Erratum

Exact Ground State Energy of the Strong-Coupling Polaron

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We are grateful to Professor Andrey V. Soldatov of the Moscow Steklov Mathematical Institute for calling our attention to an error in our paper. The commutator inequality (8) in our step **I**, namely $|k_j||\langle a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\rangle| \leq 2\langle p_j^2\rangle^{1/2}\langle a_{\mathbf{k}}^* a_{\mathbf{k}}\rangle^{1/2}$, is not correct. Rather, the right side of this inequality should be $\langle p_j^2\rangle^{1/2}(\langle a_{\mathbf{k}}^* a_{\mathbf{k}}\rangle^{1/2} + \langle a_{\mathbf{k}} a_{\mathbf{k}}^*\rangle^{1/2})$ or a related expression. The extra factor $\langle a_{\mathbf{k}} a_{\mathbf{k}}^*\rangle^{1/2}$ with the $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$ not in normal order generates uncontrolled mischief with, for example, the right side of the ultraviolet bound (10) containing an additional term $\sum_{|\mathbf{k}|\geq K} 1/2 = \infty$.

The situation is easily remedied with the help of the method introduced by Lieb and Yamazaki [14] to obtain the previous rigorous lower bound on the polaron energy. Our main result, (31), is still valid. Indeed, it is improved slightly!

Define the (vector) operator $\mathbf{Z} = (Z_1, Z_2, Z_3)$ with components

$$Z_j = (\frac{4\pi\alpha}{V})^{1/2} \sum_{|\mathbf{k}|\geq K} k_j \frac{a_{\mathbf{k}}}{|\mathbf{k}|^3} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad j = 1, 2, 3. \tag{1}$$

Then the commutator estimate (8) is replaced by

$$-(\frac{4\pi\alpha}{V})^{1/2} \sum_{|\mathbf{k}|\geq K} \left[\langle \frac{a_{\mathbf{k}}}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}\rangle + c.c.\right] \equiv -\sum_j \langle [p_j, Z_j - Z_j^*]\rangle$$

$$
\begin{aligned}
\leq 2\langle \mathbf{p}^2\rangle^{1/2}\langle -(\mathbf{Z}-\mathbf{Z}^*)^2\rangle^{1/2} &\leq 2\langle \mathbf{p}^2\rangle^{1/2}\langle 2(\mathbf{Z}^*\mathbf{Z}+\mathbf{Z}\mathbf{Z}^*)\rangle^{1/2} \\
&\leq \varepsilon\langle \mathbf{p}^2\rangle + \frac{2}{\varepsilon}\langle \mathbf{Z}^*\mathbf{Z}+\mathbf{Z}\mathbf{Z}^*\rangle. \qquad (2)
\end{aligned}
$$

Now, each component Z_j can be thought of as a single (unnormalized) oscillator mode having commutator with its adjoint, $[Z_j, Z_j^*] = (4\pi\alpha/V)\sum_{|\mathbf{k}|\geq K} k_j^2|\mathbf{k}|^{-6} \to 2\alpha/3\pi K$; moreover, Z_i and Z_j^* commute for $i \neq j$ (i.e., these modes are orthogonal). Using these facts, we have that

$$
\begin{aligned}
\frac{2}{\varepsilon}\langle \mathbf{Z}^*\mathbf{Z}+\mathbf{Z}\mathbf{Z}^*\rangle &= \frac{4}{\varepsilon}\langle \mathbf{Z}^*\mathbf{Z}\rangle + \frac{2}{\varepsilon}\left(\frac{2\alpha}{\pi K}\right) \\
&\leq \sum_{|\mathbf{k}|\geq K}\langle a_{\mathbf{k}}^* a_{\mathbf{k}}\rangle + 3/2 \qquad (3)
\end{aligned}
$$

if we choose $\varepsilon = 8\alpha/3\pi K$, which is smaller and better by a factor $1/3$ from the ε in the article. Here we have employed an orthogonal rotation of coordinates bringing $\sum_{|\mathbf{k}|\geq K} a_{\mathbf{k}}^* a_{\mathbf{k}}$ into a form $(4/\varepsilon)\mathbf{Z}^*\mathbf{Z}$+non-negative operators. (Compare Eqs.(21,22) of the article.) Combining these inequalities, we obtain

$$
-(\frac{4\pi\alpha}{V})^{1/2}\sum_{|\mathbf{k}|\geq K}\left[\langle \frac{a_{\mathbf{k}}}{|\mathbf{k}|}e^{i\mathbf{k}\cdot\mathbf{x}}\rangle + c.c.\right] \leq \varepsilon\langle \mathbf{p}^2\rangle + \sum_{|\mathbf{k}|\geq K}\langle a_{\mathbf{k}}^* a_{\mathbf{k}}\rangle + 3/2. \qquad (4)
$$

This last inequality is our replacement for the ultraviolet bound (10). It follows that $H \geq H_K - 3/2$ where H_K is as in Eq.(11), but with the coefficient of $\mathbf{p}^2$ given by $(1-8\alpha/3\pi K)$ rather than $(1-8\alpha/\pi K)$. With the choice $K = 8\alpha/3\pi$, inequality (13) becomes $H \geq -(16\alpha^2/3\pi^2) - 3/2$, a bound at least consistent with a known *upper* bound for the ground state energy linear in α.

The remainder of the article is an analysis of H_K and needs only minor modification. The coefficient of $\mathbf{p}^2$ in Eqs.(19,23,27,28,30) should be $(1 - c_1\alpha^{-1/5}/3)$ and, at the end of the article, $c_5 = (c_1/3 + 2c_4)c_P$. Due to the smaller value of ε defined above, our estimate on the coefficient of $\alpha^{9/5}$ in (31) is slightly improved to 2.337, rather than 3.822 as reported. Of course, our lower bound for the ground state energy is decreased merely by the constant $-3/2$, which is unimportant on a scale of $\alpha^{9/5}$.