ξ - ζ relation

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Abstract

In this note we prove a relation between the Riemann Zeta function, ζ and the ξ function (Krein spectral shift) associated with the Harmonic Oscillator in one dimension. This gives a new integral representation of the zeta function.

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1 Introduction

The Riemann zeta function is a well studied object, for example, Titchmarch [6] gives a detailed exposition of this function. There are several expressions for ζ, and in this note we present an integral representation for ζ, that comes from the Krein spectral shift formula of Krein [3, 4]. Recently Gesztesy-Simon [1] generalized the trace formulae for Schrödinger operators using the Krein spectral shift function, which they named the ξ function, as it is central to inverse spectral theories in one dimension and had several important applications in spectral theories of operators in one dimension. This work used the proof of the Krein formula, given in Simon [5], theorem I.10 and its generalizations. A proof of the formula for a slightly larger class is shown in Mohapatra-Sinha [2]. We refer to these papers for the history and other work on the Krein spectral shift function.

2 ξ function of the Harmonic oscillator

In this section we recall the ξ function of the Harmonic oscillator.

We consider the Harmonic oscillator, H $= \frac{1}{2}(-\frac{d^2}{dx^2} + x^2 + 1)$, acting on $L^2(\mathbb{R})$, normalized so that its spectrum is the positive integers $\mathbb{Z}^+$. We consider the operator H_∞, using the notation of [1, 5], obtained by looking at the Harmonic oscillator with Dirichlet condition at 0. Then the spectrum of H_∞ is even integers $2\mathbb{Z}^+$, with uniform multiplicity 2. The Krein spectral shift function $\xi(\lambda)$ for the pair of operators (H, H_∞) is given by

$$\xi(\lambda) = \sum_{m=1}^{\infty} \chi_{[2m-1,2m)}(\lambda)$$

where χ_X denotes the indicator function of X. In terms of the ξ function, Gesztesy-Simon [1], Simon [5] theorem I.10 (case $\alpha = \infty$), and Mohapatra-Sinha theorem 4.2 , proved the trace formula,

$$Tr(f(H) - f(H_\infty)) = -\int (f(\lambda))'\xi(\lambda)d\lambda \qquad (2.1)$$

with different smoothness and decay conditions on f. Fix $s = \sigma + it$ and consider the smooth function $f(\lambda) = \lambda^s$, on [1,∞), then by functional calculus, it follows that,

$$f(H) = \int f(\lambda)dE_H(\lambda), \quad and \quad f(H_\infty) = \int f(\lambda)dE_{H_\infty}(\lambda)$$

are both trace class for $\sigma > 2$, since

$$\sum_{m=1}^{\infty} m^{-\sigma} \quad and \quad \sum_{m=1}^{\infty} (2m)^{-\sigma}$$

converge.

Here is the primary relation between the ξ and ζ functions.

Theorem 2.1. *Let $\xi(\lambda)$ denote the Krein spectral shift function for the pair of operators (H, H_∞), defined above. Then the Riemann zeta function $\zeta(s)$ is related to ξ through the relation,*

$$(1 - 2^{(1-s)})\zeta(s) = s \int_1^{\infty} \lambda^{-s-1} \quad \xi(\lambda) d\lambda \tag{2.2}$$

valid for any $s = \sigma + it$, with $\sigma > 0$.

Proof. . We consider $s = \sigma + it$ with $\sigma > 2$, then we take f(x) = $x^{(-s)}$. Then by definition $\zeta(s) = Trace(f(H))$. Now we rewrite this as,

$$\zeta(s) = Trace(f(H) - f(H_\infty)) + Trace(f(H_\infty)$$

by the linearity of the trace. Now we notice that since the spectral multiplicity of H_∞ is 2 and the spectrum is the even integers we have,

$$Trace(f(H_\infty)) = 2\sum_{n=1}^{\infty}(2n)^{(-s)} = 2^{(1-s)}\sum_{n=1}^{\infty}(n)^{(-s)} = 2^{(1-s)}\zeta(s).$$

Using the above relations, and the trace formula in terms of the Krein spectral shift, we immediately see that for $\sigma > 2$, the theorem is valid, since then the function f(λ) = λ^{-s} is C^2 and satisfies $(1 + |\lambda|^2)f^j(\lambda) \in L^2(\mathbb{R}^+), j = 1, 2,$ while its extension to $\sigma > 0$ follows from the analyticity of the left and right hand sides of equation 2.2.

A simple change of variables $\ln \lambda = x$ in the expression for ζ given in the above theorem gives the following corollary. In the following we take,

$$\phi(x) = \sum_{n=1}^{\infty} \chi_{[\ln(2n-1), \ln(2n))}(x).$$

Corollary 2.2. *The zeta function is given, in the region $\sigma > 0$, $s = \sigma + it$, by,*

$$\zeta(s) = \frac{s}{(1 - 2^{1-s})} \int_0^{\infty} e^{-sx}\phi(x) dx$$

Remark 2.3. *In the context of inverse spectral theory of the Harmonic oscillator, we could have taken instead of H_∞ with boundary condition at 0, the operators with the Dirichlet condition at any x, however this will not lead to a closed expression for the ζ function. It may still happen for special non-zero values of x (those for which all but finitely many of the Dirichlet eigen values again fall in the spectrum of H) and it would be interesting to identify these values of x and the resulting identities for ζ, if they exist.*

References

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