Convergence of the transfer operator for rational maps

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Abstract: We prove that the transfer operator for a general class of rational maps converges exponentially fast in the supremum norm and in Hölder norms for small enough Hölder exponents to its principal eigendirection.

1 Introduction

Let $T : \mathbf{C} \to \mathbf{C}$ be a rational map of degree $d \geq 2$, and denote by J its Julia set. If $f : J \to \mathbf{R}$ is a continuous function, then we would like to consider the action of the associated transfer operator $\mathcal{L}_f$. It is well known that for real f the operator $\mathcal{L}_f$ has a largest simple eigenvalue whose associated eigenfunction and eigenfunctional define an invariant measure μ on J which is conformal with respect to $P(f) - f$, where $P(f)$ is the pressure of f. If the function f is Hölder continuous and satisfies the condition $P(f) - f > 0$, then it was shown [1] that μ is in 'fact the equilibrium state for f, that is it realises the maximum in the variational principle

$$P(f) = \sup_{\nu}(h(\nu) + \mu(f)),$$

where the supremum is over all T-invariant probability measures ν on J, and $h(\nu)$ denotes the metric entropy of ν.

For $f : J \to \mathbf{R}$, one defines the transfer operator $\mathcal{L}$ by

$$\mathcal{L}\phi(x) = \sum_{y \in T^{-1}x} e^{f(y)}\phi(y),$$

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where ϕ are functions on J and $x \in J$. In order to use the euclidean metric on $\mathbf{C}$ (rather than the spherical metric on $\bar{\mathbf{C}}$) let us assume that $\infty \notin J$, and denote by $C^\alpha(J), \alpha > 0$, the Hölder continuous functions on J with Hölder exponent α, that is, if $f \in C^\alpha(J)$ then there exists a smallest constant $|f|_\alpha$ so that $|f(x) - f(y)| \leq |f|_\alpha |x - y|^\alpha$, for all $x, y \in J$. If we denote by $|f|_\infty$ the supremum norm on J, then the natural norm on $C^\alpha(J)$ is given by $\|\cdot\|_\alpha = |\cdot|_\alpha + |\cdot|_\infty$. The main result in this paper is to show that $\mathcal{L}$ contracts to the eigenspace spanned by μ exponentially fast in the supremum norm (theorem 10) and in the Hölder norm for small enough Hölder exponents $\kappa < \alpha$ (theorem 11). It was previously shown in [2] that the contraction is subexponential at the rate $\vartheta^{\sqrt{n}}$ for some $\vartheta < 1$, where n is the number of times the operator $\mathcal{L}$ is iterated. Clearly, in general we cannot expect the convergence to be exponential in the C^α-norm, since this remains the privilege of the case when the map T is (uniformly) hyperbolic in which case the operator $\mathcal{L} : C^\alpha \to C^\alpha$ has a spectral gap which is essential to effect exponential convergence in C^α.

Quasicompactness of the transfer operator acting on Sobolev spaces has been shown by Smirnov [6].

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2 Inverse branches

We shall need the following theorem (where $|\mathcal{S}|$ denotes the cardinality of the finite set $\mathcal{S}$).

Theorem 1 *Let J not contain any critical periodic point of T and let $0 < \lambda < 1$. Then there exist $\varepsilon > 0$, $\eta \in (0, 1)$, a sequence of simply connected regions $\Omega_n, n \in \mathbf{N}$, and a disjoint decomposition of the inverse branches of T^n on Ω_n into two subsets $\mathcal{S}'_n = \mathcal{S}'_n(\lambda)$ and $\mathcal{S}''_n = \mathcal{S}''_n(\lambda)$ so that*
(a) $|\mathcal{S}''_n| \leq r\lambda^{-n}, n \in \mathbf{N}$, *for some constant r.*
(b) $|\varphi'(z)| \leq \eta^n$ *for $z \in \Omega_n$ and in particular* $\operatorname{diam}(\varphi(\Omega_n)) \leq \eta^n, \forall \varphi \in \mathcal{S}'_n, n \in \mathbf{N}$.
(c) $\operatorname{dist}(z, \Omega_n) \leq C_1 e^{-n\varepsilon}$ *for all $z \in J, n \in \mathbf{N}$, for some constant $C_1 > 0$.*

The following lemma serves to obtain in lemma 3 an estimate on the boundary behaviour of the Riemann map for a polygon which then will be used to get uniform bounds on the Köbe constant of polygonal regions (lemma 5).

Lemma 2 *Let $g(z)$ be analytic in some simply connected region Ω and $x_0 \in \partial\Omega$. Moreover, let $h(z) = (g(z) - g(x_0))^{1/\gamma}$ (with suitable branch cuts), where $0 < \gamma < 2$. Then there exists a constant $C_2 \geq 1$, so that*

$$|g(z) - g(x)| \leq C_2 |h(z) - h(x)|^\gamma.$$

Proof. We consider two cases, namely $x = x_0$ and $x \neq x_0$.
(I) Let us assume that $x = x_0$. Then we have $|g(z) - g(x)| = |h(z)|^\gamma = |h(z) - h(x)|^\gamma$, as $h(x) = h(x_0) = 0$.
(II) Let us now assume that $x \neq x_0$ and put $d = |h(x) - h(x_0)| = |h(x)|$. Let us first consider the case when $|h(z) - h(x)| \leq d/2$, that is when $h(z)| \geq d/2$. We then obtain that

$$\begin{aligned} |g(z) - g(x)| &\leq \int_{h(z)}^{h(x)} |\frac{dh^\gamma}{dh}| \, |dh| \\ &\leq \gamma \max_{\zeta \in [z,x]} |h(\zeta)|^{\gamma-1} |h(z) - h(x)| \\ &\leq \gamma 3^\gamma 2^{1-\gamma} |h(z) - h(x)|^\gamma. \end{aligned}$$

In the case $|h(z) - h(x)| > d/2$ we get

$$\begin{aligned} |g(z) - g(x)| &\leq |h(z)^\gamma - h(x_0)^\gamma| + |h(x_0)^\gamma - h(x)^\gamma| \\ &\leq |h(z)|^\gamma + d^\gamma \\ &\leq 5^\gamma |h(z) - h(x)|^\gamma, \end{aligned}$$

where we used that $|h(z)| \leq d + |h(z) - h(x)| \leq 3|h(z) - h(x)|$.

Thus the lemma is proven, if we put $C_2 = \max(5^\gamma, \gamma 3^\gamma 2^{1-\gamma})$. □

Lemma 3 *Let $\gamma > 0$ and $N \geq 3$. Then there exists a constant C_3 (depending on γ and N) so that for every polygon P with at most N vertices and interior angles $\geq \pi\gamma > 0$ the following estimate holds:*

$$\operatorname{dist}(f(z), \partial P) \leq C_3 (1 - |z|)^{\gamma^N}.$$

where $f : D \to P$ is the Riemann map from the unit disc D to P.

Proof. Let $x_j \in \partial D, j = 1, 2, \ldots, N$, be the prevertices of the polygon P (assume $0 \in P$) under the map f. We can assume that $\mathrm{Arg}(x_j) < \mathrm{Arg}(x_{j+1})$ for all $j = 1, \ldots, N$ ($x_{N+1} = x_1$). For every j, the map f has by the Schwarz reflection principle an analytic continuation to the slit plane $\mathbf{C} \setminus S_j$, which it maps conformally to P and its reflection along the edge $f(S_j)$ which lie in a branched cover of the complex plane, where $S_j \subset \partial D$ is the arcsegment whose endpoints are the prevertices x_j, x_{j+1} and which does not contain any other prevertices. Let $x \in \partial D$ and $B_\delta(x)$ be the ball of radius centred at x. Then $f(B_\delta(x)) \subset B_{2p}(0)$, $\delta < 1$, where $p = \mathrm{diam}(P)$. Without loss of generality we can assume that $x_j, j = 1, \ldots, M, M < N$ are those prevertices of P which lie in the ball $B_\delta(x)$. Put $g_0(z) = f(z)$ and define inductively

$$g_j(z) = (g_{j-1}(z) - g_{j-1}(x_j))^{1/\gamma_j}, j = 1, \ldots, M,$$

where $\pi\gamma_j$ is the interior angle at the vertex $f(x_j)$. These maps are analytic in $B_\delta(x) \cap D$ and g_M can by the Schwarz reflection principle holomorphically be extended to the entire ball $B_\delta(x)$. From the estimates

$$\sup_{B_\delta(x) \cap D} |g_j(z)| \leq 2^{1/\gamma_j} \sup_{B_\delta(x) \cap D} |g_{j-1}(z)|^{1/\gamma_j}$$

$j = 1, \ldots, M$, and the fact that $\sup_{B_\delta(x) \cap D} |g_0(z)| \leq p$ we conclude that

$$\sup_{B_\delta(x) \cap D} |g_M(z)| \leq (2^M p)^{1/\gamma^M} \leq (2^N p)^{\gamma^{-N}}.$$

Since by the reflection principle g_M can analytically be extended to the entire ball $B_\delta(x)$ we obtain $\sup_{B_\delta(x)} |g_M(z)| \leq 2(2^N p)^{\gamma^{-N}}$. For $z \in B_{\delta/2}(x)$ a Cauchy estimate provides $|g'_M(z)| \leq c_1$, where $c_1 = 4(2^N p)^{\gamma^{-N}}$, and therefore $|g_M(z) - g_M(x)| \leq c_1|x - z|$. A repeated application of lemma 2 yields the estimate

$$|g_0(z) - g_0(x)| \leq (C_2^N c_1)^{-\gamma'} |x - z|^{\gamma'},$$

where $\gamma' = \gamma_1\gamma_2 \cdots \gamma_M \geq \gamma^N$. This proves the lemma with $C_3 = (C_2^N c_1)^{-\gamma^N}$. $\square$

Lemma 4 *(Köbe distortion theorem [3]) A univalent function g on the unit-disc satisfies:*

$$\left(\frac{\delta'}{2-\delta'}\right)^4 \leq \left|\frac{g'(w)}{g'(w')}\right| \leq \left(\frac{2-\delta'}{\delta'}\right)^4$$

for all $|w|, |w'| < 1 - \delta'$, where $\delta' > 0$.

Lemma 5 *Let P (as above) be a polygon with $\leq N$ vertices whose interior angles are $\geq \pi\gamma > 0$, and let φ be a univalent function on P. Then for every (small) $\delta > 0$ we have that*

$$\frac{1}{256}\left(\frac{\delta}{C_3}\right)^{8/\gamma^N} \leq \left|\frac{\varphi'(z)}{\varphi'(z')}\right| \leq 256\left(\frac{C_3}{\delta}\right)^{8/\gamma^N}$$

for all $z, z' \in P$ for which $\mathrm{dist}(z, \partial P), \mathrm{dist}(z', \partial P) \geq \delta$, where C_3 is a constant (determined by lemma 3).

Proof. Let $f : D \to P$ be the Riemann map and put $\delta' = (\delta/C_3)^{1/\gamma^N}$. This ensures that

$$P \setminus B_\delta(\partial P) \subset f(B_{1-\delta'}(0)).$$

As $g = \varphi \circ f$ is univalent on the unitdisc D it satisfies the Köbe distortion inequality

$$\frac{\delta'^4}{16} \leq \left|\frac{g'(w)}{g'(w')}\right| \leq \frac{16}{\delta'^4}$$

for all $w, w' \in B_{1-\delta'}(0)$. Since $g'(w) = \varphi'(z)f'(w)$, where $z = f(w) \in P$, we obtain

$$\left|\frac{\varphi'(z)}{\varphi'(z')}\right| = \left|\frac{g'(w)}{g'(w')}\right| \times \left|\frac{f'(w')}{f'(w)}\right|,$$

and, since also f satisfies the distortion estimate of lemma 4, we obtain

$$\frac{\delta'^8}{256} \leq \left|\frac{\varphi'(z)}{\varphi'(z')}\right| \leq \frac{256}{\delta'^8},$$

for all $z, z' \in P \setminus B_\delta(\partial P)$. The estimate in the lemma now follows readily. □

Proof of theorem 1. Denote by Λ_n the critical values of T^n (critical points of the inverse maps) and let U be a topological disc in $\mathbf{C}$ which avoids all periodic critical points of T. We shall assume that U is a finite region. Denote by π_x the projection of $\mathbf{C}$ onto the real axis (x-axis) and by π_y the projection onto the imaginary axis (y-axis). Let (without counting muliplicities) $\{x_j : j = 1, 2, \ldots\}$ be the (finite) set $\pi_x\Lambda_n$, labelled in such a way that $x_1 < x_2 < \cdots$. Put $a_j = x_{j+1} - x_j, j = 1, 2, \ldots$.

Let $\lambda \in (0, 1)$ put $\delta = \frac{-\log\lambda}{1536}$ and choose $\varepsilon \in (0, \delta)$. Put $\varepsilon_n = e^{-\varepsilon n}$ and $\delta_n = e^{-\delta n}$.

Let us assume that $\sum_j a_j > dn^2\varepsilon_n$ (the points in $\pi_x\Lambda_n$ are spread out). In this case we can find a gap $\tilde{a} > 0$ so that $|\tilde{a} - a_j| > \varepsilon_n \forall j$. Since $|\Lambda_n| \leq dn$, we can choose $\tilde{a} \leq n\varepsilon_n$. We say two points x_j, x_{j+1} are close if $a_j = x_{j+1} - x_J < \tilde{a} - \varepsilon_n$ and are apart if $a_j > \tilde{a} + \varepsilon_n$. A set of points $\{x_j : j = k, k+1, \ldots, k+k_0\} \subset \pi_x\Lambda_n$ (some k, k_0) form a *cluster* if $a_j < \tilde{a} - \varepsilon_n$ for $j = k, \ldots, k+k_0-1$, and $a_{j-s}, ak + k_0 > \tilde{a} + \varepsilon_n$.

The set $\pi_x\Lambda_n$ consists of a finite number of (disjoint) clusters. In the case when $\sum_j a_j \leq dn^2\varepsilon_n$ (the points in $\pi_x\Lambda_n$ are closely packed) the entire set $\pi_x\Lambda_n$ might form a single cluster. We shall now do 'branch cuts' to remove all the critical values Λ_n in U by cutting out a cluster at a time.

For large enough n there exists a number $\tilde{b} \in (0,1)$ so that $|\pi_y(z) - \tilde{b}| > \varepsilon_n$ for every point $z \in \Lambda_n$. Let Γ be a cluster of critical values. If $\pi_y\Gamma \cap (-\infty, \tilde{b}) \neq \emptyset$ then we do a rectangular cut whose short side is parallel to the x-axis at hight $\max_{z\in\Gamma, \pi_y(z)<\tilde{b}} \pi_y(z) \leq \tilde{b} - \varepsilon_n$ and whose long sides are parallel to the y-axis, extend all the way in negative y-direction to the boundary of U and are made to enclose the portion of the cluster Γ which lies in the halfplane $\{(x,y) : y < \tilde{b}\}$. To remove the portion of the cluster Γ which lies in the halfplane $\{(x,y) : y > \tilde{b}\}$ we do a similar rectangular cut whose short side has the y-coordinate $\min_{z\in\Gamma, \pi_y(z)>\tilde{b}} \pi_y(z) \geq \tilde{b} + \varepsilon_n$ and whose long sides reach in the positive y-direction to the boundary ∂U. Notice that the widths of these 'cuts' are $\leq |\Lambda_n|\tilde{a} \leq dn^2\varepsilon_n$ (the absolute values denote the cardinality of the finite set).

In this way we remove every cluster of critical values and denote by U_n the region U minus the rectangular cuts. It has the property that any two points $z, z' \in U_n$ which satisfy $\mathrm{dist}(z, \partial U_n), \mathrm{dist}(z', \partial U_n) \geq \delta_n$ can be connected by a polygonal path $\mathcal{C}$ with at most three straight lines parallel to the coordinate axes and which lies inside a polygonal region $P \subset U_n$ whose edges are parallel to the coordinate axies and satisfies $\mathrm{dist}(\mathcal{C}, \partial P) \geq \frac{1}{\sqrt{2}}\delta_n$. Therefore, if φ is a univalent function of U_n, then by lemma 5

$$K_n^{-1} \leq \left| \frac{\varphi'(z)}{\varphi'(z')} \right| \leq K_n,$$

where $K_n = 2^{200}\, C_2^{384}\, \delta_n^{-384}$ (with $N \leq 6$ and $\gamma \geq 1/2$), for $z, z' \in U_n \setminus B_{\delta_n}(\partial U_n)$.

Put $\Omega_n = T(U_n \setminus B_{\delta_n}(\partial U_n))$. To construct the inverse branches of T^n on Ω_n we shall follow [1]. Let $\mathcal{S}_n$ be the (univalent) inverse branches of T^n on

Ω_n and denote by $|V|$ the normalised spherical measure of measurable set $V \subset \mathbf{C}$. Define $\mathcal{S}_n'' = \{\varphi \in \mathcal{S}_n : |\varphi(\Omega_n)| > \lambda^n\}$ and $\mathcal{S}_n' = \mathcal{S}_n \setminus \mathcal{S}_n''$ (for the set of 'good' branches). Since the multiplicity of the family $\{\varphi(\Omega_n) : \varphi \in \mathcal{S}_n\}$ is uniformly in n bounded by some number r [?] (that is for every point $z \in \mathbf{C}$ there are at most r branches $\varphi \in \mathcal{S}_n$ so that $z \in \varphi(\Omega_n)$), we obtain $|\mathcal{S}_n''| \leq r\lambda^{-n}$. This proves part (a) of the theorem.

To estimate the diameter of the sets $\varphi(\Omega_n)$ for $\varphi \in \mathcal{S}_n'$ from their spherical measures we proceed as follows:

$$\begin{aligned} |\varphi(\Omega_n)| &\geq |\Omega_n| \inf_{z\in\Omega_n} |\varphi'(z)|^2 \\ &\geq K_n^{-2}|\Omega_n| \sup_{z\in\Omega_n} |\varphi'(z)|^2. \end{aligned}$$

Together with the assumption $|\varphi(\Omega_n)| \leq \lambda^n$, this yields ($c_1 > 0$)

$$\begin{aligned} \sup_{z\in\Omega_n} |\varphi'(z)| &\leq K_n \sqrt{\frac{|\varphi(\Omega_n)|}{|\Omega_n|}} \\ &\leq c_1\, \lambda^{n/2}\delta_n^{-384}. \end{aligned}$$

Since $\delta = \frac{-\log\lambda}{1536}$ we obtain $\operatorname{diam}(\varphi(\Omega_n)) \leq \eta^n$, where $\eta = \sqrt[4]{\lambda}$. This concludes the proof of part (b) of the theorem.

To prove the last part of the theorem, let us observe that by construction of the inverse branches $\operatorname{dist}(z, U_n \setminus B_{\delta_n}(U_n)) \leq \varepsilon_n + \delta_n$, which immediately leads to the estimate $\operatorname{dist}(z, \Omega_n) \leq \sqrt{2}(\varepsilon_n + \delta_n) \leq C_1 e^{-\varepsilon n}$, for some C_1. $\square$

Remark. In the proof of theorem 1 we describe doing the thickened branch cuts parallel to the imaginary axis. Naturally the branch cuts can of course be done at any angle with respect to the imaginary axis. In particular, given any two points $x, x' \in \Omega$ which lie outside the set $B_{C_1 e^{-\varepsilon n}}(\Lambda_n)$ then we can arrange the branch cuts in such a way that they avoid x, x'—that is we can assume that $x, x' \in \Omega_n$ whenever $x, x' \notin B_{C_1 e^{-\varepsilon n}}(\Lambda_n)$

3 Main result

Theorem 6 *Let $f \in C^\alpha$ satisfy $P(f) > \sup f$, and μ be its equilibrium state on J. Let Ω_n be as in constructed in Theorem 1 and $\lambda \in (\rho, 1)$, where $\rho = e^{\sup f - P(f)}$.*

(a) $|\mu(B_n)| \leq (\rho/\lambda)^n$, *where* $B_n = \bigcup_{\varphi \in \mathcal{S}''_n(\lambda)} \varphi(\Omega_n)$.
(b) $\mu(J \setminus \Omega_n) \leq (\rho/\lambda)^n$.

Proof. The equilibrium state μ of a function $f \in C^\alpha$ is of the form $h\nu$, where h is Hölder continuous [2] and an eigenfunction to the largest eigenvalue e^P ($P = P(f)$) of the transferoperator $\mathcal{L}_f$. The measure ν is an eigenfunctional to the same eigenvalue of the dual operator, or, equivalently, an e^{f-P}-conformal measure [**?**], which means that

$$\nu(TA) = \int_A e^{P-f}\, d\nu,$$

for measurable sets A on which T is injective. Hence

$$\nu(\varphi(\Omega_n)) = \int_{\Omega_n} e^{f^n - nP}\, d\nu \leq \rho^n,$$

for all $\varphi \in \mathcal{S}$, where $f^n = f + f \circ T + \cdots + f \circ T^{n-1}$ is the nth ergodic sum of f. As h is Hölder continuous $\mu(\varphi(\Omega_n)) \leq c_1\, \rho^n$, for some $c_1 > 0$, and therefore by theorem 1 (a)

$$\mu(B_n) \leq \sum_{\varphi \in \mathcal{S}''(\lambda)_n} \mu(\varphi(\Omega_n)) \leq c_1\, |\mathcal{S}''(\lambda)_n| \rho^n \leq c_1\, r\, (\rho/\lambda)^n,$$

where ρ/λ is less than 1. This proves the theorem. □

Let $\tilde{\mathcal{S}}_n$ denote univalent extensions of the inverse branches $\mathcal{S}_n$ to simply connected regions $\tilde{\Omega}_n$ ($\Omega_n \subset \tilde{\Omega}_n$) which are the topological disk U minus a finite number of regular branch cuts.

Lemma 7 *([2]) There exists a constant* $C_4 > 1$ *and* $\gamma_0, \xi > 0$, *so that* $\operatorname{diam}(\varphi(B_\gamma(x))) \leq C_4^n \gamma^\xi$ *for all* $x \in J$, $\varphi \in \tilde{\mathcal{S}}_n$, $n \geq 0$ *and* $\gamma \leq \gamma_0$, *provided* $B_\delta(x) \subset \tilde{\Omega}_n$.

We shall need the following result which is proven in [5] and [2] and which is based on the previous lemma.

Lemma 8 *([5] and [2] (3.1)) There exist constants* C_5 *and* $\beta > 0$ *(depending on* α *and* T*) so that* $\|\mathcal{L}_f^n \psi\|_\beta \leq C_5 \|\psi\|_\alpha$, *for all* $n \in \mathbf{N}$, $\psi \in C^\alpha$.

A consequence of this result is that the eigenfunction h to the largest eigenvalue e^P of the transfer operator $\mathcal{L}_f : C^\alpha \to C^\beta$, for real $f \in C^\alpha$, is Hölder continuous with exponent β which is given by lemma 8. As h is bounded away from 0 and ∞, we can introduce a normalised transfer operator $\hat{\mathcal{L}} : C^\alpha \to C^\gamma$ by $\hat{\mathcal{L}} = e^{-P(f)}\mathcal{L}_{f+\log h - \log h\circ T}$, where the Hölder exponent γ depends on β and T and is given by lemma 8. The normalised transfer operator has the property that the principal eigenvalue has been rescaled to 1 with the associated eigenfunctions being the constants, that is $\hat{\mathcal{L}}\mathbf{1} = \mathbf{1}$.

Consider $\chi(x, x')$ as a function of the first variable $x \in J$ where the second entry $x' \in J$ is assumed to be a parameter, and let

$$\|\chi(x')\|_\gamma = \sup_x |\chi(x, x')| + \sup_{x\neq y} \frac{|\chi(x, x') - \chi(y, x')|}{|x - y|^\gamma},$$

be its γ-Hölder norm ($\gamma > 0$) for the parameter value x'. Then we denote by $\|\chi\|_\gamma = \sup_{x'} \|\chi(x')\|_\gamma$ the norm of χ.

The transfer operator $\hat{\mathcal{L}}$ acts on the function χ as:

$$\hat{\mathcal{L}}\chi(x, x') = \sum_{\varphi\in\tilde{\mathcal{S}}_1} g_1(\varphi x)\chi(\varphi x, \varphi x'),$$

where $g_1 = \frac{h}{h\circ T} e^{f-P(f)}$.

The following statement is a slight generalisation of lemma 8 and is proven in the same way. (The supremum norm estimate is straighforward and the estimates on the variation do not depend on the 'parameter' x'.)

Lemma 9 *([5], [2]) There exist constants C_6 and $\delta \in (0, \gamma)$ (depending on γ and T) so that*

$$\|\hat{\mathcal{L}}_f^n\chi\|_\delta \leq C_5\|\chi\|_\gamma,$$

for all $n \in \mathbf{N}$, where $\chi(x, x')$ is as above a function on $J \times J$ and $\|\chi\|_\gamma$ its γ-Hölder norm with respect to the variable x.

Let us now prove the main result of this paper.

Theorem 10 *Let $\psi \in C^\alpha$ and μ its equilibrium state. There exists a $\sigma < 1$ and a constant C_7, such that for all $k \geq 1$:*

$$|\hat{\mathcal{L}}^k\psi - \mu(\psi)|_\infty \leq C_7\sigma^k\|\psi\|_\alpha.$$

Proof. Without loss of generality we can assume that $\mu(\psi) = 0$. Let $\rho = e^{\sup f - P(f)}$ and choose $\lambda \in (\rho, 1)$. Let η be the contraction rate of the 'good' inverse branches of theorem 1. Let C_4 be as in lemma 7, β (depending on α and T) as given by lemma 8 and choose $\eta' \in (\eta, 1)$. Moreover choose $\sigma < 1$ so that $\sigma > \max(\vartheta, \eta^\alpha)$, where $\vartheta = \max(\eta', e^{-\delta'\varepsilon}, \frac{\rho}{\lambda})$, where $\delta' < \delta$ is an arbitrarily positive number and $\delta < \gamma$ is given by lemma 9 with $\gamma = \xi\beta$ (ξ as in lemma 7. Finally, choose a number $p > 1$ close enough to 1 so that $p < 1 + \frac{\log(\eta'/\eta)}{\beta \log |T'|_\infty}$ and $C_4^{\beta\frac{p-1}{p}} < \sigma/\vartheta$.

Let N be an integer which will be specified below. Then we define a sequence of integers $n_j = [n(1-p^{-j})]$ ($[\cdot]$ denotes integer part), $j = 1, \dots, M$, where M is so that $N \le np^{-M} \le pN$. Put $m_j = n_j - n_{j-1}$, $\psi_0 = \psi$ and $\psi_j = \hat{\mathcal{L}}^{n_j}\psi = \hat{\mathcal{L}}^{m_j}\psi_{j-1}$, $j = 1, \dots, M$. Observe that $np^{-j} \le n - n_j \le np^{-j} + 1$ and moreover $m_j \le (p-1)(n - n_j) + 1$. We also have the lower bound $m_j \ge (p-1)np^{-j} = (p-1)np^{-M}p^{M-j} \ge (p-1)Np^{M-j}$

The theorem will be proven by induction, where the induction hypothesis shall be

$$|\hat{\mathcal{L}}^k \psi|_\infty \le C_7 \sigma^k \|\psi\|_\alpha,$$

for $k = 0, 1, \dots, n-1$. We shall show that the estimate also holds for $k = n$. In particular we have $|\psi_j|_\infty \le C_7 \sigma^{n_j} \|\psi\|_\alpha$ for $j = 0, \dots, M$.

Let Ω_n be the regions for the inverse branches of T^n given by theorem 1 and let $x, x' \in J \cap \Omega_n$. Then

$$\begin{aligned}
\hat{\mathcal{L}}^n \psi(x) - \hat{\mathcal{L}}^n \psi(x') &= \sum_{\varphi \in \mathcal{S}_n} (g_n(\varphi x)\psi(\varphi x) - g_n(\varphi x')\psi(\varphi x')) \\
&= I + II,
\end{aligned}$$

where $g_n = e^{f^n + h - h\circ T^n - nP}$. In the first term,

$$I = \sum_{\varphi \in \mathcal{S}_n} g_n(\varphi x)(\psi(\varphi x) - \psi(\varphi x')),$$

we get by theorem 1(b) for the contracting branches $\varphi \in \mathcal{S}'_n$:

$$|\psi(\varphi x) - \psi(\varphi x')| \le |\psi|_\alpha |\varphi x - \varphi x'|^\alpha \le |\psi|_\alpha \eta^{n\alpha} |x - x'|^\alpha.$$

On the other hand, since $\sup_{y \in \varphi(\Omega_n)} g_n(y) \le c_1\, \mu(\varphi(\Omega_n))$, for some c_1, we obtain by theorem 6 for the non-contracting branches that

$$\sum_{\varphi \in \mathcal{S}''_n} g_n(\varphi x)|\psi(\varphi x) - \psi(\varphi x')| \le 2|\psi|_\infty c_1 \mu(B_n) \le 2|\psi|_\infty c_1 \left(\frac{\rho}{\lambda}\right)^n,$$

which implies (with some c_2):

$$\begin{aligned} I &\leq |\psi|_\alpha \eta^{n\alpha}|x-x'|^\alpha + 2|\psi|_\infty c_1 \left(\frac{\rho}{\lambda}\right)^n \\ &\leq c_2\|\psi\|_\alpha \left(\max(\eta^\alpha, \frac{\rho}{\lambda})\right)^n \\ &\leq \frac{C_7}{8}\|\psi\|_\alpha \sigma^n, \end{aligned}$$

provided n is large enough.

Let us now consider the second term for which we use the following telescoping sums:

$$\begin{aligned} II &= \sum_{\varphi\in\mathcal{S}_n} \psi(\varphi x')\left(g_n(\varphi x) - g_n(\varphi x')\right) \\ &= \sum_{j=1}^{M} \sum_{\varphi\in\tilde{\mathcal{S}}_{n-n_{j-1}}} \hat{\mathcal{L}}^{n_{j-1}}\psi(\varphi x')\left(g_{n-n_{j-1}}(\varphi x) - g_{m_j}(\varphi x')g_{n-n_j}(T^{m_j}\varphi x)\right) \\ &= \sum_{j=1}^{M} \sum_{\varphi\in\tilde{\mathcal{S}}_{n-n_{j-1}}} \hat{\mathcal{L}}^{n_{j-1}}\psi(\varphi x')g_{n-n_j}(T^{m_j}\varphi x)\left(1 - \frac{g_{m_j}(\varphi x')}{g_{m_j}(\varphi x)}\right). \end{aligned}$$

In order to estimate the second sum for a given value of j, let us consider two cases: (i) If the point x lies in $J\setminus(\Omega_{n-n_{j-1}}\setminus B_{C_1 e^{-\varepsilon(n-n_{j-1})}}(\Lambda_{n-n_{j-1}}))$ then following the remark made at the end of section 2, we can execute the branch cuts at some angle to the imaginary axis rather than parallel to it so that x and x' come to lie outside the cut in the region $\Omega_{n-n_{j-1}}$. Put $y=x$. (ii) If on the other hand $x\in(J\setminus\Omega_{n-n_{j-1}})\cap B_{C_1 e^{-\varepsilon(n-n_{j-1})}}(\Lambda_{n-n_{j-1}})$, then let us use the fact the Julia set is uniformly perfect [4] which means that there exists a constant c_3 which only deponds on the map T so that for every $r>0$ and $x\in J$ the distance $\mathrm{dist}(x, J\setminus B_r(x))$ is bounded by $c_3 r$. Since the number of critical values for $T^{n-n_{j-1}}$ is bounded by $c_4(n-n_{j-1})$ (for some constant c_4), we can find a point $y\in J\cap\Omega_{n-n_{j-1}}$ so that $|x-y|\leq 2c_3c_4(n-n_{j-1})e^{-\varepsilon(n-n_{j-1})}$.

In a similar way if $x'\in(J\setminus\Omega_{n-n_{j-1}})\cap B_{C_1 e^{-\varepsilon(n-n_{j-1})}}(\Lambda_{n-n_{j-1}})$, then there is a point $y'\in J\cap\Omega_{n-n_{j-1}}$ so that $|x'-y'|\leq 2c_3c_4(n-n_{j-1})e^{-\varepsilon(n-n_{j-1})}$ (and otherwise we put $y'=x'$).

Now, since $\hat{\mathcal{L}}^{n_{j-1}}\psi=\psi_{j-1}$ and $p>1$ was chosen so that $|T'|_\infty^{\beta m_j}\eta^{n-n_{j-1}}\leq |T'|_\infty^{\beta}\eta'^{n-n_{j-1}}$ we obtain considering only the contracting branches ($c_6 \leq$

$c_5 C_7$):

$$
\begin{aligned}
\left| \sum_{\varphi \in \mathcal{S}'_{n-n_{j-1}}} \hat{\mathcal{L}}^{n_{j-1}} \psi(\varphi y') g_{n-n_j}(T^{m_j}\varphi y) \left(1 - \frac{g_{m_j}(\varphi y')}{g_{m_j}(\varphi y)}\right) \right| & \\
\Pi \quad \leq \quad & c_5 |\psi_{j-1}|_\infty |T'|_\infty^{\beta m_j} \eta^{n-n_{j-1}} |y-y'|^\beta \sum_{\varphi \in \tilde{\mathcal{S}}_{n-n_j}} g_{n-n_j}(\varphi y) \\
\leq \quad & c_6 \|\psi\|_\alpha \sigma^{n_{j-1}} |T'|_\infty^\beta \eta'^{n-n_{j-1}} \\
\leq \quad & c_6 \|\psi\|_\alpha \sigma^n |T'|_\infty^\beta \left(\frac{\eta'}{\sigma}\right)^{n-n_{j-1}},
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\left|1 - \frac{g_m(\varphi y')}{g_m(\varphi y)}\right| &= \left|1 - e^{f^m(\varphi y') - f^m(\varphi y) + h(\varphi y') - h(\varphi y) + h(T^m \varphi y) - h(T^m \varphi y')}\right| \\
&\leq c_5 \, |T'|_\infty^{m\beta} |\varphi y - \varphi y'|^\beta,
\end{aligned}
$$

(as $|T^j \varphi y' - T^j \varphi y| \leq |T'|_\infty^j |\varphi y' - \varphi y|$) for some constant c_5 which depends on the functions f and h.

For the non-contracting branches we proceed as above in estimating term I:

$$
\left| \sum_{\varphi \in \mathcal{S}''_{n-n_{j-1}}} \hat{\mathcal{L}}^{n_{j-1}} \psi(\varphi y') g_{n-n_j}(T^{m_j}\varphi y) \left(1 - \frac{g_{m_j}(\varphi y')}{g_{m_j}(\varphi y)}\right) \right| \leq 2c_1 |\psi_{j-1}|_\infty \left(\frac{\rho}{\lambda}\right)^{n-n_{j-1}}.
$$

In order to estimate the error we made when replacing x by $y \in \Omega_{n-n_{j-1}}$, put

$$
\chi_j(x,y) = \sum_{\varphi \in \tilde{\mathcal{S}}_{m_j}} \psi_{j-1}(\varphi y) \left(g_{m_j}(\varphi x) - g_{m_j}(\varphi y)\right),
$$

(note that $\chi_j(y,y) = 0$) and apply lemma 9 to obtain

$$
\begin{aligned}
\left| \sum_{\varphi \in \tilde{\mathcal{S}}_{n-n_{j-1}}} \hat{\mathcal{L}}^{n_{j-1}} \psi(\varphi y) g_{n-n_j}(T^{m_j}\varphi x) \left(1 - \frac{g_{m_j}(\varphi y)}{g_{m_j}(\varphi x)}\right) \right| & \\
\Pi \quad = \quad & \left| \hat{\mathcal{L}}^{n-n_j} \chi_j(x,y) \right| \\
\Pi \quad \leq \quad & C_6 \|\chi_j\|_\gamma |x-y|^\delta \\
\Pi \quad \leq \quad & C_6 \|\chi_j\|_\gamma 2c_3 c_4 (n-n_{j-1}) e^{-\delta\varepsilon(n-n_{j-1})},
\end{aligned}
$$

where $\gamma = \xi\beta$ and $\delta < \gamma$ was given by lemma 9. If we let $\delta' < \delta$ then we get with some constant c_7:

$$\left|\hat{\mathcal{L}}^{n-n_j}\chi_j(x,y)\right| \leq c_7\|\chi_j\|_\gamma e^{-\delta'\varepsilon(n-n_{j-1})}.$$

It remains to estimate the γ-norm of χ_j. By the induction hypothesis: $|\chi_j|_\infty \leq 2|\psi_{j-1}|_\infty \leq 2C_7\sigma^{n_{j-1}}\|\psi\|_\alpha$. Since, by lemma 7 $|T^\ell\varphi z - T^\ell\varphi z'| \leq C_4^{m-\ell}|z-z'|^\xi$, $\ell = 0,\ldots,m$, $\varphi \in \tilde{\mathcal{S}}_m$, for any two points $z, z' \in J$ (that are close enough), we can write:

$$\begin{aligned}
|\chi_j(z,y) - \chi_j(z',y)| &\leq |\psi_{j-1}|_\infty \sum_{\varphi\in\tilde{\mathcal{S}}_{m_j}} |g_{m_j}(\varphi z) - g_{m_j}(\varphi z')| \\
&\leq |\psi_{j-1}|_\infty \sum_{\varphi\in\tilde{\mathcal{S}}_{m_j}} g_{m_j}(\varphi z)\left|1 - \frac{g_{m_j}(\varphi z')}{g_{m_j}(\varphi z)}\right| \\
\Pi &\leq c_8|\psi_{j-1}|_\infty C_4^{\beta m_j}|z-z'|^{\beta\xi},
\end{aligned}$$

for some constant c_8. One concludes that

$$\|\chi_j\|_\gamma \leq c_8 C_4^{\beta m_j}|\psi_{j-1}|_\infty \leq c_8 C_7 C_4^{\beta m_j}\sigma^{n_{j-1}}\|\psi\|_\alpha.$$

Therefore

$$\left|\hat{\mathcal{L}}^{n-n_j}\chi_j(x,y)\right| \leq c_9 C_7 C_4^{\beta m_j}\|\psi\|_\alpha\sigma^n\left(\frac{e^{-\delta'\varepsilon}}{\sigma}\right)^{n-n_{j-1}},$$

for some constant c_9, and we arrive at the estimate ($c_{10} > 0$ and recall that $m_j \leq (p-1)(n-n_j)+3$ and $n - n_j \geq np^{-j} \geq Np^{M-j} = Np^k$ if we put $k = M - j$):

$$\begin{aligned}
II &\leq (c_6|T'|_\infty^\beta + 2c_1 + c_9C_7)\|\psi\|_\alpha\sigma^n\sum_{j=1}^M C_4^{\beta m_j}\left(\frac{\vartheta}{\sigma}\right)^{n-n_{j-1}} \\
&\leq c_{10}C_7\|\psi\|_\alpha\sigma^n\sum_{k=0}^\infty\left(\frac{\vartheta C_4^{\beta(p-1)}}{\sigma}\right)^{Np^k},
\end{aligned}$$

where $\vartheta = \max(\eta', e^{-\delta'\varepsilon}, \frac{\rho}{\lambda}) < \sigma$. Since the number $p > 1$ was chosen so that the expression inside the brackets is less than 1 we can make the sum over

k arbitrarily small if we only choose N large enough. In particular we can achieve that $II \leq \frac{C_7}{8}\|\psi\|_\alpha \sigma^n$.

Finally we obtain the estimate :

$$\left|\hat{\mathcal{L}}^n\psi(x) - \hat{\mathcal{L}}^n\psi(x')\right| \leq I + II \leq \frac{C_7}{4}\|\psi\|_\alpha \sigma^n,$$

for $x, x' \in J \cap \Omega_n$.

Now suppose $y, y' \in J \setminus \Omega_n$, then, in the same way as above (according to the remark), we can assume that in fact y, y' lie in a $C_1 e^{-n\varepsilon}$-neighbourhood of the critical values Λ_n. Since J is a uniformly perfect set, we can find, as above, $x, x' \in J \cap \Omega_n$ so that $|x-y|, |x'-y'| \leq n 2 c_4 c_3 C_1 e^{-n\varepsilon}$. Therefore, as by lemma 8 $|\hat{\mathcal{L}}^n\psi|_\tau \leq C_5\|\psi\|_\alpha$, we obtain that

$$\left|\hat{\mathcal{L}}^n\psi(y) - \hat{\mathcal{L}}^n\psi(y')\right| \leq \frac{C_7}{4}\|\psi\|_\alpha \sigma^n + 4nC_5 c_3 c_4 C_1 \|\psi\|_\alpha e^{-n\varepsilon} \leq \frac{C_7}{2}\|\psi\|_\alpha \sigma^n,$$

for all n, provided $C_7 > 6C_5 c_3 c_4 C_1 \sup_{n>0} n e^{-n\varepsilon}\sigma^{-n}$. Since $\mu(\psi) = \mu(\hat{\mathcal{L}}^n\psi) = 0$ and $\hat{\mathcal{L}}^n\psi$ is continuous, this implies that $|\hat{\mathcal{L}}^n\psi|_\infty \leq C_7\|\psi\|_\alpha\sigma^n$. □

Theorem 11 *Let $\psi \in C^\alpha$ and, as above, μ the equilibrium state of the potential $f \in C^\alpha$. Let $\beta < \alpha$ be as in lemma 8 and ξ as in lemma 7. For every (positive) small enough $\kappa < \xi\beta$ there exist a $\varsigma < 1$ and a constant C_8, such that for all $n \geq 1$:*

$$\|\hat{\mathcal{L}}^n\psi - \mu(\psi)\|_\kappa \leq C_8 \varsigma^n \|\psi\|_\alpha.$$

Proof. We have to show that the Hölder constants $|\hat{\mathcal{L}}^n\psi|_\kappa$ goes exponentially fast to zero with some rate $\varsigma < 1$. For simplicity's sake we shall as in the previous theorem assume that $\mu(\psi) = 0$. To estimate the variation of $\hat{\mathcal{L}}^n\psi$ let $x, x' \in J$ and put $\epsilon = |x - x'|$. Assume $\epsilon > 0$. To get bounds on the variation for large n we shall use theorem 10 and for small n we use lemma 7. Let β, ξ and $C_4 > 1$ be as in lemmas 8 and 7 and choose $\kappa < \xi\beta$ small enough so that $\sigma C_4^{\frac{2\kappa\beta}{\xi\beta-\kappa}}$ is less than 1. Put

$$P(\epsilon) = \frac{\xi\beta - \kappa}{2\beta}\,\frac{|\log\epsilon|}{\log C_4},$$

and consider the two cases (I) and (II):

(I) If $n \geq P(\epsilon)$ then, by theorem 10,

$$\left|\hat{\mathcal{L}}^n\psi(x) - \hat{\mathcal{L}}^n\psi(x')\right| \leq 2|\hat{\mathcal{L}}^n\psi|_\infty \leq 2C_7\|\psi\|_\alpha\sigma^n \leq C_8\|\psi\|_\alpha\varsigma^n\epsilon^\kappa,$$

where the last inequality holds with $C_8 \geq 2C_7$ provided

$$\frac{\varsigma}{\sigma} \geq C_4^{\frac{2\kappa\beta}{\xi\beta-\kappa}}.$$

(II) If $n < P(\epsilon)$ then we have by lemma 7 that $|\varphi x - \varphi x'| \leq C_4^n\epsilon^\xi$ for all $\varphi \in \tilde{\mathcal{S}}_n$. This implies

$$|\psi(\varphi x) - \psi(\varphi x')| \leq |\psi|_\alpha C_4^{n\alpha}\epsilon^{\xi\alpha}$$

and

$$\left|1 - \frac{g_n(\varphi x)}{g_n(\varphi x')}\right| \leq c_1 C_4^{n\beta}\epsilon^{\xi\beta}$$

for some constant c_1 which depends on the Hölder norms of f and h. We can therefore estimate in the following (rough) manner:

$$\begin{aligned}
\left|\hat{\mathcal{L}}^n\psi(x) - \hat{\mathcal{L}}^n\psi(x')\right| &\leq \sum_{\varphi\in\tilde{\mathcal{S}}_n} g_n(\varphi x)|\psi(\varphi x) - \psi(\varphi x')| \\
&\quad + \sum_{\varphi\in\tilde{\mathcal{S}}_n} g_n(\varphi x')|\psi(\varphi x')|\left|1 - \frac{g_n(\varphi x)}{g_n(\varphi x')}\right| \\
&\leq |\psi|_\alpha C_4^{n\alpha}\epsilon^{\xi\alpha} + |\psi|_\infty c_1 C_4^{n\beta}\epsilon^{\xi\beta} \\
&\leq \|\psi\|_\alpha c_2 C_4^{n\beta}\epsilon^{\xi\beta}, \\
&\leq \|\psi\|_\alpha c_2 C_4^{-n\beta}\epsilon^{\delta},
\end{aligned}$$

with $c_2 = \max((\operatorname{diam} J)^{\alpha/\beta}, c_1)$, where in the last inequality we replaced n by P.

Cases (I) and (II) together yield

$$\left|\hat{\mathcal{L}}^n\psi(x) - \hat{\mathcal{L}}^n\psi(x')\right| \leq C_8\|\psi\|_\alpha\varsigma^n\epsilon^\kappa,$$

for all integers n, where $C_8 = \max(2C_7, c_2)$ and where by assumption

$$\varsigma = \max\left(C_4^{-\beta},\ \sigma C_4^{\frac{2\kappa\beta}{\xi\beta-\kappa}}\right)$$

is less than 1. □

Corollary 12 *Let E be a bounded measurable function on J and F be an α-Hölder continuous function on J. Then there exists a constant C_9 such that for all $n \geq 0$:*

$$|\mu(F \cdot E \circ T^n) - \mu(E)\mu(F)| \leq C_9 \sigma^n,$$

where $\sigma < 1$ is given by theorem 10.

Proof. Without loss of generality we can assume that the function F has average zero, i.e. that $\mu(F) = 0$. Then, by theorem 10, $|\hat{\mathcal{L}}^n F|_\infty \leq C_7 \|\psi\|_\alpha \sigma^n$ and therefore

$$\begin{aligned} |\mu(F \cdot E \circ T^n)| &= |\mu(\hat{\mathcal{L}}^n (F \cdot E \circ T^n))| \\ &= |\mu(E \cdot \hat{\mathcal{L}}^n F)| \\ &\leq |E|_\infty |\hat{\mathcal{L}}^n F|_\infty \\ &\leq |E|_\infty \|\psi\|_\alpha C_7 \sigma^n. \end{aligned}$$

This proves the corollary with $C_9 = |E|_\infty \|\psi\|_\alpha C_7$. □

References

[1] M Denker and M Urbanski: Ergodic theory of equilibrium states for rational maps, Nonlinearity 4 (1991), 103–134

[2] M Denker, F Przytycki and M Urbanski: On the transfer operator for rational functions on the Riemann sphere, Göttingen preprint 1994

[3] E Hille: *Analytic Function Theory*, Vol. 2, 1962

[4] A Hinkkanen: Julia sets of rational functions are uniformly perfect. Math. Proc. Cambridge Philos. Soc. 113 no. 3 543–559 (1993)

[5] F Przytycki: On the Perron-Frobenius-Ruelle operator for rational maps on the Riemann sphere and for Hölder continuous functions, Boll. Bras. Soc. Mat. 20 95–125 (1990)

[6] S K Smirnov: Spectral Analysis of Julia Sets, PhD Thesis, Caltech 1996