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A Matrix Commuting with the Square of the Almost Mathieu Operator

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Abstract. Using a transformation that reduces the almost Mathieu operator to a tridiagonal form with zero main diagonal, a unitary matrix is constructed that commutes with the square of the N-dimensional operator H_ψ

$$(H_\psi\psi)_n = \psi_{n-1} + 2\cos(\omega n+\theta)\psi_n + \psi_{n+1},$$
$$n = 0,1,\ldots,N-1; \quad \psi_{-1}=\psi_{N-1}; \quad \psi_N=\psi_0;$$

in the case when $\omega = 2\pi M/N$ (numbers M, N are relatively prime integers), N is divisible by 4, and $\cos(N\theta)=-1$.

As is well known, the problem of an electron on a two-dimensional lattice subject to a perpendicular uniform magnetic field reduces to consideration of Harper's equation (see, e.g., [1–4]) in a Hilbert space $l^2(Z)$:

$$\begin{aligned}
&(H\psi)_n = \varepsilon\psi_n;\\
&(H\psi)_n = \psi_{n-1} + 2\cos(\omega n+\theta)\psi_n + \psi_{n+1}, \qquad\qquad (1)\\
&n = \ldots,-1,0,1,\ldots, \qquad \omega,\theta\in\mathbf{R}.
\end{aligned}$$

H is a particular case of the almost Mathieu operator. This operator has been studied by many workers (see [5] for a review). When $\omega = 2\pi M/N$, with relatively prime integers M, N (that is they do not have a common divisor other than 1), H is an N-periodic tridiagonal matrix and, hence, its spectrum consists of N intervals. When $\omega/2\pi$ is irrational, the spectrum is a zero measure Cantor set. The latter fact is proved for Lebesgue a.e. ω [6]. For irrational $\omega/2\pi$ the spectrum is independent of the parameter θ. Consider the N-dimensional operator H_ψ defined as follows:

Let $\omega = 2\pi M/N$, where M and N are relatively prime, and write the equations (1) for $n = 0, 1, \ldots, N-1$ with periodic boundary conditions, that is

$$\begin{aligned} &\psi_{n-1} + (q^n e^{i\theta} + q^{-n} e^{-i\theta})\psi_n + \psi_{n+1} = \varepsilon\psi_n, \\ &n = 0, 1, \ldots, N-1; \qquad q = e^{i\omega}; \qquad \psi_{-1} = \psi_{N-1}; \quad \psi_N = \psi_0. \end{aligned} \tag{2}$$

The corresponding $N \times N$ matrix I shall denote H_ψ, where index ψ reminds of the basis in which H_ψ is defined. Later on, similar notation for the same operator in other bases will be used. Note, by the way, that the discriminant $S_N(\varepsilon)$ (see, e.g., [7,8]) associated with the N-periodic tridiagonal matrix H (1) is equal to $(-1)^N \det(H_\psi - \varepsilon I) + 2$, where I is the unity $N \times N$ matrix.

As $N \to \infty$, the almost Mathieu operator with irrational $\omega/2\pi$ is obtained from H_ψ. In [3,9], it was shown that the spectrum of matrices equivalent to H_ψ for some values of θ can be represented as a solution of Bethe-ansatz-type algebraic equations. In [10], a set of equations for arbitrary θ was derived. In the present paper, a certain operator W is indicated that commutes with H_ψ^2 at $\cos(N\theta) = -1$. Thus the spectral problem for H_ψ is closely related to the spectral problem for W. Let us now turn to exact constructions.

Substituting $\psi_n = \sum_{k=0}^{N-1} U_{nk} e^{ik\theta}\phi_k$, where $U_{nk} = q^{nk}$, $n, k = 0, 1, \ldots, N-1$, is, obviously, a unitary[1] matrix, in (2) and using the linear independence of the columns of the transformation operator, we have

$$\begin{aligned} &\phi_{k-1} + (q^k + q^{-k})\phi_k + \phi_{k+1} = \varepsilon\phi_k, \\ &k = 0, 1, \ldots, N-1; \qquad \phi_{-1} = e^{iN\theta}\phi_{N-1}; \quad \phi_N = e^{-iN\theta}\phi_0. \end{aligned} \tag{3}$$

In particular, if $\theta = 0 \pmod{2\pi}$ then we see that the commutator $[H_\psi, U] = 0$. This fact is not very valuable since $U^4 = N^2 I$ and, consequently, $U/\sqrt{N}$ may have only 4 different eigenvalues: $i, -i, 1, -1$.

[1] Henceforth unitary means unitary up to a factor, that is $UU^* = \mathrm{Const}\, I$.

Further, we shall consider the case when N is even and $N\theta = \pi \,(\mathrm{mod}\, 2\pi)$. These conditions ensure that after application of the unitary transformation

$$\phi_k = \sum_{n=0}^{N-1} q^{\frac{1}{2}(k-1/2)^2+kn}\zeta_n \tag{4}$$

to (3), we get a system with periodic boundary conditions:

$$\begin{aligned}&(1+q^{n-1})\zeta_{n-1} + (1+q^{-n})\zeta_{n+1} = \varepsilon\zeta_n,\\ &n = 0,1,\ldots,N-1; \qquad \zeta_{-1} = \zeta_{N-1}; \quad \zeta_N = \zeta_0.\end{aligned} \tag{5}$$

Due to the fact that (5) does not contain (unlike (2) and (3)) the middle term with ζ_n, the square H_ζ^2 splits into the direct sum of two matrices: one is $H_{\zeta,1}^2$ defined in the basis $\{e_{2n}\}_{n=0,1,\ldots,N/2-1}$, the other is $H_{\zeta,2}^2$ in $\{e_{2n+1}\}_{n=0,1,\ldots,N/2-1}$, where $\{e_n\}_{n=0,1,\ldots,N-1}$ is the basis in which H_ζ (or $(H_\zeta - \varepsilon I)\zeta$, i.e. (5)) is written.[2]

Consider $H_{\zeta,1}^2$ first. We shall further assume that $N/4$ is integer. Let us apply to $H_{\zeta,1}^2$ the unitary transformation $\zeta_n = \sum_{k=0}^{N/2-1} q^{n^2+2nk}\tilde{\xi}_k$ and then shift the basis by $N/4$: $\tilde{\xi}_{k+N/4} = \xi_k$. The purpose of the shift is to get a matrix with zero boundary conditions:

$$\begin{aligned}&(1+q^{-1}-q^{-2k}-q^{2k-1})\xi_{k-1} + (4-q^{2k}-q^{-2k}-q^{2k+1}-q^{-2k-1})\xi_k +\\ &(1+q-q^{-2k-1}-q^{2k+2})\xi_{k+1} = \varepsilon\xi_k, \qquad k = 0,1,\ldots,N/2-1;\end{aligned} \tag{6}$$

(We may assume any ξ_{-1} and $\xi_{N/2}$.)

Now it is straightforward to verify that application of the transformation $\xi_k = \sum_{p=0}^{N/2-1} \widetilde{W}_{kp}\varphi_p$, $\widetilde{W}_{kp} = q^{(2k+1)p+(k+p)N/2}$, $k,p = 0,1,\ldots,N/2-1$ to (6) leaves these equations unchanged. Thus $[H_{\xi,1}^2, \widetilde{W}] = 0$. (Cf. $[H_\psi, U] = 0$.) The diagonal transformation $V_{pm} = q^{pN/2}\delta_{pm}$ reduces $\widetilde{W}$ to a simpler form

$$W_{kp} = q^{(2k+1)p}, \qquad k,p = 0,1,\ldots,N/2-1. \tag{7}$$

Similarly, we find the basis in which $[H_{\gamma,2}^2, W'] = 0$, where

$$W'_{kp} = q^{(2k+3)p}, \qquad k,p = 0,1,\ldots,N/2-1. \tag{8}$$

[2]Similar to (5) representation without the middle term was first obtained in [11] for the Hamiltonian of an electron on a 2-dimensional lattice in magnetic field by changing the gauge of the field. Such a change, roughly speaking, involves (unlike (4)) not only H but rather a direct sum of operators H over θ.

Unlike U, the unitary matrices W and W' have many different eigenvalues. There exist corresponding eigenvectors (for simple eigenvalues they are uniquely defined up to a factor) that, as follows from the commutativity, are eigenvectors of H^2.

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