# Fluctuations of a surface submitted to a random average process

P.A. Ferrari L. R. G. Fontes
Instituto de Matemática e Estatística, USP

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**Abstract**

We consider a hyper surface of dimension $d$ imbeded in a $d+1$ space. For each $x \in \mathbf{Z}^d$, let $\eta_t(x) \in \mathbf{R}$ be the height of the surface at site $x$ at time $t$. At rate 1 the $x$-th height is updated to a random convex combinations of the heights of the 'neighbors' of $x$. The distribution of the convex combination is translation invariant and does not depend on the heights. This motion, named random average process (RAP), is one of the linear processes introduced by Liggett (1985). Special cases of RAP are a type of smoothing process (when the convex combination is deterministic) and the voter model (when the convex combination concentrates on one of the neighbors chosen at random). We start the heights located on a hyperplane passing through the origin but different from the trivial one $\eta(x) \equiv 0$. We show that when the convex combination is neither deterministic nor concentrating on one of the neighbors the variance of the height at the origin at time $t$ is proportional to the number of returns to the origin of a symmetric random walk of dimension $d$. Under mild conditions on the distribution of the random convex combination, this gives variance of the order of $t^{1/2}$ in dimension $d=1$, $\log t$ in dimension $d=2$ and uniformly bounded in $t$ in dimensions $d \geq 3$. We also show that for each initial hyperplane the process as seen from the height at the origin converges to an invariant measure on the hyper surfaces conserving the initial asymptotic slope. The height at the origin satisfies a weak law of large numbers and a central limit theorem. To obtain the results we use a corresponding probabilistic cellular automaton, for which similar results are derived. This automaton corresponds to the product of (infinitely dimensional) independent random matrices whose lines are independent.

**Keywords:** random average process, random surfaces, product of random matrices, linear process, voter model, smoothing process.

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# 1 Introduction

We consider a stochastic process $\eta_t$ in $\mathbf{R}^{\mathbf{Z}^d}$. At each site $i \in \mathbf{Z}^d$ at each time $t$ corresponds a height $\eta_t(i)$. These heights evolve according to the following rule. For each $i \in \mathbf{Z}^d$ let $U(i,\cdot)$ be a random probability distribution on $\mathbf{Z}^d$. Each height has an independent Poisson clock of parameter 1. When the clock rings for site $i$ at time $t$, a (independent of everything) realization $u(i,\cdot)$ of the random probability is chosen and then the height at site $i$ moves to the position

$$\sum_{j\in\mathbf{Z}^d} u(i,j)\eta_t(j).$$

In other words, at rate one each height moves to a random convex combination of the other heights. The weights of this convex combination are chosen independently at each time. We call this process random average process (RAP). The motion is well defined under suitable conditions on the distributions of $u$ and $\eta_0$. The formal generator is given by

$$Lf(\eta) = \sum_i \int dm(u(i))[f(u(i)\eta) - f(\eta)] \tag{1.1}$$

where $u(i)$ is an operator that puts in the $i$-th coordinate a convex combination of the other coordinates:

$$(u(i)\eta)(k) = \begin{cases} \eta(k) & \text{if } k \neq i \\ \sum_j u(i,j)\eta(j) & \text{if } k = i \end{cases} \tag{1.2}$$

and $m$ is the distribution of $u(i)$. If we interpret $u(i)$ as the matrix $a$ with entrances

$$a(k,j) = \begin{cases} 1 & \text{if } j = k \neq i \\ u(i,j) & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

then $u(i)\eta$ can be interpreted as $a\eta^*$, the product of the matrix $a$ and $\eta^*$, the transpose of the vector $\eta$[1]. The RAP is a special case of the linear processes of chapter IX of Liggett (1985).

Particular cases of this process are one of the smoothing processes (Andjel (1985), Liggett and Spitzer (1981) and Liggett (1985)) and the voter model (Liggett (1985) and Durrett (1996) for a recent review).

To obtain the smoothing process one imposes the distribution of the matrix $u$ to concentrate mass on a constant matrix that does not change with time. In the above papers a "more random" smoothing process was studied: at each realization of the Poisson process, the random convex combination is multiplied by an independent positive random variable $W$ of mean 1, but the initial configurations were restricted to the positive real numbers. Questions about existence and ergodicity of the system when the initial heights are positive were considered. It would be interesting to study the ergodicity questions for the case of general initial conditions. One crucial difference between the processes studied in the above papers and the RAP is that when the configurations are restricted to being positive and the process is studied from a fixed framework, the non trivial realization of the process remain bounded. This is the reason why it makes sense to multiply by a random variable with mean 1. In contrast, the typical

[1] Elsewhere in the paper, $^*$ will have the same meaning.

configurations of the RAP grow with time and are localizable only from a moving frame of reference (the height at the origin, for instance). In this case it does not make sense to multiply the absolute weights by a random variable.

To obtain the voter model we consider $u(i, \cdot)$ as a random probability vector for which only one of the coordinates $j$ is different from zero –and hence equal to 1. To get the usual voter model one has to restrict the initial configurations to those assuming for instance only two values. But the question of the behavior of the system when infinitely many initial values are allowed is of interest.

In this paper we will concentrate on the other cases, but some marginal results will concern the smoothing process and the voter model.

The starting configuration is a hyperplane passing through the origin: for a given vector $\lambda \in \mathbf{R}^d$ we set $\eta_0(i) = i\lambda^*$, the (matrix) product of $i$ and the transpose of $\lambda$ (this is the internal product). Notice that if $\lambda \equiv 0$, then $\{\eta_t \equiv 0\}$ because this initial configuration is invariant for the system. We assume that the distribution of $u(i,j)$ is translation invariant: $u(i,j)$ and $u(i+k, j+k)$ have the same distribution. Our main result is to show that when the initial configuration is a hyperplane, the variance of the height at the origin at time $t$ is proportional to the expected numbers of returns to the origin of a symmetric random walk. Denoting $\mathbf{V}$ as the variance, we show that

$$\mathbf{V}\eta_t(0) = (\sigma^2 + (\mu\lambda^*)^2) \int_0^t \mathbf{P}(D_s = 0 | D_0 = 0) ds \tag{1.3}$$

where $D_t$ is a (symmetric) random walk with transition rates

$$q(\ell, k) = \sum_j \mathbf{E}(u(0,j)u(\ell, j+k)),$$

$\mu = \mathbf{E}\left(\sum_j j u(0,j)\right)$ and $\sigma^2 = \mathbf{V}\left(\sum_j j\lambda^* u(0,j)\right)$. By definition, when the initial configuration is a hyperplane, it does not have any spatial fluctuation. In this case for $\sigma^2 > 0$, the asymptotic variance of the height at the origin is of the same order for both biased ($\mu \neq 0$) and unbiased ($\mu = 0$) distributions of the convex combinations. If the initial configuration is distributed according to a measure with spatial fluctuations, one expects the biased and unbiased cases to have different asymptotic variances: normally the height at the origin in the biased case will pick the spatial fluctuations of the initial configuration. We show this difference with one example in the one dimensional case ($d = 1$). We do not know how to treat the general case when the initial configuration is random.

Let $|j| = \sqrt{jj^*}$, the $L_2$-norm of $j \in \mathbf{Z}^d$. In the case of finite second moment, *i.e.* when $\sum_j |j|^2 \mathbf{E}u(0,j) < \infty$ our result implies that starting with the hyperplane, the fluctuations of the height at the origin behave asymptotically as follows: for $\sigma^2 + (\mu\lambda^*)^2 > 0$,

$$\mathbf{V}\eta_t(0) \text{ is of the order of } \begin{cases} \sqrt{t} & \text{if } d = 1 \\ \log t & \text{if } d = 2 \\ \text{constant} & \text{if } d \geq 3 \end{cases} \tag{1.4}$$

where $\mathbf{V}$ is variance. Here the phrase "$f(t)$ is of the order of $g(t)$" means that we can find positive constants $A$ and $B$ such that $Ag(t) \leq f(t) \leq Bg(t)$ for all $t \geq 0$. We show a central limit theorem for $\eta_t(0)$ for the deterministic hyperplane initial configuration. In one example, in the one dimensional case we show it for a random initial configuration.

The nearest neighbor one dimensional case is isomorphic to a conservative nearest particle system studied before by the authors (unpublished). In this case, the position of the height at the origin at time $t$ can be interpreted as the position of a tagged particle at the same time in the conservative nearest particle system. At the end of Section 6 we describe this correspondence and correct a statement of Ferrari (1996) about the asymptotic behavior of the tagged particle.

We show that starting with a hyperplane the distribution of the height difference $(\eta_t(i) - \eta_t(j))$ converges in probability to a limiting random variable with expectation $\eta_0(i) - \eta_0(j)$, for all $i$, $j$. In particular this implies that the process starting with a hyperplane converges weakly to an invariant measure. Furthermore we show that, under the limiting invariant measure the slope is conserved, that is $\mathbf{E}(\eta(i) - \eta(j)) = \eta_0(i) - \eta_0(j)$, and the variance of the differences has the following asymptotic behavior: for $\sigma^2 + (\mu\lambda^*)^2 > 0$

$$\mathbf{V}[\eta(i) - \eta(j)] \text{ is of the order of } \begin{cases} |i-j| & \text{if } d = 1 \\ \log|i-j| & \text{if } d = 2 \\ \text{constant} & \text{if } d \geq 3 \end{cases} \tag{1.5}$$

To prove the above in $d = 2$ we need a stronger condition: that a second-plus-delta moment is finite: $\sum_j |j|^{2+\delta}\mathbf{E}u(0,j) < \infty$ for some $\delta > 0$. Unless for a few one dimensional cases, we are unable to further describe this measure. Probably it is a Gibbs measure, but there are no reasons to expect that the measure is in general reversible for this process. Also it would be nice to show that the limiting measure obtained when the process starts with a hyperplane is the unique invariant measure with the slope of the initial hyperplane.

In dimension $d = 1$, if $u(i, i+1) = 1 - u(i, i-1)$, then the heights remain ordered if started so and form a point process. Moreover, if $u(i, i+1)$ is uniform in $[0, 1]$, for each $\lambda > 0$, the Poisson process of parameter $\lambda$ is reversible for the system. There are other cases of reversibility for suitable choices of $u$, but in general this process is not reversible and the Poisson process is not necessarily invariant. For instance in the nearest neighbor one dimensional case if $u(i, i+1) \equiv 1/2$ (this is a smoothing process), the measures concentrating mass in configurations for which the distances between successive neighboring heights are constant are translation invariant and invariant for the process. Are these all the measures in this simple case? For other choices of $U$, the ordering is not necessarily maintained and the invariant measures may be difficult to find.

The nearest neighbors one dimensional process has been studied by Kipnis, Presutti and Marchioro (1982) for $u(i, i+1) = 1 - u(i, i-1)$ uniform in $[0, 1]$. They consider $L$ differences between successive heights moving according to the random average process. They put different boundary conditions in the extremes of the box so that to force a gradient in the difference height. In the limit as $L \to \infty$ they show that the difference height profile in the normalized box (so that the length is kept to be 1), adopts a linear form interpolating between the boundary conditions. This implies that the hydrodynamic limit of the difference heights corresponds to a heat equation. If $u(i, i+k)$ and $u(i, i-k)$ have the same distribution, then the hydrodynamic equation for the RAP is

$$\left(\frac{\partial \Phi}{\partial t}\right)_h = D_h \left(\Delta\Phi\right)_h$$

where $D_h = \sum_j |j_h| u(0,j)$ and $(\cdot)_h$ is the $h$-th coordinate of the vector $(\cdot) \in \mathbf{Z}^d$. Here $\Phi(r,t)$ represents the height of the surface at the macroscopic time $t$ at the macroscopic position $r$. The equation is obtained when the time in the RAP is scaled as the square of the space.

The main tool to study this process is duality. The graphical construction of the RAP is realized as in the usual Harris construction of particle systems: a (independent of everything) Poisson processes of parameter 1 is attached to each site $i \in \mathbf{Z}^d$. To each mark of this Poisson process a (independent of everything) realization of $u(i,\cdot)$ is attached. If at time $t$ a Poisson mark appears at site $i$ and $u(i,\cdot)$ is the corresponding realization of the weights, then at time $t$ the new configuration is $u(i)\eta_{(t^-)}$, where $u(i)$ was defined in (1.2). This Harris construction works if $u$ has finite range. Durrett (1995) uses a percolation argument to show that there exists a (small) time $t_0$ such that $\mathbf{Z}^d$ is partitioned in random finite subsets with no interaction among them in the interval $[0,t_0]$. Hence the graphical construction can be performed in each of these finite sets. Repeating the argument between $t_0$ and $2t_0$ and so on, the process is constructed. The same argument should work under reasonable decay conditions on the distribution of $u$, but one can avoid this by using the almost sure duality property of this process as we see below.

Define a random walk $\tilde Y_t \in \mathbf{Z}^d$ using the same Poisson marks and realizations of $u(i,\cdot)$ used in the construction of $\eta_t$. The motion is the following: if at time $t^-$ it is $\tilde Y_{t^-} = i$, a Poisson mark of site $i$ appears at time $t$ and $u(i,\cdot)$ is the realization of the weights corresponding to this mark, then $\tilde Y_t$ will be $j$ with probability $u(i,j)$. Hence $\tilde Y_t$ has (average) rates

$$\mathbf{P}(\tilde Y_{t+h} = j \mid \tilde Y_t = i) = h\mathbf{E}u(i,j) + o(h)$$

Call $\mathcal{F}_t$ the sigma algebra generated by the Poisson marks and the realization of the $u$'s on these marks between 0 and $t$. Notice that $\tilde Y_t$ and $Y_t^{0,t}$ have the same distribution. When conditioned to $\mathcal{F}_t$, $\tilde Y_t$ can be seen as a "random walk in a space-time random environment". Provided that the Harris construction is feasible, the following duality relation holds almost surely:

$$\eta_t(x) = \mathbf{E}(\eta_0(Y_t^{x,t}) \mid \mathcal{F}_t) \tag{1.6}$$

where $Y_s^{x,t}$, $0 \le s \le t$ is the walk following the marks from $t$ to 0 backwards in time starting at the site $x$. Another possibility is to take (1.6) as the definition of the process and then to show that the resulting process has the desirable infinitesimal properties. This can be done if (1.6) is finite for all $t$. Actually we construct the process as a limit of a sequence of discrete time processes. It turns out that a sufficient condition for the existence of the process is that there exists a constant $C$ such that $\mathbf{E}\eta_0(j) \le C|j|^2$. We discuss this in Sections 2 and 7.

When $u$ is constant, that is, there exists a stochastic matrix $a$ such that $\mathbf{P}(u=a)=1$, we get

$$\eta_t(x) = \mathbf{E}(\eta_0(Y_t^{x,t})).$$

This corresponds to the potlatch process which is exactly the dual of the smoothing process. When $U$ is not allowed to have more than one positive coordinate —equal to one and the others equal to zero— $Y_t$ is just ordinary random walk with transition probabilities given by the distribution of $u$. Hence in this case

$$\eta_t(x) = \eta_0(Y_t^{x,t})$$

This corresponds to coalescing random walks, the dual of the voter model.

To show the above results we prefer to introduce a probabilistic cellular automaton (PCA). The time of the PCA is discrete and at all times each site chooses an independent random convex combination of the heights at the neighboring sites and updates its height to this new one. If we denote $X_n$ the vector indexed by $\mathbf{Z}^d$ of the heights at time $n$ and $u_n(\cdot,\cdot)$ a random sequence of iid stochastic matrices, then $X_n^* = (\prod_{k=1}^n u_{n-k})X_0^*$. The matrices $u_m$ have also the property that $\{u_m(i,\cdot) : i \in \mathbf{Z}^d\}$ is a family of iid random vectors. The results described above are shown for this PCA and then a standard approximation of the particle system by a family of PCA's is performed.

In Section 2 we introduce the PCA and prove in Proposition 2.3 the discrete-time version of (1.3). In Section 3 we prove some estimates for the discrete version of the random walk $D_t$ which lead to the asymptotics (1.4). In Section 4 we show the central limit theorem for the discrete version of $\eta_t(0)$ starting with the hyperplane. In Section 5 we show that the process starting with the hyperplane converges to an invariant measure with the same slope as the hyperplane and that under this invariant measure, the fluctuations between the heights at distant sites behave as in (1.5). In Section 6 we discuss the one dimensional case and show that, when the initial distribution of heights differences is an ergodic stationary process with fluctuations, these fluctuations appear in the variance of the height at the origin when the process is biased. In Section 7 we discuss the passage from the discrete to the continuous case and in Section 8 we discuss the hydrodynamic limit.

# 2 Discrete time: Probabilistic cellular automaton

The process we study in this Section is a discrete time system or probabilistic cellular automaton whose configurations belong to $\mathbf{R}^{\mathbf{Z}^d}$. Under this evolution at time $n$ each height chooses a random convex combination of the heights of its neighboring heights at time $n-1$ and jumps to this new height. Let $(X_n)_{n\geq 0}$ denote the height system. Let $\{u_n(i, i+\cdot),\, n \geq 1,\, i \in \mathbf{Z}^d\}$ be an i.i.d. and independent of $X_0$ family of random probability vectors distributed in $[0,1]^{\mathbf{Z}^d}$. That is, for all $i,j \in \mathbf{Z}^d$ $u(i,j) \in [0,1]$, $\mathbf{P}(\sum_j u_n(i,j) = 1) = 1$. Define

$$\pi(j) = \mathbf{E}u_n(0,j)$$

and assume

$$\sum_j |j|^2 \pi(j) < \infty, \tag{2.1}$$

where $|j| = \sqrt{jj^*}$. To avoid dealing with aperiodicity, assume that $\pi(x) > 0$ for $|x| \leq 1$. For $n \geq 1$ and $i \in \mathbf{Z}^d$ we define $X_n(i)$ as

$$X_n(i) = \sum_j u_n(i,j) X_{n-1}(j). \tag{2.2}$$

This may define a degenerate random variable. In the case $u(i,j) = 0$ if $|i-j| > M$ for some $M$ (finite range), (2.2) defines a honest Markov process on $\{0,1\}^{\mathbf{Z}^d}$. After introducing the concept of duality below we give sufficient conditions for (2.2) to define a honest Markov process under our hypothesis. Definition (2.2) means that the transpose of the height process at time $n$ is the multiplication of $n$ infinite dimensional independent identically distributed random matrices times the transpose of the heights vector at time 0:

$$X_n^* = \left(\prod_{k=1}^{n} u_{n-k}\right) X_0^*$$

Notice that not only the matrices are independent, but the rows inside each matrix are independent too.

The height at the origin at time $n$, $X_n(0)$, can be expressed as the expectation of a function of a simple random walk in (space–time) random environment. To our probability space attach a family $\{w_n(i) : n \geq 0, i \in \mathbf{Z}^d\}$ of iid random variables uniformly distributed in $[0,1]$ and independent of everything else. For $1 \leq k \leq n$, let $Y_k$ denote the position at time $k$ of a random walk in $\mathbf{Z}^d$ running up to time $n$, starting at the origin defined by $Y_0^{x,n} = x$ and for $1 \leq k \leq n$:

$$Y_k^{x,n} = \sum_j j\mathbf{1}\{w_k(Y_{k-1}) \in I_{n-k}(Y_{k-1}, j)\},$$

where for each $i \in \mathbf{Z}^d$, $\{I_k(i,j) : j \in \mathbf{Z}^d\}$ is a (random) partition of the interval $[0,1]$ with lengths $|I_k(i,j)| = u_k(i,j)$. The process $Y_n^{0,n}$ is a random walk with transition probabilities

$$\mathbf{P}(Y_k^{x,n} = j | Y_{k-1}^{x,n} = i) = \mathbf{E}u_{n-k}(i,j) = \pi(j-i)$$

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $\{u_m(i,j) : i,j \in \mathbf{Z}^d, 0 \leq m \leq n\}$. Conditioned to $\mathcal{F}_n$, $Y_k^{x,n}$ has the following transition probabilities

$$\mathbf{P}(Y_k^{x,n} = j | Y_{k-1}^{x,n} = i, \mathcal{F}_n) = u_{n-k}(i,j) := v_{k-1}(i,j)$$

In words, conditioned to $\mathcal{F}_n$, the walk at time $k$ jumps from $i$ to $j$ with probability $v_k(i,j)$. Since this walk uses the $u_n(i,j)$ backwards in time, we will call $Y_n^{x,n}$ the backwards walk.

**Lemma 2.1.** *(Duality formula) For any initial configuration $X_0$ such that $|X_0(i)| < \infty$ for all $i$ and for all $n$ it holds*

$$X_n(x) = \mathbf{E}(X_0(Y_n^{x,n})|\mathcal{F}_n).$$

*In particular, if $X_0(i) = i\lambda^*$ for some vector $\lambda$, where $\lambda^*$ is the transpose of $\lambda$, then*

$$X_n(x) = (\mathbf{E}(Y_n^{x,n}|\mathcal{F}_n))\lambda^*$$

**Proof.** Notice that $\mathcal{F}_n$ is also the $\sigma$-algebra generated by $\{v_{m,i} : i \in \mathbf{Z}, 0 \leq m \leq n\}$. Let us compute $X_2(0)$:

$$\begin{aligned} X_2(0) &= \sum_j u_2(0,j)X_1(j) \\ &= \sum_j u_2(0,j)\sum_k u_1(j,k)X_0(k) \\ &= \mathbf{E}(X_0(Y_2^{0,n})|\mathcal{F}_2). \end{aligned}$$

An induction argument works for the general case. □

The duality formula gives an easy criterium for the existence of the process:

**Corollary.** *A sufficient condition for (2.2) to define a honest Markov process is that for all* $n \geq 0$

$$\mathbf{E}|X_0(Y_n)| < \infty$$

*where* $Y_n = Y_n^{0,n}$.

**Proof.** The hypothesis plus the duality relation implies that $\mathbf{E}|X_n(0)| < \infty$. To see that this holds for the other $x$, write

$$\mathbf{E}|X_n(x)| = \mathbf{E}|\mathbf{E}X_0(Y_n^{x,n})|\mathcal{F}_n)| \leq \mathbf{E}|X_0(Y_n^{x,n})| = \mathbf{E}|X_0(Y_n + x)|$$

by translation invariance. Conditioning on the first jump of $Y_n$ we get

$$\mathbf{E}|X_0(Y_n)| \geq \mathbf{E}u(0,x)\mathbf{E}|X_0(Y_{n-1} + x)|$$

for any $x$. But we have assumed that for $|x| = 1$ $\mathbf{E}u(0,x) = \pi(x) > 0$, hence $\mathbf{E}|X_0(Y_{n-1}+x)| < \infty$ for $|x| = 1$. Using induction on $k = ||x||_1$ we prove that $\mathbf{E}|X_0(Y_n)| \geq C(x)\mathbf{E}|X_0(Y_{n-k} + x)|$ where $C(x)$ is positive. This proves the Corollary. □

The corollary allows us to give a sufficient condition for our case. If $\sum_j |j|^2\pi(j) < \infty$, then $\mathbf{E}|Y_n|^2 < \infty$. Hence, a sufficient condition for the existence of the process under this hypothesis is that

$$\mathbf{E}|X_0(j)| \leq C|j|^2 \tag{2.3}$$

for some positive constant $C$.

Let $Z_n = \mathbf{E}(Y_n^{0,n}|\mathcal{F}_n)$. For each fixed $n$, $Z_n$ has the same distribution as $\tilde{Z}_n$, the position at time $n$ of the (forward) process defined by

$$\tilde{Z}_n = \mathbf{E}(\tilde{Y}_n|\mathcal{F}_n)$$

where $\tilde{Y}_n$ is a (forward) random walk in a random environment given by $\tilde{Y}_0 = 0$ and for $k \geq 1$,

$$\tilde{Y}_{k+1} = \sum_j j\mathbf{1}\{w_k(\tilde{Y}_k) \in I_k(\tilde{Y}_k, j)\}$$

where $w_n(i)$ is the same sequence of iid random variables uniformly distributed in $[0,1]$ defined above and $\{I_k(i,j) : j\}$ is the partition of the interval $[0,1]$ with lengths $|I_k(i,j)| = u_k(i,j)$ defined above. Given $\mathcal{F}_n$, the conditional probabilities for this walk are

$$\mathbf{P}(\tilde{Y}_{k+1} = j|\tilde{Y}_k = i, \mathcal{F}_n) = u_k(i,j)$$

for $1 \leq k \leq n$. Let

$$\tilde{Z}_n = \mathbf{E}(\tilde{Y}_n|\mathcal{F}_n)$$

We have

$$\mathbf{E}\tilde{Z}_n = \mathbf{E}\tilde{Y}_n = n\sum_j j\pi(j) := n\mu.$$

**Lemma 2.2.** *The process $\tilde{Z}_n - \mu n$ is a martingale with respect to the filtration $\{\mathcal{F}_n : n \geq 0\}$.*

**Proof.** We have, for $n \geq 1$,

$$\begin{aligned}
\tilde{Z}_n &= \sum_k \mathbf{E}[\tilde{Y}_n | \tilde{Y}_{n-1} = k, \mathcal{F}_n] \mathbf{P}(\tilde{Y}_{n-1} = k | \mathcal{F}_n) \\
&= \sum_k \sum_j j u_n(k,j) \mathbf{P}(\tilde{Y}_{n-1} = k | \mathcal{F}_{n-1}) \\
&= \sum_k k \mathbf{P}(\tilde{Y}_{n-1} = k | \mathcal{F}_{n-1}) + \sum_k \sum_j (j-k) u_n(k,j) \mathbf{P}(\tilde{Y}_{n-1} = k | \mathcal{F}_{n-1}) \\
&= \tilde{Z}_{n-1} + \mathcal{W}_n,
\end{aligned}$$

where $\mathcal{W}_n = \mathbf{E}(\theta_n(\tilde{Y}_{n-1}) | \mathcal{F}_n)$, with

$$\theta_n(k) = \sum_j (j-k) u_n(k,j) \in \mathbf{Z}^d$$

for all $n$ and $k$. Iterating, we get

$$\tilde{Z}_n = \sum_{i=1}^n \mathcal{W}_i. \tag{2.4}$$

Now,

$$\begin{aligned}
\mathbf{E}(\mathcal{W}_i | \mathcal{F}_{i-1}) &= \sum_k \sum_j \mathbf{E}[(j-k) u_i(k,j) | \tilde{Y}_{i-1} = k, \mathcal{F}_{i-1}] \mathbf{P}(\tilde{Y}_{i-1} = k | \mathcal{F}_{i-1}) \\
&= \sum_j \mathbf{E}[j u_i(0,j)] \sum_k \mathbf{P}(\tilde{Y}_{i-1} = k | \mathcal{F}_{i-1})] = \mu
\end{aligned} \tag{2.5}$$

by the independence of $u_i(k, \cdot)$ from $\mathcal{F}_{i-1}$ for all $i$ and $k$. The result follows from (2.4) and (2.5). $\square$

Let $D_n$ be a random walk with the following transition probabilities:

$$\mathbf{P}(D_{n+1} = k | D_n = \ell) = \sum_j \mathbf{E}(u_1(0,j) u_1(\ell, j+k)) := \gamma(\ell, k). \tag{2.6}$$

We prove in Lemma 2.4 below that $D_n$ is symmetric, i.e. $\gamma(\ell, \ell+k) = \gamma(\ell, \ell-k)$.

**Proposition 2.3.** *Let $\lambda$ be a vector in $\mathbf{R}^d$, $\sigma^2 = \mathbf{V}[\theta_n(k)\lambda^*]$ and $\mathcal{P}_n = \sum_{i=0}^{n-1} \mathbf{P}(D_i = 0)$. Then*

$$\mathbf{V}(\tilde{\mathcal{Z}}_n) = \sigma^2 \mathcal{P}_n, \tag{2.7}$$

*where $\tilde{\mathcal{Z}}_n = \tilde{Z}_n \lambda^* \in \mathbf{R}$.*

The finiteness of $\sigma^2$ is implied by the second moment condition assumed in the beginning of the section.

**Proof.** Since $\tilde{\mathcal{Z}}_n$ is a martingale with increments $\tilde{\mathcal{W}}_n = \mathcal{W}_n \lambda^*$,

$$\mathbf{V}(\tilde{\mathcal{Z}}_n) = \sum_{i=0}^{n-1} \mathbf{V}(\tilde{\mathcal{W}}_i)$$

Now, by the identity $\mathbf{E}\theta_n(k)\lambda^* \equiv \mu\lambda^*$ and the independence of $\theta_n(k)$ and $\mathcal{F}_{n-1}$,

$$\tilde{\mathcal{W}}_i - \mathbf{E}\tilde{\mathcal{W}}_i = \sum_k [\theta_n(k) - \mathbf{E}\theta_n(k)]\lambda^* \mathbf{P}(\tilde{Y}_{n-1} = k|\mathcal{F}_{n-1}).$$

Hence we have

$$\begin{aligned} \mathbf{V}(\tilde{\mathcal{W}}_n) &= \mathbf{E}(\tilde{\mathcal{W}}_n - \mathbf{E}\tilde{\mathcal{W}}_n)^2 \\ &= \mathbf{E}\sum_{k,l}([\theta_n(k) - \mathbf{E}\theta_n(k)]\lambda^*)([\theta_n(l) - \mathbf{E}\theta_n(l)]\lambda^*) \\ &\quad \times \mathbf{P}(\tilde{Y}_{n-1} = k|\mathcal{F}_{n-1})\mathbf{P}(\tilde{Y}_{n-1} = l|\mathcal{F}_{n-1}) \\ &= \sigma^2 \mathbf{E}\sum_k \mathbf{P}^2(\tilde{Y}_{n-1} = k|\mathcal{F}_{n-1}), \end{aligned}$$

since $\mathbf{E}\{([\theta_n(k) - \mathbf{E}\theta_n(k)]\lambda^*)([\theta_n(l) - \mathbf{E}\theta_n(l))]\lambda^*)\} = 0$ for all $k \neq l$, by the independence among the $u$'s. To get the result by iteration it remains only to verify that

$$\mathbf{E}\sum_k \mathbf{P}^2(\tilde{Y}_n = k|\mathcal{F}_n) = \mathbf{P}(D_n = 0)$$

for all $n \geq 1$. For that, let $\tilde{D}_n = \tilde{Y}_n - \hat{Y}_n$, with $\hat{Y}_n$ an independent copy of $\tilde{Y}_n$ (given $\mathcal{F}_n$) starting at the origin. Then

$$\begin{aligned} \mathbf{P}(\tilde{D}_n = 0) &= \mathbf{E}\mathbf{P}(\tilde{D}_n = 0|\mathcal{F}_n) \\ &= \mathbf{E}\sum_{k,l} \mathbf{P}(\tilde{D}_n = 0|\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = l, \mathcal{F}_n)\mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = l|\mathcal{F}_n) \\ &= \mathbf{E}\sum_k \mathbf{P}(\tilde{D}_n = 0|\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k, \mathcal{F}_n)\mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k|\mathcal{F}_n) \\ &\quad + \mathbf{E}\sum_k \sum_{\ell \neq 0} \mathbf{P}(\tilde{D}_n = 0|\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell, \mathcal{F}_n) \qquad (2.8) \\ &\quad \times \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell|\mathcal{F}_n) \\ &= \mathbf{E}\sum_k \sum_j (u_n(k,j))^2 \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k|\mathcal{F}_{n-1}) \\ &\quad + \mathbf{E}\sum_k \sum_{\ell \neq 0} \sum_j u_n(k,j)u_n(k+\ell,j)\mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell|\mathcal{F}_{n-1}) \\ &= \sum_k \sum_j \mathbf{E}(u_n(k,j))^2 \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k) \\ &\quad + \sum_{\ell \neq 0} \sum_k \sum_j \mathbf{E}[u_n(k,j)u_n(k+\ell,j)]\mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell) \\ &= \sum_j \mathbf{E}(u_n(0,j))^2 \sum_k \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k) \\ &\quad + \sum_{\ell \neq 0} \sum_j \mathbf{E}[u_n(0,j)u_n(\ell,j)] \sum_k \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell) \\ &= \gamma(0,0)\sum_k \mathbf{P}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k) \\ &\quad + \sum_{\ell \neq 0} \gamma(\ell, 0) \sum_k \mathbf{E}(\tilde{Y}_{n-1} = k, \hat{Y}_{n-1} = k + \ell) \\ &= \sum_\ell \gamma(\ell, 0)\mathbf{P}(\tilde{D}_{n-1} = \ell) \qquad (2.9) \end{aligned}$$

where

$$\gamma(0,0) = \sum_j \mathbf{E}(u_1(0,j))^2, \quad \gamma(\ell,0) = \sum_j \mathbf{E}(u_1(0,j)u_1(\ell,j))$$

A similar reasoning yields

$$\mathbf{P}(\tilde{D}_n = k) \;=\; \sum_\ell \gamma(\ell,k)\mathbf{P}(\tilde{D}_{n-1} = \ell) \tag{2.10}$$

where $\gamma(\ell,k)$ is given by (2.6).

Since $D_n$ also satisfies the same relations (2.9), (2.10) and $D_n$ and $\tilde{D}_n$ start at the same place (the origin), we conclude that

$$\mathbf{P}(\tilde{D}_n = k) = P(D_n = k),$$

for all $k$, in particular $k = 0$, and this establishes (2.7). □

**Lemma 2.4.** *The transitions $\gamma(x,y)$ given by (2.6) correspond to those of a symmetric random walk.*

**Proof.** Let $k \neq 0$. By translation invariance

$$\gamma(0,k) = \sum_j \mathbf{E}(u_1(0,j-k)u_1(0,j)) = \gamma(0,-k)$$

and for $\ell \neq 0$, by translation invariance and independence of $u_1(0,\cdot)$ and $u_1(\ell,\cdot)$,

$$\begin{aligned}
\gamma(\ell,\ell+k) &= \sum_j \mathbf{E}(u_1(0,j)u_1(\ell,j+\ell+k)) \\
&= \sum_j \mathbf{E}u_1(0,j)\mathbf{E}u_1(\ell,j+\ell+k) \\
&= \sum_j \mathbf{E}u_1(0,j)\mathbf{E}u_1(0,j+k) \\
&= \sum_j \mathbf{E}u_1(0,j-k)\mathbf{E}u_1(0,j) \\
&= \gamma(\ell,\ell-k).
\end{aligned}$$

This finishes the proof. □

# 3 Random walk estimates

We show in this section various asymptotic results involving $\mathbf{P}(\tilde{D}_n = 0|\tilde{D}_0 = 0)$ and related quantities. We use these results to prove the discrete-time version of the asymptotics (1.4) in Lemma 3.3 below. Recall that $\tilde{D}_n$ was defined in (2.6). The random walk $\tilde{D}_n$ is not spatially homogeneous (but almost), the (time homogeneous) transition probabilities are distinct (only) at the origin. This poses a small technical problem for deriving the results of this section, since the supporting random walk results we need will be quoted from Spitzer (1964) which treats only the homogeneous case. The second moment condition assumed in Section 2 implies a second moment condition for the random walk $D_n$, which is then seen to be recurrent (except for trivial cases). It is also aperiodic due to the positivity assumptions made on $u(\cdot,\cdot)$. At this point we also, for simplicity, assume isotropy, that is, the distribution of $u(\cdot,\cdot)$ is assumed invariant under permutation of coordinates. This of course leads to isotropy of $D_n$.

We start by proving monotonicity of $\mathbf{P}(\tilde{D}_n = 0|\tilde{D}_0 = 0)$ with respect to $n$.

**Lemma 3.1.** $\mathbf{P}(\tilde{D}_n = 0|\tilde{D}_0 = 0)$ *is non-increasing in* $n$.

**Proof.** Equation (5.7) below says that $\mathbf{P}(\tilde{D}_{n+1} = 0|\tilde{D}_1 = 0) - \mathbf{P}(\tilde{D}_{n+1} = 0|\tilde{D}_1 = k)$ is the increment on the variance of a martingale. It thus has to be non negative. Then

$$\begin{aligned}
\mathbf{P}(\tilde{D}_{n+1} = 0|\tilde{D}_0 = 0) &= \sum_k \mathbf{P}(\tilde{D}_{n+1} = 0|\tilde{D}_1 = k)\mathbf{P}(\tilde{D}_1 = k|\tilde{D}_0 = 0) \\
&\leq \sum_k \mathbf{P}(\tilde{D}_{n+1} = 0|\tilde{D}_1 = 0)\mathbf{P}(\tilde{D}_1 = k|\tilde{D}_0 = 0) \\
&= \mathbf{P}(\tilde{D}_n = 0|\tilde{D}_0 = 0). \square
\end{aligned}$$

Our next step is to calculate the power series of $\mathbf{P}(\tilde{D}_n = 0)$ in terms of that of the related quantity $\mathbf{P}(T = n)$, where $T$ is the return time of the walk to the origin after leaving it. We will establish a comparison between the former power series and the corresponding one for the homogeneous walk, from which the asymptotics of interest of the two walks are shown to be the same.

Let

$$f(s) = \sum_{n\geq 0} \mathbf{P}(\tilde{D}_n = 0)s^n; \qquad g(s) = \sum_{n\geq 0} \mathbf{P}(H_n = 0)s^n$$

be the power series of $\mathbf{P}(\tilde{D}_n = 0)$ and $\mathbf{P}(H_n = 0)$, respectively, where $H_n$ is the homogeneous random walk with transition probability function

$$\gamma_H(\ell, k) = \sum_j \mathbf{E}(u_1(0,j))\mathbf{E}(u_1(\ell, j+k)).$$

Notice that the transition functions of $\tilde{D}_n$ and $H_n$ differ only at the origin.

**Lemma 3.2.** *f(t) is of the order of* $g(t)$.

**Proof.** Let $\{g_i,\, i = 1, 2, \ldots\}$ be the successive waiting times of the walk at the origin and $\{T_i,\, i = 1, 2, \ldots\}$ the successive return times to the origin after leaving it and let $G_i$ and $\tau_i$ denote their partial sums, respectively.

We then have

$$\mathbf{P}(\tilde{D}_n = 0) = \sum_{i\geq 0} \mathbf{P}(n \in [G_i + \tau_i, G_{i+1} + \tau_i)), \tag{3.1}$$

where $G_0 = \tau_0 = 0$.

The last probability can be written as

$$\sum_{r\geq 0} \mathbf{P}(G_i + \tau_i = r)\mathbf{P}(n \in [r, r + g_{i+1})) = \sum_{r=0}^{n} \mathbf{P}(G_i + \tau_i = r)\mathbf{P}(g_{i+1} > n - r)$$

$$= \sum_{r=0}^{n} \gamma^{n-r} \mathbf{P}(G_i + \tau_i = r),$$

where $\gamma = \gamma(0,0)$.

Thus

$$\mathbf{P}(\tilde{D}_n = 0) = \sum_{r=0}^{n} \gamma^{n-r} \sum_{i \geq 0} \mathbf{P}(G_i + \tau_i = r) \tag{3.2}$$

and forming the power series, we get

$$\begin{aligned} f(s) &:= \sum_{n \geq 0} \mathbf{P}(\tilde{D}_n = 0) s^n \\ &= \left( \sum_{n \geq 0} \gamma^n s^n \right) \left( \sum_{n \geq 0} \sum_{i \geq 0} \mathbf{P}(G_i + \tau_i = n) s^n \right) \\ &= (1 - \gamma s)^{-1} \sum_{i \geq 0} \mathbf{E}(s^{G_i + \tau_i}) \\ &= (1 - \gamma s)^{-1} \sum_{i \geq 0} [\mathbf{E}(s^{g_1}) \mathbf{E}(s^T)]^i \\ &= \frac{1}{1 - \gamma s} \frac{1}{1 - \frac{(1-\gamma)s}{1-\gamma s} \psi_T(s)} \\ &= \frac{1}{1 - s + (1-\gamma) s [1 - \psi_T(s)]}, \end{aligned} \tag{3.3}$$

where the second identity follows from the fact that the right hand side in (3.2) represents the general term in a power series gotten out of the multiplication of the two power series in the right hand side of that identity. $\psi_T(s)$ denotes $\mathbf{E}(s^T)$.

Now

$$\mathbf{E}(s^T) = \sum_{x \neq 0} \mathbf{E}(s^{T^x}) p_x,$$

where $T^x$ is the hitting time of the origin of the walk starting at $x$ and $p_x$, $x \neq 0$, is the distribution of the jump from the origin.

Notice that $T^x$ is stochastically smaller than the sum of $||x||_1$ independent copies of $T^{\mathsf{e}}$, where $\mathsf{e}$ is any of the origin's nearest neighbors[2] and that $p_{\mathsf{e}} > 0$.

Thus,

$$[1 - \mathbf{E}(s^{T^{\mathsf{e}}})] p_{\mathsf{e}} \leq 1 - \psi_T(s) = \sum_{x \neq 0} [1 - \mathbf{E}(s^{T^x})] p_x \tag{3.4}$$

$$\leq \sum_{x \neq 0} \{1 - [\mathbf{E}(s^{T^{\mathsf{e}}})]^{||x||_1}\} p_x \leq [1 - \mathbf{E}(s^{T^{\mathsf{e}}})] \nu, \tag{3.5}$$

where $\nu = \sum ||x||_1 p_x < \infty$. We conclude that

$$\frac{1}{1 - s + \nu(1-\gamma)s[1 - \psi_{T_{\mathsf{e}}(s)}]} \leq f(s) \leq \frac{1}{1 - s + p_{\mathsf{e}}(1-\gamma)s[1 - \psi_{T_{\mathsf{e}}(s)}]}. \tag{3.6}$$

[2]Here is the only place in the argument where the assumed isotropy is used. See Remark 3.1.

We have similar bounds for $g(s)$ (with different $\gamma < 1$, $\nu < \infty$ and $p_{\mathsf{e}} > 0$). The result follows immediately.. $\square$

**Remark 3.1.** *It is not difficult to check that, denoting* $(1-\gamma)\sum_x a(x)p_x$*, where* $a(x)$ *is refered to in the proof of Proposition 5.1 below, by* $C \equiv C(\gamma, (p_x))$*,*

$$f(s)/g(s) \to C'/C \tag{3.7}$$

*as* $s \to 1$*, where* $C' = C(\gamma', (p'_x))$ *and the primes denote parameters of the homogeneous random walk* $H_n$*. This does not require isotropy and thus eliminates the need for its assumption.*

Next, we will use the behavior of $f(s)$ as $s \to 1$ to read the behavior of $\sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0)$ as $n \to \infty$.

**Lemma 3.3.** *Assume* $\sigma^2 > 0$*. Then in* $d = 1$ *and* $2$*,* $\sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0)$ *is of the order of* $\sqrt{n}$ *and* $\log n$*, respectively. In* $d \geq 3$*, it is bounded.*

**Proof.** By the previous lemma, $f(s)$ behaves the same as $g(s)$. Spitzer (1964), **P7.9** (on p.75) says that $\mathbf{P}(H_n = 0)$ is $O(n^{d/2})$. This implies (we leave this as an exercise to the reader) that $g(s)$ is $O[(1-s)^{-1/2}]$ as $s \to 1$ for $d = 1$, $O[\log(1-s)^{-1}]$ for $d = 2$ and constant for $d \geq 3$. Then so does $f(s)$.

This in turn gives information on $\sum_{i=1}^{n} \mathbf{P}(\tilde{D}_i = 0)$, the expected number of visits to the origin, from the fact that

$$\frac{e}{e+1} f(1-1/n) \leq \sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0) \leq 2ef(1-1/n), \tag{3.8}$$

for all large enough $n$. To get the lower bound above, we write

$$f(1-1/n) = \sum_{i=0}^{\infty} \mathbf{P}(\tilde{D}_i = 0)(1-1/n)^i \leq \sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0) + \sum_{i=n}^{\infty} \mathbf{P}(\tilde{D}_i = 0)(1-1/n)^i. \tag{3.9}$$

Then the monotonicity of Lemma 3.1 allows us to bound the last term from above by

$$(1-1/n)^n n\mathbf{P}(\tilde{D}_n = 0) \leq e^{-1} \sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0), \tag{3.10}$$

where the inequality is justified again by the monotonicity of $\mathbf{P}(\tilde{D}_i = 0)$. The bound follows.

For the upper bound, we write

$$f(1-1/n) \geq (1-1/n)^n \sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0) \tag{3.11}$$

The factor in front of the above sum is bounded below by $(2e)^{-1}$ for all large enough $n$ and the bound follows.

In $d = 1$ and $2$, we conclude that $\sum_{i=0}^{n-1} \mathbf{P}(\tilde{D}_i = 0)$ is of the order of $\sqrt{n}$ and $\log n$, respectively. In $d \geq 3$, it is bounded. $\square$

**Corollary 3.4.** *Let $\lambda$ be a non null vector in $\mathbf{Z}^d$ and assume $\sigma^2 > 0$. Then*

$$\mathbf{V}(\tilde{\mathcal{Z}}_n) \text{ is of the order of } \begin{cases} \sqrt{n} & \text{if } d=1 \\ \log n & \text{if } d=2 \\ \text{constant} & \text{if } d \geq 3, \end{cases} \tag{3.12}$$

*where $\tilde{\mathcal{Z}}_n = \tilde{Z}_n \lambda^*$ was defined in Proposition 2.3.*

**Proof.** Follows from Lemma 3.3 and Proposition 3.2. □

**Remark 3.2.** Corollary 3.4 is the discrete-time version of (1.4). Notice however that there is a difference. While in the discrete time one multiplies by $\sigma^2$, in the continuous time the constant is $\sigma^2 + (\mu\lambda^*)^2$. In particular this means that for the smoothing process, for which $\sigma^2 = 0$, if $\mu \neq 0$, we get non trivial fluctuations for the continuos time case and no fluctuations for the discrete time. In the latter case the motion reduces to a hyperplane that moves deterministically at speed $\mu$. These fluctuations come from the fluctuations of the Poisson processes.

**Remark 3.3.** Exact expressions for the constants of proportionality in 1 and 2 dimensions follow from those mentioned in Remark 3.1 and from strengthenings of (3.8). For the latter, in 2 dimensions, we can use the lower and upper bounds $f(1-\log n/n)$ and $f(1-1/n\log n)+const.$, respectively, instead on (3.8). In 1 dimension, we can use **P20.2** in Spitzer(1964), on p.225. We leave the details for the interested reader.

# 4 Central limit theorem

In this section we prove a central limit theorem for the forward walk $\tilde{Y}_n$ when conditioned to the values of $\mathcal{F}_n$, the sigma algebra generated by $u_k(\cdot,\cdot)$ for $k \leq n$. Of course it will then hold also for the the coordinate heights of the RAP starting with a planar configuration. For simplicity, we study only the one dimension nearest neighbor case, where $u_1(i,j) = 0$ for $|i-j| > 1$. Other cases can be similarly treated. We also assume that $\sigma^2$ as defined in Proposition 2.3 is strictly positive (which amounts to asking that $u_n(\cdot,\cdot)$ be non deterministic).

**Theorem 4.1.** *As $n \to \infty$, $\mathcal{P}_n^{-1/2}(\tilde{Z}_n - n\mu)$ converges to a centered normal random variable with variance $\sigma^2$.*

Notice that in the case we are considering $\mathcal{P}_n$ is of the order of $\sqrt{n}$ as argued at the end of the previous section. By definition, $\tilde{Z}_n$ is the height at zero at time $n$ when the starting configuration of heights is the hyperplane $X_0(i) = i$ for all $i$. Hence Theorem 4.1 shows also a central limit theorem for the height at the origin when the initial configuration is a hyperplane.

**Proof.** To establish this CLT, it is enough to check the two conditions of the corollary to Theorem 3.2 in Hall and Heyde (1980) (identifying $X_{ni}$ there with $\mathcal{P}_n^{-1/2}\mathcal{W}_i$ and $\mathcal{F}_{ni}$ with $\mathcal{F}_i$). The first condition is trivially satisfied while the second one can be written as

$$V_n^2 = \mathcal{P}_n^{-1} \sum_{i=1}^{n} \Big\{ \mathbf{E}[\mathcal{W}_i^2 | \mathcal{F}_{i-1}] - \mathbf{E}[\mathcal{W}_i^2] \Big\} \to 0 \tag{4.1}$$

in probability as $n \to \infty$. Calculating the expectations above by further conditioning on $Y_{i-1}$ (as in many of the calculations so far), we can write (4.1) as

$$\sigma^2 \mathcal{P}_n^{-1} \sum_{i=0}^{n-1} \left[ \mathbf{P}(\tilde{D}_i = 0 | \mathcal{F}_i) - \mathbf{P}(\tilde{D}_i = 0) \right] \to 0. \tag{4.2}$$

It suffices to prove that the variance of the left hand side of (4.2) goes to 0 as $n \to \infty$. For that, write the variance of the sum in (4.2) as

$$\sum_{j=1}^{n-1} \mathbf{E} \left\{ \sum_{i=0}^{n-1} \left[ \mathbf{P}(\tilde{D}_i = 0 | \mathcal{F}_{i \wedge j}) - \mathbf{P}(\tilde{D}_i = 0 | \mathcal{F}_{i \wedge (j-1)}) \right] \right\}^2. \tag{4.3}$$

Some terms in the sum inside the expectation sign cancel to yield

$$\sum_{i=j}^{n-1} \left[ \mathbf{P}(\tilde{D}_i = 0 | \mathcal{F}_j) - \mathbf{P}(\tilde{D}_i = 0 | \mathcal{F}_{j-1}) \right]. \tag{4.4}$$

We look now into the summands above. We first condition in $\tilde{D}_j$ to get

$$\sum_k \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = k) \left[ \mathbf{P}(\tilde{D}_j = k | \mathcal{F}_j) - \mathbf{P}(\tilde{D}_j = k | \mathcal{F}_{j-1}) \right]. \tag{4.5}$$

We further condition in $\tilde{Y}_{j-1}$ and $\hat{Y}_{j-1}$ to get

$$\begin{aligned} &\sum_{k,l,l'} \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = k) \\ &\quad \times \left[ \mathbf{P}(\tilde{D}_j = k | \tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l', \mathcal{F}_j) - \mathbf{P}(\tilde{D}_j = k | \tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l') \right] \\ &\quad \times \mathbf{P}(\tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l' | \mathcal{F}_{j-1}). \end{aligned} \tag{4.6}$$

Notice that the possible values for $k$ are $l-1$, $l$ and $l+1$ and that

$$\begin{aligned} \mathbf{P}(\tilde{D}_j = l-1 | \tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l', \mathcal{F}_j) &= (1 - u_{j,l'+l}) u_{j,l'}, \\ \mathbf{P}(\tilde{D}_j = l | \tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l', \mathcal{F}_j) &= (1 - u_{j,l'+l})(1 - u_{j,l'}) + u_{j,l'+l} u_{j,l'} \quad \text{and} \\ \mathbf{P}(\tilde{D}_j = l+1 | \tilde{Y}_{j-1} = l' + l, \hat{Y}_{j-1} = l', \mathcal{F}_j) &= (1 - u_{j,l'}) u_{j,l'+l}. \end{aligned}$$

Substituting in (4.6), we get, after some more manipulation,

$$\begin{aligned} &\sum_{l,l'} \left[ \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = l+1) - \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = l) \right] \\ &\quad \times \{ u_{j,l'+l}(1 - u_{j,l'}) - \mathbf{E}[u_{j,l'+l}(1 - u_{j,l'})] \} \mathbf{P}(\tilde{Y}_{j-1} = l' + l | \mathcal{F}_{j-1}) \mathbf{P}(\tilde{Y}_{j-1} = l' | \mathcal{F}_{j-1}) \\ &+ \sum_{l,l'} \left[ \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = l-1) - \mathbf{P}(\tilde{D}_i = 0 | \tilde{D}_j = l) \right] \\ &\quad \times \{ u_{j,l'}(1 - u_{j,l'+l}) - \mathbf{E}[u_{j,l'+l}(1 - u_{j,l'})] \} \} \mathbf{P}(\tilde{Y}_{j-1} = l' + l | \mathcal{F}_{j-1}) \mathbf{P}(\tilde{Y}_{j-1} = l' | \mathcal{F}_{j-1}). \end{aligned} \tag{4.7}$$

We will analyze explicitly only the first sum above by taking the sum over $i$, squaring and taking expectations. The second one has the same behavior.

To alleviate notation, let us define

$$\begin{aligned}
A^n_{j,l} &:= \sum_{i=j}^{n-1} \Big[\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_j = l+1) - \mathbf{P}(\tilde{D}_i = 0|\tilde{D}_j = l)\Big] &\quad (4.8)\\
\overline{u_{j,l'+l}(1-u_{j,l'})} &:= u_{j,l'+l}(1-u_{j,l'}) - \mathbf{E}[u_{j,l'+l}(1-u_{j,l'})] &\quad (4.9)\\
p_j(l) &:= \mathbf{P}(\tilde{Y}_{j-1} = l|\mathcal{F}_{j-1}). &\quad (4.10)
\end{aligned}$$

We want then to estimate

$$\mathbf{E}\left\{\sum_{l,l'} A^n_{j,l}\overline{u_{j,l'+l}(1-u_{j,l'})}p_j(l'+l)p_j(l')\right\}^2. \tag{4.11}$$

We will show below that (4.11) is (at most) of the order of $\mathbf{P}(\tilde{D}_j = 0)$. Substituting in (4.3) and performing the sum in $j$, we get a term of the order of $\mathcal{P}_n$. To find (an upper bound to) the variance of (4.2), we multiply by constant times $\mathcal{P}_n^{-2}$ (already taking into account the second sum in (4.7)). Since $\mathcal{P}_n \to \infty$ as $n \to \infty$, the last variance then goes to 0 as $n \to \infty$ as desired.

We rewrite (4.11) as

$$\mathbf{E}\left\{\sum_{l,l',k,k'} A^n_{j,l}A^n_{j,k}\mathbf{E}[\overline{u_{j,l'+l}(1-u_{j,l'})}\,\overline{u_{j,k'+k}(1-u_{j,k'})}]p_j(l'+l)p_j(l')p_j(k'+k)p_j(k')\right\}, \tag{4.12}$$

where we use the independence between the $u$'s and $p$'s (the $A$'s are deterministic).

Now we bound (4.12) by constant times

$$\mathbf{E}\left\{\sum_{l,l',k,k'} \left|\mathbf{E}[\overline{u_{j,l}(1-u_{j,l'})}\,\overline{u_{j,k}(1-u_{j,k'})}]\right| p_j(l)p_j(l')p_j(k)p_j(k')\right\}, \tag{4.13}$$

since $|A^n_{j,l}|$ is uniformly bounded in $l$, $j$ and $n$ (we will prove this result below).

The expectation inside the sum above vanishes if $\{l,l'\}\cap\{k,k'\} = \emptyset$. The sum over pairs with full intersection is bounded above by

$$2\mathbf{E}\left\{\sum_{l,l'} |\mathbf{E}[\overline{u_{j,l}(1-u_{j,l'})}]^2| p_j^2(l)p_j^2(l')\right\}. \tag{4.14}$$

The expectation inside the sum is constant for $l \neq l'$ and separately for $l = l'$. Thus it is bounded above uniformly by a constant, so this part of the sum is bounded above by constant times

$$\mathbf{E}\left\{\sum_{l,l'} p_j^2(l)p_j^2(l')\right\} = \mathbf{E}\left\{\sum_l p_j^2(l)\right\}^2 \leq \mathbf{E}\left\{\sum_l p_j^2(l)\right\} = \mathbf{P}(\tilde{D}_j = 0). \tag{4.15}$$

For pairs with intersection at only one point, we argue similarly to bound the sum by constant times

$$\mathbf{E}\left\{\sum_{l,l',k} p_j^2(l)p_j(l')p_j(k)\right\} = \mathbf{E}\left\{\sum_l p_j^2(l)\right\} = \mathbf{P}(\tilde{D}_j = 0). \tag{4.16}$$

All cases have been considered and we thus have the result. □

**Lemma 4.2.** $|A_{j,l}^n|$ *is uniformly bounded in* $l$, $j$ *and* $n$.

**Proof.** By the time translation invariance of the model, it suffices to prove the uniform boundedness of

$$\sum_{i=0}^{n}\left[\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = l) - \mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = l+1)\right] \tag{4.17}$$

for $l \geq 0$. Consider the $\sum_{i=0}^{n}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = l+1)$ and condition on the time $T$ that the walk starting in $l+1$ first hits $l$ to get

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_j = l)\mathbf{P}(T = j) = \sum_{i=1}^{n}\sum_{j=1}^{i}\mathbf{P}(\tilde{D}_{i-j} = 0|\tilde{D}_0 = l)\mathbf{P}(T = j), \tag{4.18}$$

by translation invariance again. Reversing the order of the sum in (4.18)

$$\sum_{j=1}^{n}\sum_{i=j}^{n}\mathbf{P}(\tilde{D}_{i-j} = 0|\tilde{D}_0 = l)\mathbf{P}(T = j) \tag{4.19}$$

and the variable $i$ to $k = i - j$

$$\sum_{j=1}^{n}\sum_{k=0}^{n-j}\mathbf{P}(\tilde{D}_k = 0|\tilde{D}_0 = l)\mathbf{P}(T = j) = \mathbf{E}\sum_{i=0}^{(n-T)^+}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = l). \tag{4.20}$$

Thus (4.17) becomes

$$\mathbf{E}\sum_{i=(n-T)^+}^{n}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = l). \tag{4.21}$$

The probabilities inside the sum are maxima when $l = 0$ (by the martingale argument already used in the proof of Lemma 3.1). Rewrite (4.21) (for $l = 0$) as

$$\sum_{j=0}^{n}\sum_{i=(n-j)}^{n}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = 0)\mathbf{P}(T = j) + \mathbf{P}(T > n)\sum_{i=0}^{n}\mathbf{P}(\tilde{D}_i = 0|\tilde{D}_0 = 0). \tag{4.22}$$

Changing variables $i$ to $k = n - i$ in the first sum, it becomes

$$\sum_{j=0}^{n}\sum_{k=0}^{j}\mathbf{P}(\tilde{D}_{n-k} = 0|\tilde{D}_0 = 0)\mathbf{P}(T = j) \tag{4.23}$$

and changing the order of summation,

$$\sum_{k=0}^{n}\sum_{j=k}^{n}\mathbf{P}(\tilde{D}_{n-k}=0|\tilde{D}_0=0)\mathbf{P}(T=j)=\sum_{k=0}^{n}\mathbf{P}(\tilde{D}_{n-k}=0|\tilde{D}_0=0)\mathbf{P}(n\geq T\geq k). \tag{4.24}$$

Summing the second term in (4.22), we finally get

$$\sum_{k=0}^{n}\mathbf{P}(\tilde{D}_{n-k}=0|\tilde{D}_0=0)\mathbf{P}(T\geq k). \tag{4.25}$$

Now since $\tilde{D}_i$ is a one dimensional simple symmetric random walk (it is homogeneous off the origin and $T$ does not depend on the transitions from the origin), we have that $\mathbf{P}(T\geq n)$ is of the order of $n^{-1/2}$ (see Spitzer (1964) **P32.1** on p.378 and **P32.3** on p.381). By the arguments at the end of the previous section, so is $\mathbf{P}(\tilde{D}_n=0|\tilde{D}_0=0)$. This implies that (4.24) is bounded in $n$. The argument is finished. □

# 5 Convergence to an invariant distribution

In this section we prove that the process as seen from the height at the origin starting with a deterministic hyperplane converges weakly. Let

$$\hat{Z}_n := Z_n - Z_n(0), \tag{5.1}$$

with $\hat{Z}_n(x) := Z_n(x) - Z_n(0)$ for all $x$, where $Z_n(x) = \mathbf{E}(Y_n^{x,n}|\mathcal{F}_n)$. Proposition 5.1 below contains also the discrete-time version of the asymptotics announced in (1.5). In Proposition 5.2 we show that the weak limit is invariant for the process.

**Proposition 5.1.** *$\hat{Z}_n$ converges weakly. Let $\hat{Z}_\infty$ denote the weak limit. If $\sigma^2>0$, then*

$$\mathbf{V}[\hat{Z}_\infty(i)-\hat{Z}_\infty(j)] \text{ is of the order of } \begin{cases} |i-j| & \text{if } d=1,\\ \log|i-j| & \text{if } d=2,\\ \text{constant} & \text{if } d\geq 3.\end{cases} \tag{5.2}$$

*If $\sigma^2=0$, then $\hat{Z}_n(i)-\hat{Z}_n(j)=\hat{Z}_0(i)-\hat{Z}_0(j)$ for all $n\geq 0$.*

**Proof.** $\hat{Z}_n$ has the same distribution as $\bar{Z}_n := \tilde{Z}_n - \tilde{Z}_n(0)$. We start proving the proposition by showing that for each $x\in\mathbf{Z}^d$, $\mathbf{E}(\bar{Z}_n(x)-x)^2$ is uniformly bounded in $n$. Since it is a martingale, it then converges almost surely and in $L^2$ (see Theorem 2.5 and Corollary 2.2 in Hall and Heyde (1980) for instance). Given $x$,

$$\mathbf{V}(\bar{Z}_n(x)-x) = 2\mathbf{V}(\tilde{Z}_n(0))^2 - 2\mathbf{C}(\tilde{Z}_n(0),\tilde{Z}_n(x)-x), \tag{5.3}$$

where $\mathbf{C}$ denotes the covariance sign, since $\tilde{Z}_n(x)-x$ and $\tilde{Z}_n(0)$ are equally distributed by the translation invariance of the model. We already have an expression for the first term on the right hand side from Proposition 2.3. We need to derive one for the last one, which we do now. We already have from the proof of Proposition 2.3 that

$$\tilde{Z}_n = \sum_{i=1}^{n}\mathcal{W}_i, \tag{5.4}$$

with $\mathcal{W}_i = \mathbf{E}(\theta_{i,\tilde{Y}_{i-1}}|\mathcal{F}_i)$. Similarly,

$$\tilde{Z}_n(x) - x = \sum_{i=1}^{n} \mathcal{W}_i^x, \tag{5.5}$$

with $\mathcal{W}_i^x = \mathbf{E}(\theta_{i,\tilde{Y}_{i-1}^x}|\mathcal{F}_i)$, where $\tilde{Y}_n^x$ is a random walk in $\mathcal{F}_n$ like $\tilde{Y}_n$, but starting from $x$. Thus,

$$\begin{aligned}
\mathbf{C}(\tilde{Z}_n(0), \tilde{Z}_n(x) - x) &= \mathbf{E}\left(\sum_{i,j=1}^{n} [\mathbf{E}(\theta_{i,\tilde{Y}_{i-1}}|\mathcal{F}_i) - \mathbf{E}(\theta_{i,\tilde{Y}_{i-1}})][\mathbf{E}(\theta_{j,\tilde{Y}_{j-1}^x}|\mathcal{F}_j) - \mathbf{E}(\theta_{j,\tilde{Y}_{j-1}^x})]\right) \\
&= \mathbf{E}\sum_{i,j}\sum_{k,l}(\theta_{i,k} - \mathbf{E}\theta_{i,k})(\theta_{j,l} - \mathbf{E}\theta_{j,l})\mathbf{P}(\tilde{Y}_{i-1} = k|\mathcal{F}_{i-1})\mathbf{P}(\tilde{Y}_{j-1}^x = l|\mathcal{F}_{j-1}) \\
&= \sigma^2 \sum_i \mathbf{E}\sum_k \mathbf{P}(\tilde{Y}_{i-1} = k|\mathcal{F}_{i-1})\mathbf{P}(\tilde{Y}_{i-1}^x = k|\mathcal{F}_{i-1}),
\end{aligned}$$

the last equality due to the independence of at least one of the $\theta$'s (the one with higher subscript) to the conditional probabilities generally and to the other $\theta$ when $i \neq j$ and the null mean of the $\theta$'s. $\sigma^2$ is the second moment of the $\theta$'s.

Now, reasoning as in the proof of Proposition 2.3,

$$\mathbf{E}\sum_k \mathbf{P}(\tilde{Y}_{i-1} = k|\mathcal{F}_{i-1})\mathbf{P}(\tilde{Y}_{i-1}^x = k|\mathcal{F}_{i-1}) = \mathbf{P}(D_{i-1}^x = 0), \tag{5.6}$$

where $D_i^x$ is a random walk like $D_i$ but starting in $x$.

From the above discussion, we conclude that

$$\mathbf{E}(\bar{Z}_{n+1}(x) - x)^2 = 2\sigma^2 \sum_{i=0}^{n-1} \left(\mathbf{P}(D_i = 0) - \mathbf{P}(D_i^x = 0)\right). \tag{5.7}$$

After a similar reasoning as used in the proof of Lemma 4.2 above, one sees that the last expression equals

$$\sum_{k=0}^{n} \mathbf{P}(\tilde{D}_{n-k} = 0|\tilde{D}_0 = 0)\mathbf{P}(T_x \geq k), \tag{5.8}$$

where $T_x$ is the hitting time of the origin by the walk $\tilde{D}_i$ starting at $x$.

Since the above expression is the variance of a martingale, it is monotone in $n$. To show it is bounded, it suffices to consider its power series (in the variable $0 \leq s < 1$) and show it is bounded as $s \to 1$ when multiplied by $1 - s$. The limiting value of this procedure gives the limit in $n$ of (5.8).

Now

$$\begin{aligned}
F(s) &:= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathbf{P}(\tilde{D}_{n-k} = 0|\tilde{D}_0 = 0)\mathbf{P}(T_x \geq k)s^n \\
&= f(s)g(s),
\end{aligned}$$

where $f(s)$ is the power series of $\mathbf{P}(\tilde{D}_n = 0|\tilde{D}_0 = 0)$ and $g(s)$ that of $\mathbf{P}(T_x \geq n)$.

We have $g(s) = (1-s)^{-1}(1 - s\psi_{T_x}(s))$ (using the notation of Section 3) and from (3.3), we get

$$(1-s)F(s) = \frac{1 - s\psi_{T_x}(s)}{1 - s + (1-\gamma)s[1 - \psi_T(s)]}.$$

Now similar arguments as in the proof of Lemma 3.2 apply and we conclude that since this expression is bounded as $s \to 1$ for $\gamma$ and $(p_y)$ of the homogeneous case (as follows from Spitzer(1964) **T32.1** on p.378), then so it is for the non-homogeneous case of our model.

In 3 or more dimensions, the expression in (5.7) is the difference of two positive quantities, the first of which is bounded in $n$ (of course then also the second, which is smaller), as follows from Spitzer(1964), **P7.9** on p.75. We conclude that it is bounded in $n$ and $x$.

Now we prove the asymptotics (5.2). Assume $\sigma^2 > 0$. In this case, the space scalings of the invariant measure follow from the dependence in $x$ of the limit of (5.7).

We first remark that in 3 or more dimensions there is no space scaling, since (5.7) is bounded in $x$ as well as in $n$ as seen above.

In 1 and 2 dimensions, the spatial scaling is given by a function $a(x)$, as readily seen from Spitzer(1964), **P32.1** on p.378 and (5.8).

In 1 dimension $a(x)$ scales as $|x|$ (Spitzer(1964) **P28.4** on p.345) under our hypotheses $\mathbf{E}(\sum_j ju(0,j))^2 < \infty$.

In 2 dimensions, in order to use **P12.3** on p.124 of Spitzer(1964) to state that $a(x)$ scales as $\log|x|$, we need to make the stronger assumption of finiteness of any moment of $\sum_j ju(0,j)$ greater than 2.

The case $\sigma^2 = 0$ follows from (5.7). □

**Remark 5.1.** The case $\sigma^2 = 0$ corresponds to the discrete time smoothing process with no randomness ($W \equiv 1$ in Definition (0.1) of Liggett (1985)). In this case the hyperplanes are invariant configurations.

**Proposition 5.2.** *Let $\hat{X}_n$ defined by $\hat{X}_n(x) = X_n(x) - X_n(0)$ be the RAP as seen from the height at the origin. Then $\hat{Z}_\infty$ is invariant for $\hat{X}_n$.*

**Proof.** Let $U_0$ denote the matrix that has all the lines identical to the vector $\{U(0,x),\, x\}$ and let $\hat{U} = U - U_0$. To prove the proposition, it suffices then to show that $\hat{U}\hat{Z}^*_\infty = \hat{Z}^*_\infty$ in distribution, with $U$ independent from $\hat{Z}_\infty$. But that is clear from the facts that $\hat{U}\hat{Z}^*_n = \hat{Z}^*_{n+1}$ in distribution and that $\hat{U}\hat{Z}^*_n$ converges weakly to $\hat{U}\hat{Z}^*_\infty$. The latter point is argued from the $L^2$ convergence of $U\bar{Z}^*_n$ and $U_0\bar{Z}^*_n$ as follows. Given $x$,

$$\begin{aligned}
\mathbf{E}[U\bar{Z}^*_n(x) - U\bar{Z}^*_\infty(x)]^2 &= \mathbf{E}[U(\bar{Z}^*_n - \bar{Z}^*_\infty)(x)]^2 && (5.9)\\
&= \mathbf{E}[\sum_y u(x,y)(\bar{Z}_n(y) - \bar{Z}_\infty(y))]^2 && (5.10)\\
&\leq \mathbf{E}\sum_y u(x,y)[\bar{Z}_n(y) - \bar{Z}_\infty(y)]^2 && (5.11)\\
&= \sum_y \pi(y-x)\mathbf{E}[\bar{Z}_n(y) - \bar{Z}_\infty(y)]^2 && (5.12)\\
&= \mathbf{E}[\bar{Z}_n(0) - \bar{Z}_\infty(0)]^2, && (5.13)
\end{aligned}$$

where the inequality follows by Jensen and the last identity is due to space translation invariance of the distribution of $\bar{Z}_n(y) - \bar{Z}_\infty(y)$. The same inequality is true with $U$ replaced by $U_0$ since it amounts to replace $x$ by 0. One concludes

$$\mathbf{E}[\hat{U}\bar{Z}_n^*(x) - \hat{U}\bar{Z}_\infty^*(x)]^2 \leq 4\mathbf{E}[\bar{Z}_n(0) - \bar{Z}_\infty(0)]^2. \tag{5.14}$$

and $L^2$ convergence follows. □

**Remark 5.2.** *Along the same lines of Remark 3.3, here one can find, in d=1,2, the exact limiting constant of proportionality of the spatial fluctuations with respect to their order of magnitude, with the extra information from the exact asymptotics of $a(x)$ in Spitzer(1964).*

# 6 The one dimensional case

In the one dimensional case we are able to treat more general initial conditions. Let $(X_n)_{n\geq 0}$ denote the height system and $(x_i)_{i\in\mathbf{Z}^d}$ be its initial configuration. Thus we have

$$X_0 = (x_i); \qquad x_0 = 0.$$

Assume $x_{i+1} - x_i = y_i$ with $y_i$ a stationary ergodic process with $\mathbf{E}y_i = 1/\alpha > 0$ and $\mathbf{V}y_i = \beta^2$. Notice that this distribution satisfies the sufficient condition for the existence of the process given by (2.3). Denote $y_i - 1/\alpha$ by $\bar{y}_i$ and let $\bar{X}_0(i) = X_0(i) - (1/\alpha)i$ and $S_n = \mathbf{E}(\bar{X}_0(Y_n)|\mathcal{F}_n)$. We then have

$$X_n(0) = S_n + (1/\alpha)Z_n. \tag{6.1}$$

where, as before, $Z_n = \mathbf{E}(Y_n|\mathcal{F}_n)$.

**Proposition 6.1.** *$S_n$ and $Z_n$ are uncorrelated for all $n$. If $\mu = \sum j\mathbf{E}u_n(j) = 0$, then $\mathbf{V}S_n$ is of order $\sqrt{n}$. If $\mu \neq 0$, then $\mathbf{V}S_n$ is of the order of $n$.*

The proposition above allows us to obtain different behavior of the fluctuations of the height at the origin in the biased and unbiased cases. The corollary below shows that in the unbiased case the fluctuations are sub diffusive (variance proportional to the square root of time) but in the biased case they are diffusive (variance proportional to the time).

**Corollary.** *If $\mu = 0$, then $\mathbf{V}X_n(0)$ is of order $\sqrt{n}$. If $\mu \neq 0$, then $\mathbf{V}X_n$ is of the order of $n$.*

**Proof of Corollary.** Since $S_n$ and $Z_n$ are uncorrelated, $\mathbf{V}X_n(0) = \mathbf{V}Z_n + \mathbf{V}S_n$. We have shown in Corollary 3.4 that $\mathbf{V}\mathbf{Z}_n$ is of the order of $\sqrt{n}$ in dimension one. Then, Proposition 6.1 shows that $\mathbf{V}X_n$ is of the order of $\mathbf{V}S_n$. □

**Proof.** In this proof we abuse notation by writing $Y_n$ instead of $Y_n^{0,n}$.

$$\begin{aligned}
\mathbf{E}(S_nZ_n) &= \mathbf{E}\left[\mathbf{E}(\bar{X}_0(Y_n)|\mathcal{F}_n)\mathbf{E}(Y_n|\mathcal{F}_n)\right] \\
&= \mathbf{E}\left[\sum_k \bar{X}_0(k)\mathbf{P}(Y_n = k|\mathcal{F}_n)\sum_k k\mathbf{P}(Y_n = k|\mathcal{F}_n)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sum_{k,l} \bar{X}_0(k) l \mathbf{P}(Y_n = k|\mathcal{F}_n)\mathbf{P}(Y_n = l|\mathcal{F}_n)\right] \\
&= \sum_{k,l} \mathbf{E}(\bar{X}_0(k)) l \mathbf{E}[\mathbf{P}(Y_n = k|\mathcal{F}_n)\mathbf{P}(Y_n = l|\mathcal{F}_n)] = 0,
\end{aligned}
$$

since $E(\bar{X}_0(k)) = 0$ for all $k$. This proves uncorrelatedness.

$$
\begin{aligned}
S_n &= \sum_k \bar{X}_0(k)\mathbf{P}(Y_n = k|\mathcal{F}_n) \\
&= \sum_{k<0}\sum_{i=k}^{-1} \bar{y}_i \mathbf{P}(Y_n = k|\mathcal{F}_n) + \sum_{k>0}\sum_{i=1}^{k} \bar{y}_{i-1}\mathbf{P}(Y_n = k|\mathcal{F}_n) \\
&= \sum_{i>0} \bar{y}_{i-1}\mathbf{P}(Y_n \geq i|\mathcal{F}_n) - \sum_{i<0} \bar{y}_i \mathbf{P}(Y_n \leq i|\mathcal{F}_n) \\
&= \sum_{i\geq 0} \bar{y}_i \mathbf{P}(Y_n > i|\mathcal{F}_n) - \sum_{i<0} \bar{y}_i \mathbf{P}(Y_n \leq i|\mathcal{F}_n)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{E}(S_n^2) &= \mathbf{E}\left[\sum_{i\geq 0} \bar{y}_i \mathbf{P}(Y_n > i|\mathcal{F}_n)\right]^2 \\
&\quad + \mathbf{E}\left[\sum_{i<0} \bar{y}_i \mathbf{P}(Y_n \leq i|\mathcal{F}_n)\right]^2 \\
&\quad - 2\mathbf{E}\left[\sum_{i\geq 0} \bar{y}_i \mathbf{P}(Y_n > i|\mathcal{F}_n) \sum_{i<0} \bar{y}_i \mathbf{P}(Y_n \leq i|\mathcal{F}_n)\right]
\end{aligned}
$$

The first expectation on the right equals

$$
\sum_{i,j\geq 0} \mathbf{E}(\bar{y}_i\bar{y}_j)\mathbf{E}[\mathbf{P}(Y_n > i|\mathcal{F}_n)\mathbf{P}(Y_n > j|\mathcal{F}_n)] \tag{6.2}
$$

which is then seen to be

$$
\beta^2 \sum_{i\geq 0} \mathbf{E}[\mathbf{P}^2(Y_n > i|\mathcal{F}_n)], \tag{6.3}
$$

since $\mathbf{E}(\bar{y}_i\bar{y}_j) = \mathbf{E}(\bar{y}_i)\mathbf{E}(\bar{y}_j) = 0$ for all $i \neq j$ and $\mathbf{E}\bar{y}_i^2 = \beta^2$ for all $i$. Similarly, the second expectation equals

$$
\beta^2 \sum_{i<0} \mathbf{E}[\mathbf{P}^2(Y_n \leq i|\mathcal{F}_n)] \tag{6.4}
$$

and the last one vanishes.

We get the following upper and lower bounds for $\mathbf{E}(S_n^2)$, (the bounds are based on $\mathbf{P}^2(\cdots) \leq \mathbf{E}(\mathbf{P}^2(\cdots|\mathcal{F}_n)) \leq \mathbf{P}(\cdots)$)

$$
\beta^2 \left(\sum_{i\geq 0} \mathbf{P}(Y_n > i)] + \sum_{i<0} \mathbf{P}(Y_n \leq i)\right) = \beta^2\mathbf{E}(|Y_n|) \tag{6.5}
$$

and

$$\beta^2 \left( \sum_{i\geq 0} \mathbf{P}^2(Y_n > i)] + \sum_{i<0} \mathbf{P}^2(Y_n \leq i) \right), \tag{6.6}$$

respectively. When $\mu = \mathbf{E}\sum_j ju(0,i) = 0$, the random walk $Y_n$ has zero mean and since the variance is positive and finite these bounds are both of order $\sqrt{n}$. When $\mu \neq 0$, the random walk $Y_n$ is not centered and the bounds are of order $n$. □

# 7 Continuous time

In this Section we discuss the continuous time process described in the introduction. Each site in $\mathbf{Z}^d$ is provided with an exponential clock of rate 1. Every time it rings, the corresponding height jumps to a random convex combination of its neighboring heights. Let $\eta_t$, $t \geq 0$ denote this process. The way we choose to show the results for this process is to approximate it by a family of discrete time processes, looking much the same as the discrete process we considered before. A direct approach should also work.

Let $T(i,n)$ be the successive times of the Poisson process of site $i$, $n \geq 1$ and let $u_k(i,\cdot)$ the $k$-th (independent) realization of the random vector $u(i,\cdot)$. In addition to the setting of that process, consider further a family $\{\beta_{n,i},\ n \geq 1,\ i \in \mathbf{Z}\}$ of (i.i.d. Bernoulli) random variables (with parameter $\mathbf{P}(T(i,1) \leq 1/N) = 1 - \exp(-1/N)$), where $N$ is a positive integer (scaling parameter), defined by $\beta_{n,i} = \mathbf{1}\{T(i,k) \in [(n-1)/N, n/N)$ for some $k \geq 1\}$ and let $K_n = \sum_{i=1}^n \beta_{k,i}$. For a fixed initial configuration $X_0 \in \mathbf{R}^{\mathbf{Z}^d}$ define the process $(X_n^N)_{n\geq 0}$ by

$$\begin{aligned} X_0^N(i) &= X_0(i), \\ X_n^N(i) &= \beta_{n,i} \sum_j u_{K_n}(i,j) X_{n-1}^N(j) + (1-\beta_{n,i}) X_{n-1}^N(i), \quad n \geq 1. \end{aligned}$$

Let $\mathcal{F}_n^N$ be the sigma algebra generated by $\{\lambda_{k,i}, u_k(i,j), 1 \leq k \leq n, i \in \mathbf{Z}^d\}$. Again $X_n^N(0)$ is described as the conditional expectation of a function of a random walk:

$$X_n^N(0) = E(X_0^N(Y_n^{N,0,n}) \mid \mathcal{F}_n^N), \tag{7.1}$$

where $Y_k^{N,0,n}$, $k = 0, \ldots, n$, is the backwards random walk starting at the origin and for $0 \leq k \leq n$,

$$\begin{aligned} \mathbf{P}(Y_k^{N,0,n} = j | Y_{k-1}^{N,0,n} = i) &= \beta_{n-k,i} u_{K_{n-k}}(i,j) + (1-\beta_{n-k,i}) \mathbf{1}\{i = j\} \\ &:= u_{n-k}^N(i,j), \end{aligned}$$

Notice that $\{u_k^N(i,\cdot) : k \geq 0, i \in \mathbf{Z}^d\}$ is a family of iid vectors.

The same arguments of Section 2 yield

$$\mathbf{V} X_n^N(0) = \sigma_N^2 \mathbf{E} M^N(n)$$

where $M^N(n)$ is the number of returns to the origin in the time interval $[0,t]$ of a (symmetric) random walk $D_t^N$ with transition probabilities

$$\gamma^N(\ell,k) = \sum_j \mathbf{E}(u_1^N(0,j) u_1^N(\ell, j+k))$$

and, reminding that $u_1(i,j)$ has the same distribution as $u(i,j)$,

$$\begin{aligned}\sigma_N^2 &= \mathbf{V}\left[\sum_j \lambda j^* u_1^N(0,j)\right] \\ &= \frac{1}{N}\left(\mathbf{V}\left[\sum_j \lambda j^* u(0,j)\right] + \left[\mathbf{E}\sum_j \lambda j^* u(0,j)\right]^2\right) - \frac{\mathbf{E}\left[\sum_j \lambda j^* u(0,j)\right]^2}{N^2} \\ &= \frac{1}{N}\left(\sigma^2 + (\lambda\mu^*)^2\right) + o(1/N)\end{aligned}$$

where $\sigma^2$ is defined in Proposition 2.3 and $\mu = \mathbf{E}\left(\sum_j j u(0,j)\right)$. It is a standard matter to prove that the process $Y_s^{x,t}$, $s \in [0,t]$ defined by the almost sure coordinatewise limit

$$Y_s^{x,t} = \lim_{N\to\infty} Y_{\lfloor Ns\rfloor}^{N,x,\lfloor Nt\rfloor}$$

is a random walk with transition rates $\{u.(\cdot,\cdot)\}$. This walk uses the Poisson marks and the realizations of the $u$ backwards in time from $t$ to 0.

Using the duality formula (7.1), we prove below (see Lemma 7.1) that $\mathbf{E}(Y_{\lfloor Nt\rfloor}^{N,x,\lfloor Nt\rfloor}|\mathcal{F}_{\lfloor Nt\rfloor}^N)$ converges in $L^1$ to $\mathbf{E}(Y_t^{x,t}|\mathcal{F}_t)$, where $\mathcal{F}_t$ is the sigma algebra generated by the Poisson processes and the realizations of the $\{u.(\cdot,\cdot)\}$ corresponding to the Poisson marks in the interval $[0,t]$. It is then easy to show that the limiting process

$$\eta_t = \lim_{N\to\infty} X_{\lfloor Nt\rfloor}^N \quad \text{and} \quad X_0^N = \eta_0 \tag{7.2}$$

exists, has generator (1.1) and

$$\eta_t(x) = \mathbf{E}(\eta_0(Y_t^{x,t})|\mathcal{F}_t). \tag{7.3}$$

**Lemma 7.1.** *If $\mathbf{E}\eta_0(j) \le const.\ |j|^2$ for all $j$ then $X_{\lfloor Nt\rfloor}^N$ converges to $\eta_t$ in $L^1$ as $N \to \infty$ for all fixed $t$. If $\mathbf{E}\eta_0(j)^2 \le const.\ |j|^2$ for all $j$ then there is also convergence in $L^2$.*

**Proof.** Consider the event

$$\begin{aligned}A_t^N = \{&Y_s^{x,t}\text{ jumps twice in a time interval } [(n-1)/N, n/N) \text{ for some } 1 \le n \le \lfloor Nt\rfloor \\ &\textit{or} \text{ jumps once in } [\lfloor Nt\rfloor, \lfloor Nt\rfloor + 1/N)\}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}|X_{\lfloor Nt\rfloor}^N - \eta_t| &\le \mathbf{E}|\eta_0(Y_t^{0,t}) - \eta_0(Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor})| &(7.4)\\ &= \mathbf{E}[|\eta_0(Y_t^{0,t}) - \eta_0(Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor})|; A_t^N]. &(7.5)\end{aligned}$$

Notice that $\mathcal{F}_{\lfloor Nt\rfloor}^N \subset \mathcal{F}_t$ for all $t$. The latter expectation can be written, disregarding constant factors,

$$\begin{aligned}&\sum_{i,j} \mathbf{E}|\eta_0(i) - \eta_0(j)|\mathbf{P}(Y_t^{0,t} = i, Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor} = j, A_t^N) &(7.6)\\ \le\ &\sum_{i,j}(\mathbf{E}|\eta_0(i)| + \mathbf{E}|\eta_0(j)|)\mathbf{P}(Y_t^{0,t} = i, Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor} = j, A_t^N) &(7.7)\\ \le\ &\sum_{i,j}(i^2 + j^2)\mathbf{P}(Y_t^{0,t} = i, Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor} = j, A_t^N) &(7.8)\\ =\ &\mathbf{E}[(Y_t^{0,t})^2; A_t^N] + \mathbf{E}[(Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor})^2; A_t^N] &(7.9)\end{aligned}$$

and the argument for convergence in $L^1$ closes with the observation that $(Y_t^{0,t})^2$ and $(Y_{\lfloor Nt\rfloor}^{N,0,\lfloor Nt\rfloor})^2$ are uniformly integrable and $\mathbf{P}(A_t^N) \to 0$ as $N \to \infty$ for all fixed $t$, both of which assertions are not difficult to check. Convergence in $L^2$ follows the same steps, with the $|\cdots|$ expressions in (7.4-7.7) replaced by $[\cdots]^2$. (The corresponding of inequality (7.4) follows by Jensen.) □

Now, by Lemmas 2.1 (duality formula) and 7.1, when $\eta_0(x) \equiv x$

$$\begin{aligned}\mathbf{V}\eta_t(0)) &= \lim_{N\to\infty} \mathbf{V}X_{\lfloor Nt\rfloor}^N(0) \\ &= \lim_{N\to\infty} \sigma_N^2 \mathbf{E}M^N(\lfloor Nt\rfloor) \\ &= (\sigma^2 + (\lambda\mu^*)^2)\mathbf{E}M(t),\end{aligned}$$

where $M(t)$ is the time spent in the origin up to $t$ by a continuous time random walk $D_t$ with transition rates given by

$$q(0,k) = \sum_j \mathbf{E}(u(0,j)u(0,j+k)) \tag{7.10}$$

and, for $\ell \neq 0$,

$$q(\ell, \ell+k) = (\pi_k + \pi_{-k}). \tag{7.11}$$

Now the asymptotic variance for the continuous case follows from the asymptotics of the number of visits of a continuous time symmetric random walk with a inhomogeneity at the origin. The asymptotics of Section 2 can be obtained also for the continuous time walk.

The convergence to an invariant measure follows for the continuous case similarly as above, by using the discrete time approximation and the fact that $\mathbf{E}(Y_t \mid \mathcal{F}_t)$ is a martingale. Here $Y_t$ is the limiting process obtained from $Y_{\lfloor Nt\rfloor}^N$, the walk with transitions

$$\begin{aligned}\mathbf{P}(Y_k^N = j | Y_{k-1}^N = i) &= \beta_{k,i} u_{K_k}(i,j) + (1-\beta_{k,i})\mathbf{1}\{i=j\} \\ &=: u_k^N(i,j).\end{aligned}$$

We have

$$\begin{aligned}\mathbf{V}[\eta_t(x) - \eta_t(0)] &= \lim_{N\to\infty} 2\sigma_N^2[\mathbf{E}M^N(\lfloor Nt\rfloor) - \mathbf{E}M_x^N(\lfloor Nt\rfloor)] \\ &= (\sigma^2 + (\lambda\mu^*)^2)[\mathbf{E}M(t) - \mathbf{E}M_x(t)],\end{aligned}$$

where the subscripts $x$ mean that the corresponding random walk starts at $x$. Now the fact that the last quantity is bounded and the martingale property imply the convergence. The fact that the limit is invariant follows as in Proposition 5.2.

We leave the extension of the results of Sections 4 and 5 to the reader, who may employ the approximation above.

**Comparison with Liggett's results.** In display (0.2) of Chapter IX, Liggett (1985) defines linear systems using families of random operators $A_z(x,y)$ and linear coefficients $a(x,y)$. Our process corresponds to the choice $a(x,y) = 0$ for all $x$, $y$ and

$$A_z(x,y) = \begin{cases} u(x,y) & \text{if } x = z \\ \mathbf{1}\{x=y\} & \text{if } x \neq z \end{cases}.$$

Then conditions (1.3) and (1.4) of Liggett are satisfied. On the other hand the hypothesis of his Lemma 1.6 reads

$$\sup_x \mathbf{E} \sum_y u(x,y) = \sup_x \mathbf{E} \sum_y u(0, y-x) = \mathbf{E} \sum_y u(0,y) = 1$$

by translation invariance. This implies that also Liggett's (1.5) is satisfied and his construction works for our case. We have proposed two somehow simpler constructions. One of them, sketched in the introduction exploits the existence of a almost sure duality relation. The other is a standard passage from discrete to continuous time but without hard estimates.

It is also interesting to notice that when computing the two point correlation functions in the translation invariance case of his Theorem 3.1, a function called there $q_t(x,y)$ plays a fundamental role (see page 445). While in the general case $\sum_y q_t(x,y)$ is not expected to be one, in the RAP case $q_t(x,y)$ are the probability transition functions at time $t$ of the random walk with rates $q(x,y)$ given by (7.10) and (7.11).

His Theorem 3.17 can be applied to our process to yield weak convergence to constant positive configurations, when starting with translation invariant positive initial conditions. He does not treat initial conditions we considered in Theorem 5.1 (because his work has a different perspective presumably).

**The nearest neighbors case. Conservative nearest particle systems.** Consider $d = 1$ and $u(i,j) = 0$ if $|i-j| > 1$. In this case if the initial heights are ordered, that is $\eta_0(i) < \eta_0(i+1)$ for all $i$, then the same will occur for latter times. If we project the heights on a vertical line, we obtain a point process. We can interpret that at each event of this point process there is a particle. If $\eta_0(0) = 0$, then the initial point process has a particle at the origin. The dynamics obtained by this projection can be described as follows. Each particle, after an exponential time chooses a random position between the neighboring particles and jump to it. Since the interaction occurs only with the two nearest neighboring particles and the number of particles is conserved, we called this motion as conservative nearest particle system. See Liggett (1985) for examples of (non conservative) nearest particle systems.

To study the height at the origin $\eta_t(0)$ in the height system is equivalent to tag the particle at the origin and follow it in the particle system. The particle interpretation also allows us to study the projection of configurations that otherwise will be not localized. In particular, when $u(i,i+1) = 1 - u(i,i-1)$ is a uniform random variable in $[0,1]$, any homogeneous Poisson process is reversible for the particle system, when we disregard the labels of the particles. To see this, notice that if $X$ and $Y$ are independent exponential random variables of the same parameter and $U$ is an uniform in $[0,1]$ independent of the above random variable, then $u(X+Y)$ and $(1-U)(X+Y)$ are again independent exponential random variables of same parameter. The reversibility of the Poisson process for the particle system implies that the particle system as seen from the tagged particle has as reversible measure the Poisson process conditioned to have a particle at the origin. This in particular implies that if initially the difference between successive heights are iid exponentially distributed with some parameter $\rho$, then this distribution is reversible for the process as seen from the height at the origin.

Another consequence of this isomorphism between the particle process and the height system is that we get the asymptotic behavior for the motion of the tagged particle in the conservative nearest particle system. The continuous time version of Corollary to Proposition 6.1 implies that the motion of the tagged particle in the non biased case ($\mu = 0$) is sub diffusive. That is, starting with the Poisson process, the variance of $\eta_t(0)$ will behave asymptotically as $\sqrt{t}$. This corrects Theorem 9.1 in Ferrari (1996) which states wrongly that the behavior was diffusive.

# 8 Hydrodynamics

Let $\phi : \mathbf{R}^d \to \mathbf{R}$ be a continuous function. Let $\phi_n : \mathbf{Z}^d \to \mathbf{R}$ be defined by

$$\phi_n(i) = \phi(i/n)$$

If the distribution of $u(i,j)$ is symmetric in the sense that $u(i,j)$ and $u(j,i)$ have the same distribution, then the following Hydrodynamic limit holds:

$$\lim_{n\to\infty} \mathbf{E}(X_{n^2t}(nr)|X_0 = \phi_n) = \Phi(r,t)$$

where $\Phi$ is the solution of the heat equation with initial condition $\phi$. In other words, $\Phi$ is the solution of

$$\left(\frac{\partial \Phi}{\partial t}\right)_h = D_h\,(\Delta\Phi)_h\,,$$

where $D_h = \sum_j |j_h| u(0,j)$, where $j_h$ is the $h$-th coordinate of the vector $j \in \mathbf{Z}^d$. To show the above one just computes the derivative of the $h$-th coordinate of $\mathbf{E}X_{n^2t}(rn)$ as follows

$$\begin{aligned}
&\lim_{n\to\infty} n^2\,\mathbf{E}(X_{n^2t}(rn) - X_{n^2t-1}(rn))_h \\
&\quad = \lim_{n\to\infty} n^2 \sum_j \mathbf{E}(u_{n^2t}(rn, rn+j))_h \mathbf{E}(X_{n^2t-1}(rn+j) - X_{n^2t-1}(rn))_h \\
&\quad = \lim_{n\to\infty} n^2 \sum_{j:j_h>0} \mathbf{E}(u_{n^2t}(rn, rn+j))_h \mathbf{E}(X_{n^2t-1}(rn+j) + X_{n^2t-1}(rn-j) - X_{n^2t-1}(rn))_h
\end{aligned}$$

where $(\cdot)_h$ is the $h$-th coordinate of the vector $(\cdot)$. When $n\to\infty$ this gives the desired result. Presumably both the law of large numbers and local equilibrium hold.

We close this Section giving a comparison between the hydrodynamics of the $\eta_t$ process and the $\xi_t$ process studied by Kipnis, Presutti and Marchioro (1982). Consider the continuous time $\eta_t$ process in one dimension with nearest neighbor interaction: $u(i,i+1) = 1 - u(i,i-1)$ uniformly distributed in $[0,1]$ and $u(i,j) = 0$ otherwise. For this process the Poisson Processes of rate $\rho$ on the line are invariant (and reversible) measures. The $\xi_t$ process is just defined by the differences:

$$\xi_t(i) = \eta_t(i+1) - \eta_t(i)$$

Since the Poisson process is invariant for the $\eta_t$ process, the measure defined as product of independent exponentials of parameter $\rho$ is invariant for the $\xi_t$ process.

Kipnis, Presutti and Marchioro (1982) consider this process but in a finite box $\rho_L = \{1,\dots,L\}$ with the following boundary conditions: at the times of a Poisson process of rate 1, the value at site 1 is updated independently of everything by substituting whatever is in the site by an exponential random variable of mean $\rho_-$; at the times of a (independent of everything) Poisson process of rate 1, the value at site $L$ is updated independently of everything by substituting whatever is in the site by an exponential random variable of mean $\rho_+$. They studied the unique invariant measure around site $rL$, $r \in [0,1]$ and obtained that, as $L \to \infty$, this measure converges to a product of exponentials with parameter $\varphi(r) = r\rho_- + (1-r)\rho_+$. This is called local equilibrium. The function $\rho(s)$ is the solution of the heat equation

$$\frac{\partial^2 \varphi}{\partial r^2} = 0$$

with boundary conditions

$$\varphi(0) = \rho^-; \qquad \varphi(1) = \rho^+.$$

The corresponding solution for the hydrodynamic limit of the corresponding $\eta_t$ process is a non equilibrium stationary profile growing with time: the solution of the equation

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial r^2}$$

with Neumann conditions

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=0} = \rho_-; \qquad \left.\frac{\partial \Phi}{\partial r}\right|_{r=1} = \rho_+.$$

This solution is

$$\Phi(r,t) = (\rho_+ - \rho_-)\frac{r^2}{2} + r\rho_- + t(\rho_+ - \rho_-).$$

And the relation is

$$\frac{\partial \Phi}{\partial r} = \varphi.$$

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Instituto de Matemática e Estatística — Universidade de São Paulo
Cx. Postal 66281 — 05315-970 São Paulo SP — Brasil
<pablo@ime.usp.br>, <lrenato@ime.usp.br>