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On the Torus Quantization of Two Anyons with Coulomb Interaction in a Magnetic Field

Luca Salasnich[1]

Dipartimento di Matematica Pura ed Applicata
Università di Padova, Via Belzoni 7, I 35131 Padova, Italy
and
Istituto Nazionale di Fisica Nucleare, Sezione di Padova,
Via Marzolo 8, I 35131 Padova, Italy

Abstract

We study two anyons with Coulomb interaction in a uniform magnetic field B. By using the torus quantization we obtain the modified Landau and Zeeman formulas for the two anyons. Then we derive a simple algebraic equation for the full spectral problem up to the second order in B.

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[1]Electronic address: salasnich@math.unipd.it

In 1977 Leinaas and Myrheim[1)] suggested the possible existence of arbitrary statistics between the Bose and the Fermi case in space dimensions less than three. A model of particles with arbitrary, or fractional, statistics was introduced by Wilczek[2)], who called such objects as *anyons*. Anyons are thought to be good candidates for the explanation of such important collective phenomena in condensed matter physics as the fractional quantum Hall effect and high temperature superconductivity (for recent reviews see Ref. 3–5).

In this paper we study two anyons with Coulomb interaction in a uniform magnetic field. The eigenvalue problem for this system was solved numerically by Myrheim, Halvorsen and Vercin[6)] by using a finite difference method to discretize the Schrödinger equation. Here, instead, we obtain some analytical formulas of the energy spectrum by using the torus quantization[7),8)].

The Lagrangian of two anyons with Coulomb interaction in two dimensions is given by

$$L_0 = \frac{m}{2}(\dot{\mathbf{r}}_1^2 + \dot{\mathbf{r}}_2^2) + \frac{e^2}{||\mathbf{r}_2 - \mathbf{r}_1||} + \alpha\hbar\dot{\theta} \qquad (1)$$

where $\mathbf{r}_1 = (x_1, y_1)$, $\mathbf{r}_2 = (x_2, y_2)$ and α is the statistical parameter: Bose statistics is recover for $\alpha = 0$ and Fermi statistics for $\alpha = 1$. The azimuthal angle is defined by $\theta = \arctan(\frac{y_2 - y_1}{x_2 - x_1})$.

The separation of the center of mass motion can be achieved through the change of variables $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}_G = \mathbf{r}_2 + \mathbf{r}_1$. In this way we obtain

$$L_0 = \frac{\mu}{2}\dot{\mathbf{r}}_G^2 + \frac{\mu}{2}\dot{\mathbf{r}}^2 + \frac{e^2}{r} + \alpha\hbar\dot{\theta} \ , \qquad (2)$$

where $\mu = m/2$ is the reduced mass and $r = ||\mathbf{r}||$. Neglecting the motion of the center of mass $\mathbf{r}_G$, the Lagrangian can be written in polar coordinates

$\mathbf{r} = (r\cos\theta, r\sin\theta)$ as

$$L_0 = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{e^2}{r} + \alpha\hbar\dot{\theta}\ . \tag{3}$$

Let us introduce a uniform magnetic field $\mathbf{B}$ along the z–axis. Choosing the symmetric gauge, the Lagrangian of interaction is given by

$$L_I = -\frac{eB}{2c}r^2\dot{\theta}\ , \tag{4}$$

so that the total lagrangian reads

$$L = L_0 + L_I = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{e^2}{r} + (\alpha\hbar - \frac{eB}{2c}r^2)\dot{\theta}\ . \tag{5}$$

Introducing the generalized momenta

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}\ , \tag{6}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} + (\alpha\hbar - \frac{eB}{2c}r^2)\ , \tag{7}$$

the Hamiltonian $H = p_r\dot{r} + p_\theta\dot{\theta} - L$ can be written as

$$H = \frac{p_r^2}{2\mu} + \frac{(p_\theta - \alpha\hbar)^2}{2\mu r^2} - \frac{e^2}{r} + \frac{e^2B^2}{8\mu c^2}r^2 + \frac{eB}{2\mu c}(p_\theta - \alpha\hbar) = E\ . \tag{8}$$

The angular momentum p_θ is a constant of motion because the Hamiltonian is independent of θ. It is readily verified that H and p_θ are in involution, so this two–degrees of freedom system is integrable. We observe that $\frac{eB}{2\mu c}(p_\theta - \alpha\hbar)$ is constant and we study the reduced Hamiltonian

$$\bar{H} = \frac{p_r^2}{2\mu} + W(r) = \bar{E}, \tag{9}$$

where

$$W(r) = \frac{(p_\theta - \alpha\hbar)^2}{2\mu r^2} - \frac{e^2}{r} + \frac{e^2B^2}{8\mu c^2}r^2\ . \tag{10}$$

Also the Hamiltonian $\bar{H}$ is integrable and each classical trajectory is confined on a 2–dimensional torus. The action variables on the torus are given by

$$I_\theta = \frac{1}{2\pi} \oint p_\theta d\theta = p_\theta \ , \tag{11}$$

$$I_r = \frac{1}{2\pi} \oint \sqrt{2\mu(\bar{E} - W(r))} dr \ , \tag{12}$$

and by inverting the last relation we have $\bar{E} = \bar{H}(I_r, I_\theta)$.

The energy spectrum of the system can be obtained by using the *torus quantization*. The torus quantization goes back to the early days of quantum mechanics and was developed by Bohr and Sommerfeld for separable systems, it was then generalized for integrable (but not necessarily separable) systems by Einstein[7)]. In fact, Einstein's result was corrected for the phase changes due to caustics by Maslov[8)]. The torus quantization is just the first term of a certain $\hbar$-expansion of the Schrödinger equation[8),9)] but it is of extreme importance since in many cases this is the only approximation known in the form of an explicit formula[10),11)]. For our system the torus quantization reads

$$I_\theta = n_\theta \hbar \ , \tag{13}$$

$$I_r = (n_r + \frac{1}{2})\hbar \ , \tag{14}$$

where the radial quantum number n_r is a non–negative integer, whilst the angular quantum number n_θ can be also negative.

In this way the quantized spectrum is given by

$$E(n_r, n_\theta) = \bar{E}(n_r, n_\theta) + \frac{eB\hbar}{2\mu c}(n_\theta - \alpha) \ , \tag{15}$$

where $\bar{E} = \bar{E}(n_r, n_\theta)$ is obtained by solving the equation

$$\frac{1}{2\pi} \oint \sqrt{-F + \frac{2G}{r} - \frac{M}{r^2} - Nr^2} dr = (n_r + \frac{1}{2})\hbar \ , \tag{16}$$

with $F = 2\mu(-\bar{E})$, $G = \mu e^2$, $M = (n_\theta - \alpha)^2\hbar^2$ and $N = e^2B^2/(4c^2)$.

This integral equation can be evaluated analytically in some limit cases. In particular, if B is very strong then the Coulomb interaction is negligible ($G = 0$). In this case, if r is represented in the complex plane, the integrand can be pictured on a Riemann surface of two sheets with branch points at the roots r_1 and r_2 ($r_1 < r_2$) of the radicand[12)]. The path of integration encloses the line joining the two roots but, with the inverse direction of rotation, the path encloses the poles of the integrand: $r = 0$ and $r = \infty$. It follows from the *theorem of residues* that the integral is equal to the negative sum of the residues Res_0 and Res_∞ of the integrand in these poles

$$\oint \sqrt{-F - \frac{M}{r^2} - Nr^2}dr = -2\pi i(Res_0 + Res_\infty) = -2\pi(\sqrt{M} + \frac{F}{2\sqrt{N}}) \ , \quad (17)$$

where Res_0 is the coefficient of $1/r$ in the Laurent expansion of the integrand in the neighbourhood of the pole $r = 0$. Instead Res_∞ is calculated as the Res_0 of the integrand arising from the substitution $z = 1/r$. In this way we obtain the modified Landau formula for the two anyons

$$E_L(n_r, n_\theta) = \frac{eB\hbar}{2\mu c}(n_r + \frac{1}{2} + |n_\theta - \alpha| + (n_\theta - \alpha)) \ . \quad (18)$$

Now we consider the case of a week magnetic field. In the simplest case B^2 is negligible ($N = 0$) and we can apply again the theorem of residues

$$\oint \sqrt{-F + \frac{2G}{r} - \frac{M}{r^2}}dr = -2\pi(\sqrt{M} - \frac{G}{\sqrt{F}}) \ , \quad (19)$$

from which we obtain the modified Zeeman formula for the anyons

$$E_Z(n_r, n_\theta) = \frac{-\mu e^4}{2\hbar^2[(n_r + \frac{1}{2}) + |n_\theta - \alpha|]^2} + \frac{eB\hbar}{2\mu c}(n_\theta - \alpha) \ . \quad (20)$$

Let us study the case $N \neq 0$ where Nr^2 represents a correction term. We expand the square root of Eq. (16) in powers of N. At first order in N (second order in B) we get

$$\oint \sqrt{-F + \frac{2G}{r} - \frac{M}{r^2} - Nr^2}dr = \oint \sqrt{-F + \frac{2G}{r} - \frac{M}{r^2}}dr - \frac{N}{2} \oint \frac{r^3}{\sqrt{-Fr^2 + 2Gr - M}}dr \; . \tag{21}$$

The first integral has been obtained previously and the second one gives

$$\oint \frac{r^3}{\sqrt{-Fr^2 + 2Gr - M}}dr = -2\pi(3M - \frac{5G^2}{F})\frac{G}{2F^2\sqrt{F}} \; . \tag{22}$$

The quantization formula, up to the second order in B, is given by Eq. (15) where $\bar{E}$ is obtained by solving the following equation

$$\bar{E}^7 = \frac{-\mu e^4}{2\hbar^2[(n_r + \frac{1}{2}) + |n_\theta - \alpha|]^2}\Big(\bar{E}^3 + \frac{3e^2B^2(n_\theta - \alpha)^2\hbar^2}{64\mu^2c^2}\bar{E} + \frac{5e^6B^2}{128\mu c^2}\Big)^2 \; . \tag{23}$$

This algebraic equation, which can be easily solved numerically by using the Newton–Raphson method, gives the first correction to the modified Zeeman formula. To get higher order corrections it is sufficient to calculate other terms in the expansion of the square root of Eq. (16).

In summary, we have obtained some analytical formulas for the energy spectrum of two anyons with Coulomb interaction in a uniform magnetic field B. The system has been quantized by using the torus quantization and the integrals which derive from it have been calculated in the complex plane by using the theorem of residues. In the limit of strong magnetic field we have derived a modified Landau formula. Instead, for a weak magnetic field we have deduced a modified Zeeman formula and then its correction up to the second order in B.

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