) \quad &\leq \sum_{\substack{\lambda:\delta\lambda=\{t,t'\}\\ \lambda\subset\mathcal{E}^*(\Lambda_L^*)}} q_L^*(\lambda) = \langle\sigma(t)\sigma(t')\rangle_{\Lambda_L^*} \\ &\leq \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}^*} = \sum_{\substack{\lambda:\delta\lambda=\{t,t'\}\\ \lambda\subset\mathcal{E}^*(\mathbb{L}^*)}} q_{\mathbb{L}^*}^*(\lambda)\,.\end{aligned} \tag{2.44}$$

2.4 Wall free energy

The last thermodynamical quantity, which enters into the description of the properties of the interface, is the wall free energy. We define the difference of the contributions of the wall to the free energy when the bulk phase is the + phase, respectively the − phase, as[9]

$$\hat{\tau}_{\mathrm{bd}} = \hat{\tau}_{\mathrm{bd}}(\beta,h) := -\lim_{L\to\infty}\frac{1}{2L+1}\log\frac{Z^{00}(\Lambda_L)}{Z^{-}(\Lambda_L)}\,, \tag{2.45}$$

where $Z^{00}(\Lambda_L)$ is the partition function with $a = b = 0$. There is a proposition analogous to Proposition 2.2, which relates $\hat{\tau}_{\mathrm{bd}}$ to the decay–rate of the boundary two–point function of the dual model (see [PV1])

Proposition 2.4 *Let $\beta > \beta_c$. Let $t, t' \in \mathbb{L}^*$, $t(2) = t'(2) = -1/2$. Then*

$$-\lim_{n\to\infty}\frac{1}{n}\log\langle\sigma(nt)\sigma(nt')\rangle_{\mathbb{L}^*}(\beta^*,h^*) = \|t-t'\|\cdot\hat{\tau}_{\mathrm{bd}}(\beta,h)\,. \tag{2.46}$$

The quantity $\hat{\tau}_{\mathrm{bd}}(\beta,h)$ allows to detect the wetting transition through Cahn's criterium (see [FP1] and [FP2]). Since $h > 0$, $0 < \hat{\tau}_{\mathrm{bd}}(\beta,h) \leq \hat{\tau}(\beta)$. There is partial wetting of the wall if and only if $\hat{\tau}_{\mathrm{bd}}(\beta,h) < \hat{\tau}(\beta)$; this occurs if and only if $h < h_w$, where h_w has been computed by Abraham [A]. The transition value $h_w(\beta)$ is the solution of the equation

$$\exp\{2\beta\}\{\cosh 2\beta - \cosh 2\beta h_w(\beta)\} = \sinh 2\beta\,. \tag{2.47}$$

By duality we show in subsection 6.2 that we get finite size effects for the two–point function when $h^* > h_c^*$, where $h_c^*(\beta^*) := (h_w(\beta))^*$[10].

[8] Notice that Proposition 2.3 does not imply Proposition 2.2, see discussion in subsection 6.2

[9] The definition of $\hat{\tau}_{\mathrm{bd}}$ differs from the analogous quantity used in [PV1] or [PV2], because in these papers the reference bulk phase is the + phase and here it is the − phase.

[10] In this paper we always assume that $0 < h < \infty$, so that we also have $0 < h^* < \infty$. Notice that $0 < h_w(\beta) < 1$ for any $0 < \beta < \beta_c$; consequently $1 < h_c^*(\beta^*) < \infty$ for any $\beta_c < \beta^* < \infty$.

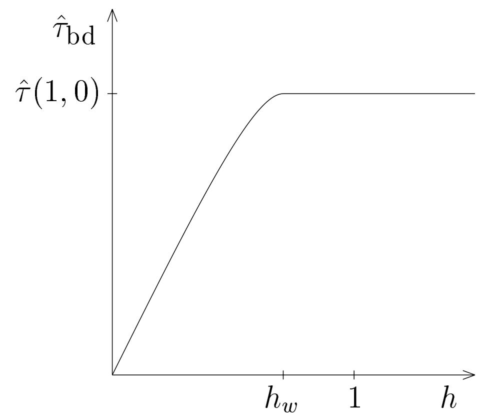

Figure 1: $\hat\tau_{\mathrm{bd}}$ as a function of the magnetic field h, for $\beta = 1.4\beta_c$.

3 The variational problem

The interface is a macroscopic non–random object, whose properties are described by a functional involving the surface tension. In Q the interface is a simple rectifiable curve $\mathcal{C}$ with end–points $A := (-1/2, a)$, $0 < a < 1$, and $B := (1/2, b)$, $0 < b < 1$. We denote by $w_Q = \{\, x \in Q : x(2) = 0 \,\}$ the wall and by $|\mathcal{C} \cap w_Q|$ the length of the portion of the interface in contact with the wall w_Q. Suppose that $[0, w] \to Q$, $s \mapsto \mathcal{C}(s) = (u(s), v(s))$, is a parameterization of the interface. The free energy of the interface $\mathcal{C}$ can be written

$$\mathtt{W}(\mathcal{C}) := \int_0^w \hat\tau(\dot u(s), \dot v(s))ds + |\mathcal{C} \cap w_Q| \cdot \Big[\hat\tau_{\mathrm{bd}} - \hat\tau(1,0)\Big] , \tag{3.1}$$

(because the function $\hat\tau(x(1), x(2))$ is positively homogeneous and $\hat\tau(x(1), x(2)) = \hat\tau(-x(2), x(1))$). The interface at equilibrium is the minimum of this functional. Therefore we have to solve the

Variational problem: *Find the minimum of the functional* $\mathtt{W}$ *among all simple rectifiable open curves in* Q *with extremities* $A = (-1/2, a)$ *and* $B = (1/2, b)$.

Let $\mathcal{D}$ be the straight line from A to B and $\mathcal{W}$ be the curve composed of the following three straight line segments: from A to a point w_1 on the wall, then along the wall from w_1 to w_2, and finally from w_2 to B. The points w_1 and w_2 are such that the angles between the first segment and the wall and between the last segment and the wall are equal, chosen[11] in the interval $[0, \pi/2]$, and solutions of

$$\cos\theta_Y\, \tau(\theta_Y) - \sin\theta_Y\, \tau'(\theta_Y) = \hat\tau_{\mathrm{bd}} , \tag{3.2}$$

which is known as the Herring-Young equation. (For the case under consideration the existence of θ_Y is an immediate consequence of the Winterbottom construction).

Proposition 3.1 *Let* θ_Y *be the solution of the Herring-Young equation (3.2).*

1. *If* $\tan\theta_Y \le a + b$ *then the minimum of the variational problem is given by the curve* $\mathcal{D}$.

[11] This choice leads to a different sign at the right–hand side of the Herring-Young equation (3.2) than in [PV2] formulae (1.5) or (4.60); in these latter references we use $\pi - \theta$ instead of θ.

2. *If* $\pi/2 > \theta_Y > \arctan(a+b)$ *then the minimum of the variational problem is given by* $\mathcal{D}$ *if* $\mathtt{W}(\mathcal{D}) < \mathtt{W}(\mathcal{W})$, *by* $\mathcal{W}$ *if* $\mathtt{W}(\mathcal{D}) > \mathtt{W}(\mathcal{W})$ *and by both* $\mathcal{D}$ *and* $\mathcal{W}$ *if* $\mathtt{W}(\mathcal{D}) = \mathtt{W}(\mathcal{W})$.

Proof. The proof is an easy consequence of the two following lemmas. Lemma 3.1 states that the minimum is a polygonal line.

Lemma 3.1 *Let* $\mathcal{C}$ *be some simple rectifiable parameterized curve with initial point* A *and final point* B.

If $\mathcal{C}$ *does not intersect the wall, then*

$$\mathtt{W}(\mathcal{C}) \geq \mathtt{W}(\mathcal{D}) \tag{3.3}$$

with equality if and only if $\mathcal{C}=\mathcal{D}$.

If $\mathcal{C}$ *intersects the wall, let* t_1 *be the first time* $\mathcal{C}$ *touches the wall and* t_2 *the last time* $\mathcal{C}$ *touches the wall. Let* $\widehat{\mathcal{C}}$ *be the curve given by three segments from* A *to* $\mathcal{C}(t_1)$, *from* $\mathcal{C}(t_1)$ *to* $\mathcal{C}(t_2)$ *and from* $\mathcal{C}(t_2)$ *to* B. *Then*

$$\mathtt{W}(\mathcal{C}) \geq \mathtt{W}(\widehat{\mathcal{C}})\,. \tag{3.4}$$

Equality holds if and only if $\mathcal{C} = \widehat{\mathcal{C}}$.

Proof. Since $\hat{\tau}$ is convex and homogeneous, we have in the first case by Jensen's inequality

$$\mathtt{W}(\mathcal{C}) = w\frac{1}{w}\int_0^w \hat{\tau}(\dot{u}(s),\dot{v}(s))ds \geq w\hat{\tau}(\frac{1}{w}\int_0^w \dot{u}(s)ds, \frac{1}{w}\int_0^w \dot{v}(s)ds) = \mathtt{W}(\mathcal{D})\,. \tag{3.5}$$

The inequality is strict if $\mathcal{C} \neq \mathcal{D}$ as is easily seen using the sharp triangle inequality (2.20).

In the second case we apply Jensen's inequality to the part of $\mathcal{C}$ between A and $\mathcal{C}(t_1)$ and between $\mathcal{C}(t_2)$ and B to compare with the corresponding straight segments of $\widehat{\mathcal{C}}$. Combining Jensen's inequality and the fact that $\hat{\tau}_{\mathrm{bd}} \leq \hat{\tau}$, we can also compare the part of $\mathcal{C}$ between $\mathcal{C}(t_1)$ and $\mathcal{C}(t_2)$ with the corresponding straight segment of $\widehat{\mathcal{C}}$. □

Lemma 3.2 *Let* $\widehat{\mathcal{C}}$ *be a polygonal line from* A *to* $\hat{w}_1 \in w_Q$, *then from* $\hat{w}_1$ *to* $\hat{w}_2 \in w_Q$, *and finally from* $\hat{w}_2$ *to* B. *Let* θ_Y *be the solution of the Herring-Young equation (3.2). If* $\pi/2 > \theta_Y > \arctan(a+b)$ *then*

$$\mathtt{W}(\widehat{\mathcal{C}}) \geq \mathtt{W}(\mathcal{W})\,, \tag{3.6}$$

with equality if and only if $\widehat{\mathcal{C}} = \mathcal{W}$.
If $\arctan(a+b) \geq \theta_Y$

$$\mathtt{W}(\widehat{\mathcal{C}}) > \mathtt{W}(\mathcal{D})\,. \tag{3.7}$$

Proof.

Let $\theta_1 \in (0, \pi/2)$ be the angle of the straight segment of $\widehat{\mathcal{C}}$, from A to $\hat{w}_1$, with the wall w_Q, and $\theta_2 \in (0, \pi/2)$ be the angle of the straight segment of $\widehat{\mathcal{C}}$, from $\hat{w}_2$ to B, with the wall w_Q. We have

$$\begin{aligned} \mathtt{W}(\widehat{\mathcal{C}}) &= \tau(\theta_1)\frac{a}{\sin\theta_1} + \hat{\tau}_{\mathrm{bd}}(1 - \frac{a}{\tan\theta_1} - \frac{b}{\tan\theta_2}) + \tau(\theta_2)\frac{b}{\sin\theta_2} \\ &= g(\theta_1, a) + g(\theta_2, b)\,, \end{aligned} \tag{3.8}$$

where we have introduced

$$g(\theta, x) := \tau(\theta)\frac{x}{\sin\theta} + \hat{\tau}_{\mathrm{bd}}(1/2 - \frac{x}{\tan\theta})\,. \tag{3.9}$$

Let θ_Y be defined as the solution of the Herring-Young equation (3.2), so that

$$\frac{\partial}{\partial\theta} g(\theta_Y, x) = \frac{x}{\sin^2\theta_Y}(\sin\theta_Y\,\tau'(\theta_Y) - \cos\theta_Y\,\tau(\theta_Y) + \hat{\tau}_{\mathrm{bd}}) = 0\,. \tag{3.10}$$

The second derivative of $g(\theta, x)$ is

$$\frac{\partial^2}{\partial\theta^2} g(\theta, x) = \frac{x(\tau(\theta) + \tau''(\theta))}{\sin\theta} - \frac{2}{\tan\theta}\frac{\partial}{\partial\theta} g(\theta, x)\,. \tag{3.11}$$

Therefore, for $\theta \in (0, \pi/2)$, we have

$$\frac{\partial}{\partial\theta} g(\theta, x) = x\int_{\theta_Y}^{\theta} \exp\{-\int_{\gamma}^{\theta}\frac{2}{\tan\alpha}\,d\alpha\}\,\frac{\tau(\gamma) + \tau''(\gamma)}{\sin\gamma}\,d\gamma\,. \tag{3.12}$$

Since τ has positive stiffness, i.e. $\tau(\theta) + \tau''(\theta) > 0$, (3.12) implies that θ_Y is an absolute minimum of $g(\theta, x)$ over the interval $(0, \pi/2)$, and that g is strictly monotonous over the intervals $(\theta_Y, \pi/2)$ and $(0, \theta_Y)$.

A necessary and sufficient condition, that we can construct a simple polygonal line $\widehat{\mathcal{C}}$ as above, is

$$\frac{a}{\tan\theta_1} + \frac{b}{\tan\theta_2} \leq 1\,. \tag{3.13}$$

In particular $\theta_1 \in [\theta_a, \pi/2]$ where $\theta_a := \arctan a$, and $\theta_2 \in [\theta_b, \pi/2]$ where $\theta_b := \arctan b$. Similarly $\mathcal{W}$ is a simple curve in Q if and only if

$$\theta_Y \in [\arctan a + b, \pi/2)\,. \tag{3.14}$$

From the preceding results we have

$$\mathtt{W}(\widehat{\mathcal{C}}) \geq g(\theta_1^*, a) + g(\theta_2^*, b)\,, \tag{3.15}$$

with

$$\begin{aligned} \theta_1^* &= \begin{cases} \theta_Y & \text{if } \theta_Y \in [\theta_a, \pi/2], \\ \theta_a & \text{otherwise,} \end{cases} \\ \theta_2^* &= \begin{cases} \theta_Y & \text{if } \theta_Y \in [\theta_b, \pi/2], \\ \theta_b & \text{otherwise.} \end{cases} \end{aligned} \tag{3.16}$$

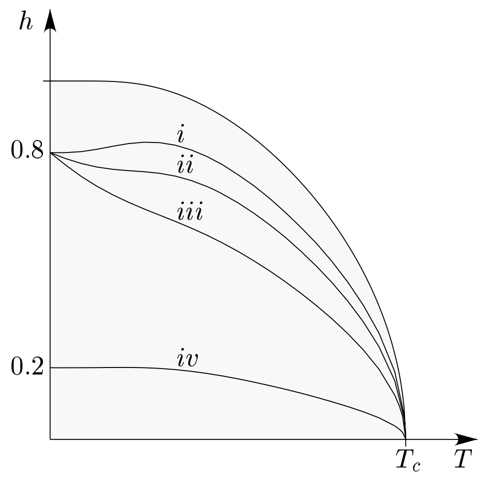

Figure 2: A sequence of phase coexistence lines, separating the phase in which the interface is a straight line and the phase in which it is pinned to the wall, for different values of the parameters a and b. For each of the four curves, for all values of the parameters β and h above the curve, the interface is the straight line and for all values of the parameters below the curve the interface is pinned to the wall. The shaded area correspond to the value of (β, h) so that $\hat{\tau}_{\mathrm{bd}}(\beta, h) < \hat{\tau}((1,0);\beta)$. The four curves correspond to: i) $a = 0.1$, $b = 0.1$; ii) $a = 0.1$, $b = 0.2$; iii) $a = 0.1$, $b = 0.4$; iv) $a = 0.4$, $b = 0.4$. Observe that the system in case 1) exhibits reentrance: if we fix the value of the magnetic field near 0.8 and increase the temperature from 0 to T_c, the system changes from phase I to phase II and then to phase I again (see also Fig. 3).

If (3.14) holds, then (3.15) implies $\mathtt{W}(\widehat{\mathcal{C}}) \geq \mathtt{W}(\mathcal{W})$. If (3.14) does not hold, then the two segments from A to the wall, and from B to the wall intersect at some point P. Let $\widehat{\mathcal{W}}$ be the simple polygonal curve going from A to P, then from P to B. We have (this follows from Lemma 3.1 and $\hat{\tau}(1,0) \geq \hat{\tau}_{\mathrm{bd}}$)

$$g(\theta_Y, a) + g(\theta_Y, b) \geq \mathtt{W}(\widehat{\mathcal{W}})\,. \tag{3.17}$$

Applying again Lemma 3.1 we get

$$\mathtt{W}(\widehat{\mathcal{W}}) > \mathtt{W}(\mathcal{D})\,. \tag{3.18}$$

□

4 Concentration properties

By duality, properties of interfaces at temperatures below T_c are related to properties of the random–line representation of the two–point function of the Ising model at temperatures above T_c. The results of this section are given for the two–point function and are valid for all temperatures above the critical temperature T_c. Our concentration results are based on the sharp triangle inequality, which allows us to improve Propositions 6.1 and 6.2 of [PV1]. The results are essentially optimal.

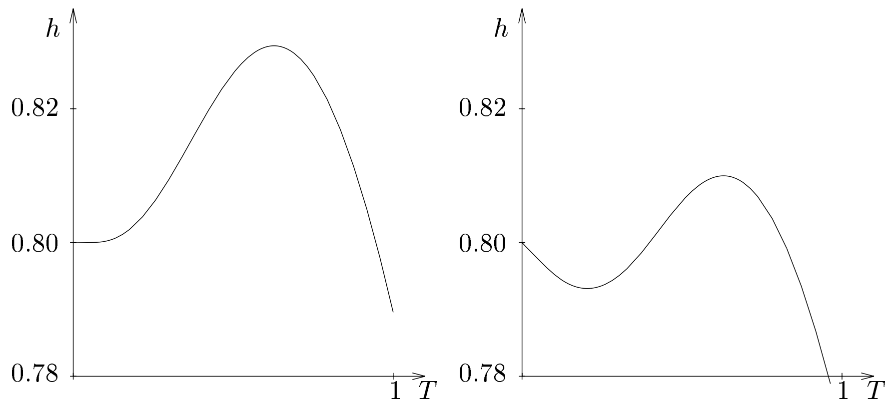

Figure 3: This figure shows part of the phase coexistence line for $a = 0.1, b = 0.1$ (left), and $a = 0.1, b = 0.12$ (right). For values of the parameters β and h below these curves the interface is pinned, while it is a straight line above these curves. In case 2) the system has even one more transition in temperature for h slightly smaller than 0.8.

The random–line representation for the two–point function $\langle\sigma(t)\sigma(t')\rangle_{\Lambda_L^*}$ is the formula

$$\langle\sigma(t)\sigma(t')\rangle_{\Lambda_L^*} = \sum_{\lambda:t\to t'} q_L^*(\lambda)\,. \tag{4.1}$$

On the set of all open contours λ, such that $\delta\lambda = \{t, t'\}$, $q_L^*(\lambda)$ defines a measure whose total mass is $\langle\sigma(t)\sigma(t')\rangle_{\Lambda_L^*}$. The same is true for the similar representations of $\langle\sigma(t)\sigma(t')\rangle(\beta^*)$ or $\langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}^*}$. It is therefore important to have good upper and lower bounds for these quantities. We recall some basic results. Proposition 4.1 is proven in [MW] and the last part in [PV1]; Proposition 4.2 is proven in [PV1].

We set $\Sigma^* := \{x \in \mathbb{L}^* : x(2) = -1/2\,\}$ and $\Sigma_L^* := \Sigma^* \cap \Lambda_L^*$.

Proposition 4.1 *Let $\beta^* < \beta_c$.*

1. *There exists K such that* [12]

$$\langle\sigma(x)\sigma(y)\rangle \geq K\frac{\exp\{-\hat\tau(y-x)\}}{\|x-y\|^{1/2}}\,. \tag{4.2}$$

2. *Let $\hat\tau(1,0) = \hat\tau_{\mathrm{bd}}$ and $x, y \in \Sigma^*$. Then there exists K' such that*

$$\langle\sigma(x)\sigma(y)\rangle_{\mathbb{L}^*} \geq K'\frac{\exp\{-\hat\tau_{\mathrm{bd}}(y-x)\}}{\|x-y\|^{3/2}}\,. \tag{4.3}$$

3. *Let $\hat\tau(1,0) > \hat\tau_{\mathrm{bd}}$ and $x, y \in \Sigma^*$. Then there exists $C > 0$ such that*

$$\langle\sigma(x)\sigma(y)\rangle_{\mathbb{L}^*} \geq C\exp\{-\hat\tau_{\mathrm{bd}}\|x-y\|\}\,. \tag{4.4}$$

Proposition 4.2 *Let $\beta^* < \beta_c$ and $0 < h^* < \infty$.*

[12]We need the bound (4.2) for all temperatures $\beta^* < \beta_c$. Such results have been obtained perturbatively in [Pl], [BF] and [DKS] for example. These lower bounds are essential in the proof of Proposition 5.1.

1. *Let x, y be two different points of $\mathbb{Z}^{2*}$. Then*
$$\langle \sigma(x)\sigma(y)\rangle = \sum_{\lambda: x\to y} q^*(\lambda) \leq \exp\{-\hat{\tau}(y-x)\}\,. \tag{4.5}$$

2. *Let x, y be two different points of Σ_L^*. Then*
$$\langle \sigma(x)\sigma(y)\rangle_{\Lambda_L^*} \leq \langle \sigma(x)\sigma(y)\rangle_{\mathbb{L}^*} = \sum_{\lambda: x\to y} q^*_{\mathbb{L}^*}(\lambda) \leq \exp\{-\hat{\tau}_{\mathrm{bd}} \cdot \|y-x)\|\}\,. \tag{4.6}$$

3. *Let x, y be two different points of Λ_L^*. Then*
$$\sum_{\substack{\lambda: x\to y\\ \lambda\cap\mathcal{E}^*(\Sigma^*)=\emptyset}} q^*_L(\lambda;\beta^*,h^*) \leq \langle \sigma(x)\sigma(y)\rangle(\beta^*)\,. \tag{4.7}$$

4. *Let x, y, z be three different points of Λ_L^*. Then*
$$\sum_{\substack{\lambda: x\to z\\ y\ni\lambda}} q^*_L(\lambda) \leq \sum_{\lambda: x\to y} q^*_L(\lambda) \sum_{\lambda: y\to z} q^*_L(\lambda)\,. \tag{4.8}$$

Remark: Statement 4. of Proposition 4.2 can be written as
$$\sum_{\substack{\lambda: x\to z\\ y\ni\lambda}} q^*_L(\lambda) \leq \langle \sigma(x)\sigma(y)\rangle_{\Lambda_L^*}\,\langle \sigma(y)\sigma(z)\rangle_{\Lambda_L^*}\,. \tag{4.9}$$

There are variants of 4.. Let $\lambda : x \to z$ and $y \in \lambda$; let λ_1 be the part of λ from x to y and λ_2 be the part of λ from y to z. Then
$$\sum_{\substack{\lambda: x\to z\,, y\ni\lambda\\ \lambda_1\cap\mathcal{E}^*(\Sigma^*)=\emptyset}} q^*_L(\lambda;\beta^*,h^*) \leq \langle \sigma(x)\sigma(y)\rangle_{\Lambda_L^*}(\beta^*,1)\,\langle \sigma(y)\sigma(z)\rangle_{\Lambda_L^*}(\beta^*,h^*)\,. \tag{4.10}$$
or
$$\sum_{\substack{\lambda: x\to z\,, y\ni\lambda\\ \lambda_1\cap\mathcal{E}^*(\Sigma^*)=\emptyset\\ \lambda_2\cap\mathcal{E}^*(\Sigma^*)=\emptyset}} q^*_L(\lambda;\beta^*,h^*) \leq \langle \sigma(x)\sigma(y)\rangle_{\Lambda_L^*}(\beta^*,1)\,\langle \sigma(y)\sigma(z)\rangle_{\Lambda_L^*}(\beta^*,1)\,. \tag{4.11}$$

We prove Propositions 4.3 to 4.5, which state concentration properties of the random lines contributing to the two-point function. Given x, $y \in \mathbb{Z}^{2*}$ and $\rho > 0$ we set
$$\mathcal{S}(x,y;\rho) := \{t \in \mathbb{Z}^{2*}\ :\ \|t-x\| + \|t-y\| \leq \|y-x\| + \rho\}\,, \tag{4.12}$$
and $\partial\mathcal{S}(x,y;\rho)$ is the set of $z \in \mathbb{Z}^{2*}\backslash\mathcal{S}(x,y;\rho)$ such that there exists an edge $\langle z,z'\rangle$ with $z' \in \mathcal{S}(x,y;\rho)$.

Proposition 4.3 *Let x, $y \in \Lambda_L^*$ and $\rho > 0$. Let $\mathcal{S}_1 := \mathcal{S}(x,y;\rho)$. Then ($\Delta$ is defined in (2.20) and (2.21))*
$$\sum_{\substack{\lambda:\, x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_1)\\ \lambda\cap\mathcal{E}^*(\mathcal{S}_1)\cap\mathcal{E}^*(\Sigma_L^*)=\emptyset}} q^*_L(\lambda) \leq |\partial\mathcal{S}_1|\exp\{-\Delta\rho\}\exp\{-\hat{\tau}(y-x)\} \tag{4.13}$$
$$+\sum_{z_1,z_2\in\Sigma_L^*} \exp\{-\hat{\tau}(z_1-x)\}\exp\{-\hat{\tau}_{\mathrm{bd}}(z_2-z_1)\}\exp\{-\hat{\tau}(y-z_2)\}\,.$$

Proof. Let $s \mapsto \lambda(s)$ be a parameterization of the open contour λ from x to y. Let s_1 be the first time (if any) such that $\lambda(s_1) \in \Sigma_L^*$ and s_2 the last time such that $\lambda(s_2) \in \Sigma_L^*$.

$$\begin{aligned}\sum_{\substack{\lambda:\, x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_1)\\ \lambda\cap\mathcal{E}^*(\mathcal{S}_1)\cap\mathcal{E}^*(\Sigma_L^*)=\emptyset}} q_L^*(\lambda) \;&\leq\; \sum_{t\in\partial\mathcal{S}_1}\sum_{\substack{\lambda:x\to y\,,\lambda\ni t\\ \lambda\cap\mathcal{E}^*(\Sigma_L^*)=\emptyset}} q_L^*(\lambda) \\ &\quad+\sum_{z_1,z_2\in\Sigma_L^*}\sum_{\substack{\lambda:x\to y\\ \lambda(s_1)=z_1,\lambda(s_2)=z_2}} q_L^*(\lambda)\,.\end{aligned} \tag{4.14}$$

Using Proposition 4.2, the remark following it and the sharp triangle inequality, we get

$$\begin{aligned}\sum_{t\in\partial\mathcal{S}_1}\sum_{\substack{\lambda:x\to y\,,\lambda\ni t\\ \lambda\cap\mathcal{E}^*(\Sigma_L^*)=\emptyset}} q_L^*(\lambda) \;&\leq\; \sum_{t\in\partial\mathcal{S}_1}\langle\sigma(x)\sigma(t)\rangle_{\Lambda_L^*}(\beta^*,1)\,\langle\sigma(t)\sigma(y)\rangle_{\Lambda_L^*}(\beta^*,1) \\ &\leq \sum_{t\in\partial\mathcal{S}_1}\exp\{-(\hat\tau(t-x)+\hat\tau(y-t))\} \\ &\leq \sum_{t\in\partial\mathcal{S}_1}\exp\{-(\hat\tau(y-x)+\Delta\rho)\} \\ &\leq |\partial\mathcal{S}_1|\exp\{-\Delta\rho\}\exp\{-\hat\tau(y-x)\}\,,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\sum_{z_1,z_2\in\Sigma_L^*}\sum_{\substack{\lambda:x\to y\\ \lambda(s_1)=z_1,\lambda(s_2)=z_2}} q_L^*(\lambda) \;&\leq\; \sum_{z_1,z_2\in\Sigma_L^*}\exp\{-\hat\tau(z_1-x)\} \\ &\quad\cdot\exp\{-\hat\tau_{\mathrm{bd}}(z_2-z_1)\}\exp\{-\hat\tau(y-z_2)\}\,.\end{aligned} \tag{4.16}$$

□

Remarks.

1. If $h^* \leq 1$, then the statement (4.13) simplifies,

$$\sum_{\substack{\lambda:\, x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_1)}} q_L^*(\lambda) \leq |\partial\mathcal{S}_1|\exp\{-\Delta\rho\}\exp\{-\hat\tau(y-x)\}\,. \tag{4.17}$$

2. At the thermodynamical limit we can improve (4.17) using Proposition 4.1.

$$\sum_{\substack{\lambda:x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_1)}} q^*(\lambda) \leq \frac{1}{K}|\partial\mathcal{S}_1|\|x-y\|^{1/2}\exp\{-\Delta\rho\}\langle\sigma(x)\sigma(y)\rangle\,. \tag{4.18}$$

The replacement of $\exp\{-\hat\tau(y-x)\}$ by $\langle\sigma(x)\sigma(y)\rangle$ is significant since $\langle\sigma(x)\sigma(y)\rangle$ is the total mass of the measure defined by $q^*(\lambda)$ on the set of all λ with $\delta\lambda = \{x, y\}$.

3. If $1 < h^* < h_c^*$, then the sharp triangle inequality applied two times to (4.13) gives

$$\sum_{\substack{\lambda:\, x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_1)\\ \lambda\cap\mathcal{E}^*(\mathcal{S}_1)\cap\mathcal{E}^*(\Sigma_L^*)=\emptyset}} q_L^*(\lambda) \leq \exp\{-\hat\tau(y-x)\}\Big(|\partial\mathcal{S}_1|\exp\{-\Delta\rho\} \tag{4.19}$$

$$+\sum_{z_1,z_2\in\Sigma_L^*}\exp\{-\Delta(\|z_1-x\|+\|z_2-z_1\|+\|y-z_2\|-\|x-y\|)\}\Big).$$

Proposition 4.4 *Let x, $y\in\Lambda_L^*$ with $x(2)=y(2)=-1/2$, and $\rho>0$. Let $\mathcal{S}_2:=\mathcal{S}(x,y;\rho)$. If $\hat\tau(1,0)=\hat\tau_{\mathrm{bd}}$, that is $h^*\le h_c^*$, then*

$$\sum_{\substack{\lambda:x\mapsto y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_2)}} q_L^*(\lambda)\le O(\|x-y\|+\rho)|\partial\mathcal{S}_2|\exp\{-\Delta\rho\}\exp\{-\hat\tau(y-x)\}\,. \tag{4.20}$$

Proof. If $\hat\tau(1,0)=\hat\tau_{\mathrm{bd}}$ and u, $v\in\mathbb{L}^*$ with $u(2)=v(2)=-1/2$, then

$$\hat\tau_{\mathrm{bd}}\|u-v\|=\hat\tau(u-v)\,. \tag{4.21}$$

Let $\lambda:x\mapsto y$, with $\lambda\ni t$, $t\in\partial\mathcal{S}_2$. We consider λ as a parameterized curve, $s\mapsto\lambda(s)$, from x to y. We set $t=\lambda(s^*)$; we denote by s_1 the last time before s^* such that $\lambda(s_1)\in\Sigma_L^*$; we denote by s_2 the first time after s^* such that $\lambda(s_2)\in\Sigma_L^*$. We have

$$\begin{aligned}\sum_{\substack{\lambda:x\to y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_2)}} q_L^*(\lambda) &\le \sum_{t\in\partial\mathcal{S}_2}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda) \\ &\le \sum_{t\in\partial\mathcal{S}_2}\sum_{u,v\in\Sigma_L^*}\sum_{\substack{\lambda:x\to y,\,t\in\lambda\\ \lambda(s_1)=u,\lambda(s_2)=v}} q_L^*(\lambda)\,.\end{aligned} \tag{4.22}$$

Using (4.21), Proposition 4.2, GKS inequalities and the sharp triangle inequality, we get

$$\begin{aligned}\sum_{u,v}\sum_{\substack{\lambda:x\to y\\ \lambda\ni u,t,v}} q_L^* &\le \sum_{u,v}\langle\sigma(x)\sigma(u)\rangle_{\mathbb{L}^*}\langle\sigma(u)\sigma(t)\rangle\langle\sigma(t)\sigma(v)\rangle\langle\sigma(v)\sigma(y)\rangle_{\mathbb{L}^*} \\ &\le \sum_{u,v}\exp\{-\hat\tau(u-x)-\hat\tau(t-u)\}\exp\{-\hat\tau(v-t)-\hat\tau(y-v)\} \\ &\le \sum_{u,v}\exp\{-\hat\tau(t-x)\}\exp\{-\Delta(\|x-u\|+\|u-t\|-\|x-t\|)\} \\ &\quad\cdot\exp\{-\hat\tau(y-t)\}\exp\{-\Delta(\|y-v\|+\|v-t\|-\|y-t\|)\} \\ &\le O(\|x-y\|+\rho)\exp\{-(\hat\tau(t-x)+\hat\tau(y-t))\}\,.\end{aligned}$$

Then the proof is as in (4.15). □

In the case of partial wetting, i.e. $\hat\tau(1,0)>\hat\tau_{\mathrm{bd}}(1,0)$, the previous proposition can be improved to reflect the fact that the contours stick to the wall, even microscopically.

Proposition 4.5 *Suppose $h^*>h_c^*$ (partial wetting for the dual model). Let x, $y\in\Lambda_L^*$ with $x(2)=y(2)=-1/2$, $x(1)<y(1)$, and let $\rho_i>0$, $i=1,2$. Let*

$$\mathcal{S}_3:=\{t\in\Lambda_L^*\,:\,x(1)-\rho_1\le t(1)\le y(1)+\rho_1\,,\,-1/2\le t(2)\le\rho_2\}\,. \tag{4.23}$$

Then, there exists a constant $\overline{C}(\beta)>0$ such that

$$\sum_{\substack{\lambda:x\to y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_3)}} q_L^*(\lambda)\le|\partial\mathcal{S}_3|\Big(\exp\{-2\rho_1\hat\tau_{\mathrm{bd}}\}+|\rho_2|\exp\{-\overline{C}\rho_2\}\Big)\exp\{-\hat\tau_{\mathrm{bd}}\|y-x\|\}\,. \tag{4.24}$$

Proof. Let $\partial\mathcal{S}_3 := \{t \in \mathcal{S}_3 \,:\, t(2) = \rho_2 \text{ or } t(1) = x(1) - \rho_1 \text{ or } t(1) = y(1) + \rho_1\}$; we can write

$$\begin{aligned}\sum_{\substack{\lambda:x\to y\\ \lambda\not\subset\mathcal{E}^*(\mathcal{S}_3)}} q_L^*(\lambda) \;&\le\; \sum_{t\in\partial\mathcal{S}_3}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda) \\ &\le\; \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)<\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda) + \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda)\,.\end{aligned} \tag{4.25}$$

We treat these sums separately. By symmetry and GKS inequalities

$$\begin{aligned}\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)<\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda) \;&\le\; 2\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(1)=x(1)-\rho_1}} \langle\sigma_x\sigma_t\rangle_{\mathbb{L}^*}\langle\sigma_t\sigma_y\rangle_{\mathbb{L}^*} \\ &=\; 2\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(1)=x(1)-\rho_1}} \langle\sigma_{\overline{x}}\sigma_t\rangle_{\mathbb{L}^*}\langle\sigma_t\sigma_y\rangle_{\mathbb{L}^*} \\ &\le\; 2\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(1)=x(1)-\rho_1}} \langle\sigma_{\overline{x}}\sigma_y\rangle_{\mathbb{L}^*} \\ &\le\; 2\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(1)=x(1)-\rho_1}} \exp\{-2\rho_1\hat{\tau}_{\mathrm{bd}}\}\exp\{-\hat{\tau}_{\mathrm{bd}}\|y-x\|\}\,,\end{aligned} \tag{4.26}$$

where $\overline{x}$ is the image of x under a reflection of axis $\{u \,:\, u(1) = t(1)\}$.

Let $t \in \lambda$ and $t(2) = \rho_2$. As above λ is considered as a parameterized curve, and we set $t = \lambda(s^*)$; we denote by s_1 the last time before s^* such that $\lambda(s_1) \in \Sigma_L^*$; we denote by s_2 the first time after s^* such that $\lambda(s_2) \in \Sigma_L^*$. As above we get $(u = \lambda(s_1),\, v = \lambda(s_2))$

$$\begin{aligned}\sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{\substack{\lambda:x\to y\\ \lambda\ni t}} q_L^*(\lambda) \;&\le\; \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{u,v}\langle\sigma_x\sigma_u\rangle_{\mathbb{L}^*}\langle\sigma_u\sigma_t\rangle\langle\sigma_t\sigma_v\rangle\langle\sigma_v\sigma_y\rangle_{\mathbb{L}^*} \\ &\le\; \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{u,v}\exp\{-\hat{\tau}_{\mathrm{bd}}(\|u-x\|+\|y-v\|)\} \\ &\qquad\cdot\exp\{-\hat{\tau}(t-u)-\hat{\tau}(v-t)\} \\ &\le\; \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{u,v}\exp\{-\hat{\tau}_{\mathrm{bd}}(\|u-x\|+\|y-v\|)\} \\ &\qquad\cdot\exp\{-\hat{\tau}(u-v)\}\exp\{-\Delta(\|u-t\|+\|t-v\|-\|u-v\|)\} \\ &\le\; \sum_{\substack{t\in\partial\mathcal{S}_3\\ t(2)=\rho_2}}\sum_{u,v}\exp\{-\hat{\tau}_{\mathrm{bd}}\|x-y\|\}\exp\{-(\hat{\tau}(1,0)-\hat{\tau}_{\mathrm{bd}})\|u-v\|\} \\ &\qquad\cdot\exp\{-\Delta(\|u-t\|+\|t-v\|-\|u-v\|)\}\,,\end{aligned} \tag{4.27}$$

The conclusion follows from the observations: 1. the summation is over the base of the triangle uvt; 2. the term $\exp\{-(\hat{\tau}(1,0)-\hat{\tau}_{\mathrm{bd}})\|u-v\|\}$ allows to control the triangles with a large base, while the term $\exp\{-\Delta(\|u-t\|+\|t-v\|-\|u-v\|)\}$ can be used to control the terms in which the base is far from the point t. $\square$

5 Probability of the phase separation line

We study the probability of the phase separation line by making a coarse–grained description of it. We estimate in terms of its surface tension[13] the probability that a given coarse–grained description occurs.

We first prove an essential lower bound and then proceed with the main estimate.

Proposition 5.1 *Let $\beta > \beta_c$, $h > 0$, $0 < a < 1$, $0 < b < 1$ and $\mathtt{W}^* = \mathtt{W}^*(\beta, h)$ be the minimum of the functional $\mathtt{W}$. Then there exists $C > 0$ and $L_0 = L_0(\beta, h)$ such that, for all $L \geq L_0$,*

$$Z^{ab}(\Lambda_L) \geq \frac{\exp\{-\mathtt{W}^* L\}}{L^C} Z^-(\Lambda_L)\,. \tag{5.1}$$

Remark. The dual statement of Proposition 5.1 is

$$\langle \sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*} \geq \frac{\exp\{-\mathtt{W}^* L\}}{L^C}\,. \tag{5.2}$$

Proof. We can write, using Proposition 2.3,

$$Z^{ab}(\Lambda_L) = Z^-(\Lambda_L) \sum_{\substack{\lambda: \\ \delta\lambda = \{t_l^L, t_r^L\}}} q_L^*(\lambda)\,. \tag{5.3}$$

Let $\mathcal{C}^*$ be the simple rectifiable curve in Q which realizes the minimum of the variational problem (or one of the minima in case of degeneracy). Let $K_1 > 0$, which will be chosen large enough below.

We first consider the case $\mathcal{C}^* = \mathcal{D}$. Let u_l^L and u_r^L be the points of Λ_L^* with $u_l^L(1) = t_l^L(1) + [K_1 \log L]$, $u_r^L(1) = t_r^L(1) - [K_1 \log L]$, which are closest to the straight line from t_l^L to t_r^L.

We need the following result

Lemma 5.1 *Let $\mathcal{B}$ be a rectangular box in Λ_L^*, and x, y two points on its boundary. Let $d > 0$ and u, v be two points in $\mathcal{B}$ such that u and v are closest to the straight line from x to y and $\|x-u\| = \|y-v\| = d$. If the distance of $\mathcal{B}$ to Σ_L^* is larger than $C' \log L$, with $C' > 2$, then*

$$\sum_{\substack{\lambda: x \to y \\ \lambda \subset \mathcal{E}^*(\mathcal{B})}} q_L^*(\lambda) \geq \exp\{-O(\log L)\} \sum_{\substack{\lambda: u \to v \\ \lambda \subset \mathcal{E}^*(\mathcal{B})}} q^*(\lambda)\,. \tag{5.4}$$

Proof. Following the proof of Proposition 6.1 of [PV1], from (6.38) to (6.42), we get

$$\sum_{\substack{\lambda: x \to y \\ \lambda \subset \mathcal{E}^*(\mathcal{B})}} q_L^*(\lambda) \geq \exp\{-O(\log L)\} \sum_{\substack{\lambda: u \to v \\ \lambda \subset \mathcal{E}^*(\mathcal{B})}} q_L^*(\lambda)\,. \tag{5.5}$$

[13] In our problem it is sufficient to give an extremely rough description of the phase separation line, because we do not need to control the volume under the phase separation line, as it was the case in [PV1].

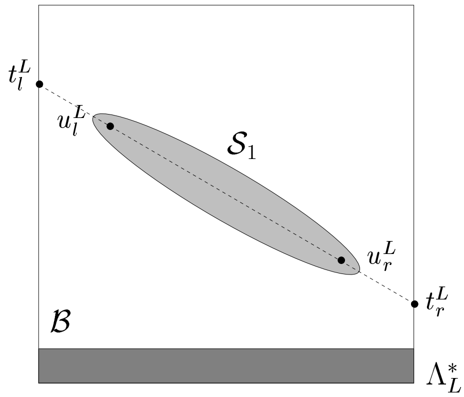

Figure 4: The ellipse $\mathcal{S}_1$; the box $\mathcal{B}$ is the whole box minus the bottom strip.

Since the distance of $\mathcal{B}$ to Σ^* is larger than $C' \log L$ and $C' > 2$, then it follows from point 4. of Lemma 5.3 in [PV1] that there exists a constant $\tilde{C} > 0$ such that for L large enough and all $\lambda \subset \mathcal{E}^*(\mathcal{B})$

$$q_L^*(\lambda) \geq q_{\mathbb{L}^*}^*(\lambda) \geq \tilde{C} q^*(\lambda)\,. \tag{5.6}$$

This proves the lemma. □

Let $C' > 2$. We introduce $\mathcal{B} := \{t \in \Lambda_L^* \,:\, t(2) > [C' \log L]\}$. Let $\mathcal{S}_1$ be the elliptical set of Proposition 4.3 with $x = u_l^L$, $y = u_r^L$, $\rho = O(K_1 \log L)$ so that $\mathcal{E}^*(\mathcal{S}_1) \subset \mathcal{E}^*(\mathcal{B})$ (see Fig. 4). From Lemma 5.1, applied to the box $\mathcal{B}$ with $x = t_l^L$, $y = t_r^L$, $u = u_l^L$ and $v = v_l^L$, and the second remark following Proposition 4.3, we get

$$\begin{aligned}
\frac{Z^{ab}(\Lambda_L)}{Z^-(\Lambda_L)} &\geq \sum_{\substack{\lambda : t_l^L \to t_r^L \\ \lambda \subset \mathcal{E}^*(\mathcal{B})}} q_L^*(\lambda) \\
&\geq \exp\{-O(\log L)\} \sum_{\substack{\lambda : u_l^L \to u_r^L \\ \lambda \subset \mathcal{E}^*(\mathcal{S}_1)}} q^*(\lambda) \\
&\geq \exp\{-O(\log L)\} \Big(\sum_{\lambda : u_l^L \to u_r^L} q^*(\lambda) - \sum_{\substack{\lambda : u_l^L \to u_r^L \\ \lambda \not\subset \mathcal{E}^*(\mathcal{S}_1)}} q^*(\lambda) \Big) \\
&\geq \exp\{-O(\log L)\} \langle \sigma(u_l^L) \sigma(u_r^L) \rangle \Big(1 - \frac{1}{K} |\partial \mathcal{S}_1| \, \|u_l^L - u_r^L\|^{1/2} \exp\{-\Delta\rho\} \Big) \\
&\geq \frac{\exp\{-\hat{\tau}(t_l^L - t_r^L)\}}{L^C} \,,
\end{aligned} \tag{5.7}$$

for some positive constant C, by taking K_1 large enough.

We now consider the case $\mathcal{C}^* = \mathcal{W}$. Since $h > 0$, the angle θ_Y satisfies $0 < \theta_Y < \pi/2$. Denote by w_1^L and w_2^L the two points on Σ_L^* which are closest to the corners of the polygonal line $\mathcal{W}$ scaled by L.

We define three rectangular boxes (see Fig. 5)

$$\mathcal{B}_1 = \{t \in \Lambda_L^* \,:\, t(1) \leq w_1^L(1)\,,\, t(2) > [C' \log L]\}\,, \tag{5.8}$$

$$\mathcal{B}_2 = \{t \in \Lambda_L^* \,:\, w_1^L(1) \leq t(1) \leq w_2^L(1)\}\,, \tag{5.9}$$

$$\mathcal{B}_3 = \{t \in \Lambda_L^* \,:\, w_2^L(1) \leq t(1)\,,\, t(2) > [C' \log L]\}\,. \tag{5.10}$$

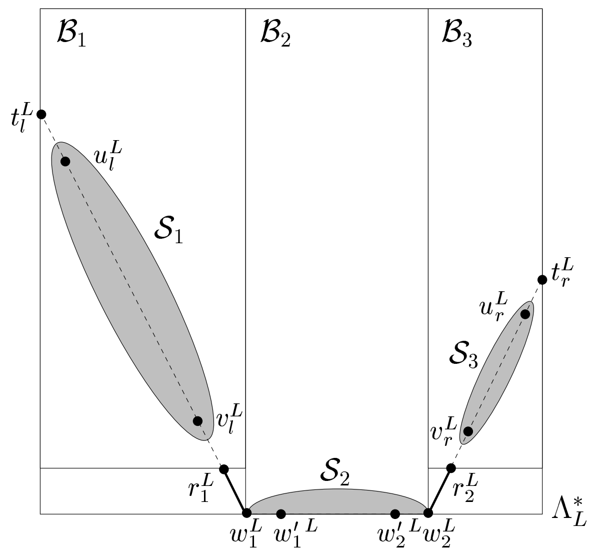

Figure 5: The three boxes $\mathcal{B}_i$, $i = 1, \dots, 3$ and their elliptical subsets; the two bold segments represent the two shortest contours $\overline{\lambda}_i$, $i = 1, 2$.

Moreover let r_1^L, resp. r_2^L, be the point of Λ_L^* closest to the straight line through t_l^L and w_1^L, resp. t_r^L and w_2^L, such that $r_1^L(2) = [C' \log L] + 1/2$, resp. $r_2^L(2) = [C' \log L] + 1/2$. Let $\overline{\lambda}_1$, resp. $\overline{\lambda}_2$, be a shortest open contour from r_1^L to w_1^L, resp. from r_2^L to w_2^L. We define $\mathcal{A}$ as the set $\mathcal{A}$ of open contours $\lambda = \lambda_1 \cup \overline{\lambda}_1 \cup \lambda_2 \cup \overline{\lambda}_2 \cup \lambda_3$ such that

- $\lambda_i \subset \mathcal{E}^*(\mathcal{B}_i)$, $i = 1, \dots, 3$;
- $\lambda_1 : t_l^L \to r_1^L$;
- $\lambda_2 : w_1^L \to w_2^L$;
- $\lambda_3 : r_2^L \to t_r^L$.

We can write

$$\begin{aligned} \frac{Z^{ab}(\Lambda_L)}{Z^-(\Lambda_L)} &\geq \sum_{\lambda \in \mathcal{A}} q_L^*(\lambda) \\ &\geq \exp\{-O(\log L)\} \prod_i \sum_{\substack{\lambda_i \subset \mathcal{E}^*(\mathcal{B}_i): \\ \lambda_i \text{ as above}}} q_L^*(\lambda_i)\,. \end{aligned} \tag{5.11}$$

We apply Lemma 5.1 to the sum over λ_1 with $\mathcal{B} = \mathcal{B}_1$, $x = t_l^L$, $y = r_1^L$, $d = K_1 \log L$; we denote by u_l^L and v_l^L the corresponding points u and v. We apply the same lemma to the sum over λ_3 with $\mathcal{B} = \mathcal{B}_2$, $x = r_2^L$, $y = t_r^L$, $d = K_1 \log L$; we denote by u_r^L and v_r^L the corresponding points u and v. The sum over λ_2 is taken care of by the following

Lemma 5.2 *Let $w_1'^L := w_1^L + ([K_1 \log L], 0)$, $w_2'^L := w_2^L - ([K_1 \log L], 0)$. Then*

$$\sum_{\substack{\lambda : w_1^L \to w_2^L \\ \lambda \subset \mathcal{E}^*(\mathcal{B}_2)}} q_L^*(\lambda) \geq \exp\{-O(\log L)\} \sum_{\substack{\lambda : w_1'^L \to w_2'^L \\ \lambda \subset \mathcal{E}^*(\mathcal{B}_2)}} q_{\mathbb{L}^*}^*(\lambda)\,. \tag{5.12}$$

Proof. Same proof of that of Proposition 6.2 in [PV1]. □

We finally introduce three elliptical sets. $\mathcal{S}_1$ is constructed as in Proposition 4.3 with $x = u_l^L$, $y = v_l^L$ and $\rho = O(K_1 \log L)$ such that $\mathcal{E}^*(\mathcal{S}_1) \subset \mathcal{E}^*(\mathcal{B}_1)$; $\mathcal{S}_3$ is constructed as in Proposition 4.3 with $x = v_r^L$, $y = u_r^L$ and $\rho = O(K_1 \log L)$ such that $\mathcal{E}^*(\mathcal{S}_3) \subset \mathcal{E}^*(\mathcal{B}_3)$; $\mathcal{S}_2$ is constructed as in Proposition 4.4 with $x = {w'_1}^L$, $y = {w'_2}^L$ and $\rho = [K_1 \log L]$ (and therefore $\mathcal{E}^*(\mathcal{S}_2) \subset \mathcal{E}^*(\mathcal{B}_2)$).

Applying Propositions 4.3 and 4.4 as before gives the conclusion,

$$\frac{Z^{ab}(\Lambda_L)}{Z^-(\Lambda_L)} \geq \frac{\exp\{-L\mathcal{W}^*\}}{L^C} \tag{5.13}$$

for some positive constant C. □

We prove using the above proposition, that the surface tension of a (very) coarse-grained version of the phase separation line cannot be too large compared to $\mathtt{W}^*$.

Let λ be the open contour. We construct a polygonal line approximation $\mathcal{P} := \mathcal{P}(\lambda)$ of λ. Let $s \mapsto \lambda(s)$ be a unit-speed parameterization of λ. If $\lambda(s)(2) > -1/2$ for all s, then let $\mathcal{P}$ be the straight line from t_l^L to t_r^L. Otherwise, let s_1 be the first time such that $\lambda(s)(2) = -1/2$ and s_2 the last time such that $\lambda(s)(2) = -1/2$; we write $\hat{w}_i^L = \lambda(s_i)$, $i = 1, 2$. We also introduce $[\mathcal{P}] := \{\omega : \mathcal{P}(\lambda(\omega)) = \mathcal{P}\}$.

By construction, if $s < s_1$ or $s > s_2$ then $\lambda(s)(2) \neq -1/2$. We can therefore apply Proposition 4.2 to estimate the probability of the event $[\mathcal{P}]$,

$$P_L^{ab}[\,[\mathcal{P}]\,] \leq \exp\{-\mathtt{W}(\mathcal{P})\}\,. \tag{5.14}$$

Proposition 5.2 *Let $\beta > \beta_c$, $h > 0$, $0 < a < 1$, $0 < b < 1$. Then there exists $L_0 = L_0(\beta, h)$ such that, for all $L \geq L_0$ and $T > 0$,*

$$P_L^{ab}[\,\{\mathtt{W}(\mathcal{P}(\lambda)) \geq \mathtt{W}^* L + T\}\,] \leq \exp\{-T + O(\log L)\}\,. \tag{5.15}$$

Proof. Let

$$\mathcal{I}(T) := \{\lambda \subset \mathcal{E}^*(\Lambda_L^*) \,:\, \delta\lambda = \{t_l^L, t_r^L\}\,,\, \mathtt{W}(\mathcal{P}(\lambda)) \geq \mathtt{W}^* L + T\}\,. \tag{5.16}$$

Then, from Propositions 5.1, 2.3 and (5.14),

$$\begin{aligned} P_L^{\pm}[\,\{\mathtt{W}(\mathcal{P}(\lambda)) \geq \mathtt{W}^* L + T\}\,] &= \frac{Z^-(\Lambda_L)}{Z^{ab}(\Lambda_L)} \sum_{\lambda \in \mathcal{I}(T)} q_L^*(\lambda) \\ &\leq \exp\{\mathtt{W}^* L + O(\log L)\} \sum_{\lambda \in \mathcal{I}(T)} q_L^*(\lambda) \\ &\leq \exp\{\mathtt{W}^* L + O(\log L)\} \sum_{\substack{\mathcal{P}: \\ \mathtt{W}(\mathcal{P}) \geq \mathtt{W}^* L + T}} \; \sum_{\substack{\lambda: \\ \mathcal{P}(\lambda) = \mathcal{P}}} q_L^*(\lambda) \\ &\leq \exp\{\mathtt{W}^* L + O(\log L)\} \sum_{\substack{\mathcal{P}: \\ \mathtt{W}(\mathcal{P}) \geq \mathtt{W}^* L + T}} \exp\{-\mathtt{W}(\mathcal{P})\} \\ &\leq \exp\{-T + O(\log L)\}\,. \end{aligned} \tag{5.17}$$

(The number of different coarse–grained polygonal lines is bounded by $O(L^2)$.) □

6 Pinning transition

The main result of the paper is a statement about concentration properties of the probability of the phase separation line. An immediate consequence of Theorem 6.1 is that at a suitable scale, when $L \to \infty$, the phase separation line defines a non–random object, the interface, which is characterized as the solution of the variational problem discussed in section 3, that is, the interface in Q is either the straight line $\mathcal{D}$ or the broken line $\mathcal{W}$. We obtain an essentially optimal description of the location of the interface up to the scale of normal fluctuations of the phase separation line.

6.1 Main result

The weight of a separation line λ in Λ_L^*, going from t_l^L to t_r^L, is given by $q_L(\lambda;\beta,h) = q_L^*(\lambda;\beta^*,h^*)$. These weights define a measure on the the set of the phase separation lines, such that the total mass is

$$\sum_{\substack{\lambda\subset\mathcal{E}(\Lambda_L^*):\\ \delta\lambda=\{t_l^L,t_r^L\}}} q_L^*(\lambda;\beta^*,h^*) = \langle\sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}(\beta^*,h^*)\,. \tag{6.1}$$

Consequently

$$P_L^{ab}[\lambda] = \frac{q_L^*(\lambda)}{\langle\sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}}\,. \tag{6.2}$$

Let $\mathcal{D}$ and $\mathcal{W}$ be the curves in Q introduced in section 3. We set

$$I_i^L := \{\, x\in\Sigma_L^* :\ \|x-w_i^L\| \le (ML\log L)^{1/2}\,\} \quad , \quad i=1,2\,, \tag{6.3}$$

and

$$\rho_L := M\log L\,. \tag{6.4}$$

We define two sets of contours. The set $\mathcal{T}_{\mathcal{D}}$ contains all $\lambda\subset\mathcal{E}^*(\Lambda_L^*)$ such that

a_1. $\delta\lambda = \{t_l^L, t_r^L\}$;

a_2. λ is inside $\mathcal{S}(t_l^L, t_r^L;\rho_L)$.

The set $\mathcal{T}_{\mathcal{W}}$ contains all $\lambda\subset\mathcal{E}^*(\Lambda_L^*)$, considered as parameterized curves $s\mapsto\lambda(s)$, such that

b_1. $\delta\lambda = \{t_l^L, t_r^L\}$, $\lambda(0) := t_l^L$;

b_2. $\exists s_1$ such that $\lambda(s_1)\in I_1^L$ and for all $s<s_1$, $\lambda(s)\cap\Sigma_L^* = \emptyset$;

b_3. $\lambda_1 := \{\lambda(s):\ s\le s_1\}$ is inside $\mathcal{S}(t_l^L,\lambda(s_1);\rho_L)$;

b_4. $\exists s_2$ such that $\lambda(s_2)\in I_2^L$ and for all $s_2<s$, $\lambda(s)\cap\Sigma_L^* = \emptyset$;

b_5. $\lambda_3 := \{\lambda(s):\ s_2\le s\}$ is inside $\mathcal{S}(\lambda(s_2), t_r^L;\rho_L)$;

b_6. $\lambda_2 := \{\lambda(s) : s_1 \leq s_2 \leq s\}$ is inside

$$\{x \in \Lambda_L^* : x(2) \leq \rho_L \,,\ \lambda(s)(1) - \rho_L \leq x(1) \leq \lambda(s)(2) + \rho_L\}\,.$$

Theorem 6.1 *Let $\beta > \beta_c$, $h > 0$, $0 < a < 1$, $0 < b < 1$. There exist $M > 0$ and $L_0 = L_0(h, \beta, M)$ such that, for all $L \geq L_0$, the following statements are true.*

1. *Suppose that the solution of the variational problem in Q is the curve $\mathcal{D}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{D}}] \geq 1 - L^{-O(M)}\,. \tag{6.5}$$

2. *Suppose that the solution of the variational problem in Q is the curve $\mathcal{W}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{W}}] \geq 1 - L^{-O(M)}\,. \tag{6.6}$$

3. *Suppose that the solution of the variational problem in Q is the either the curve $\mathcal{D}$ or the curve $\mathcal{W}$. Then*

$$P_L^{ab}[\mathcal{T}_{\mathcal{D}} \cup \mathcal{T}_{\mathcal{W}}] \geq 1 - L^{-O(M)}\,. \tag{6.7}$$

Comment: The results of Theorem 6.1 are, in some sense, optimal. Indeed, at a finer scale we do not expect the phase separation line to converge to some non–random set, but rather to some random process. It is known that fluctuations of a phase separation line of length $O(L)$, which is not in contact with the wall, are $O(L^{1/2})$ (see [Hi] and [DH]). On the other hand, if the phase separation line is attracted by the wall on a length $O(L)$, then we expect that its excursions away from the wall have a size typically bounded by $O(\log L)$.

Proof.

1. Suppose that the minimum of the variational problem is given by $\mathcal{D}$, $\mathtt{W}(\mathcal{D}) = \mathtt{W}^*$. Let $\mathtt{W}^{**}$ be the minimum of the functional over all simple curves in Q, with end–points A and B, and which touch the wall w_Q. By hypothesis there exists $\delta > 0$ with $\mathtt{W}^{**} = \mathtt{W}^* + \delta$.

We set $\mathcal{S}_1 := \mathcal{S}(t_l^L, t_r^L; \rho_L)$; for L large enough $\mathcal{S}_1 \cap \Sigma_L^* = \emptyset$ since $a > 0$ and $b > 0$.

We apply Proposition 5.1. We have

$$\begin{aligned} P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{D}}\}] &= \frac{1}{\langle \sigma(t_l^L)\sigma(t_r^L)\rangle_{\Lambda_L^*}} \sum_{\lambda \notin \mathcal{T}_{\mathcal{D}}} q_L^*(\lambda) \\ &\leq L^C \exp\{\mathtt{W}^* L\} \sum_{\lambda \notin \mathcal{T}_{\mathcal{D}}} q_L^*(\lambda)\,. \end{aligned} \tag{6.8}$$

We apply Proposition 4.3.

$$\sum_{\lambda \notin \mathcal{T}_{\mathcal{D}}} q_L^*(\lambda) \leq |\partial \mathcal{S}_1| \exp\{-\Delta\rho\} \exp\{-\hat{\tau}(t_r^L - t_l^L)\} \tag{6.9}$$

$$+ \sum_{z_1, z_2 \in \Sigma_L^*} \exp\{-\hat{\tau}(z_1 - t_l^L)\} \exp\{-\hat{\tau}_{\mathrm{bd}}(z_2 - z_1)\} \exp\{-\hat{\tau}(t_r^L - z_2)\}\,.$$

We can bound above the last sum by $O(L^2)\exp\{-L\mathtt{W}^{**}\}$. Therefore

$$P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{D}}\}] \leq \frac{O(L^{C+1})}{L^{\Delta M}} + O(L^{C+2})\exp\{-L\delta\}\,. \tag{6.10}$$

This proves the first statement.

2. Suppose that the minimum of the variational problem is given by $\mathcal{W}$, $\mathtt{W}(\mathcal{W}) = \mathtt{W}^*$. Then there exists $\delta > 0$ such that $\mathtt{W}(\mathcal{D}) = \mathtt{W}^* + \delta$. We estimate $P_L^{ab}[\{\lambda \notin \mathcal{T}_{\mathcal{W}}\}]$ in several steps. Notice that condition b_1 is always satisfied.
1. The probability that condition b_2 is satisfied, but not b_3, can be estimated by Proposition 4.3; it is smaller than $O(L^{C+1})/L^{\Delta M}$.
2. The probability that condition b_4 is satisfied, but not b_5, is estimated in the same way; it is smaller than $O(L^{C+1})/L^{\Delta M}$.
3. The probability that conditions b_2 and b_4 are satisfied, but not b_6, can be estimated by Proposition 4.5; it is smaller than $O(L^{C+1})/L^{\overline{C}\Delta M}$.
4. We estimate the probability that condition b_2 is not satisfied. The case with condition b_5 is similar. If λ does not intersect Σ_L^*, then this probability is smaller than $O(L^C)\exp\{-\delta L\}$, since $\mathtt{W}(\mathcal{D}) = \mathtt{W}^* + \delta$. Suppose that there exist s_1 and s_2, with $\lambda(s_i) \in \Sigma_L^*$, $\lambda(s) \cap \Sigma_L^* = \emptyset$ for all $s < s_1$ and $\lambda(s) \cap \Sigma_L^* = \emptyset$ for all $s_2 < s$. Let $p_i^L := \lambda(s_i)$, $i = 1, 2$. Under these conditions, b_2 is not satisfied if and only if $p_1^L \notin I_1^L$. Let $\mathcal{C}(p_1^L, p_2^L)$ be the polygonal curve from t_l^L to p_1^L, then from p_1^L to p_2^L and finally from p_2^L to t_r^L. Then the probability of this event is bounded above by

$$\sum_{\substack{p_1^L \in \Sigma_L^*:\\ p_1^L \notin I_1^L}} \sum_{p_2^L \in \Sigma_L^*} \exp\{-\mathtt{W}(\mathcal{C}(p_1^L, p_2^L))\} \leq \tag{6.11}$$

$$O(L^2)\min\{\exp\{-\mathtt{W}(\mathcal{C}(p_1^L, p_2^L))\} \,|\, p_1^L \in \Sigma_L^* \backslash I_1^L\,,\, p_2^L \in \Sigma_L^*\}\,.$$

Suppose that $\mathcal{C}$ denotes the polygonal line giving the minimum; scaled by $1/L$ we get a polygonal line in Q, denoted by $\mathcal{C}^*$, from A to some point P_1^*, then from P_1^* to P_2^* and finally from P_2^* to B. Let θ^* be the angle between the straight line from A to P_1^* with the wall. We have

$$\mathtt{W}(\mathcal{C}) = L\mathtt{W}(\mathcal{C}^*) \geq L(g(\theta^*, a) + g(\theta_Y, b))\,. \tag{6.12}$$

By hypothesis

$$|\theta^* - \theta_Y| \geq \frac{1}{L^{1/2}} O((M\log L)^{1/2})\,. \tag{6.13}$$

Therefore (use a Taylor expansion of g around θ_Y and the monotonicity of $g(\theta, x)$ on $[0, \theta_Y]$, respectively $[\theta_Y, \pi/2]$) there exists a positive constant α such that

$$\begin{aligned} \mathtt{W}(\mathcal{C}^*) &\geq g(\theta_Y, a) + g(\theta_Y, b) + \frac{\alpha M \log L}{L} \\ &= \mathtt{W}^* + \frac{\alpha M \log L}{L}\,. \end{aligned} \tag{6.14}$$

We conclude that the probability, that condition b_2 is not satisfied, is bounded above by $O(L^{c+2})/L^{\alpha M}$. If M is large enough, the second statement of the theorem is true.

3. The proof of the third statement of the theorem is similar. □

6.2 Finite size effects for the correlation length

By duality we can interpret Proposition 5.1 and Theorem 6.1 at temperatures above T_c. The fact that t_l^L and t_r^l are points on the boundary of Λ_L^* is not important for the dual model above the critical temperature, as one can check easily from the proofs of sections 5 and 6.

Let $\beta^* < \beta_c$ and Λ'_L be the subset of $\mathbb{Z}^{2*}$ obtained by translating Λ_L^* by $(0, -L/2)$, that is,

$$\Lambda'_L := \{\, t \in \mathbb{Z}^{2*} : \, t + (0, L/2) \in \Lambda_L^* \,\} \, . \tag{6.15}$$

We consider the Ising model with free b.c. on Λ'_L with coupling constants

$$J^*(t,t') := \begin{cases} h^* > 0 & \text{if } t(2) = t'(2) = -1/2 - L/2, \\ 1 & \text{otherwise.} \end{cases} \tag{6.16}$$

The corresponding Gibbs state is denoted by $\langle \cdot \rangle_{\Lambda'_L}$ or by $\langle \cdot \rangle_{\Lambda'_L}(\beta^*, h^*)$. When $L \to \infty$ the states $\langle \cdot \rangle_{\Lambda'_L}(\beta^*, h^*)$ converge to the unique infinite Gibbs state $\langle \cdot \rangle(\beta^*)$ of the model with coupling constant 1, independently of the value of h^*, since $\beta^* < \beta_c$. The (horizontal) correlation length $\xi(\beta^*)$ is therefore independent of h^* and is given by the formula

$$\xi(\beta^*) := - \lim_{L \to \infty} \frac{1}{L} \log \langle \, \sigma(t) \sigma(t + (L, 0)) \, \rangle(\beta^*) \, . \tag{6.17}$$

Theorem 6.1, as well as Proposition 5.1, show that in general we do not get the same result if we take the thermodynamical limit and the limit in (6.17) together. Indeed, we can find h^* and t_L such that the distance of t_L to the boundary of the box Λ'_L is $O(L)$ and

$$- \lim_{L \to \infty} \frac{1}{L} \log \langle \, \sigma(t_L) \sigma(t_L + (L, 0)) \, \rangle_{\Lambda'_L}(\beta^*, h^*) \neq \xi(\beta^*) \, , \tag{6.18}$$

because in the random–line representation of the two–point correlation function the random lines are concentrated near a part of the boundary of the box Λ'_L. Borrowing the terminology of [SML] about the long–range order, we can say that there is no equivalence in general between the "short" correlation length $\xi(\beta^*)$ and a "long" correlation length like in (6.18). Proposition 2.2 states that this equivalence holds when $h = h^* = 1$, the correlation length $\xi(\beta^*)$ being equal to the surface tension of an horizontal interface of the dual model. The reason for the validity of Proposition 2.2 can be formulated in physical terms: the dual model is in the complete wetting regime.

7 Sharp triangle inequality

The main property of the surface tension, which we use in this paper, is the sharp triangle inequality (STI). We recall some basic facts about the Wulff shape and prove that STI is equivalent to the property that the curvature of the Wulff shape is bounded above. This slightly extends the result of Ioffe [I]. In particular we do not suppose that the surface tension is differentiable. Our approach is geometrical.

7.1 Convex body and support function

Let $\hat{\tau} : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto \hat{\tau}(x)$, be a positively homogeneous convex function, which is strictly positive at $x \neq 0$. In this section $\langle y^*, x\rangle$ denotes the Euclidean scalar product of $y^* \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$. The conjugate function $\hat{\tau}^*$,

$$\hat{\tau}^*(y^*) := \sup_x \{\langle y^*, x\rangle - \hat{\tau}(x)\}\,, \tag{7.19}$$

is the indicator function of a convex set $W \subset \mathbb{R}^2$ defined by $\hat{\tau}$[14],

$$\hat{\tau}^*(y^*) = \begin{cases} 0 & \text{if } y^* \in W \\ \infty & \text{otherwise.} \end{cases} \tag{7.20}$$

Because $\hat{\tau}$ is strictly positive at $x \neq 0$, the interior of W is non–empty (W is a convex body) and contains 0. The function $\hat{\tau}$ is the support function of W[15],

$$\hat{\tau}(x) = \sup_{y^* \in W} \langle y^*, x\rangle\,. \tag{7.21}$$

Given x, we define the half–space $H(x)$,

$$H(x) := \{y^* : \langle y^*, x\rangle \leq \hat{\tau}(x)\}\,. \tag{7.22}$$

We have the important relation,

$$W = \{x^* \in \mathbb{R}^2 : \langle x^*, y\rangle \leq \hat{\tau}(y)\ ,\ \forall y \neq 0\} = \bigcap_{y \neq 0} H(y)\,. \tag{7.23}$$

A pair of points $(y^*, x) \in \mathbb{R}^2 \times \mathbb{R}^2$ is in **duality**[16] if and only if

$$\begin{aligned} \langle y^*, x\rangle &= \hat{\tau}(x) + \hat{\tau}^*(y^*) \\ &= \hat{\tau}(x)\,. \end{aligned} \tag{7.24}$$

If (y^*, x) are in duality and $x \neq 0$, then $y^* \in \partial W$, the boundary of W. Moreover, in such a case $(y^*, \lambda x)$ are in duality for any positive scalar λ. In the following $\hat{x}$ is always a unit vector in $\mathbb{R}^2$; there is at least one $x^* \in \partial W$, which is in duality with $\hat{x}$ for any $\hat{x}$. The geometrical meaning of $\hat{x}$ is the following: there is a support plane for W at x^* normal to $\hat{x}$. By convention the pair $(x^*, \hat{x})$ is in duality, so that $\langle x^*, \hat{x}\rangle = \hat{\tau}(\hat{x})$. We may have $y_1^* \neq y_2^*$, such that $(y_1^*, \hat{x})$ and $(y_2^*, \hat{x})$ are in duality.

Lemma 7.1 *1. Suppose that y_1^* and y_2^* are two different points of*

$$\mathcal{F}(\hat{x}) := \{y^* \in \mathbb{R}^2 : (y^*, \hat{x}) \text{ are in duality}\}\,. \tag{7.25}$$

Then $y^* = \alpha y_1^* + (1-\alpha) y_2^* \in \mathcal{F}(\hat{x})$, for all $\alpha \in [0, 1]$.

2. The set $\mathcal{F}(\hat{x})$ is equal to the set of subdifferentials of $\hat{\tau}$ at $\hat{x}$,

$$\partial\hat{\tau}(\hat{x}) := \{y^* \in \mathbb{R}^2 : \hat{\tau}(z + \hat{x}) \geq \hat{\tau}(\hat{x}) + \langle y^*, z\rangle \quad \forall z \in \mathbb{R}^2\}\,. \tag{7.26}$$

[14] In Statistical Mechanics, when $\hat{\tau}$ is the surface tension, W is called the **Wulff shape** and (7.23) the Wulff construction.

[15] $\hat{\tau}$ is a norm if and only if $W = -W$.

[16] Duality of Convexity Theory.

Proof. 1. follows from (7.24); 2. is proven e.g. in [PV2] section 4.1. □

The set $\mathcal{F}(\hat{x})$ of Lemma 7.1 is a **facet** of W with (outward) normal $\hat{x}$ (0 belongs to interior of W). Therefore, existence of a facet is equivalent to non–uniqueness of the subdifferentials of $\hat{\tau}$ at $\hat{x}$ or to non differentiability of $\hat{\tau}$. For a given $y^* \in \partial W$ we may have two different vectors $\hat{x}_1$ and $\hat{x}_2$, such that $(y^*, \hat{x}_1)$ and $(y^*, \hat{x}_2)$ are in duality. This situation corresponds to the existence of a **corner** of W at y^*.

Lemma 7.2 *There is a corner of W at y^* if and only if there exists a segment $[\hat{x}_1, \hat{x}_2] := \{x : x = \hat{x}_1 + t(\hat{x}_2 - \hat{x}_1), t \in [0,1]\}$, with $\hat{x}_1 \neq \hat{x}_2$, on which $\hat{\tau}$ is affine.*

Proof. Suppose that there is a corner at y^*. Then

$$\begin{aligned} \langle y^*, \hat{x}_1 + t(\hat{x}_2 - \hat{x}_1)\rangle &= (1-t)\hat{\tau}(\hat{x}_1) + t\hat{\tau}(\hat{x}_2) \\ &\leq \sup_{y^* \in W} \langle y^*, \hat{x}_1 + t(\hat{x}_2 - \hat{x}_1)\rangle \\ &= \hat{\tau}((1-t)\hat{x}_1 + t\hat{x}_2)\,. \end{aligned} \tag{7.27}$$

Since $\hat{\tau}$ is convex we have equality in (7.27).

Suppose that $\hat{\tau}$ is affine on $[\hat{x}_1, \hat{x}_2]$. Let $x_{1/2} := 1/2(\hat{x}_1 + \hat{x}_2)$. If $y^* \in \partial\hat{\tau}(x_{1/2})$, then $y^* \in \partial\hat{\tau}(\hat{x}_k)$, $k = 1, 2$. Indeed, for all z

$$\hat{\tau}(z) - \hat{\tau}(x_{1/2}) - \langle y^*, z - x_{1/2}\rangle \geq 0\,. \tag{7.28}$$

But, if $\hat{\tau}$ is affine on $[\hat{x}_1, \hat{x}_2]$, then

$$\frac{1}{2}\sum_{k=1}^{2}\{\,\hat{\tau}(\hat{x}_k) - \hat{\tau}(x_{1/2}) - \langle y^*, \hat{x}_k - x_{1/2}\rangle\,\} = 0\,. \tag{7.29}$$

Therefore

$$\hat{\tau}(\hat{x}_k) = \hat{\tau}(x_{1/2}) + \langle y^*, \hat{x}_k - x_{1/2}\rangle\,. \tag{7.30}$$

From this it follows that $y^* \in \partial\hat{\tau}(\hat{x}_k)$,

$$\hat{\tau}(z) \geq \hat{\tau}(x_{1/2}) + \langle y^*, z - x_{1/2}\rangle = \hat{\tau}(x_k) + \langle y^*, z - \hat{x}_k\rangle \quad \forall z\,, \tag{7.31}$$

which implies in our case that $(y^*, \hat{x}_k)$ are in duality. □

7.2 Curvature

We recall the notion of curvature of W at x^*. Let U be an open neighbourhood of x^*. Let $\mathcal{T}_i(x^*, U)$ be the family of discs $\mathcal{D}$ with the following properties

1. $\partial\mathcal{D}$ is tangent[17] to ∂W at x^*;

[17] The precise definition is the following: there is a common support plane at x^* for W and $\mathcal{D}$.

2. $W \cap U \supset \mathcal{D} \cap U$.

We allow the degenerate cases where the disc is a single point or a half–plane. Consequently $\mathcal{T}_i(x^*, U) \neq \emptyset$. We denote by $\rho(\mathcal{D})$ the radius of the disc $\mathcal{D}$ and set

$$\underline{\rho}(x^*, U) := \sup\{\rho(\mathcal{D}) : \mathcal{D} \in \mathcal{T}_i(x^*, U)\}\,. \tag{7.32}$$

Clearly $\underline{\rho}(x^*, U_1) \leq \underline{\rho}(x^*, U_2)$ if $U_1 \supset U_2$. The **lower radius of curvature** at x^* is defined as

$$\underline{\rho}(x^*) := \sup\{\underline{\rho}(x^*, U) : U \text{ open neighb. of } x^*\}\,. \tag{7.33}$$

Similarly, we introduce $\mathcal{T}_s(x^*, U) \neq \emptyset$ as the family of discs $\mathcal{D}$ with the following properties

1. $\partial\mathcal{D}$ is tangent to ∂W at x^*;
2. $W \cap U \subset \mathcal{D} \cap U$.

We set

$$\overline{\rho}(x^*, U) := \inf\{\rho(\mathcal{D}) : \mathcal{D} \in \mathcal{T}_s(x^*, U)\}\,. \tag{7.34}$$

The **upper radius of curvature** at x^* is defined as

$$\overline{\rho}(x^*) := \inf\{\overline{\rho}(x^*, U) : U \text{ open neighb. of } x^*\}\,. \tag{7.35}$$

Given $x^*, y^* \in \partial W$, $x^* \neq y^*$, let $\mathcal{C}(x^*; \rho_{y^*})$ be the circle of radius ρ_{y^*}, which is tangent to ∂W at x^* and goes through y^*[18]. If $y^* \in U$, then

$$\underline{\rho}(x^*, U) \leq \rho_{y^*} \leq \overline{\rho}(x^*, U)\,. \tag{7.36}$$

If $\rho(x^*) := \underline{\rho}(x^*) = \overline{\rho}(x^*)$, then the **radius of curvature** at x^* is $\rho(x^*)$ and the **curvature** at x^* is $\kappa(x^*) := 1/\rho(x^*)$ (see Chapter 1 of [S]). From (7.36) we get[19]

$$\underline{\rho}(x^*) = \overline{\rho}(x^*) \Longleftrightarrow \lim_{y^* \to x^*} \rho_{y^*} = \rho(x^*)\,. \tag{7.37}$$

If x^* is a corner, then $\underline{\rho}(x^*, U) = 0$ for any open neighbourhood $U \ni x^*$ and $\overline{\rho}(x^*, U)$ is as small as we wish, provided U is small enough. Therefore $\rho(x^*) = 0$. However, we may have $\rho(x^*) = 0$ when x^* is not a corner, as the following example shows. We define a convex body by its boundary,

$$\{\, z^*(1)(t) := \cos(t)|\cos(t)|^{.6}\,,\ z^*(2)(t) := \sin(t)|\sin(t)|^{.6}\ ,\ t \in [0, 2\pi]\,\}\,. \tag{7.38}$$

There is a unique support line at every point of the boundary. At the four points ($t = k\pi/2$, $k = 0, \ldots 3$) it is elementary to verify that $\rho(x^*) = 0$.

Lemma 7.3 *Let W be a convex compact body such that its lower radius of curvature is bounded below uniformly by $K_0 > 0$. Then, given $\rho < K_0$, there is a circle $\mathcal{C}(x^*; \rho) \subset W$ of radius ρ, which is tangent to ∂W at x^* for any x^*.*

[18] We suppose that $\mathcal{C}(x^*; \rho_{y^*})$ intersects the interior of W.

[19] If $\lim_{y^* \to x^*} \rho_{y^*} = \rho$, then for every $\varepsilon > 0$ there exists a neighbourhood U such that for all $y^* \in U$, $|\rho_{y^*} - \rho| \leq \varepsilon$. Therefore $\rho - \varepsilon \leq \underline{\rho}(x^*, U) \leq \overline{\rho}(x^*, U) \leq \rho + \varepsilon$.

Proof. Since the lower radius of curvature is positive, there is no corner. Consequently, for any $y^* \in \partial W$ there is unique $\hat{y}$ in duality with y^*. The hypothesis also implies that at every $x^* \in \partial W$ there is a disk $\mathcal{D}(x^*)$ with the properties: the radius $\rho(\mathcal{D}(x^*))$ of $\mathcal{D}(x^*)$ is non-zero, $\mathcal{D}(x^*) \subset W$ and $\partial\mathcal{D}(x^*)$ is tangent to ∂W at x^*. Since W is convex, the convex envelope of all these discs is a subset of W. Therefore, since W is compact, we can find $\delta > 0$, such that $\rho(\mathcal{D}(x^*)) \geq \delta$ for any x^*.

Let $x^* \in \partial W$ and $\hat{y}$ be given. Let $\mathcal{D}(x^*, \hat{y}) \subset H(\hat{x}) \cap H(\hat{y})$ be the largest disc, which is tangent to $\partial H(\hat{x})$ at x^*. If $\hat{x} = \hat{y}$, then the radius $r(x^*, \hat{y})$ of $\mathcal{D}(x^*, \hat{y})$ is infinite, otherwise it is finite. Since $\hat{\tau}(\,\cdot\,)$ is continuous, $r(x^*, \hat{y})$ is a continuous function of $\hat{y}$ at any $\hat{y} \neq \hat{x}$. We set

$$r(x^*) := \inf_{\hat{y}} r(x^*, \hat{y})\,. \tag{7.39}$$

Let $\{\hat{y}_n\}$ be a minimizing sequence, such that $\lim_n r(x^*, \hat{y}_n) = r(x^*)$ and $\lim_n \hat{y}_n =: \hat{y}$. There are two cases: $\hat{y} = \hat{x}$ and $\hat{y} \neq \hat{x}$.

If $\hat{y} = \hat{x}$, then $r(x^*) \geq K_0$. Suppose the converse, $r(x^*) < K_0$. Then, for any n such that $r(x^*, \hat{y}_n) < K_0$, we can find a disc $\mathcal{D}_n$ and a neighbourhood U_n of x^*, such that $\mathcal{D}_n$ is tangent to ∂W at x^* and

$$W \cap U_n \supset \mathcal{D}_n \cap U_n \supset \mathcal{D}(x^*, \hat{y}_n) \cap U_n\,. \tag{7.40}$$

Let z_n^* be the point of contact of $\mathcal{D}(x^*, \hat{y}_n)$ with $\partial H(\hat{y}_n)$. Since W is convex, ∂W intersects $\partial\mathcal{D}(x^*, \hat{y}_n)$ at some point t_n^* belonging to the circle arc of $\partial\mathcal{D}(x^*, \hat{y}_n)$ from x^* to z_n^*. Since $r(x^*, \hat{y}_n) < K_0$ and $\hat{x} = \lim_n \hat{y}_n$, we also have $\lim_n z_n^* = x^*$ and thus $\lim_n t_n^* = x^*$. But this contradicts (7.36). Thus $r(x^*) \geq K_0$; for any $\rho < K_0$ there is a circle $\mathcal{C}(x^*; \rho)$ of radius ρ, which is tangent to ∂W at $x^* \in \partial W$; $\mathcal{C}(x^*; \rho) \subset W$ by (7.23).

If $\hat{y} \neq \hat{x}$, then $r(x^*, \hat{y}) = r(x^*) \geq \delta$ and the disc $\mathcal{D}(x^*, \hat{y}) \subset W$ by (7.23). Let $r := \inf_{x^*} r(x^*)$; we claim that $r \geq K_0$. Suppose the converse, $r < K_0$. Let $\{z_n^*\}$ be a minimizing sequence such that $r(z_n^*) < K_0$, $\lim_n r(z_n^*) = r$ and $\lim_n z_n^* =: z^*$. For every n there exists $\hat{y}_n$ such that $r(z_n^*, \hat{y}_n) = r(z_n^*)$ and $\mathcal{D}(z_n^*, \hat{y}_n)$ is the largest disc in W, which is tangent to ∂W at z_n^*. Since $r < K_0$, there exists $\hat{y} \neq \hat{z}$, so that

$$r = r(z^*, \hat{y})\,. \tag{7.41}$$

Indeed, if $\mathcal{D}(z^*) \subset W$ is a disc tangent to ∂W at z^*, then by convexity the convex envelope of $\mathcal{D}(z^*)$ and $\mathcal{D}(z_n^*, \hat{y}_n)$ is a subset of W. If $\rho(\mathcal{D}(z^*)) > r$, then the discs $\mathcal{D}(z_n^*, \hat{y}_n)$ are not the largest discs in W which are tangent to ∂W at z_n^*, when n is sufficiently large. The existence of $\mathcal{D}(z^*, \hat{y})$ and the convexity of W imply the existence of an open set U, such that

$$\partial W \cap U = \partial\mathcal{D}(z^*, \hat{y}) \cap U\,. \tag{7.42}$$

Indeed, $\mathcal{D}(z^*, \hat{y})$ is tangent to ∂W at z^* and also at some y^* in duality with $\hat{y}$; moreover, at any point $x^* \in \partial W$ there exists a disc of radius r contained in W, tangent to ∂W at x^*. But this contradicts $r < K_0$. $\square$

7.3 STI

Proposition 7.1 *Let W be a convex compact body and $\hat{\tau}$ be its support function. Then the following statements are equivalent.*

1. *The lower radius of curvature of ∂W is uniformly bounded below by $K_0 > 0$.*
2. *There exists a positive constant K_1 such that for any $\hat{x}$ and $\hat{y}$*
$$\langle x^* - y^*, \hat{x} \rangle \geq K_1 \|\hat{x} - \hat{y}\|^2 \,. \tag{7.43}$$
3. *There exists a positive constant K_2 such that for any x, $y \in \mathbb{R}^2$*
$$\hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(x+y) \geq K_2(\|x\| + \|y\| - \|x+y\|) \,. \tag{7.44}$$

Remarks: 1. Suppose that the curvature is bounded above everywhere by κ. Then 1. holds with $K_0 = 1/\kappa$. 1. implies 2. with $K_1 = 1/2\kappa$; this follows by modifying slightly the proof given below: if $\langle x^* - y^*, \hat{x} \rangle \leq 4K_1$, then there exists $\hat{v}$ such that $\langle x^* - y^*, \hat{x} \rangle = 2K_1 \|\hat{x} - \hat{v}\|^2$ and $\langle \hat{x}, \hat{y} \rangle \geq \langle \hat{x}, \hat{v} \rangle$. 2. implies 3. with $K_2 = 1/\kappa$.

2. The validity of the sharp triangle inequality implies absence of corner for W, since it prevents $\hat{\tau}$ to be affine on segments $[\hat{x}_1, \hat{x}_2]$ with $\hat{x}_1 \neq \hat{x}_2$. However, the example before Lemma 7.3 shows that the converse is not true.

Proof. We prove $1 \Rightarrow 2$. Let $x^*, y^* \in \partial W$, $x^* \neq y^*$ and $0 < 2K_1 < K_0$. The circle $\mathcal{C}(x^*; 2K_1)$ of radius $2K_1$, center d^*, which is tangent to ∂W at x^*, is a subset of W (Lemma 7.3). If
$$\langle x^* - y^*, \hat{x} \rangle \geq 2K_1 \,, \tag{7.45}$$
then for any $\hat{y}$
$$\langle x^* - y^*, \hat{x} \rangle \geq K_1/2 \|\hat{x} - \hat{y}\|^2 \,. \tag{7.46}$$
Suppose that
$$\langle x^* - y^*, \hat{x} \rangle \leq 2K_1 \,. \tag{7.47}$$
We can find $z^* \in \mathcal{C}(x^*; 2K_1)$ such that, if $\hat{v} := (z^* - d^*)/\|z^* - d^*\|$ and φ is the angle between the unit vectors $\hat{x}$ and $\hat{v}$, then $\langle \hat{x}, \hat{v} \rangle \geq 0$ and
$$\langle x^* - y^*, \hat{x} \rangle = 2K_1(1 - \cos\varphi) = K_1 \|\hat{x} - \hat{v}\|^2 \,. \tag{7.48}$$
We claim that
$$\langle \hat{x}, \hat{y} \rangle \geq \langle \hat{x}, \hat{v} \rangle \,. \tag{7.49}$$
Suppose the converse. Let $\mathcal{C}(y^*; 2K_1) \subset W$ be the circle of radius $2K_1$, which is tangent to ∂W at y^*. By hypothesis the line perpendicular to $\hat{v}$ at z^* and the support line at y^* perpendicular to $\hat{y}$ intersects at an interior point of $H(\hat{x})$; this implies that $\mathcal{C}(y^*; 2K_1) \not\subset H(\hat{x})$, which is in contradiction with $W \subset H(\hat{x})$. Therefore
$$\|\hat{x} - \hat{y}\| \leq \|\hat{x} - \hat{v}\| \tag{7.50}$$
and
$$\langle x^* - y^*, \hat{x} \rangle = K_1 \|\hat{x} - \hat{v}\|^2 \geq K_1 \|\hat{x} - \hat{y}\|^2 \,. \tag{7.51}$$

We prove 2 $\Rightarrow$ 1. Suppose that

$$\langle x^* - y^*, \hat{x}\rangle \geq K_1 \|\hat{x} - \hat{y}\|^2 . \tag{7.52}$$

Let $\mathcal{C}(x^*; \rho_{y^*})$ be the circle of radius ρ_{y^*}, which is tangent to ∂W at x^* and goes through y^*; let c^* be its center and $\hat{u} := (y^* - c^*)/\|y^* - c^*\|$. Assume furthermore that $\langle \hat{x}, \hat{u}\rangle \geq 0$. Let $v := \hat{x} + \hat{u}$ and $\hat{v} = v/\|v\|$. Then

$$\frac{\rho_{y^*}}{2} \|\hat{x} - \hat{u}\|^2 = \langle x^* - y^*, \hat{x}\rangle \tag{7.53}$$

and $\langle x^* - y^*, \hat{v}\rangle = 0$. Since ∂W is convex, there exists z^* "between" x^* and y^* such that $\hat{z} = \hat{v}$ and

$$\|\hat{x} - \hat{z}\| \leq \|\hat{x} - \hat{y}\| . \tag{7.54}$$

On the other hand,

$$\|\hat{x} - \hat{u}\| \leq \|\hat{x} - \hat{v}\| + \|\hat{v} - \hat{u}\| \tag{7.55}$$

and

$$\|\hat{x} - \hat{v}\| = \|\hat{v} - \hat{u}\| = \|\hat{x} - \hat{z}\| . \tag{7.56}$$

If $\hat{x} = \hat{y}$, then $\rho_{y^*} = \infty$; otherwise, using (7.53) to (7.56),

$$\begin{aligned} 2\rho_{y^*} \|\hat{x} - \hat{v}\|^2 &\geq \frac{\rho_{y^*}}{2} \|\hat{x} - \hat{u}\|^2 \\ &= \langle x^* - y^*, \hat{x}\rangle \\ &\geq K_1 \|\hat{x} - \hat{y}\|^2 \\ &\geq K_1 \|\hat{x} - \hat{v}\|^2 . \end{aligned} \tag{7.57}$$

Since this holds for any y^* in a neighbourhood of x^*, we have $\underline{\rho}(x^*) \geq K_1/2$.

We prove 2 $\Rightarrow$ 3. We set

$$z := x + y \quad , \quad z^* := (x+y)^* \quad , \quad \hat{z} := \frac{x+y}{\|x+y\|} . \tag{7.58}$$

We have, using (7.21) and (7.24),

$$\begin{aligned} \hat{\tau}(x) + \hat{\tau}(y) - \hat{\tau}(z) &= \langle x^*, x\rangle + \langle y^*, y\rangle - \langle z^*, z\rangle \\ &= \langle x^* - z^*, x\rangle + \langle y^* - z^*, y\rangle \\ &= \|x\| \, \langle x^* - z^*, \hat{x}\rangle + \|y\| \, \langle y^* - z^*, \hat{y}\rangle . \end{aligned} \tag{7.59}$$

By elementary trigonometry

$$\|x\| + \|y\| - \|z\| = \frac{1}{2} \Big(\|x\| \cdot \|\hat{x} - \hat{z}\|^2 + \|y\| \cdot \|\hat{y} - \hat{z}\|^2 \Big) , \tag{7.60}$$

so that (7.43) implies (7.44).

We prove 3 $\Rightarrow$ 2. Let $(x^*, \hat{x})$ and $(y^*, \hat{y})$ be given; we set $z := \hat{x} + \hat{y}$. Using (7.60), $\|\hat{x} - \hat{z}\| = \|\hat{y} - \hat{z}\|$ and $\|\hat{x} - \hat{y}\| \leq \|\hat{x} - \hat{z}\| + \|\hat{z} - \hat{y}\|$ we have

$$\begin{aligned} \langle x^* - y^*, \hat{x}\rangle &= \langle x^*, \hat{x}\rangle + \langle y^*, \hat{y}\rangle - \langle y^*, \hat{x} + \hat{y}\rangle \\ &\geq \hat{\tau}(\hat{x}) + \hat{\tau}(\hat{y}) - \hat{\tau}(z) \\ &\geq K_2 (\|\hat{x}\| + \|\hat{y}\| - \|z\|) \\ &= K_2 \|\hat{x} - \hat{z}\|^2 \\ &\geq \frac{K_2}{4} \|\hat{x} - \hat{y}\|^2 . \end{aligned} \tag{7.61}$$

□

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