Some Schrödinger operators with power-decaying potentials and pure point spectrum

Christian Remling
Universität Osnabrück
Fachbereich Mathematik/Informatik
D-49069 Osnabrück
GERMANY

E-mail: cremling@mathematik.uni-osnabrueck.de

Current address (until May 31, 1997):
253-37 Department of Mathematics
California Institute of Technology
Pasadena, CA 91125
U.S.A.

(to appear in *Comm. Math. Phys.*)

April 25, 1997

Abstract

We construct (deterministic) potentials $V(x) = O(x^{-c})$ such that the Schrödinger equation $-y'' + Vy = Ey$ on $x \in [0, \infty)$ has dense pure point spectrum in $(0, \infty)$ for almost all boundary conditions at $x = 0$. As a by-product, we also obtain power-decaying potentials for which the spectrum

is purely singular continuous on $(0, \infty)$ for all boundary conditions.

1 Introduction

In this paper, we will construct power-decaying potentials $V(x) = O(x^{-c})$ $(c > 0)$ such that the Schrödinger equation for positive energies $E = k^2$

$$-y''(x) + V(x)y(x) = k^2 y(x) \tag{1}$$

on the half-axis $x \in [0, \infty)$ has square integrable solutions for almost all $k > 0$ with respect to Lebesgue measure. By well-known arguments (using spectral averaging, see e.g. [16]), this implies that the corresponding Hamiltonian $H_\alpha = -\frac{d^2}{dx^2} + V(x)$ with boundary condition $y(0)\cos\alpha + y'(0)\sin\alpha = 0$ has dense pure point spectrum in $(0, \infty)$ (i.e. $\sigma_{ess} = [0, \infty), \sigma_c(H_\alpha) = \emptyset$) for almost all $\alpha \in [0, \pi)$.

As far as I am aware, this is even the first explicit example of a Schrödinger operator with this spectral type and $V \to 0$. The existence of such potentials, however, is not a new result: The work on decaying random potentials [8, 14] has shown that, roughly speaking, random potentials decaying as $V_\omega(x) \sim x^{-c}$ with $c < 1/2$ almost surely lead to dense pure point spectrum if the values of the potential at different sites are independent with zero expectation. (In fact, $c = 1/2$ is generally believed to be the borderline between absolutely continuous and singular spectrum in the sense that $V(x) = O(x^{-c})$ with $c > 1/2$ should imply $\sigma_{ac} = [0, \infty)$. Kiselev has recently proved this for $c > 2/3$ [6], but in the region $1/2 < c \leq 2/3$, this question is open.)

There is, of course, still much point in looking for deterministic examples, since the proofs in the random case strongly depend on averaging procedures and independence assumptions and therefore do not give much insight into the mechanisms with which a fixed potential generates point spectrum. The first deterministic potentials with pure point spectrum were constructed in [3, 5]. However, these examples rely on tunneling through high barriers and therefore

$\limsup V(x) = \infty$. Later, these methods were extended to handle the case of wide barriers (see [19] and also [5] where results of this type are stated without proof), thereby obtaining potentials with $\limsup V < \infty$ and pure point spectrum in $(\liminf V, \limsup V)$ (a typical example is $V(x) = \cos x^\gamma$ $(0 < \gamma < 1)$; see also [17]).

Although not directly related to this work, I should also mention the construction of potentials with dense point spectrum and $|V(x)| \leq f(x)/x$ ($f \to \infty$ arbitrarily slowly) in [10, 15]. However, here one has $\sigma_{ac} = [0, \infty)$.

As an almost immediate consequence of our construction, we will also obtain power-decaying potentials with purely singular continuous spectrum in $(0, \infty)$ for all boundary conditions. Apparently, only one other class of decaying potentials with this spectral behavior is known at present, namely the type of sparse potentials introduced by Pearson in his classic [11], and although the conditions on the barrier separations could be considerably weakened recently [7], the decay of these potentials is still slower than (say) $(\ln x)^{-1/2}$.

The main tools we will use in this paper are the (modified) Prüfer transformation and transfer matrices. In some respects, our general strategy will be similar to the proof of [3, Theorem 2]. For other papers discussing some of the techniques used here, see, for instance, [7, 8, 10, 12].

This paper is organized as follows. In Section 2, we describe the model and state the result we will prove. We state some preliminary observations and explain the general strategy in Section 3. The technical details are presented in the following three Sections. We conclude with the modification which yields singular continuous spectrum.

I would like to thank A. Gordon, B. Simon and T. Wolff for useful discussions, the Deutsche Forschungsgemeinschaft for financial support, and I am grateful for the hospitality of Caltech where this work was done.

2 Description of the model

Let $L_n \in \mathbb{N}, g_n > 0, l_n \geq 0$, and let $W(x)$ be a bounded, measurable and periodic function with period p. Define $a_1 = 0$,

$$b_n = a_n + p\frac{L_n}{g_n}, \quad a_{n+1} = b_n + l_n \quad (n \geq 1).$$

The potential we will study is given by

$$V(x) = \begin{cases} 0 & \text{if } x \in (b_n, a_{n+1}) \\ W_{g_n}(x - a_n) & \text{if } x \in (a_n, b_n) \end{cases}$$

where W_g is the rescaled function $W_g(x) = g^2 W(gx)$.

Theorem 2.1 *L_n, g_n, l_n and $W(x)$ can be chosen such that:*
1) $V(x) = O(x^{-c})$ for some $c > 0$,
2) For almost all $k > 0$, (1) has an L_2-solution.

It follows from $V \to 0$ that $\sigma_{ess} = [0, \infty)$. Moreover, as pointed out above, 2) implies that for almost all boundary conditions α, the corresponding half-line operator H_α has no continuous spectrum, hence it must have dense pure point spectrum in $(0, \infty)$. Note also that by recently discovered general principles on the instability of point spectrum [1], there is a dense G_δ set of boundary conditions for which the spectrum is purely singular continuous in $(0, \infty)$!

3 Preliminary observations

Write

$$Y(x,k) := \begin{pmatrix} y(x,k) \\ y'(x,k)/k \end{pmatrix}$$

where y is a solution of (1) satisfying an initial condition of the form $y(0,k) = -\sin\alpha, y'(0,k) = \cos\alpha$. We will use the modified Prüfer variables R, φ defined by the equation

$$Y(x,k) = R(x,k) \begin{pmatrix} \sin\varphi(x,k) \\ \cos\varphi(x,k) \end{pmatrix}$$

and by requiring that φ be continuous and $R > 0$. The transfer matrix $T(t, s; k)$ is defined by the property that it takes $Y(s, k)$ to $Y(t, k)$, i.e. $Y(t, k) = T(t, s; k)Y(s, k)$. Hence it is given by

$$T(t, s; k) = \begin{pmatrix} u(t, k) & kv(t, k) \\ u'(t, k)/k & v'(t, k) \end{pmatrix} \qquad (2)$$

where u, v are the solutions of (1) with $u(s) = v'(s) = 1, u'(s) = v(s) = 0$. If the potential is periodic, then obviously $T(t+p, s+p; k) = T(t, s; k)$, (this observation is roughly one half of what is usually called Floquet theory). Abbreviating $T(k) := T_W(p, 0; k)$ (the transfer matrix over one period for the potential W), we get for the transfer matrix for the potential V described in the preceding Section

Lemma 3.1

$$T(a_n + m_2 p g_n^{-1}, a_n + m_1 p g_n^{-1}; k) = T(k g_n^{-1})^{m_2 - m_1}, \qquad m_i \in \{0, 1, \ldots, L_n\}$$

Proof. By "Floquet theory", it suffices to prove the assertion for $m_1 = 0, m_2 = 1$. Moreover, $T(a_n + p g_n^{-1}, a_n; k) = \hat{T}(p g_n^{-1}, 0; k)$ where $\hat{T}$ is the transfer matrix of $-y'' + W_{g_n}(x)y = k^2 y$.

For a function $y(x)$ and $g > 0$, let $w(t) := y(t/g)$. Then we have: y solves $-\frac{d^2 y}{dx^2} + W_g(x)y = k^2 y$ and satisfies $(y, dy/dx)(0) = (a, b)$ if and only if w solves $-\frac{d^2 w}{dt^2} + W(t)w = (k/g)^2 w$ and satisfies $(w, dw/dt)(0) = (a, g^{-1}b)$. Hence, if u_g, v_g denote the solutions of $-y'' + W_g y = k^2 y$ with $u_g(0) = v_g'(0) = 1, u_g'(0) = v_g(0) = 0$, then

$$u_g(x, k) = u_1(gx, k/g), \qquad v_g(x, k) = g^{-1} v_1(gx, k/g)$$
$$u_g'(x, k) = g u_1'(gx, k/g), \qquad v_g'(x, k) = v_1'(gx, k/g).$$

Now the assertion follows from (2). □

Next, we recall some basic facts about periodic potentials; for further background information on this topic, please consult [2]. Since $\det T(k) = 1$ by constancy of the Wronskian, the eigenvalues of T are determined by its trace

$$D(k) := \text{tr } T(k) = u(p, k) + v'(p, k),$$

(u, v solve $-y'' + Wy = k^2 y$ and $u(0) = v'(0) = 1, u'(0) = v(0) = 0$). More explicitly, the eigenvalues are μ, μ^{-1} with

$$\mu(k) = \frac{D(k)}{2} \pm \sqrt{\frac{D(k)^2}{4} - 1}. \tag{3}$$

We choose the sign so that $|\mu| \geq 1$. μ is real and larger than 1 in magnitude if and only if $|D| > 2$; if $|D| < 2$, then $|\mu| = 1, \mu \neq \pm 1$. In either case, the eigenvalues are distinct and hence T is diagonalizable. Let $U(k)$ be a diagonalizing matrix, i.e.

$$U^{-1}(k)T(k)U(k) = \begin{pmatrix} \mu(k) & 0 \\ 0 & \mu^{-1}(k) \end{pmatrix}.$$

This representation leads to a simple estimate on the norm of the transfer matrix. In the following Lemma, we use the l_2-norm $\|(v_1, v_2)^t\|^2 = |v_1|^2 + |v_2|^2$.

Lemma 3.2 *a) If $|D(k)| < 2$, then*

$$\|T(k)^m v\| \geq \frac{\|v\|}{\|U(k)\| \, \|U^{-1}(k)\|}.$$

b) If $|D(k)| > 2$, then there is an angle $\beta(k) \in [0, \pi)$ such that $|\varphi - \beta(k) - n\pi| \geq \epsilon \quad \forall n \in \mathbb{Z}$ implies

$$\|T(k)^m e_\varphi\| \geq \frac{2\epsilon}{\pi} |\mu(k)|^m$$

(where $e_\varphi = (\sin\varphi, \cos\varphi)^t$).

Proof. Obviously,

$$T(k)^m = U(k) \begin{pmatrix} \mu(k)^m & 0 \\ 0 & \mu(k)^{-m} \end{pmatrix} U^{-1}(k). \tag{4}$$

Hence $\|T^{-m} w\| \leq |\mu|^m \|U\| \, \|U^{-1}\| \, \|w\|$ which implies a) since $|\mu| = 1$ in this case. In order to prove b), notice that because $T(k)$ is real and has real, distinct eigenvalues if $|D(k)| > 2$, we may take U in the form

$$U = \begin{pmatrix} \sin\alpha & \sin\beta \\ \cos\alpha & \cos\beta \end{pmatrix}$$

with $\alpha, \beta \in [0, \pi)$, $\alpha \neq \beta$. Now an elementary calculation using (4) leads to (writing $\gamma = \alpha - \beta$)

$$\|T^m e_\varphi\|^2 = \sin^{-2}\gamma\left(\mu^{2m}\sin^2(\varphi-\beta) + \mu^{-2m}\sin^2(\varphi-\alpha) - 2\cos\gamma\sin(\varphi-\beta)\sin(\varphi-\alpha)\right).$$

Viewing the right-hand side as a (quadratic) function of $\sin(\varphi - \alpha)$ and determining the minimum shows $\|T^m e_\varphi\|^2 \geq \mu^{2m}\sin^2(\varphi-\beta)$. This yields the desired estimate, since $|\sin x| \geq 2|x|/\pi$ if $|x| \leq \pi/2$. $\square$.

Now we are in a position to explain the general idea of our construction. By Lemma 3.1, the transfer matrix for $x \in (a_n, b_n)$ is given by powers of $T(k/g_n)$. We will take a function W with infinitely many "gaps" (i.e. intervals with $|D| > 2$), and we will choose the g_n such that for every k, k/g_n is in some gap for sufficiently many n. Then Lemma 3.2b) shows that for those n the solution grows by a factor $|\mu|^{L_n}$ unless the phase $\varphi(a_n)$ is close to β. If k/g_n is not in a gap, then the solution can be controlled with the aid of Lemma 3.2a). The point of this argument is that the estimate of Lemma 3.2a) is independent of m. Therefore, after everything else has been chosen, we can take the L_n sufficiently large to obtain increasing solutions. Of course, one then needs an additional argument to deduce the existence of L_2-solutions. To this end, we will use a Ruelle type Theorem (see [13]) recently proved in [9].

We will make no attempt to optimize the exponent c from Theorem 2.1 because it does not seem possible to reach the critical value $1/2$. In particular, we will sometimes use rather crude estimates when this does not destroy the power decay.

The following Lemma allows us to adjust the phases $\varphi(a_n, k)$ if necessary. For $k > 0$ with $|D(k)| > 2$, define $S_\epsilon(k) = (\beta(k) - \epsilon, \beta(k) + \epsilon)$ with $\beta(k)$ from Lemma 3.2b).

Lemma 3.3 *Suppose $V(x)$ has been chosen for $x \leq b_n$. Let M be a measurable subset of $\{k \in [k_1, k_2] : |D(k/g_{n+1})| > 2\}$. Then there is an $l_n \in [0, \pi/k_2]$ such* *that*

$$|\{k \in M : \varphi(a_{n+1}, k) \in S_\epsilon(k/g_{n+1})\}| \leq \frac{2k_2}{\pi k_1}|M|\epsilon.$$

Of course, the phase φ has to be evaluated modulo π here.

Proof. First note that since $V = 0$ on (b_n, a_{n+1}), we have $\varphi(a_{n+1}, k) = \varphi(b_n, k) + (a_{n+1} - b_n)k = \varphi(k) + l_n k$ (using the abbreviation $\varphi(k) = \varphi(b_n, k)$). Denote the set defined in the Lemma by $J(l_n)$ and integrate $|J(l_n)|$ with respect to dl_n. Fubini-Tonelli shows

$$\int_0^{\pi/k_2} |J(l_n)|\, dl_n = \int_M dk \int_0^{\pi/k_2} dl_n\, \chi_{S_\epsilon(k/g_{n+1})}(\varphi(k) + kl_n).$$

(φ, β are continuous in k, so there is certainly no problem with measurability.) χ_S denotes the characteristic function of the set S. The second integral on the right-hand side can be estimated by $k_1^{-1}2\epsilon$, because $\varphi(k) + lk$ $(0 \leq l < \pi/k_2)$ takes on every value in $[0, \pi)$ at most once. Hence $\int_0^{\pi/k_2} |J(l_n)|\, dl_n \leq k_1^{-1}|M|2\epsilon$, and therefore the claimed inequality must hold for some l_n. □

4 Asymptotics

According to the above remarks, we need to study the asymptotics of $D(k)$ as $k \to \infty$. Since the higher order terms in this asymptotic expansion depend in a complicated way on the Fourier coefficients and also on higher moments of W such as $\int_0^p W(x) x e^{in\pi x/p}\, dx$ for a general W (cf. [2]), we now specialize to

$$W(x) = \begin{cases} 0 & (0 < x < 1) \\ 1 & (1 < x < 2) \end{cases},$$

$p = 2$. Then an elementary calculation shows

$$T(k) = \begin{pmatrix} \cos\omega\cos k - k\omega^{-1}\sin\omega\sin k & \cos\omega\sin k + k\omega^{-1}\sin\omega\cos k \\ -\cos\omega\sin k - \omega k^{-1}\sin\omega\cos k & \cos\omega\cos k - \omega k^{-1}\sin\omega\sin k \end{pmatrix}$$

where $\omega^2 = k^2 - 1$. In particular,

$$D(k) = 2\cos(\omega + k) - \frac{(k-\omega)^2}{\omega k}\sin\omega\sin k.$$

The computations in [2, Section 4.5] imply that $|D(k)| > 2$ can only hold if $k = n\pi/2 + (2n\pi)^{-1} + \delta$ with $n \in \mathbb{N}$ and $\delta = O(n^{-2})$. For these k, a Taylor expansion leads to (we omit this lengthy, but completely elementary computation)

$$D(k) = -2 + 4\delta^2 - \frac{4}{n^4\pi^4} + O(n^{-6}) \tag{5}$$

if n is odd and

$$D(k) = 2 - 4\delta^2 - \frac{8\delta^2}{n^2\pi^2} + \frac{1}{n^6\pi^6} + O(n^{-7}) \tag{6}$$

if n is even. With these formulae, we can now show

Lemma 4.1 *Let* $k_n = n\pi/2 + (2n\pi)^{-1}$ $(n \in \mathbb{N})$ *and*

$$\delta_n = \begin{cases} (n\pi)^{-2} & (n \textit{ odd}) \\ 2^{-1}(n\pi)^{-3} & (n \textit{ even}) \end{cases} .$$

Then, for sufficiently large n*:*
a) If $k_{n-1} + \delta_{n-1} + (n-1)^{-7/2} \leq k \leq k_n - \delta_n - n^{-7/2}$*, then* $|D(k)| \leq 2 - \pi^{-2}n^{-13/2}$.
b) If $|k - k_n| \leq \delta_n - n^{-7/2}$*, then* $|D(k)| \geq 2 + \pi^{-2}n^{-13/2}$.

Remark. Subsequently, we will only need that $|D| - 2$ can be estimated by some power of n if the distance of k to the band edges is $\geq Cn^{-3-\epsilon}$ $(\epsilon > 0)$. Thus, for simplicity, we did not distinguish between odd and even n in the statement; in fact, for odd n, the exponent in b) is $11/2$ rather than $13/2$. Note also that both estimates are very crude if k is not close to one of the band edges.

Proof. The Lemma is an easy consequence of (5), (6). The details are left to the reader. In order to establish a), recall that $D(k)$ is monotone in intervals where $|D| < 2$ [2, Theorem 2.3.1]. $\square$

In order to be able to use Lemma 3.2, we also need to know the asymptotics of the diagonalizing transformations $U(k)$:

Lemma 4.2 *If* k *satisfies either of the assumptions of Lemma 4.1, then* $U(k)$ *can be chosen such that* $\|U(k)\|\,\|U^{-1}(k)\| \leq Cn^{13/2}$.

Proof. Write $T(k) = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$. If $t_2 \neq 0$, then

$$U = \left(t_2(\mu^{-1} - \mu)\right)^{-1/2} \begin{pmatrix} t_2 & t_2 \\ \mu - t_1 & \mu^{-1} - t_1 \end{pmatrix}$$

is a diagonalizing transformation. Notice that $|\mu - \mu^{-1}|^2 = |4 - D^2| \geq cn^{-13/2}$ because of Lemma 4.1. Since $|t_i|, |\mu|$ are bounded from above, we get

$$\|U\|^2 \leq C_1 \frac{n^{13/4}}{|t_2|}.$$

Moreover, $\det U = 1$ implies $\|U^{-1}\| = \|U\|$. Similarly, if $t_3 \neq 0$, then

$$U = \left(t_3(\mu - \mu^{-1})\right)^{-1/2} \begin{pmatrix} \mu - t_4 & \mu^{-1} - t_4 \\ t_3 & t_3 \end{pmatrix}$$

is also a diagonalizing transformation with $\det U = 1$ and $\|U\|^2 \leq C_2 n^{13/4} |t_3|^{-1}$. So it suffices to show that $\max\{|t_2|, |t_3|\} \geq C_3 n^{-13/4}$.

In the situation of Lemma 4.1a), the conditions $t_1 t_4 - t_2 t_3 = 1$, $|t_1 + t_4| \leq 2 - \pi^{-2} n^{-13/2}$ imply that $|t_2 t_3| \geq Cn^{-13/2}$, hence $\max\{|t_2|, |t_3|\} \geq C' n^{-13/4}$, as desired. If the hypothesis of Lemma 4.1b) holds, consider

$$t_2 + t_3 = \left(\frac{k}{\omega} - \frac{\omega}{k}\right) \sin\omega \cos k.$$

A Taylor expansion similar to that leading to (5), (6) shows $t_2 + t_3 = -2(n\pi)^{-3} + O(n^{-4})$, so in particular $\max\{|t_2|, |t_3|\} \geq Cn^{-3}$. □

5 Choosing the parameters

Up to now, we have fixed the basic periodic potential W. We proceed by choosing the g_n. As discussed above, the choice has to be made so that for every k, k/g_n is in some gap for sufficiently many n.

Lemma 5.1 *Let $[k_1, k_2]$ be a subinterval of $(0, \infty)$. Then there are constants $G_0, C > 0$ such that for any $G \in (0, G_0]$ we can find numbers $g_1, \ldots, g_N$ with* *the following properties:*

1) $N \leq CG^{-3}$

2) $G/2 \leq g_i \leq G$ *for all* $i = 1, \ldots, N$

3) For all $k \in [k_1, k_2]$*, there is an* $i \in \{1, \ldots, N\}$ *so that* $|k/g_i - k_r| \leq \delta_r - r^{-7/2}$ *for some* $r \in \mathbb{N}$.

Proof. In this proof, we write

$$\begin{aligned} c_i &= k_{2i+1} - \delta_{2i+1} + (2i+1)^{-7/2}, \\ d_i &= k_{2i+1} + \delta_{2i+1} - (2i+1)^{-7/2}, \end{aligned}$$

(we work with the gaps corresponding to odd n only, because the gaps with even n are by a factor $\sim n^{-1}$ smaller, see Lemma 4.1.) For $G > 0$, let

$$\begin{aligned} r_1 &= \max\{i \in \mathbb{N} : Gc_i \leq k_1\}, \\ r_2 &= \min\{i \in \mathbb{N} : Gd_i \geq k_2\}. \end{aligned}$$

Obviously, $r_i \to \infty$ as G tends to zero.

¿From the definition of r_i (and that of k_i, δ_i) one deduces easily that $r_i = k_i(\pi G)^{-1} + O(1)$ where $O(1)$ denotes an expression that remains bounded as $G \to 0$. We take $g_1 = G$ and $g_n = g_{n-1}c_{r_2}/d_{r_2}$ for $n = 2, \ldots, m$ where m is the smallest integer with $g_m c_{r_2} \leq Gd_{r_2-1}$. By construction, this covers $I := [Gc_{r_2-1}, Gd_{r_2}]$ in the sense that for all $k \in I$ there is an $i \leq m$ such that $k/g_i \in [c_{r_2}, d_{r_2}]$. Moreover,

$$g_i = G\left(1 - \frac{1}{2\pi^3 r_2^3} + O(r_2^{-4})\right)^{i-1} \tag{7}$$

provided that r_2 is large enough, i.e. G has to be sufficiently small. In this case, one infers easily from (7) that m can be estimated by $m \leq 4\pi^3 r_2^2$. Using this and again (7) shows $g_i \geq G(1 - r_2^{-1}) \geq G/2$ if, again, G is small enough.

The intervals $[Gc_i, Gd_{i+1}]$ with $r_1 \leq i \leq r_2 - 2$ can now be covered in the same way by appropriately scaled copies of $[c_{i+1}, d_{i+1}]$. Note that the total number of g_i's is

$$N \le 4\pi^3 \left((r_1+1)^2 + \ldots + r_2^2\right) \le 4\pi^3 r_2^3.$$

Hence this set of g_i's has the properties stated in the Lemma. $\square$

In order to treat the whole half-axis $k \in (0,\infty)$, let $G_0(m), C(m)$ be the constants of Lemma 5.1 for $[k_1,k_2] = [m^{-1}, m]$ ($m \in \mathbb{N}$). Given any functions $F, G > 0$ with $F(s) \to \infty, G(s) \to 0$, we can find integers $m(s) \to \infty$ such that $m(s) \le F(s)$ and $C(m(s)) \le F(s), G_0(m(s)) \ge G(s)$ for all $s \in \mathbb{N}$ with $m(s) > 1$, (take $m(1) = \ldots = m(s_0 - 1) = 1$, then $m(s_0) = 2$ where s_0 is sufficiently large so that the inequalities hold etc.).

We can now choose the whole sequence $(g_n)_{n=1}^\infty$: Let $F(s), G(s) > 0$ be prescribed sequences with $F \to \infty, G \to 0$. Determine $m(s)$ as explained in the preceding paragraph, and for $s = 1, 2, \ldots$, pick $g_{j(s)+1}, \ldots, g_{j(s)+N(s)}$ (where $j(s) = \sum_{t=1}^{s-1} N(t)$) so that these g's and $N = N(s)$ satisfy 1)-3) of Lemma 5.1 for $G = G(s)$ and $[k_1, k_2] = [m(s)^{-1}, m(s)]$. These definitions of $j(s), N(s), g_n$ as well as the meaning of $F(s), G(s), m(s)$ should be kept in mind throughout the rest of this paper. We will also write $s(n)$ for the unique s determined by $j(s) + 1 \le n \le j(s) + N(s)$.

Next, we investigate the set of k for which k/g_n is close to one of the band edges infinitely often (abbreviated as i.o. in the sequel), i.e. for infinitely many n. More precisely, consider

$$A_n = \{k > 0 : |k/g_n - k_i - \delta_i| < i^{-7/2} \text{ or } |k/g_n - k_i + \delta_i| < i^{-7/2} \text{ for some } i\}.$$

Fix $[k_1, k_2] \subset (0,\infty)$. If n is sufficiently large, then by construction of the g_n and by Lemma 5.1 2), the condition defining A_n can only hold for $i \ge C_0(k_1)/G(s)$ (writing $s = s(n)$ for brevity). Hence

$$|A_n \cap [k_1, k_2]| \le 4G(s) \sum_{i \ge C_0/G(s)} i^{-7/2} \le CG(s)^{7/2}. \qquad (8)$$

Lemma 5.1 1) says that there are at most $F(s)G(s)^{-3}$ indices n with $s(n) = s$. Thus summing (8) over n yields

$$\sum_{n=1}^{\infty} |A_n \cap [k_1, k_2]| \leq C \sum_{s=1}^{\infty} F(s)G(s)^{1/2}.$$

Now the Borel-Cantelli Lemma implies

Lemma 5.2 *Assume* $\sum_{s=1}^{\infty} F(s)G(s)^{1/2} < \infty$. *Then* $|\{k > 0 : k \in A_n \text{ i.o.}\}| = 0$.

In the same way, we can control the measure of the set where $|D| > 2$ but at the same time the phase is close to β (recall the definitions from Section 3). For a sequence $\epsilon(s) > 0$, let (again, with $s = s(n)$)

$$B_n = \{k \in [m(s)^{-1}, m(s)] : |k/g_n - k_i| \leq \delta_i - i^{-7/2} \text{ for some } i, \varphi(a_n, k) \in S_{\epsilon(s)}(k/g_n)\}.$$

Lemma 3.3 says that $|B_n| \leq \frac{2}{\pi} F(s)^2 |M| \epsilon(s)$ for appropriate $l_{n-1} \in [0, \pi/m(s)]$. Here,

$$M = \{k \in [m(s)^{-1}, m(s)] : |k/g_n - k_i| \leq \delta_i - i^{-7/2} \text{ for some } i\}.$$

In order to estimate $|M|$, proceed as above. This leads to $|B_n| \leq CF(s)^3 G(s)^2 \epsilon(s)$, and summing over n proves

Lemma 5.3 *Assume* $\sum_{s=1}^{\infty} F(s)^4 G(s)^{-1} \epsilon(s) < \infty$. *Then there are* $l_n \to 0$ *so that* $|\{k > 0 : k \in B_n \text{ i.o.}\}| = 0$.

Note the somewhat complicated logic behind this statement: If the summability assumption holds, then for any choice of the L_n (see Section 2), we can find l_n with the stated properties. In particular, recall that l_n depends on the values of the potential $V(x)$ for $x \leq b_n$. However, this does not cause any difficulties because we may as well think of the parameters L_n, g_n as being fixed from the beginning (with the values given below, of course). The representation given here tries to avoid unmotivated choices.

There will be no further conditions on F, G, ϵ, therefore we now fix these parameters. We take $F(s) = \ln s, G(s) = s^{-3}, \epsilon(s) = s^{-5}$ (we remark that powers are by no means the only possible choice), hence $N(s) \leq s^9 \ln s$ by Lemma 5.1 1). L_n will also depend on s only, thus we can write $L(s)$.

6 Conclusion of the proof

We are now in a position to estimate the solutions of (1). More specifically, we consider $R_{ni} := R(x_{ni}, k)$ at the points $x_{ni} = a_n + 2i/g_n (n \in \mathbb{N}, i = 0, 1, \ldots, L_n)$. Fix a k which is not in the exceptional sets of Lemmas 5.2, 5.3, and let n_0 be sufficiently large so that $k \in [m(s)^{-1}, m(s)]$ if $s \geq s(n_0)$. Furthermore, take n_0 also greater than the finitely many n for which $k \in A_n \cup B_n$, and let $i_0 \in \{0, 1, \ldots, L_{n_0}\}$. For some of the deductions below, it may be necessary to restrict n_0 to even larger values; we will not mention this explicitly in the sequel. Finally, write $s_0 = s(n_0), R_0 = R_{n_0 i_0}$.

By combining our previous results, we will obtain the following estimate: If $x_{ni} \geq x_{n_0 i_0}$, then

$$\ln R_{ni} - \ln R_0 \geq K_1 \sum_{t=s_0+1}^{s(n)-1} L(t) t^{-39/4} - K_2 \sum_{t=s_0}^{s(n)} t^9 \ln^2 t. \tag{9}$$

Here, $K_{1/2}$ are positive constants which are independent of n_0, i_0, n, i. If $x_{ni} < a_{j(s_0+2)+1}$, then the first sum is zero by convention.

In order to establish (9), we argue as follows: Let m be an index with $n_0 \leq m \leq n$. By Lemma 3.1, we have $T(x_{m,i+j}, x_{mi}; k) = T(k/g_m)^j$. Furthermore, R is constant on $(b_m, a_{m+1}) = (x_{mL_m}, x_{m+1,0})$ since $V = 0$ on these intervals. Because of the definition of the sets A_m, one of the two parts of Lemma 4.1 applies to k/g_m (instead of k).

In the situation of Lemma 4.1a), we deduce from Lemma 3.2a) and Lemma 4.2 that $\|T(k/g_m)^j v\| \geq C r^{-13/2} \|v\|$ where r is the unique integer for which

$$k_{r-1} + \delta_{r-1} + (r-1)^{-7/2} \leq k/g_m \leq k_r - \delta_r - r^{-7/2}$$

holds. Since $g_m \geq G(s(m))/2$ (see Lemma 5.1 2)), $r \leq C(k)s(m)^3$. In terms of R, we thus get

$$\ln R_{m,i+j} - \ln R_{mi} \geq -C \ln s(m), \tag{10}$$

where C is independent of m, i, j. In particular, taking $i = 0, j = L_m$ we see that

$$\ln R(a_{m+1}) - \ln R(a_m) \geq -C \ln s(m). \tag{11}$$

Summing up the contributions from (10), (11) gives the second term of (9).

Assume now that Lemma 4.1b) applies to k/g_m. Observe that in this case $\varphi(a_m, k) \notin S_{\epsilon(s(m))}(k/g_m)$ because of the definition of B_m (compare Lemma 5.3). Moreover, (3) together with Lemma 4.1b) implies that $|\mu(k/g_m)| \geq 1 + \pi^{-1} r^{-13/4}$ where r is determined by $|k/g_m - k_r| \leq \delta_r - r^{-7/2}$, hence, estimating r as above, $|\mu(k/g_m)| \geq 1 + Cs(m)^{-39/4}$. Now Lemma 3.2b) shows

$$\ln R(a_{m+1}) - \ln R(a_m) \geq C_1 L(s(m)) s(m)^{-39/4} - C_2 \ln s(m),$$

and since for all t, there is at least one $m \in \{j(t)+1, \ldots, j(t)+N(t)\}$ for which k/g_m is in a gap (by Lemma 5.1 3)), we also get the first sum of (9).

It remains to treat the contribution to (9) that comes from the beginning of the interval under consideration, namely $\ln R_{n_0, i_0+j} - \ln R_0$, for the case when k/g_{n_0} is in a gap (i.e. Lemma 4.1b) applies to k/g_{n_0}). This term is not included in the above treatment because Lemma 5.3 gives no information on the phase at $x_{n_0 i_0}$. However, Lemma 3.2b) does show that

$$R_{n_0, i_0+j} \geq \frac{2}{\pi} \epsilon(s_0) |\mu(k/g_{n_0})|^{i_0+j} R(a_{n_0}).$$

On the other hand, $R_0 \leq \|T(k/g_{n_0})^{i_0}\| R(a_{n_0})$ and therefore, taking $\|T^m\| \leq |\mu|^m \|U\| \, \|U^{-1}\|$ into account (see the proof of Lemma 3.2),

$$R_{n_0, i_0+j} \geq \frac{2\epsilon(s_0) |\mu(k/g_{n_0})|^j}{\pi \|U(k/g_{n_0})\| \, \|U(k/g_{n_0})^{-1}\|} R_0.$$

$\|U\| \, \|U^{-1}\|$ can be controlled with the aid of Lemma 4.2. In conclusion, we get again an estimate of the form of (10). Note that k/g_n might also happen to be in a gap (n is from (9)), but in this case it is obvious from Lemma 3.2b) that (10) holds. Thus the proof of (9) is complete.

If we now take, say, $L(s) = [s^{79/4}]$ ($[x]$ is the largest integer $\leq x$), then the first sum in (9) dominates the second one, i.e. we get increasing solutions. Simple estimates allow us to rewrite (9) in the following more useful form:

$$\ln R_{ni} - \ln R_0 \geq \begin{cases} -C_1 s_0^9 \ln^2 s_0 & s(n) = s_0, s_0+1 \\ C_2 s(n)^{10} & s(n) \geq s_0 + 2 \end{cases} . \tag{12}$$

In order to establish the existence of L_2-solutions, we use the following version of [9, Theorem 8.1]. This (slight) reformulation takes into account that we do not have accurate *upper* bounds for the norm of the transfer matrices.

Lemma 6.1 ([9]) *Assume that there is an increasing sequence $x_n \to \infty$ such that:*

1) $\|T(x_{n+1}, x_n; k)\| \leq C(k)$ for all n

2) There is a vector $v \in \mathbb{R}^2$ with $\sum \|T(x_n, x_0; k)v\|^{-2} < \infty$.

Then either

a) There exists another vector $u \neq 0$ such that (writing $\rho_n = \|T(x_n, x_0; k)u\|, R_n = \|T(x_n, x_0; k)v\|$)

$$\rho_n^2 \leq C_1(k) \left[R_n^{-2} \left(\sum_{i=n}^{\infty} (R_n/R_i)^2 \right)^2 + R_n^{-2} \right].$$

or

b) $\rho_n/R_n \to \infty$ if u is not a constant multiple of v.

Proof. Follow the proof of [9, Theorem 8.1]. If the vector u_∞ constructed there is not a constant multiple of v, then a) follows as in [9] (with $u = u_\infty$). If $u_\infty = cv$, then b) follows directly from eq. (8.2) of [9] and the fact that the vector u from that statement is exactly u_∞. □

The hypotheses of this Lemma are satisfied for almost every k if we let x_n be the sequence of the points $x_{ni} \geq x_{n_0 i_0}$ defined above: Since $\int_{x_{ni}}^{x_{n,i+1}} |V(x)|\, dx =$

g_n, the standard Gronwall estimate [4, Section III.1] proves 1) (in fact, it proves much more, namely that $T(x_{n+1}, x_n; k) \to 1$ uniformly on compact k sets). Furthermore, (12) obviously implies 2).

Now let M be the set of those k for which conclusion b) of the Lemma holds. Clearly, b) in particular implies that no solution of (1) is polynomially bounded (as a function of x), hence M is a set of spectral measure zero for all boundary conditions α by Schnol's Theorem (see [18, Proposition 9]). Therefore, using spectral averaging, $|M| = \int_0^\pi \rho_\alpha(M)\, d\alpha = 0$.

Using (12), we can estimate R_n^{-2} as well as $\sum_{i\geq n}(R_n/R_i)^2$. Fixing n_0, i_0, we immediately get (with positive constants C_i, as usual)

$$R_{ni}^2 \geq C_1 e^{C_2 s(n)^{10}}.$$

In order to bound $\sum_{x_{mj}\geq x_{ni}}(R_{ni}/R_{mj})^2$ (of course, the sum is over m, j, and n, i are fixed), we also use (12), but now with R_{ni} taking the role of R_0. Here it is important that the constants of (12) are independent of n_0, i_0. As in the preceding section, we translate the sum over m, j into a sum over s and estimate the number of terms with the aid of Lemma 5.1 1). The result of these by now familiar arguments is

$$\sum_{x_{mj}\geq x_{ni}} (R_{ni}/R_{mj})^2 \leq C_3 e^{C_4 s(n)^9 \ln^2 s(n)}.$$

Thus Lemma 6.1a) implies that for almost all $k > 0$, there is a solution y satisfying

$$(y^2 + (y'/k)^2)(x_{ni}) = \rho_{ni}^2 \leq C e^{-C' s^{10}}. \qquad (13)$$

Because of Gronwall's Lemma (compare the argument above), this estimate also holds for $x \in [x_{ni}, x_{n,i+1}]$ (with a possibly larger C). The number of indices n, i corresponding to the same s and the length of the intervals $[x_{ni}, x_{n,i+1}]$ can both be bounded by a power of s, hence (13) guarantees that $y \in L_2$.

Finally, for $x \geq 0$, let s be the index corresponding to x, i.e. $a_{j(s)+1} \leq x <$

$a_{j(s+1)+1}$. Then by construction of V (see Section 2)

$$x < a_{j(s+1)+1} \leq \sum_{t=1}^{s} N(t) \left(\frac{4L(t)}{G(t)} + \pi \right) \leq Cs^{33}.$$

The second term in the sum estimates the contribution from the l_n. On the other hand, $0 \leq V(x) \leq s^{-6}$, so indeed $V(x) = O(x^{-2/11})$. This completes the proof of Theorem 2.1.

7 Singular continuous spectrum

We modify V as follows. For a growing sequence $s_n \to \infty$, let

$$V_{aux}(x) = \begin{cases} V(x) & (a_{j(s_n)+1} \leq x \leq b_{j(s_n)+N(s_n)}) \\ 0 & \text{else} \end{cases} .$$

Now the modified potential $\tilde{V}$ is obtained from V_{aux} by readjusting the l_n (more precisely, the subsequence of l_n corresponding to the sequence s_n) according to Lemma 5.3. The remark following Lemma 5.3 shows that this may be necessary.

Since the transfer matrix for zero potential is unitary, the above arguments also apply to $\tilde{V}$. In particular, an estimate similar to (9) still holds, the sums now being over the subsequence s_n. Consequently, one can show along the lines of the preceding Section that for almost all k

$$R(b_{j(s_n)+N(s_n)}, k) \geq C_1(k) \exp\left(C_2(k) \sum_{i=1}^{n} s_i^{10} \right). \tag{14}$$

A version of Schnol's Theorem (namely [18, Proposition 9]) states that if $h \in L_2 \cap L_\infty$, then $h(\cdot)v(\cdot, k^2) \in L_2$ for spectrally almost all $E = k^2$. Here v is a solution of the Schrödinger equation satisfying the boundary condition at $x = 0$. Let $D_n = a_{j(s_{n+1})+1} - b_{j(s_n)+N(s_n)}$ and $h(x) = (n^2 D_n)^{-1/2}$ if $b_{j(s_n)+N(s_n)} \leq x \leq a_{j(s_{n+1})+1}$, and $h(x) = 0$ otherwise. Then (14) shows that $hv \notin L_2$ for almost all k, hence the spectrum is supported by a set of Lebesgue measure zero and therefore $\sigma_{ac} = \emptyset$.

On the other hand, if D_n grows sufficiently fast (which can be achieved by taking a sparse sequence s_n), then we can also exclude the existence of L_2-solutions. More specifically, choose the parameters by the following inductive procedure: Take $s_1 = 1$ and pick the corresponding l_n's. Next, determine D_1 large enough so that for all $k \in [1^{-1}, 1]$ the following holds: If y is any solution with $R(0,k) = 1$, then $D_1 R^2(b_{j(s_1)+N(s_1)}, k) \geq 1$. Pick any s_2 so that $a_{j(s_2)+1} \geq b_{j(s_1)+N(s_1)} + D_1$ (for instance, take the smallest number with this property) and adjust the l_n's corresponding to this s_2. Again, choose D_2 so that $D_2 R^2(b_{j(s_2)+N(s_2)}, k) \geq 1$, but this time for all $k \in [2^{-1}, 2]$ etc. We can also be more explicit; for instance, it is not hard to show (using Gronwall's Lemma again) that the sequence s_n defined by $s_1 = 1$,

$$s_{n+1} = 2^{(2^{s_n})}$$

has the required properties.

Finally, note that since the original V satisfies $V(x) = O(x^{-c})$, $\tilde{V}$ is also of order $O(x^{-c})$. We summarize our result as

Theorem 7.1 *The potential $\tilde{V}$ constructed above satisfies:*
1) $\tilde{V}(x) = O(x^{-c}) \quad (c > 0)$
2) For all boundary conditions at $x = 0$, the spectrum is purely singular continuous in $(0, \infty)$.

References

[1] del Rio, R., Makarov, N., Simon, B.: Operators with Singular Continuous Spectrum, II. Rank One Operators. Comm. Math. Phys. **165**, 59–67 (1994)

[2] Eastham, M.S.P.: The Spectral Theory of Periodic Differential Equations. London: Scottish Academic Press 1973.

[3] Gordon, A.Y., Molchanov, S.A., Tsagani, B.: Spectral Theory of One-Dimensional Schrödinger Operators with Strongly Fluctuating Potentials.

Funct. Anal. Appl. **25**, 236–238 (1992) and Gordon, A.Y.: private communication

[4] Hartman, P.: Ordinary Differential Equations. Boston: Birkhäuser 1982

[5] Kirsch, W., Molchanov, S.A., Pastur, L.: One-dimensional Schrödinger operators with high potential barriers. Operator Theory: Advances and Applications, vol. 57, 163–170. Birkhäuser Verlag, Basel 1992

[6] Kiselev, A.: Preservation of the absolutely continuous spectrum of Schrödinger equation under perturbations by slowly decreasing potentials and a.e. convergence of integral operators. Preprint (1996)

[7] Kiselev, A., Last, Y., Simon, B.: Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators. Preprint (1996)

[8] Kotani, S., Ushiroya, N.: One-dimensional Schrödinger operators with random decaying potentials. Comm. Math. Phys. **115**, 247–266 (1988)

[9] Last, Y., Simon, B.: Eigenfunctions, Transfer Matrices, and Absolutely Continuous Spectrum of One-Dimensional Schrödinger Operators. Ann. Math. (submitted)

[10] Naboko, S.N.: Dense Point Spectra of Schrödinger and Dirac Operators. Theor. and Math. Phys. **68**, 646–653 (1986)

[11] Pearson, D.B.: Singular Continuous Measures in Scattering Theory. Comm. Math. Phys. **60**, 13–36 (1978)

[12] Remling, C.: A probabilistic approach to one-dimensional Schrödinger operators with sparse potentials. Comm. Math. Phys. (to appear)

[13] Ruelle, D.: Ergodic theory of differentiable dynamical systems. Publ. IHES **50**, 275–306 (1979)

[14] Simon, B.: Some Jacobi matrices with decaying potentials and dense point spectrum. Comm. Math. Phys. **87**, 253–258 (1982)

[15] Simon, B.: Some Schrödinger operators with dense point spectrum. Proc. Amer. Math. Soc. (to appear)

[16] Simon, B., Wolff, T.: Singular Continuous Spectrum under Rank One Perturbations and Localization for Random Hamiltonians. Comm. Pure Appl. Math. **39**, 75–90 (1986)

[17] Simon, B., Zhu, Y.: The Lyapunov Exponents for Schrödinger Operators with Slowly Oscillating Potentials. J. Funct. Anal. **140**, 541–556 (1996)

[18] Stolz, G.: Localization for random Schrödinger operators with Poisson potential. Ann. Inst. H. Poincaré **63**, 297–314 (1995)

[19] Stolz, G.: Localization for Schrödinger operators with effective barriers, Preprint (1995)