Non-Integrability and Infinite Branching of Solutions of 2DOF Hamiltonian Systems in Complex Plane of Time *

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Abstract

It has been proved by S.L.Ziglin [1], for a large class of 2-degree-of-freedom (d.o.f) Hamiltonian systems, that transverse intersections of the invariant manifolds of saddle fixed points imply infinite branching of solutions in the complex time plane and the non–existence of a second analytic integral of the motion. Here, we review in detail our recent results, following a similar approach to show the existence of infinitely–sheeted solutions for 2 d.o.f. Hamiltonians which exhibit, upon perturbation, subharmonic bifurcations of resonant tori around an elliptic fixed point [2]. Moreover, as shown recently, these Hamiltonian systems are non–integrable if their resonant tori form a dense set. These results can be extended to the case where the periodic perturbation is not Hamiltonian.

Contents

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1 Introduction

In the last 15 years, there has been active interest in the study of integrability (or absence thereof) of nonlinear dynamical systems based on the analysis of their singularities in the complex time t–plane [1–6].

Singularity analysis is the study of the behavior of the solutions of differential equations around their singularities in complex time. While, any anlytic system of differential equation is locally integrable, the different local patches do not, in general, fit together globally. The main idea of the singularity analysis is to obtain global information on the integrability of a system through the local analysis of the solution in the complex plane. Its origins can be found in Kowaleskaya's classical work and on the Painlevé 's classification of second-order ordinary differential equations (see, e.g.[3]). However, for a long time, Kowaleskaya's work and Painlevé 's theory were consider interesting, if not old fashioned, masterpieces in the theory of special functions and little attention was paid to them late 1970's, when it was noticed that they were intimately related to the theory of solitons. The various Painlevé tests for ODEs which followed this discovery [3–5] are based on the formal existence of Laurent expansions for the solutions around the movable singularities of the solution in the complex plane.

According to this approach one seeks to establish conditions such that all **movable** (i.e initial condition dependent) singularities of the solutions of the equations of motion of the system

$$\frac{dx}{dt} = \dot{x} = f(x,t) \qquad x = (x_1(t), \cdots, x_n(t)) \tag{1.1}$$

are isolated poles, i.e that (1.1) possesses the so–called Painlevé property [3–5]. If this is the case, then all solutions of (1.1) are meromorphic (single–valued and analytic everywhere except at the poles) and the system is often integrable, in the sense of having global, single–valued integrals of the motion.

On the other hand, if infinitely multi–valued solutions are found, one expects that such integrals do not exist and the system is called non–integrable. The presence of infinitely–sheeted solutions (so–called I.S.S property) can be deduced either analytically, by showing e.g. that their series expansions near a singularity contain logarithmic terms, or, numerically, by integrating (1.1) along contours enclosing one or more singularities in $t \in \mathbb{C}$.

One rigorous approach to the connection between non–integrability and the I.S.S property was introduced, a number of years ago, by Ziglin [1]. He showed, under some general assumptions, that 2–d.o.f. Hamiltonian systems of the form

$$\mathcal{H} = \mathcal{H}_0(x_1, x_2, I) + \epsilon \mathcal{H}_1(x_1, x_2, I, \phi) \tag{1.2}$$

with $\mathcal{H}_1$ 2π–periodic in ϕ, possessing, for $\epsilon = 0$, a closed homoclinic (resp. heteroclinic) orbit, which joins one saddle fixed point to itself (resp. two saddle fixed points), exhibit, for a large class of such perturbations and $0 < |\epsilon| \ll 1$, infinitely–sheeted solutions. Ziglin's remarkable discovery was that he was able to relate directly this I.S.S. property, using Mel'nikov's theory to the transversal intersections of the stable and unstable manifolds of the homo(or hetero)clinic orbit.

Furthermore, Ziglin proved that these perturbed Hamiltonians are non–integrable in the sense that they cannot have a global, single–valued integral of the motion, other than the Hamiltonian itself.

The purpose of this paper is first to review, Ziglin's results on transverse intersections of the invariant manifolds of saddle fixed points and how these imply infinite branching of solutions in the complex time plane and the non–existence of a second analytic integral of the motion. Then we shall prove as reported [2], the I.S.S property for Hamiltonian systems of the form (1.2) around *an elliptic* fixed point and thus establish a connection between infinite branching of solutions and the non–integrability of such systems, as long as they exhibit subharmonic bifurcations on *a dense* set of resonant tori [7].

Our approach follows closely that of Ziglin, in that we use one of Mel'nikov 's theorems to show that I.S.S is a direct consequence of subharmonic bifurcations (at $\epsilon \neq 0$) of resonant invariant curves of the unperturbed ($\epsilon = 0$) system. Moreover, we also make the crucial assumption that, for $\epsilon = 0$, (1.2) possesses only meromorphic solutions in the complex domain. Unlike Ziglin, however, we do not assume the presence of a saddle fixed point, whose invariant manifolds govern the dynamics.

In section 2, we present some useful backgraund information on the structure and importance of Riemann surfaces.

In section 3, we state Ziglin's theorm on the transversal intersection of invariant manifolds in the perturbed 2 d.o.f Hamiltonian (1.2). For $\epsilon = 0$, these invariant manifolds join "smoothly" in a single separatrix, or homo(hetero)clinic orbit of the completely integrable unperturbed problem, whose slutions have only poles in complex t. For $\epsilon \neq 0$, however, these manifolds interesct at infinitely many points and infinitely branched multi–valued solutions appear, as predicted by Ziglin, with e.g. logarithmic singularities.

In section 4, we state our main theorem, as described in [2], prove that the series expansions near a singularity contain logatithmic terms using the theory of Abelian integrals and we illustrate our results on the example of a driven Duffing oscillator.

In section 5, we extend and apply our results to the case of non–Hamiltonian perturbations. Finally, in section 6, we end with some concluding remarks.

2 Riemann Surfaces

A Riemann surface X is a connected two–dimensional topological manifold with a complex–analytic structure on it. The latter implies that for each point $P \in X$ there is a homeomorphism $\phi : U \longrightarrow V$ of some neighborhood $U \ni P$ onto an open set $V \in \mathbb{C}$, and is defined so that any two such homeomorphisms ϕ, $\hat{\phi}$ with $U \cap \hat{U} \neq \emptyset$ are holomorphically compatible, i.e., the mapping $\phi \circ \hat{\phi}^{-1} : \hat{\phi}(U \cap \hat{U}) \longrightarrow \phi(U \cap \hat{U})$, called a transition function, is holomorphic. In what follows, the homeomorphism ϕ will be referred to as a local parameter. Any set ϕ_i of holomorphically compatible local parameters such that the appropriate neighborhoods U_i cover the entire manifold X is called a complex atlas of the Riemann surface X. The union of the atlases that correspond to the same complex–analytic structure on the manifold X, i.e., to the same Riemann surface X, is again an atlas. This property is violated if the atlases making up a union belong to different complex–analytic structures or, which is equivalent, to different, yet topologically identical Riemann surfaces.

The simplest examples of a Riemann surface are: any open subset of a complex plane $\mathbb{C}$, $\mathbb{C}$ itself or an extended complex plane $\Sigma = \mathbb{C} \cup \infty$. The two canonical examples for the occurence of multi–valuedness are the log function (for each $z \in \mathbb{C} - 0$, $log(z)$ are the solution of $e^w = z$) and the function $z^{\frac{1}{q}}$ ($q \in \mathbb{N}$, $q \geq 2$, solution of $w^q = z$). For these functions, the points 0 and ∞ are critical, that is, there is no meromorhic continuation around these points. However, in any regions $D_J = \{z = re^{i\theta}; r \geq 0, \theta \in [a, b]\}$ with $a \leq b \leq a + 2\pi$, the function f is single–valued and analytic. To define these functions one introduces cuts in the complex plane, across which the function cannot be continued. This approach is not satisfactory since the functions are not continuous on ∂D_J. The solution to this problem is to extend the domain of definition, rather than restricting tha values of the function. It is exactly this domain, in which these functions are single–valued, which forms a sheet of the Riemann surface, in the covering–space of $\mathbb{C} - 0$. For example, each sheet of the surface corresponds to a particular branch of $log(z)$.

Meromorhic functions, i.e., non–constant holomorhic mappings $f : X \longrightarrow \Sigma$, constitute a meaningful object of analysis on Riemann surfaces. The local notation $f(z) = f \circ \phi^{-1}(z)$ of a meromorhic function f, for any local parameters ϕ is a meromorhic function of the variable $z \in \phi(U)$ in the usual sense. The meromorhic function f takes evey value $c \in \Sigma$ the same finite number of times (with the multiplicity taken into account). A point $P_0 \in f^{-1}(\infty)$ is said to be pole of the function f. In the neighborhood of any $P_0 \in X$, a meromorhic function f can be represented a convergent Laurent series:

$$\sum_{j=-N}^{\infty} c_j (z - z_0)^j, \quad z \equiv \phi(P), \quad z_0 = \phi(P_0) \tag{2.1}$$

where ϕ is a local parameter, the number $N > -\infty$ and does not depend on a specific choice of ϕ.

In addition to the notion of a function on a Riemann surface, we introduce the notion of an Abelian differential. An Abelian differential on the Riemann surface X is a meromorhic 1–form ω, given on X. This implies that we can write ω locally as $f(z)dz$, where $f(z)$ is a meromorphic function of z in its domain. For any Abelian differential, the notion of a pole and that of a zero are precisely defined, along with the notions of multiplicities and that of a residue:

$$Res(\omega; P_0) = c_{-1}, \quad \omega(P) = \sum c_j (z - z_0)^j dz \tag{2.2}$$

Abelian differentials are usually divided into three kinds: holomorphic differentials (first kind), meromorphic differentials with residues equal to zero at all singular points (second kind), and meromorhic differentials of the general form (third kind).

Any Abelian differential ω on the Riemann surface X satisfies the closure condition

$$d\omega = 0 \tag{2.3}$$

where $d\omega$ denotes the total derivative of the 1–form ω. Therefore, locally, a primitive function for the differential ω always exists and can be defined by the formula

$$\Omega = \int_{Po}^{P} \omega \tag{2.4}$$

for any simply–connected domain on X that involves (in the case of third-kind differentials) no singularities of the differential ω. Formula (2.4) considered on the whole surface X, defines, in general, a multivalued function called an Abelian integral. The division of Abelian differentials into the three kinds can be extended to Abelian integrals. Locally, Abelian integrals of the first kind are holomorhic functions, Abelian integrals of the second kind are meromorphic functions, and Abelian integrals of the third kind have logarithmic singularities:

$$\omega = (\cdots + \frac{c_{-1}}{z} + \cdots)dz \longrightarrow \Omega = \cdots + c_{-1} lnz + \cdots \tag{2.5}$$

We will use expression (2.5) to prove that the series expansions of solutions of (1.2) near a singularity contain logarithmic terms, (see Section 4).

3 Ziglin's theorem on a 2 d.o.f Hamiltonian System

In this section, we shall state Ziglin's theorem [1] on the splitting of separatrices in 2 d.o.f Hamiltonian systems. Consider the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0(x_1, x_2, I) + \epsilon \mathcal{H}_1(x_1, x_2, I, \phi) \tag{3.1}$$

where (x, y) and (I, ϕ) are canonically conjuagate pairs of momentum–position and action–angle variables respectively. With Ziglin [1], we now make the following assumptions about (3.1):

I $\mathcal{H}$ is real analytic in some domain of $\mathbf{x} = (x, y)$, $|I - I_0| < \mu$, $|\epsilon| < \mu$ and 2π - periodic in ϕ.

II For $\epsilon = 0$, $I = I_0$, (3.1) has two hyperbolic fixed points $\mathbf{x}_+$, $\mathbf{x}_-$ joined by a doubly asymptotic solution $\hat{\mathbf{x}}(t) \longrightarrow \mathbf{x}_\pm$ as $t \longrightarrow \pm\infty$.

III $\partial_I \mathcal{H}_0(\hat{\mathbf{x}}(t), I_0) \geq c \geq 0$ for all t and the solution

$$\hat{\mathbf{z}}(t) = (\hat{\mathbf{x}}(t), I_0, \hat{\phi}(t)), \quad \hat{\phi}(t) = \int^{t} \frac{\partial \mathcal{H}_0}{\partial I} \, dt' \tag{3.2}$$

can be analytically continued to the strip $\Pi : 0 \leq Imt \leq 2\pi/\lambda_+$ (where λ_+ is the positive eigenvalue of the linearized system about $\mathbf{x}_+$) and has no more than a finite number of singular points in Π.

IV $\mathcal{H}(\mathbf{z}, \epsilon)$ can be analytically continued for complex $\mathbf{z}$ and $\frac{\partial \mathcal{H}_0}{\partial I}(\hat{\mathbf{z}}(t))$, $\frac{\partial \mathcal{H}_1}{\partial \phi}(\hat{\mathbf{z}}(t, \phi_0))$ are single valued in Π, for all $\phi_0 \in \mathbb{R}$, where $\hat{\mathbf{z}}(t, \phi_0)$ denotes the solution $\hat{\mathbf{z}}(t))$ of (3.2) with $\hat{\phi}$ replaced by $\hat{\phi} + \phi_0$.

Theorem 3.1 *(Ziglin [1]). Under the above assumptions and if $\frac{\partial \mathcal{H}_1}{\partial \phi}(\hat{\mathbf{z}}(t, \phi_0))$ has nonzero sum of residues (2.2) in Π (for at least one ϕ_0), the system (3.1) possesses multiple–valued solutions $I(t) = I_0 + \epsilon I_1 + \cdots$ since*

$$\Delta I_1 = \oint_\Gamma \dot{I}_1 \, dt = -\oint_\Gamma \partial_\phi \mathcal{H}_1(\hat{\mathbf{z}}(t, \phi_0)) \, dt \neq 0 \tag{3.3}$$

for some contour $\Gamma \in \Pi$. In fact, since for any given ϕ_0, going around Γ changes the value of I_1 by the same amount ΔI_1, we conclude that $I(t)$ is infinitely branched in the complex t–plane, much like a logt function.

The connection between (3.3) above and the splitting of separatrices comes from Ziglin's proof [1] that (3.3) implies that the following integral does not vanish identically:

$$J(\phi_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\hat{\mathbf{z}}(t), \phi + \phi_0) \, dt \neq 0 \tag{3.4}$$

where $\{.,.\}$ denotes the Poisson bracket, and H_0, H_1 are related to the original Hamiltonians by solving (3.1) for a (single–valued) I on a constant energy surface $-I = H_0(x,y) + \epsilon H_1(x,y,\phi) + \cdots$. Ziglin also proves that (3.3) implies that the Hamiltonian system (3.1) does not possess a second analytic integral indepedent of $\mathcal{H}$ for any sufficiently small $|\epsilon| \neq 0$ [1].

Theorem 3.1 can be generalized to the non–Hamiltonian case of a periodically driven system

$$\dot{x}_i = f_i(x_1, x_2) + \epsilon g_i(x_1, x_2, t), \quad i = 1,2$$

where $g_i(x_1, x_2, t) = g_i(x_1, x_2, t+T)$, f_i, g_i are analytic in x_1, x_2 and the unperturbed ($\epsilon = 0$) equations have single–valued solutions and a smooth separatrix joining two fixed saddle points (see [12]). The main idea is that if the Mel'nikov integral is not identically zero, one can always find an analytic function of x_1, x_2 which is infinitely–sheeted in the complex t–plane.

Of course, splitting of separatrices does not necessarily mean the appearance of chaos, since, in a non–Hamiltonian system, invariant manifolds of saddle points can split, for $\epsilon \neq 0$, without intersecting ($J(\phi_0) \neq 0$ for all ϕ_0). Splitting does mean, however, non–integrability, in the sense of the appearence of I.S.S with the type of infinite multivaluedness one finds in functions with logarithmic singularities.

4 Infinitely Multivalued Solutions and Subharmonic Bifurcations

Let $U = D \times (I_0 - \mu, I_0 + \mu) \times S^1$ be the direct product of a domain $D \subset \mathbb{R}^2$ with coordinates $x = (x^1, x^2)$ and I be an action variable in the interval $I_0 - \mu < I < I_0 + \mu$ with an angular coordinate ϕ on the circle S^1. Consider Hamiltonian of the form (3.1)

$$\mathcal{H}(z,\epsilon) = \mathcal{H}_0(x,I) + \epsilon \mathcal{H}_1(x,I,\phi), \qquad z = (x,I,\phi) \tag{4.1}$$

which is real–analytic in $x = (x_1, x_2), I, \phi$, $\mathcal{H}_0(x,I) = F(x) + G(I)$ for $x \in D$, $|I - I_0| < \mu$, $|\epsilon| < \mu$ and is 2π - periodic in ϕ. For $\epsilon = 0$, we have the unperturbed system:

$$\dot{x}_1 = \partial_{x_2}\mathcal{H}_0(x,I) \qquad \dot{x}_2 = -\partial_{x_1}\mathcal{H}_0(x,I) \tag{4.2a}$$

$$\dot{\phi} = \partial_I \mathcal{H}_0(x,I) \qquad \dot{I} = -\partial_\phi \mathcal{H}_0(x,I) = 0 \tag{4.2b}$$

We now make the following assumptions :

A1. The system (4.2) possesses an elliptic fixed point at (0,0) and a family of doubly periodic (in $t \in \mathbb{C}$) solutions $q^a(t)$, which rotate about (0,0) in the (x_1, x_2) plane. Let $a \in [-1,1]$ with $q^{-1}(t) = (0,0)$ and $q^a(t) \longrightarrow Q(t)$ as $a \longrightarrow 1$, where $q^1(t) = Q(t)$ is a boundary of D.

A2. Region D of the phase plane of the (uncoupled) $F(x)$ system is filled with periodic orbits $q^\alpha(t)$, whose period T_α varies continuously with respect to the energy $\mathcal{H}(z,0) = h$. Each such orbit is a level of $F : F(x) = h_\alpha$ provided, for energy $h > h_\alpha$, the unperturbed ($\epsilon = 0$) system has a corresponding closed orbit given by $G^{-1}(h - h_\alpha) \equiv I_\alpha$.

A3. The Hamiltonian $\mathcal{H}(z,\epsilon)$ can be continued analytically, but in general, not single–valuedly to a domain in complex (z,ϵ) - space.

A4. The functions

$$\partial_I \mathcal{H}_0(z(t)), \qquad \partial_\phi \mathcal{H}_1(z(t,\bar{\phi})) \tag{4.3}$$

with $z(t) = (q^a(t), I)$ and $z(t,\bar{\phi}) = (q^a(t), I, \phi + \bar{\phi})$ are single–valued for every $\bar{\phi} \in \mathbb{R}$.

For some I_0 and sufficiently small $|I - I_0|$, $|x - q^0|$ and $|\epsilon|$ the equation of the isoenergy surface,

$$\mathcal{H}(x,I,\phi,\epsilon) = h \tag{4.4.a}$$

has a single–valued solution for I,

$$-I = H(x,\phi,\epsilon;h) = H_0(x) + \epsilon H_1(x,\phi;h) + \cdots \tag{4.4.b}$$

with

$$\mathcal{H}_0(x, -H_0(x)) = h \qquad H_0(x) = G^{-1}(h - F(x)) \tag{4.4.c}$$

$$H_1(x,\phi) = \frac{\mathcal{H}_1(x,-H_0(x),\phi)}{\partial_I \mathcal{H}_0(x,-H_0(x))} \equiv \frac{\mathcal{H}_1(x,-H_0(x),\phi)}{\Omega(H_0(x))} \tag{4.4.d}$$

This allows us to go, on the surface (4.4.a) (for sufficiently small $|I - I_0|$, $|x - q^0|$ and $|\epsilon|$), from system (4.1) to the reduced system :

$$x_1' = \partial_{x_2} H(x,\phi,\epsilon) \qquad x_2' = -\partial_{x_1} H(x,\phi,\epsilon) \tag{4.5}$$

(where primes denote differentiation with respect to ϕ). We now consider system (4.5) in the extended phase space, $V = D^1 \times S^1$, which is the direct product of a domain $D^1 \subset D$ containing the periodic orbit $q^a(t)$ and S^1 (with angular coordinate ϕ) and define the Poincaré map:

$$P_\epsilon : (x_1(\phi), x_2(\phi)) \longrightarrow (x_1(\phi + 2\pi), x_2(\phi + 2\pi)) \tag{4.6}$$

on the solutions of (4.5), i.e $P_\epsilon : D^1 \longrightarrow D^1$.

Before we proceed to our main theorem, we need to establish an important proposition which is analogous to Mel'nikov 's second theorem on subharmonic bifurcations, but does not assume the presence of a saddle fixed point governing the dynamics of the system.

For fixed h, eqs (4.5) take the form of a periodically perturbed planar system since H_0 depends only on (x_1, x_2) while H_1 has an explicit ϕ–depedence. The unperturbed Hamiltonian (4.1) has the form $\mathcal{H}_0(x, I) = F(x) + G(I)$ and the period of periodic orbits for this system satisfies the resonance relationship

$$T_\alpha = \frac{2\pi m}{n\Omega(h - h_\alpha)} \tag{4.7}$$

for relatively prime integer pairs m, n. In terms of the "new time" ϕ, however, the period $\hat{T}_\alpha$ of the H system is given by

$$\hat{T}_\alpha = \frac{2\pi m}{n}$$

Proposition 4.1 (see [2])
Let $\{H_0, H_1\}$ denote the Poisson bracket of H_0, H_1 with respect to x_1, x_2 and consider a periodic orbit of the unperturbed system $q^\alpha(\phi)$ with period $\hat{T}_\alpha = 2\pi m/n$ on the resonant closed curve $\mathcal{H}_0 = h_\alpha$ of the unperturbed Poincaré map P_0, cf. (4.6). Fix $h > 0$, m, n integer relatively prime and choose ϵ sufficiently small. Then, if the subharmonic Mel'nikov function

$$M^{m/n}(\bar{\phi}) = \int_0^{2\pi m} \{H_0, H_1\}(q^a(\phi - \bar{\phi}), \phi; h)\, d\phi \tag{4.8}$$

has j simple zeros as a function of $\bar{\phi} \in [0, 2\pi m/n)$, the resonant torus given by $(q^\alpha(\phi - \bar{\phi}), \phi)$ breaks into $2k = j/m$ distinct $2\pi m$–periodic orbits, and there are no other $2\pi m$ periodic orbits in its neighborhood.

We now proceed to our main theorem [2]:

Theorem 4.1 *Consider a real analytic 2 d.o.f Hamiltonian system (4.1), which can be reduced to the form (4.4), by virtue of assumptions A1–A4, and let $q^\alpha(t)$ be a periodic solution of the unperturbed problem, $\epsilon = 0$, with $\mathcal{H}_0 = h_\alpha$. Assume, furthermore, that $q^\alpha(t)$ is a meromorphic function of $t \in \mathbb{C}$, doubly periodic with real and imaginary periods T_α, iT'_α respectively and can be analytically continued in the strip*

$$L = \{t \in \mathbb{C} : 0 < \Im t < T'_\alpha\}$$

Finally, suppose that $\partial_\phi \mathcal{H}_1(q^\alpha(t), \bar{\phi})$ (where by $\bar{\phi}$ we abbreviate $\phi + \bar{\phi}$) has a finite number of poles, inside a closed contour $\Gamma \subset \Pi$, where Π is the period parallelogram

$$\Pi = \{t \in \mathbb{C} : 0 < \Re t < T_\alpha, \quad 0 < \Im t < T'_\alpha\}$$

Then, if for some $\bar{\phi}$, the sum of the residues at these poles

$$S = \sum_{t_j \in \Pi} Res\{\partial_\phi \mathcal{H}_1(q^\alpha(t_j), \bar{\phi})\} \neq 0 \tag{4.9}$$

and ϵ is small enough:

- **(i)** The $\mathcal{H}_0 = h_\alpha$ invariant curve undergoes a subharmonic bifurcation.
- **(ii)** The system possesses $\mathcal{O}(\epsilon)$ solutions, which are infinitely–sheeted in $t \in \mathbb{C}$.

Remark 4.1 The second result, (ii), follows immediately from Hamilton's equations and assumption (4.9): Evaluating the solution $I(t) = I_0 + \epsilon I_1(t) + \cdots$ along a closed contour $\Gamma \subset \Pi$ we have to $\mathcal{O}(\epsilon)$:

$$\Delta I_1 = \oint_\Gamma \dot{I}_1 \, dt = -\oint_\Gamma \partial_\phi \mathcal{H}_1(q^\alpha(t), \bar{\phi}) \, dt = -2\pi i S \neq 0$$

Hence, $I(t)$ increases, after every circuit around Γ, by a fixed amount $\Delta I(\epsilon)$ such that

$$\lim_{\epsilon \to 0} \frac{\Delta I(\epsilon)}{\epsilon} \neq 0$$

and the system possesses the I.S.S property. The quantity ΔI_1 is an Abelian integral on a Riemann surface Σ (see section 2) and we note that the 1–form ω (third kind), given by $\omega = \partial_\phi \mathcal{H}_1(q^\alpha(t), \bar{\phi})dt$, has a finite number of poles and the Abelian integral ΔI_1 a logarithmic singularities. Then $I(t)$ is infinitely branched in the complex plane of time much like a $logt$ function.

Remark 4.2 Finally, we can show, using the results of [7], that under the above conditions, the reduced system (4.5) is non–integrable. This is done by noting that due to (4.7), our system exhibits subharmonic bifurcations on a dense (n, m) set of invariant tori and hence according to the theorems given in [7], it cannot possess a second analytic, single–valued integral of the motion.

Let us now illustrate Theorem 4.1 on the example of a Duffing oscillator, which does not have a fixed saddle point and is perturbed by a periodic function, which preserves the Hamiltonian formulation of the system. More specifically, we consider the system of o.d.es

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1 - {x_1}^3 + \epsilon cos\omega t \tag{4.10}$$

Note that we can always introduce an angle variable $\phi = \omega t$ and a conjugate action variable I, such that the Hamiltonian of this system can be written in the form (4.4b), i.e

$$H = H_0 + \epsilon H_1 = \frac{{x_2}^2}{2} + \frac{{x_1}^2}{2} + \frac{{x_1}^4}{4} - \epsilon x_1 cos\phi + I\omega \tag{4.11}$$

It is easy to verify that the unperturbed system ($\epsilon = 0$) possesses a family of periodic orbits around the elliptic fixed point $(0, 0)$, given in terms of the Jacobi elliptic functions [10, 11]

$$q^k(t) = A(cn(\lambda t, k), -\lambda sn(\lambda t, k)dn(\lambda t, k)) \tag{4.12}$$

with $A^2 = 2\lambda^2 k^2$ and $\lambda^2(2k^2 - 1) = -1$. Clearly, this family of periodic orbits can be analytically continued in the strip :

$$L = \{t \in \mathbb{C} : 0 \leq \Im t \leq 2K'\}$$

where K, K' are the elliptic integrals of first and second kind. Since within any closed contour in L, the number of poles is finite, this satisfies the corresponding requirement of Theorem 4.1, cf.(4.9).

As is well–known [8, 9] subharmonic bifurcations occur in this problem when the Mel'nikov integral

$$M^{m/n}(\bar{\phi}) = \int_0^{nT_\alpha} \{H_0, H_1\}(q^k(t), t + \bar{\phi}) \, dt \tag{4.13}$$

has simple zeros, with $nT_a = 2\pi m$. We now compute the integral (4.13) by the method of residues and find that

$$M^{m/n}(t_0) = \begin{cases} \dfrac{(-1)^m \pi^2 m}{K\sqrt{2 - 4k^2}} e^{(-\frac{\pi m K'}{2K})} sin\omega t_0 & \text{if } n = 1 \\ \sqrt{2}\pi\omega(2sin\omega t_0 - cos\omega t_0) sinhK'\beta & \text{if } n \neq 1 \end{cases} \tag{4.14}$$

$$S = \sum_{t_j \in \Pi} Res\{\partial_\phi \mathcal{H}_1(q^k(t_j), I_0)\} = \sum_{t_j \in \Pi} Res(cn(\lambda t_j, k) sin\phi) \neq 0$$

where Π is the period parallelogram of the cn function. Thus, according to Theorem 4.1, $\Delta I_1 = -2\pi i S \neq 0$ and the infinitely–sheeted property of the solution $I(t) = I_0 + \epsilon I_1(t) + \cdots$, to $\mathcal{O}(\epsilon)$, has been established. We recall that the existence of such infinitely–sheeted solutions for a Duffing oscillator similar to (4.10) has been explicitly demonstrated elsewhere [12, 13], in the form of so–called psi series expansions, involving logarithmic singularities.

5 The case of non–Hamiltonian perturbations

Consider a Hamiltonian system of the form (4.1), (4.2) perturbed by dissipative terms as follows:

$$\dot{x}_1 = \partial_{x_2} F(x) + \epsilon \partial_{x_2} \mathcal{H}_1 + \epsilon \gamma_1 f_1 \tag{5.1a}$$

$$\dot{x}_2 = -\partial_{x_1} F(x) - \epsilon \partial_{x_1} \mathcal{H}_1 + \epsilon \gamma_2 f_2 \tag{5.1b}$$

$$\dot{\phi} = \Omega(I) + \epsilon \partial_I \mathcal{H}_1 + \epsilon \delta_1 g_1 \tag{5.1c}$$

$$\dot{I} = -\epsilon \{\partial_\phi \mathcal{H}_1 - \delta_2 g_2\} \tag{5.1d}$$

where F, G and $\mathcal{H}_1$ are as defined in section 4 and f_i, g_i are analytic functions of (x_1, x_2, I) and 2π–periodic in ϕ. Furthermore we assume that the f_i, g_i are such that the above perturbation is of non–Hamiltonian character.

In this case, the function $\mathcal{H} = \mathcal{H}_0(x_1, x_2, I) + \epsilon \mathcal{H}_1(x_1, x_2, I, \phi)$, with $\mathcal{H}_0 = F(x_1, x_2) + G(I)$, is no longer conserved, as it satisfies

$$\dot{\mathcal{H}} = \epsilon \left(\gamma_1 f_1 \partial_{x_1} \mathcal{H} + \gamma_2 f_2 \partial_{x_2} \mathcal{H} + \delta_2 g_2 \partial_I \mathcal{H} + \delta_1 g_1 \partial_\phi \mathcal{H} \right) \tag{5.2}$$

or to order ϵ,

$$\dot{\mathcal{H}} = \epsilon h(x_1, x_2, I) + \mathcal{O}(\epsilon^2) \tag{5.3}$$

with $h(x_1, x_2, I) = \gamma_1 f_1 \partial_{x_1} F + \gamma_2 f_2 \partial_{x_2} F + \delta_2 g_2 \Omega(I)$.

We now apply the implicit function theorem and solve the equation $\mathcal{H}(x, I, \phi) = \mathcal{H}$ for I

$$I = H_0(x_1, x_2; \mathcal{H}) + \epsilon H_1(x_1, x_2, \phi; \mathcal{H}) + \cdots$$

whence, after some calculations, the reduced system has the form:

$$\dot{x}_1 = -\frac{\partial H_0}{\partial x_2} - \epsilon \left(\frac{\partial H_1}{\partial x_2} - \frac{\gamma_1 f_1}{\Omega} - \frac{\partial H_0}{\partial x_2} \frac{\delta_1 g_1}{\Omega} \right) + \mathcal{O}(\epsilon^2) \tag{5.4a}$$

$$\dot{x}_2 = \frac{\partial H_0}{\partial x_1} + \epsilon \left(\frac{\partial H_1}{\partial x_1} - \frac{\gamma_2 f_2}{\Omega} - \frac{\partial H_0}{\partial x_1} \frac{\delta_1 g_1}{\Omega} \right) + \mathcal{O}(\epsilon^2) \tag{5.4b}$$

$$\dot{\mathcal{H}} = \epsilon \Omega \left(\delta_2 g_2 - \frac{\partial H_0}{\partial x_1} \gamma_1 f_1 - \frac{\partial H_0}{\partial x_2} \gamma_2 f_2 \right) + \mathcal{O}(\epsilon^2) \tag{5.4c}$$

Let $d = (\gamma_1, \gamma_2, \delta_1, \delta_2)$ denote the dissipation coefficients in our system. Also, let $P_{\epsilon,d}$ denote the Poincaré map associated with the system (5.4). As before, we consider a function $\mathcal{H} = \mathcal{H}(x_1, x_2, I, \phi, \epsilon)$ which can be continued analytically, but not, in general, single–valuedly to a domain in complex space, while the function

$$R(q^\alpha(t), I, \phi + \bar{\phi}) = \{\frac{\partial \mathcal{H}_1}{\partial \phi} - \delta_2 g_2\}(q^\alpha(t), I, \phi + \bar{\phi}) \tag{5.5}$$

is single–valued for every $\bar{\phi}$.

We now extend the main theorem of section 4 to (5.1), assuming that the function (5.5) has a finite number of poles inside a closed contour $\Gamma \subset \Pi$, such that

$$S = \sum_{t_j \in \Pi} Res\left(\frac{\partial \mathcal{H}_1}{\partial \phi} - \delta_2 g_2\right) \neq 0 \tag{5.6}$$

Thus, we conclude that the non–Hamiltonian system (5.1) possesses infinitely sheeted solutions to $\mathcal{O}(\epsilon)$, of the form $I(t) = I_0 + \epsilon I_1(t) + \cdots$, since

$$\Delta I_1 = \oint_\Gamma \dot{I}_1 \, dt = -\oint_\Gamma \{\partial_\phi \mathcal{H}_1(q^\alpha(t), \bar{\phi}) - \delta_2 g_2(q^\alpha(t), \bar{\phi})\} \, dt = -2\pi i S \neq 0 \tag{5.7}$$

Finally, it is easy to verify, using the ideas of Theorem 4.1, that condition (5.6) also implies that the Mel'nikov function of system (5.4)

$$M_d^{m/n}(\bar{\phi}) = M^{m/n}(\bar{\phi}) + \frac{1}{\Omega}\int_0^{2\pi m} \left(\frac{\partial H_0}{\partial x_2}\gamma_2 f_2 - \frac{\partial H_0}{\partial x_1}\gamma_1 f_1\right)(q^\alpha(\phi - \bar{\phi}), \phi) \, d\phi \tag{5.8}$$

cf. (4.8) is not identically zero, for general perturbations f_i, g_i. Hence, if (5.8) turns out to have simple zeros, the perturbed system will exhibit subharmonic bifurcations as expected.

Applying this approach to non–Hamiltonian perturbations (5.1) of the example of section 4, we find that the second term of the r.h.s of (5.8) yields:

$$\hat{M}_{\gamma_2}^{m/n} = -\lambda\gamma_2 \int_0^{4nK} sn^2(u,k) dn^2(u,k) \, du \tag{5.9}$$

where we have taken $f_1 = 0$, $f_2 = x_2$, $g_1 = g_2 = 0$ cf. (4.12). Performing the integrations in (5.9) we finally find

$$\hat{M}_{\gamma_2}^{m/n} = \gamma_2 \frac{4}{3k^2\sqrt{1-2k^2}}[E(k)(1-2k^2) - k'^2 K(k)] \tag{5.10}$$

When added to (4.14), the constant $\hat{M}_{\gamma_2}^{m/n}$ for $n = 1$, obtained above for suitable choices of γ_2

$$\gamma_2 \leq R^m(\omega) = \frac{J_1(m,1)}{(-1)^m\sqrt{2}\pi\omega exp(-\frac{m\pi K'}{2K})} \tag{5.11}$$

$J_1(m,1) = \hat{M}_{\gamma_2}^{m/1}/\gamma_2$ can prevent the total $M_d^{m/1}(\bar{\phi}, \gamma_2)$ of (5.8) from having simple zeros, in $\bar{\phi} = \omega t_0$, thus eliminate the occurrence of subharmonic bifurcations in this example and the unperturbed periodic orbit (4.12) satisfying the resonance relation

$$4K(k)\sqrt{1-2k^2} = \frac{2\pi m}{\omega}$$

6 Conclusions

In this paper, we have shown that 2 d.o.f Hamiltonians which exhibit, upon perturbation, subharmonic bifurcations of their resonant invariant curves around an elliptic fixed point possess solutions which are infinitely–sheeted in the complex domain. Such systems are known to be non–integrable when these bifurcating tori form a dense set.

Our analysis follows Ziglin 's approach to similar systems, with the important difference that, in this case, the unperturbed Hamiltonian possesses a homoclinic orbit. In our systems, we do not require the existence of such an orbit and make crucial use of Mel'nikov 's subharmonic function. We have applied our results to a conservative Duffing oscillator, showing that this classical example exhibits infinitely–sheeted solutions near an elliptic fixed point.

Finally, we have shown that our results can be extended to the case of non–Hamiltonian perturbations, and have illustrated the analysis by applying to a class of non–Hamiltonian perturbation of Duffing's oscillator.

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