Time and Temperature Dependent Correlation Functions of $1D$ Models of Quantum Statistical Mechanics

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We consider gapless models of statistical mechanics. At zero temperatures correlation functions decay asymptotically as powers of distance in these models. Temperature correlations decay exponentially. We used an example of solvable model to find the formula, which describes long distance and large time asymptotic of correlation function of local fields. The formula describes correlation at any temperature and arbitrary coupling constant.

PACS numbers: 05.30.-d, 05-50.+q, 65.50.+m, 67.40.Db

We consider quantum models of statistical mechanics in one space and one time dimension. We shall study models with one type of excitation (particle).

First let us discuss zero temperature. Ground states of the models, in which we are interested are Fermi spheres. All possible states of the particle are filled from $-k_F$ to k_F. We shall use k for the momentum of the particle and ε for the energy. Fermi velocity v_F is defined as:

$$v_F = \left.\frac{\partial \varepsilon}{\partial k}\right|_{k=k_F}. \tag{1}$$

At large distance x correlation function of local quantum fields $\psi(x,t)$ decay as a power law:

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle \underset{\substack{x\to\infty \\ t\to\infty}}{\longrightarrow} \frac{1}{(x-v_Ft)^{2\Delta^+}(x+v_Ft)^{2\Delta^-}}. \tag{2}$$

The powers $\Delta^\pm$ can be called conformal dimensions [1]. At very small temperatures T Fermi velocity still can be defined, and correlation function decay exponentially

$$\langle \psi(0,0)\psi^\dagger(x,t)\rangle_T \underset{\substack{x\to\infty \\ t\to\infty}}{\longrightarrow} \exp\left\{-\frac{2\Delta^+\pi T}{v_F}|x-v_Ft| - \frac{2\Delta^-\pi T}{v_F}|x+v_Ft|\right\}. \tag{3}$$

Increasing temperature will destroy Fermi surface and temperature dependence of the correlation function will become more complicated. In this paper we found a universal formula (21) which describes asymptotical correlations at any temperature. We consider an example of Bose gas with delta interaction. Our description of the model will follow Chapters I and XVIII of the book [2].

In order to define the model let us consider many-body quantum mechanics of N identical particles. The Hamiltonian of Bose gas can be represented as

$$\mathcal{H}_N = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{N\ge j>k\ge 1}\delta(x_j-x_k). \tag{4}$$

Coupling constant c is positive. The model also can be reformulated as a model of quantum field theory. It can be described by the canonical quantum

Bose fields $\psi(x,t)$. Commutation relations are standard

$$\begin{aligned} [\psi(x,t), \psi^\dagger(y,t)] &= \delta(x-y), \\ [\psi(x,t), \psi(y,t)] &= 0. \end{aligned} \tag{5}$$

Now the Hamiltonian of the same model can be written as an operator in the Bosonic Fock space:

$$H = \int_{-\infty}^{\infty} dx \left(\partial_x \psi^\dagger(x) \partial_x \psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) \right). \tag{6}$$

In this form the model is known as Quantum Nonlinear Schrödinger equation.

Bethe Ansatz for the model was discovered by E. H. Lieb and W. Liniger [3]. Physics of interacting bosons ($c > 0$) is different from physics of free bosons ($c = 0$). For interacting bosons Pauli principle is valid in momentum space [4]. At zero temperature ground state is Fermi sphere. In order to describe the ground state exactly it is more convenient to use spectral parameter λ instead of physical momentum of the particle k. Spectral parameter λ coincide with momentum only for zero density of the gas. At positive density momentum k can be expressed as a function of spectral parameter (see (10)). Ground state can be described by Lieb–Liniger integral equation:

$$\rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda,\mu)\rho(\mu)\, d\mu = \frac{1}{2\pi}. \tag{7}$$

Here q is the value of the spectral parameter on the Fermi surface, and

$$K(\lambda,\mu) = \frac{2c}{c^2 + (\lambda-\mu)^2}. \tag{8}$$

The density function $\rho(\lambda)$ shows distribution of particles in the Fermi sphere: $L\rho(\lambda)\, d\lambda$ gives the number of particles in the interval $[\lambda, \lambda + d\lambda]$. By L we denoted the length of the box. Excitation energy $\varepsilon(\lambda)$ (energy of the particle) can be described by similar integral equation

$$\varepsilon(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda,\mu)\varepsilon(\mu)\, d\mu = \lambda^2 - h. \tag{9}$$

It vanishes on the Fermi surface $\varepsilon(\pm q) = 0$. Here h is chemical potential. We consider $h > 0$ case.

Momentum of the particle can be expressed as a function of spectral parameter:

$$k(\lambda) = \lambda + \int_{-q}^{q} \theta(\lambda - \mu)\rho(\mu)\, d\mu. \tag{10}$$

Here

$$\theta(\lambda) = i \ln \left(\frac{ic + \lambda}{ic - \lambda} \right). \tag{11}$$

Fermi velocity is equal to

$$v_F = \left. \frac{\partial \varepsilon(\lambda)}{\partial k(\lambda)} \right|_{\lambda = q} = \frac{\varepsilon'(q)}{k'(q)}. \tag{12}$$

Denominator can be expressed in terms of ρ

$$\frac{\partial k(\lambda)}{\partial \lambda} = 2\pi \rho(\lambda). \tag{13}$$

This derivative on the Fermi surface is called dressed charge

$$Z = 2\pi \rho(q). \tag{14}$$

It defines conformal dimensions $\Delta^{\pm}$ from (2)

$$2\Delta^{+} = 2\Delta^{-} = \frac{1}{4Z^2}. \tag{15}$$

Luttinger liquid derivation of these formulæ was first found by F. D. M. Haldane [5].

At very small temperature formula (3) describes correlations. We calculated correlation function at arbitrary temperature. Before formulating our results let us describe thermodynamics.

At positive temperatures it is necessary to introduce two density functions $\rho_p(\lambda)$ and $\rho_t(\lambda)$. The number of particles in the interval $[\lambda, \lambda + d\lambda]$ is equal to $L\rho_p(\lambda)\, d\lambda$. The joint number of particles and holes in $[\lambda, \lambda + d\lambda]$ is equal to $L\rho_t(\lambda)\, d\lambda$. C. N. Yang and C. P. Yang discovered [6] a system of equations which describe the state of thermodynamic equilibrium

$$\begin{aligned} &2\pi \rho_t(\lambda) = 1 + \int_{-\infty}^{\infty} K(\lambda, \mu)\rho_p(\mu)\, d\mu, \\ &\frac{\rho_p(\lambda)}{\rho_t(\lambda)} = \frac{1}{1 + e^{\frac{\varepsilon(\lambda)}{T}}}, \\ &\varepsilon(\lambda) = \lambda^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} K(\lambda, \mu) \ln \left(1 + e^{-\frac{\varepsilon(\mu)}{T}} \right) d\mu. \end{aligned} \tag{16}$$

The existence of stable excitation (particle) also was discovered in [6]. This is a consequence of integrability of the model. Energy of this excitation is $\varepsilon(\lambda)$ and momentum is

$$k(\lambda) = \lambda + \int_{-\infty}^{\infty} \theta(\lambda - \mu)\rho_p(\mu)\, d\mu. \tag{17}$$

Velocity of the excitation is

$$v(\lambda) = \frac{\partial\varepsilon(\lambda)}{\partial k(\lambda)} = \frac{\varepsilon'(\lambda)}{k'(\lambda)}. \tag{18}$$

Denominator in (18) is

$$\frac{\partial k(\lambda)}{\partial\lambda} = 2\pi\rho_t(\lambda). \tag{19}$$

Correlation function at positive temperature is defined as

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = \frac{\mathrm{tr}\left(e^{-\frac{H}{T}}\psi(0,0)\psi^\dagger(x,t)\right)}{\mathrm{tr}\left(e^{-\frac{H}{T}}\right)}. \tag{20}$$

Here trace is taken over all space of states. We evaluated long distance and large time asymptotic of this correlation function

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T \underset{\substack{x\to\infty \\ t\to\infty}}{\longrightarrow} \exp\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi\rho_t(\lambda)}|x - v(\lambda)t| \ln\left|\frac{e^{\frac{\varepsilon(\lambda)}{T}} - 1}{e^{\frac{\varepsilon(\lambda)}{T}} + 1}\right|\right\}. \tag{21}$$

This is our main result.

The key-point of our derivation is representation of correlation function as a determinant of Fredholm integral operator [7]. On the basis of this representation a system of integrable equations was derived [8]. These equations can be solved by Riemann–Hilbert problem and long distance asymptotic can be evaluated. This method of evaluation of correlation functions is described in detail in Parts III and IV of the book [2].

Acknowledgments

This work was supported by NSF Grant PHY-9321165, RFBR Grant 96-01-00344 and INTAS-01-166-ext.

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