α-continuity properties of one-dimensional quasicrystals

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Abstract

We apply the Jitomirskaya-Last extension of the Gilbert-Pearson theory to discrete one-dimensional Schrödinger operators with potentials arising from generalized Fibonacci sequences. We prove for certain rotation numbers that for every value of the coup ling constant, there exists an $\alpha > 0$ such that the corresponding operator has purely α-continuous spectrum. This result follows from uniform upper and lower bounds for the $\|\cdot\|_L$-norm of the solutions corresponding to energies from the spectrum of the operator.

1 Introduction

In this paper we consider a discrete one-dimensional Schrödinger operator on $l^2(\mathbf{Z})$ with potential arising from a generalized Fibonacci sequence already studied in [2, 10], namely,

$$(H_{\lambda,\theta,\beta}u)(n) = u(n+1) + u(n-1) + \lambda v_{\theta,\beta}(n)u(n), \tag{1}$$

where

$$v_{\theta,\beta}(n) = \chi_{[1-\theta,1)}(n\theta + \beta),$$

$\lambda > 0$, $\theta \in (0,1)$ irrational and $\beta \in [0,1]$, together with the corresponding eigenvalue equation

$$H_{\lambda,\theta,\beta}u = Eu. \tag{2}$$

In [2] it is shown by Bellissard et al. that $H_{\lambda,\theta,\beta}$ has purely singular spectrum which is supported on a Cantor set of zero Lebesgue measure (see also [21] for the golden case). They show that the spectrum of $H_{\lambda,\ theta,\beta}$ is independent of β and study

$$H_{\lambda,\theta} \equiv H_{\lambda,\theta,0}.$$

In this case the potential has the form

$$v_{\theta,0} = [(n+1)\theta] - [n\theta] \tag{3}$$

and one can prove recursive relations for the transfer matrices at certain sites. One can then show that the spectrum coincides with the set of energies where the Lyapunov exponent vanishes. By [16], this set has zero Lebesgue measure.

Let us remark that the absence of absolutely continuous spectrum also follows in a more general context for minimally ergodic operators with potentials taking finitely many values by combining the results of Kotani

[16] and Last-Simon [19].

For the concrete operator $H_{\lambda,\theta}$ we can also show continuity of the spectrum:

Theorem 1 *The point spectrum of $H_{\lambda,\theta}$ is empty for all λ, θ.*

Remarks

1. This result follows quite easily from the symmetry properties of the potential which were deduced in [2]. However, this generalization of the corresponding result in the golden case [20] was not stated in [2].

2. Related results are obtained in [4, 8].

Thus, $H_{\lambda,\theta}$ has purely singular continuous spectrum for all parameter values. The natural question arises whether estimates on the Hausdorff dimensional properties of the spectral measures of $H_{\lambda,\theta}$ can be obtained, which in t urn give information about the dynamical properties of the associated quantum systems as shown by Last [18]. The related question about control of the Hausdorff dimension of the spectrum as a set was stated as an open problem in [2]. Recently, Jitomirskaya and Last provided an approach for such investigations in terms of estimates on the behavior of the solutions u of (2) [12, 13]. This approach is a natural extension of the Gilbert-Pearson theory relating the existence (resp. non-existence) of subordinate solutions to supports of the singular (resp. absolutely continuous) part of a given Schrödinger operator [7, 6, 15]. Consequently, using an upper bound result from [10] we apply this approach to the operator $H_{\ lambda,\theta}$. We therefore provide an extension of a similar result about the original Fibonacci operator which was announced in [12]. The organization is as follows: After collecting known properties of $H_{\lambda,\theta}$ we prove in Section 2 for certain rotation numbers θ uniform bounds on $\|u\|_L \equiv (\sum_{n=0}^{L} |u(n)|^2)^{\frac{1}{2}}$ for every solution u of (2) which imply Theorem 1. In Section 3 we show how α-continuity of the spectral measures of $H_{da,\theta}$ follows from these bounds, proving the following

Theorem 2 *Suppose θ is a bounded density number. Then for every $\lambda > 0$ there exists an $\alpha_{\lambda,\theta} > 0$ such that $H_{\lambda,\theta}$ has purely $\alpha_{\lambda,\theta}$-continuous spectrum, namely, the spectral measures do not give wei ght to sets of zero $h^{\alpha_{\lambda,\theta}}$ measure.*

Remarks

1. The α-dimensional Hausdorff measure h^α is defined by

$$h^\alpha(S) \equiv \lim_{\delta\to 0} \inf_{\delta-\text{covers of } S} \sum_{n=1}^{\infty} |b_n|^\alpha,$$

where a δ-cover of a set $S \subseteq \mathbf{R}$ is a countable collection of intervals $\{b_n\}_{n\in\mathbf{N}}$ with $S \subseteq \bigcup_{n=1}^{\infty} b_n$ and $|b_n| < \delta \ \forall n$, $|\cdot|$ denoting Lebesgue measure. See [5] for a gen eral exposition on this topic and [18] for decompositions of singular continuous spectral measures with respect to Hausdorff measures.

2. Theorem 6.1 of [18] gives bounds on the time evolution of initial states with nontrivial $\alpha_{\lambda,\theta}$-continuous component. Therefore, these bounds apply to every state in our model.

3. If the characteristic function $\chi_{[1-\theta,1)}$ in the potential is replaced by a nicer function, for example a trigonometric polynomial, then the spectral measures are purely zero-dimensional for λ large enough [11].

4. In particular, this result implies that for this set of θ's the spectrum $\sigma(H_{\lambda,\theta})$ as a set has positive Hausdorff dimension.

5. The set of θ's satisfying the assumption of Theorem 2 contains all constant type numbers (golden mean, silver mean, ...) and is of zero Lebesgue measure [17].
6. Extending this result to a larger class of θ's essentially amounts to extending the upper bound result of [10] since the lower bound result which will be derived in Section 2 holds for almost every θ.

Using the constancy of the spectrum we have the following

Corollary 1 *Suppose that θ is a bounded density number. Then for every λ and every β the spectrum of $H_{\lambda,\theta,\beta}$ has positive Hausdorff dimension.*

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2 Uniform bounds for $\|u\|_L$, proof of Theorem 1

Consider the continued fraction expansion of θ:

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \equiv [a_1, a_2, \ldots].$$

Then, best rational approximations $\frac{p_k}{q_k}$ of θ are given by (see, e.g., [17])

$$p_{k+1} = a_{k+1} p_k + p_{k-1}, \;\; p_0 = 0, \; p_1 = 1 \tag{4}$$

$$q_{k+1} = a_{k+1} q_k + q_{k-1}, \;\; q_0 = 1, \; q_1 = a_1. \tag{5}$$

θ is called a *bounded density number* if the following condition holds

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n < \infty.$$

As already remarked in [2], $v_{\theta,0}$ has the following symmetry properties:

$$v_{\theta,0}(n + q_k) = v_{\theta,0}(n) \;\; 1 \le n < q_{k+1} - 1 \tag{6}$$

$$v_{\theta,0}(-n) = v_{\theta,0}(n-1) \;\; n \ge 2. \tag{7}$$

Let $T_{\lambda,\theta}(n, E)$ be the transfer matrix

$$T_{\lambda,\theta}(n,E) \equiv \begin{pmatrix} E - \lambda v_{\theta,0}(n) & -1 \\ 1 & 0 \end{pmatrix}$$

which provides the standard reformulation of the eigenvalue equation (2), namely

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = T_{\lambda,\theta}(n,E) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}$$

for any solution u of (2). Leaving the dependence on λ, θ and E implicit for notational convenience, we define $M(n) \equiv T_{\lambda,\theta}(n,E) \times \ldots \times T_{\lambda,\theta}(1,E)$ and $M_n \equiv M(q_n)$. From (**??**) we obtain recursive relations for the matrices M_n:

$$M_{n+1} = M_{n-1} M_n^{a_{n+1}}. \tag{8}$$

We would like to remark at this point that for the original Fibonacci operator ($a_n = 1 \; \forall n$), this follows more directly from an alternative description of the potential by means of a substitution rule. The recursive relation (8) in this case arises naturally from the self-similarity of the potential which is induced by the substitution rule; compare, for example, [3]. The potential can also be generated by a substitution rule for other values of θ; compare [1] for a r elated class of operators.

The recursion (8) induces a recursion for the traces $x_n \equiv \mathrm{tr}(M_n)$ of the transfer matrices at sites q_n. In [2] it is shown that E is in the spectrum of $H_{\lambda,\theta}$ if and only if the sequence $(|x_n|)_{n\in\mathbf{N}}$ is bounded. Moreover, for every λ there exists a uniform bound c_λ for all sequences $(|x_n|)_{n\in\mathbf{N}}$ corresponding to energies $E \in \sigma(H_{\lambda,\theta})$. Using this result, the authors of [10] derive upper bounds fo r solutions u of (2) in the case that θ is a bounded density number. Keeping track of the constants and using the uniform bounds on the traces, one verifies

$$\|u\|_L \leq L^{\alpha_{\lambda,\theta}^{(2)}} \tag{9}$$

for all normalized solutions u corresponding to $E \in \sigma(H_{\lambda,\theta})$ with a suitably chosen $\alpha_{\lambda,\theta}^{(2)}$. We will now prove a similar lower bound:

Proposition 2.1 *Suppose the sequence (q_n) associated to θ satisfies $q_n \leq B_\theta^n$. Then for every $\lambda > 0$, there exist $\alpha_{\lambda,\theta}^{(1)} > 0$ and $L_0 \geq 1$ such that*

$$\|u\|_L \geq L^{\alpha_{\lambda,\theta}^{(1)}} \;\; \forall L \geq L_0 \tag{10}$$

for all solutions u corresponding to $E \in \sigma(H_{\lambda,\theta})$ with normalization $|u(0)|^2 + |u(1)|^2 = 1$.

Remark The set of θ's satisfying the assumption has full Lebesgue measure [14].

This proposition will be proved in three steps:

- We consider $\|u\|_{q_{5n}}$ and show that this quantity grows at least exponentially in n.
- We show that the claimed lower bound holds at the sites q_{5n}.
- The lower bound also holds at all other sites.

Lemma 2.2 *For every irrational θ and every λ we have*

$$\|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-5}}$$

for any solution u corresponding to $E \in \sigma(H_{\lambda,\theta})$ with normalization $|u(0)|^2 + |u(1)|^2 = 1$.

Proof The lemma will be proved by exploiting squares in the transfer matrix arrangements. We will use the following mass-reproduction technique: Suppose the 2-vector

$$U(k) \equiv \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix}$$

sees a square in the transfer matrix arrangement, that is

$$U(k+2l) = A_l^2 U(k).$$

Then the following inequality holds (compare, e.g., [20]):

$$\max(\|U(k+2l)\|, |\mathrm{tr}\, A_l| \, \|U(k+l)\|) \geq \tfrac{1}{2}\|U(k)\|.$$

Typically, A_l will be a cyclic permutation of a product

$$T_{\lambda,\theta}(l,E) \times \ldots \times T_{\lambda,\theta}(1,E),$$

where l is equal to some q_n. Thus $\mathrm{tr}\, A_l = \mathrm{tr}\, M_n = x_n$ and we can use the uniform upper bound for these traces. This technique will be applied to a certain number of consecutive sites. Hence, there exists a constant $C_\lambda > 1$ such that if the sites $k = 1, \ldots, q_{n_1}$ see the squares of cyclic permutations of a matrix product M_{n_2}, we have

$$\|u\|_{q_{n_1}+2q_{n_2}} \geq C_\lambda \|u\|_{q_{n_1}}.$$

We have to consider three cases on different levels.
Case 1: $a_n \geq 3$

$$\begin{aligned} &\Rightarrow \quad M_n = M_{n-2}M_{n-1}^{a_n} = \widehat{M_n} M_{n-1}^3 \\ &\Rightarrow \quad \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-1}} \end{aligned}$$

and we are done in this case.
Case 2: $a_n = 2$

$$\begin{aligned} &\Rightarrow \quad M_n = M_{n-2}M_{n-1}^2 \\ &\Rightarrow \quad \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-2}} \end{aligned}$$

since we can apply our mass-reproduction technique to the sites $1, \ldots, q_{n-2}$. Thus, we are also done in this case.
Case 3: $a_n = 1$

$$\Rightarrow M_n = M_{n-2}M_{n-1} = M_{n-2}M_{n-3}M_{n-2}^{a_{n-1}}.$$

Now, we have to consider the three cases on the next level:
Case 3.1: $a_{n-1} \geq 3$

$$\begin{aligned} &\Rightarrow \quad M_n = \widetilde{M_n} M_{n-2}^3 \\ &\Rightarrow \quad \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-2}} \end{aligned}$$

and we are done in this case.
Case 3.2: $a_{n-1} = 2$

$$\begin{aligned} &\Rightarrow \quad M_n = M_{n-2}M_{n-3}M_{n-2}^2 \\ &\Rightarrow \quad \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-3}} \end{aligned}$$

by the argument from Case 2.
Case 3.3: $a_{n-1} = 1$

$$\Rightarrow M_n = M_{n-2}M_{n-3}M_{n-2}.$$

Again, we have to consider the three cases on the next level:
Case 3.3.1: $a_{n-2} \geq 3$

$$\begin{aligned} &\Rightarrow \quad M_n = \overline{M_n} M_{n-3}^3 \\ &\Rightarrow \quad \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-3}} \end{aligned}$$

Case 3.3.2: $a_{n-2} = 2$

$$\begin{aligned} \Rightarrow \quad & M_n = M_{n-2}M_{n-3}M_{n-4}M_{n-3}^2 \\ \Rightarrow \quad & \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-4}} \end{aligned}$$

Case 3.3.3: $a_{n-2} = 1$

$$\Rightarrow \; M_n = M_{n-2}M_{n-3}M_{n-4}M_{n-3}.$$

For the last time we have to consider the three cases on the next level:
Case 3.3.3.1: $a_{n-3} \geq 3$

$$\begin{aligned} \Rightarrow \quad & M_n = M_n' M_{n-4}^3 \\ \Rightarrow \quad & \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-4}} \end{aligned}$$

Case 3.3.3.2: $a_{n-3} = 2$

$$\begin{aligned} \Rightarrow \quad & M_n = M_{n-2}M_{n-3}M_{n-4}M_{n-5}M_{n-4}^2 \\ \Rightarrow \quad & \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-5}} \end{aligned}$$

Case 3.3.3.3: $a_{n-3} = 1$

$$\begin{aligned} \Rightarrow M_n \quad &= \quad M_{n-2}M_{n-1} = M_{n-4}M_{n-3}^2M_{n-2} \\ &= \quad M_{n-4}M_{n-5}M_{n-4}M_{n-3}M_{n-2} = M_{n-4}M_{n-5}M_{n-2}^2 \end{aligned}$$

$$\Rightarrow \; \|u\|_{q_n} \geq C_\lambda \|u\|_{q_{n-5}}$$

Hence, we obtain the desired inequality in all cases. □

Proof of Theorem 1 From Lemma 2.2, we see that for all parameters λ, θ, there exist no l^2-solutions. □

Now, we need an estimate on the growth of the q_n in the case that θ is a bounded density number in order to show that these numbers satisfy the assumption of Proposition 2.1.

Lemma 2.3 *Suppose that θ is a bounded density number. Then there exists a constant C such that*

$$q_n \leq C^n.$$

Proof We compare the sequence (q_n) with the sequence (r_n) which is generated by the recursion

$$r_{n+1} = 2a_{n+1}r_n$$

with initial condition $r_1 = a_1$. We have $q_n \leq r_n$ and

$$r_n = \prod_{i=1}^{n} 2a_i,$$

yielding

$$\ln(r_n)^{\frac{1}{n}} = \frac{1}{n}\sum_{i=1}^{n} \ln 2a_i.$$

The right hand side is bounded by assumption. Hence, the assertion follows. □

Lemma 2.4 *Suppose the sequence (q_n) associated to θ satisfies $q_n \leq B_\theta^n$. Then for every $\lambda > 0$, there exists a $\gamma_{\lambda,\theta} > 0$ such that*

$$\|u\|_{q_{5n}} \geq q_{5n}^{\gamma_{\lambda,\theta}}.$$

Proof By assumption we have

$$C_{\theta,1}^n \leq q_{5n} \leq C_{\theta,2}^n$$

and, by Lemma 2.2,

$$\|u\|_{q_{5n}} \geq C_\lambda^n.$$

Choose $\gamma_{\lambda,\theta} > 0$ such that

$$C_{\theta,2}^{\gamma_{\lambda,\theta}} \leq C_\lambda.$$

We have

$$\frac{\|u\|_{q_{5n}}}{q_{5n}^{\gamma_{\lambda,\theta}}} \geq \left(\frac{C_\lambda}{C_{\theta,2}^{\gamma_{\lambda,\theta}}}\right)^n \geq 1$$

and the proof is complete. □

Proof of Proposition 2.1 Choose $\varepsilon \in (\frac{\ln C_{\theta,2} - \ln C_{\theta,1}}{\ln C_{\theta,2}} \gamma_{\lambda,\theta}, \gamma_{\lambda,\theta})$. We have $\alpha_{\lambda,\theta}^{(1)} \equiv \gamma_{\lambda,\theta} - \varepsilon > 0$ and

$$\frac{C_{\theta,2}^{\alpha_{\lambda,\theta}^{(1)}}}{C_{\theta,1}^{\gamma_{\lambda,\theta}}} < 1.$$

Choose n_0 such that

$$\left(\frac{C_{\theta,2}^{\alpha_{\lambda,\theta}^{(1)}}}{C_{\theta,1}^{\gamma_{\lambda,\theta}}}\right)^{n_0} \leq C_{\theta,2}^{-\alpha_{\lambda,\theta}^{(1)}}$$

and let $L_0 \equiv q_{5n_0}$. Now, let $L \geq L_0$. Then $q_{5n} \leq L < q_{5(n+1)}$ for a suitably chosen $n \geq n_0$. With these definitions the desired lower bound follows:

$$\begin{array}{rcl} \|u\|_L & \geq & \|u\|_{q_{5n}} \\ & \geq & q_{5n}^{\gamma_{\lambda,\theta}} \\ & \geq & C_{\theta,1}^{n\gamma_{\lambda,\theta}} \\ & \geq & C_{\theta,2}^{(n+1)\alpha_{\lambda,\theta}^{(1)}} \\ & \geq & q_{5(n+1)}^{\alpha_{\lambda,\theta}^{(1)}} \\ & \geq & L^{\alpha_{\lambda,\theta}^{(1)}} \end{array}$$

and the proof is complete. □

3 Non-existence of α-subordinate solutions, proof of Theorem 2

In the last section we have only considered the behavior of eigenfunctions on the right half-line. This is due to two reasons. The reflection symmetry (7) of the potential gives rise to the same recursive relations for the transfer matrices on the left half-line, as already remarked in [2]. Thus, the same strategy leads to upper and lower bounds for the $l^2(-L, 0)$-norm of eigenfunctions. On the other hand, it is sufficient to have such bounds on one half-line since we want to prove α-continuity of the spectral measures. The continuity of either half-line problem provides a lower bound for the continuity of the whole-line problem which can be seen from the dependence of the spectral measure in the whole-line case on the spectral measures of the two half-line operators. Hence, by [12] it suffices to show:

There exists an $\alpha_{\lambda,\theta} > 0$ such that for every pair (u_1, u_2) of normalized solutions of (2) corresponding to $E \in \sigma(H_{\lambda,\theta})$, the following holds:

$$\liminf_{L\to\infty} \frac{\|u_1\|_L}{\|u_2\|_L^{\frac{\alpha_{\lambda,\theta}}{2-\alpha_{\lambda,\theta}}}} > 0.$$

Using the uniform bounds we obtain for $L \geq L_0$,

$$\frac{\|u_1\|_L}{\|u_2\|_L^{\frac{\alpha}{2-\alpha}}} \geq \frac{L^{\alpha^{(1)}_{\lambda,\theta}}}{(L^{\alpha^{(2)}_{\lambda,\theta}})^{\frac{\alpha}{2-\alpha}}} = L^{\alpha^{(1)}_{\lambda,\theta} - \alpha^{(2)}_{\lambda,\theta}\frac{\alpha}{2-\alpha}}.$$

Let

$$\alpha_{\lambda,\theta} \equiv \frac{2\alpha^{(1)}_{\lambda,\theta}}{\alpha^{(1)}_{\lambda,\theta} + \alpha^{(2)}_{\lambda,\theta}}.$$

This yields the uniform positivity of the $\liminf$ over the spectrum and proves Theorem 2.

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