Examples of discrete operators with a pure point spectrum of finite multiplicity

L. Amour * and J. C. Guillot †

Abstract

One constructs operators acting on $l^2(\mathbf{Z}^m)$ (or $l^2(\mathbf{Z}^m)^p$), $m, p \geq 1$, with a real pure point spectrum of finite multiplicity by perturbing diagonal matrices using a KAM procedure. The point spectrum can be dense on an interval or a Cantor set of measure zero. The basic fact here is to remark that for perturbations built up with an infinite number of block diagonals, regularly separated, it is possible to deal with eigenvalues of multiplicity strictly greater than one. Examples of discrete operators associated with discretization of systems of partial differential equations are given.

1 Introduction.

It is known that given a sequence of real numbers $(d_i)_{i \in \mathbf{Z}^m}$ and a sufficiently small perturbation P it is possible to construct a diagonal matrix D such that the discrete operator $D + P$ has the d_i's as eigenvalues (see [C], [P]).

*Département de Mathématiques, U.R.A. C.N.R.S. 1870, Université de Reims, Moulin de la Housse, B.P. 1039, 51687 Reims Cedex 2, France. E-mail: laurent.amour@univ-reims.fr

†C.M.A.P., Ecole Polytechnique, U.R.A. C.N.R.S. 756, 91128 Palaiseau Cedex, France and Département de Mathématiques, U.R.A. C.N.R.S. 742, Université Paris-Nord, 93430 Villetaneuse, France. E-mail: guillot@cmapx.polytechnique.fr and guillot@math.univ-paris13.fr

For this construction one use a KAM procedure and it is crucial that the divisors $d_i - d_j$ do not get too small as $|i-j|$ is large. A typical condition is

$$|d_i - d_j| \geq c|i-j|^{-\alpha}, \quad i \neq j, \tag{1}$$

where α and c are positive real numbers. Each eigenvalue d_i is simple.

In this paper our aim is precisely to avoid the restrictions on the multiplicity of the eigenvalues. Therefore we replace the conditions (1) by weaker assumptions, balanced by a stronger hypothesis on the perturbation P. This additional assumption on the perturbation is on its block structure. Of course, to see that the simplicity of the eigenvalues is not an essential condition for the construction of D, it suffices to look at diagonal perturbations.

For $a, b \in \mathbf{Z}^m$ we use the notation

$$a < b \Leftrightarrow a_\nu < b_\nu, \ \forall \ \nu \in \{1, \dots m\}$$

and

$$|a| = \sum_{\nu=1}^{m} |a_\nu|.$$

Let $\mathbf{N}$ be the space of positive integers. For $a \in \mathbf{N}^n$ set

$$\pi_a = \prod_{\nu=1}^{n} a_\nu.$$

In $\mathbf{Z}^m$ and $\mathbf{N}^n$ we use the pointwise addition and multiplication. Then for $a, b \in \mathbf{N}^n$, we say that a is a divisor of b if there exists $k \in \mathbf{N}^n$ such that $a = bk$. Then we write $a|b$ and $k = a/b$.

In what follows we consider perturbations of diagonal matrices acting on $l^2(\mathbf{Z}^m)$ that are sufficiently small in a suitable space.

Let N and $N^\sharp$ be two $m-$integers such that $N \geq 1$, $N^\sharp$ is a divisor of N and $N^\star = N/N^\sharp$.

The perturbations are real matrices $P = (P(i,j))_{(i,j) \in \mathbf{Z}^{2m}}$ that are decomposed into $N \times N$ submatrices such that each $N \times N$ submatrix is $N^\sharp \times N^\sharp$

block diagonal. Thus the perturbations are matrices with an infinite number of $N^\sharp \times N^\sharp$ block diagonals regularly separated.

Precisely let J be the following subset of $\mathbf{Z}^{2m}$

$$J = \Big\{ (i,j) \in \mathbf{Z}^{2m};\ \text{there exist } l, l' \in \mathbf{Z}^m \text{ and } k \in \mathbf{N}^m$$

with $k < N^\star$ such that

$$0 \le i - lN - kN^\sharp, j - l'N - kN^\sharp \le N^\sharp \Big\}.$$

Then the set of perturbations P is denoted by $\mathcal{A}$ with

$$\mathcal{A} = \Big\{ P = (P(i,j))_{(i,j)\in \mathbf{Z}^{2m}}\,;\, P(i,j) = 0 \text{ if } (i,j) \notin J \Big\}.$$

For $N, N^\sharp \ge 1$ and $N^\sharp \ne N$ and small perturbations in $\mathcal{A}$ we will construct operators on $l^2(\mathbf{Z}^m)$ with a pure point spectrum of multiplicity strictly greater than one. These results are extensions of the ones obtained by [C] and [P] for simple eigenvalues.

In what follows the infinite matrices that we study can be associated to kernels of integral operators on $l^2(\mathbf{Z}^m)$.

A similar result holds for operators on $l^2(\mathbf{Z}^m)^N$ where N is a n-integer. In that case the operators can be discretizations of systems of partial differential equations. Let us give examples of such systems of partial differential equations. Here $n = 1$.

$$\mathcal{H}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} P_{11} + \hat{D}_1 & P_{22} & 0 & 0 \\ P_{21} & P_{22} + \hat{D}_2 & 0 & 0 \\ 0 & 0 & P_{33} + \hat{D}_3 & P_{34} \\ 0 & 0 & P_{43} & P_{44} + \hat{D}_4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Each P_{ij} is the infinite matrix of a discretized partial differential operator and each $\hat{D}_i$ is an infinite diagonal matrix associated to a discretized potential. In this case the multiplicity of the chosen eigenvalues can be two.

Acknowledgment. It is a great pleasure to thank L.H. Eliasson, F. Klopp and J. Ralston for helpful discussions.

2 Matrices on $l^2(\mathbf{Z}^m)$.

As in [P] we consider the Banach algebra $\mathcal{M}$ of real m-dimensional sequences $a = (a_i)_{i\in\mathbf{Z}^m}$ with the norm $\|a\|_\infty = \sup\limits_{i\in\mathbf{Z}^m} |a_i|$ and the operations of pointwise addition and multiplication of sequences. $\mathcal{M}$ is invariant under translation, i.e., $T_k a \in \mathcal{M}$ for $a \in \mathcal{M}$ and where $(T_k a)_i = a_{i+k}$.

For sake of completeness let us recall according to [P] the basic facts concerning the so-called approximation functions first introduced by [R]. A continuous function $\Omega : [0,\infty) \to [1,\infty)$ is an approximation function if both

$$\Phi(\sigma) = \sigma^{-4m} \sup_{r\geq 0} \Omega(r) e^{-\sigma r},\ \sigma \geq 0$$

and

$$\Psi(\sigma) = \inf_{S_\sigma} \prod_{\nu=0}^{\infty} \Phi(\sigma_\nu)^{2^{-u-1}},\ \sigma \geq 0$$

are finite for $\sigma > 0$, where S_σ is the set of all sequences $\sigma_0 \geq \sigma_1 \geq \cdots \geq 0$ with $\sum \sigma_\nu \leq \sigma$. See [R] and the appendix of [P].

A real sequence $d = (d_i)_{i\in\mathbf{Z}^m}$ is a distal sequence for $\mathcal{M}$ if

$$(d - T_k d)^{-1} \in \mathcal{M},\ \|(d - T_k d)^{-1}\|_\infty \leq \Omega(|k|),\ 0 \neq k \in \mathbf{Z}^m \tag{2}$$

for some approximation function $\Omega(\,\cdot\,)$.

A pair of arbitrary real sequences $d = (d_i)_{i\in\mathbf{Z}^m}$ and $\tilde{d} = (\tilde{d}_i)_{i\in\mathbf{Z}^m}$ is called a distant pair for $\mathcal{M}$ if

$$(d - T_k\tilde{d})^{-1} \in \mathcal{M},\ \|(d - T_k\tilde{d})^{-1}\|_\infty \leq \Omega(|k|),\ k \in \mathbf{Z}^m \tag{3}$$

for some approximation function.

Let M be the space of all matrices $P = (P(i,j))_{(i,j)\in\mathbf{Z}^{2m}}$ satisfying

$$P_k = (P(i, i+k))_{i\in\mathbf{Z}^m} \in \mathcal{M}.$$

for every $k \in \mathbf{Z}^m$.

As in [P] we define a scale of Banach spaces

$$M^s = \{P \in M; \|P\|_s < \infty\}$$

where

$$\|P\|_{M^s} = \sup_{k \in \mathbf{Z}^m} \|P_k\|_\infty e^{|k|s}, \; 0 \le s \le \infty.$$

In what follows we consider a subset of diagonal matrices D and we find a diagonal matrix $\hat{D}$ such that $\hat{D} + P$ is similar to D when P is a sufficiently small perturbation. Thus we construct a subset of matrices each one being isospectral to D. The spectrum of D is pure point and can be dense in an interval or a Cantor set of measure zero.

Our starting point is a set of real sequences $d = (d_i)_{i \in \mathbf{Z}^m}$ in $\mathcal{M}$ such that some subsequences satisfy diophantine conditions.

$N, N^\sharp$ and $N^\star = N/N^\sharp$ are chosen as above. Consider a real sequence $d = (d_i)_{i \in \mathbf{Z}^m}$ in $\mathcal{M}$. For every $\alpha \in \mathbf{N}^m$ with $0 \le \alpha < N$ define $d_\alpha = (d_{\alpha + iN})_{i \in \mathbf{Z}^m}$. d_α is a subsequence of d. There are π_N such subsequences and $d = \cup_{0 \le \alpha < N} d_\alpha$.

The set of real sequence $d \in \mathcal{M}$ that we consider is denoted $\mathcal{D}_{N,N^\sharp}$. $\mathcal{D}_{N,N^\sharp}$ is the set of sequences d such that

(i) For every $\alpha \in \mathbf{N}^m$ with $0 \le \alpha < N$, d_α is distal for $\mathcal{M}$ and for an approximation function denoted by $\Omega_\alpha(\,\cdot\,)$.

(ii) For every pair $(\alpha, \beta) \in \mathbf{N}^m \times \mathbf{N}^m$ satisfying $0 \le \alpha < \beta < N$ and $0 \le \alpha - kN^\sharp, \beta - kN^\sharp < N^\sharp$ for $k \in \mathbf{N}^m$ with $0 \le k < N^\star$ the pair (d_α, d_β) is distant for $\mathcal{M}$ and for an approximation function denoted by $\Omega_{\alpha\beta}$.

The diagonal matrix in $l^2(\mathbf{Z}^m)$ defined by $d \in \mathcal{D}_{N,N^\sharp}$ is denoted by D.

$\Psi_\alpha(\,\cdot\,)$ (resp. $\Psi_{\alpha\beta}(\,\cdot\,)$) is associated with $\Omega_\alpha(\,\cdot\,)$ (resp. $\Omega_{\alpha\beta}(\,\cdot\,)$) and set

$$\Psi(\sigma) = \sup\Big(\sup_\alpha \Psi_\alpha(\sigma), \sup_{\alpha,\beta} \Psi_{\alpha\beta}(\sigma)\Big), \; \sigma \ge 0.$$

We then have the following extension of theorem A of [P].

Theorem 1 *Let D be a diagonal matrix associated with $d \in \mathcal{D}_{N,N^\sharp}$.*

Set

$$\rho = \frac{s}{1 + \inf\limits_{1 \leq i < m} (N_i - N_i^\sharp)}, \;\; 0 < s < \infty$$

and $\rho = \infty$ if $s = \infty$.

Let $P \in M^\rho \cap \mathcal{A}$ such that

$$\|P\|_{M^\rho} < \lambda \Psi(\frac{\sigma}{2})^{-1}$$

where $0 < \sigma \leq \inf(1, s/2)$ and $\lambda > 0$ depends only on m and N.

Then there exists a diagonal matrix $\hat{D}$ and an invertible matrix V such that

$$V^{-1}(P + \hat{D})V = D. \tag{4}$$

Remark 1.

In theorem 1 it possible to consider the given sequence $d = (d_i)_{i \in \mathbf{Z}^m}$ with each d_i at most repeated $\pi_{N^\star}$ times. Define

$$I_k = \Big\{ \alpha \in \mathbf{N}^m, \; 0 \leq \alpha - kN^\sharp < N^\sharp \Big\}$$

where $0 \leq k < N^\star$.

If $k \neq k'$, $I_k \cap I_{k'} = \emptyset$. Thus the pair $(\alpha, \beta) \in I_k \times I_{k'}$ with $k \neq k'$ is not concerned with hypothesis (ii) that needs not to be checked. Therefore, it is possible to choose $d_\alpha = d_\beta$ and more generally it possible to choose

$$d_{\alpha^1} = d_{\alpha^2} = \;\cdots\; = d_{\alpha^{\pi_{N^\star}}},$$

for any $(\alpha^1, \alpha^2, \cdots, \alpha^{\pi_{N^\star}}) \in I_{k^1} \times I_{k^2} \times \cdots \times I_{k^{\pi_{N^\star}}}$ with $k^i \neq k^j$ for $i \neq j$.

Remark 2.

When $N = (1, \cdots, 1)$ no pair (α, β) is concerned with the hypothesis (ii) that needs not to be checked. $\alpha = (0, \cdots, 0)$ is the only m-integer concerned with the hypothesis (i). Thus $d_\alpha = d$ and theorem 1 is nothing else than the theorems of Craig and Pöschel(cf [C] and [P]).

Remark 3.

Given N it is with $N^\sharp = (1, \cdots, 1)$ that the chosen eigenvalues d_i can have the highest multiplicity which is always π_N by remark 2. In that case the perturbations are matrices with an infinite number of diagonals regularly separated by $N - (1, \cdots, 1)$ zero coefficients.

Remark 4.

In the situation described in remark 1 we can choose d_{α^i} dense in an interval or a Cantor set of measure zero as in [P]. Consequently the sequence d will be dense in the same interval or Cantor set.

3 Results for discretized systems.

In this section we use the Hilbert space $l^2(\mathbf{Z}^m)^N$ where N is a n-integer. u is in $l^2(\mathbf{Z}^m)^N$ if and only if $u = (u_\alpha)_{0 \le \alpha < N}$ where u_α is in $l^2(\mathbf{Z}^m)$ for every $n-$integer α.

A similar result to theorem 1 can be stated for discrete operators on $l^2(\mathbf{Z}^m)^N$. Such an operator is a matrix $Q = (Q_{\alpha\beta})_{0 \le \alpha, \beta < N}$ such that

$$Qu = \left(\sum_{0 \le \beta < N} Q_{\alpha\beta} u_\beta \right)_{0 \le \alpha < N} \in l^2(\mathbf{Z}^m)^N$$

for every $u = (u_\alpha)_{0 \le \alpha N}$ in $l^2(\mathbf{Z}^m)^N$.

Set

$$\Lambda_N = \Big\{ Q = (Q_{\alpha\beta})_{0 \le \alpha, \beta < N} ;\ Q_{\alpha\beta} \in M \text{ for every } (\alpha, \beta) \Big\}.$$

and

$$\Delta_{N,N^\sharp} = \Big\{ (\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n ;\ 0 \le \alpha, \beta < N \text{ and } \exists k \in \mathbf{N}^n \text{ with } 0 \le k < N^\star$$

$$\text{such that } 0 \le \alpha - kN^\sharp, \beta - kN^\sharp < N^\sharp \Big\}$$

We then set

$$\Lambda_{N,N^\sharp} = \Big\{ Q \in \Lambda_N;\ Q_{\alpha\beta} = 0 \text{ for } (\alpha,\beta) \notin \Delta_{N,N^\sharp} \Big\}.$$

Let $\| \cdot \|_{N\times N}$ be a norm on the space of real matrices $R = (R(i,j))_{0\le i,j<N}$, $i, j \in \mathbf{N}^n$ satisfying

$$\|RS\|_{N\times N} \le \|R\|_{N\times N} \|S\|_{N\times N}.$$

We then set

$$\|Q\|_{\Lambda^s_N} = \Big\| \left(\|Q_{\alpha\beta}\|_{M^s} \right)_{0\le\alpha,\beta<N} \Big\|_{N\times N}.$$

where $0 < s \le \infty$ and define the following Banach spaces

$$\Lambda^s_N = \{ Q \in \Lambda_N,\ \|Q\|_{\Lambda^s_N < \infty} \}$$

and

$$\Lambda^s_{N,N^\sharp} = \{ Q \in \Lambda_{N,N^\sharp},\ \|Q\|_{\Lambda^s_N < \infty} \}.$$

We finally introduce the space

$$\Lambda^D_N = \Big\{ Q \in \Lambda_N;\ Q_{\alpha\beta} = 0 \text{ for } \alpha \ne \beta \text{ and } Q_{\alpha\alpha} \text{ is diagonal for every } \alpha \Big\}.$$

When $Q \in \Lambda^D_N$ we say that Q is diagonal.

The set of real diagonal matrices that we consider is denoted by $\mathcal{E}_{N,N^\sharp}$. $\mathcal{E}_{N,N^\sharp}$ is the set of diagonal matrices $D \in \Lambda^D_N$ such that

(i) for every $\alpha \in \mathbf{N}^m$ with $0 \le \alpha < N$ the sequence $d_\alpha = (D_{\alpha\alpha}(i,i))_{i\in\mathbf{Z}^m}$ is distal for $\mathcal{M}$ and for some approximation function $\Omega_\alpha(\,\cdot\,)$.

(ii) for every $(\alpha,\beta) \in \Delta_{N,N^\sharp}$ the pair (d_α, d_β) is distant for $\mathcal{M}$ and for some approximation function $\Omega_{\alpha\beta}(\,\cdot\,)$.

There are π_N subsequences d_α.

As in section 2 $\Psi_\alpha(\,\cdot\,)$ (resp. $\Psi_{\alpha\beta}(\,\cdot\,)$) is associated with $\Omega_\alpha(\,\cdot\,)$ (resp. $\Omega_{\alpha\beta}(\,\cdot\,)$) and

$$\Psi(\sigma) = \sup\Big(\sup_\alpha \Psi_\alpha(\sigma), \sup_{\alpha,\beta} \Psi_{\alpha\beta}(\sigma)\Big),\ \sigma \geq 0.$$

Theorem 2 *Let D be a diagonal matrix in $\mathcal{E}_{N,N^\sharp}$. Let Q be in $\Lambda^s_{N,N^\sharp}$ such that*

$$\|Q\|_{\Lambda^s_{N,N^\sharp}} < \delta\Psi(\frac{\sigma}{2})^{-1}$$

where $0 < \sigma \leq \inf(1, \frac{s}{2})$ and δ is a constant depending only on m and N.

Then there exists a diagonal matrix $\hat{D}$ and an invertible matrix V such that

$$V^{-1}(Q + \hat{D})V = D. \tag{5}$$

4 The canonical map.

In this section $n = m$ and $N, N^\sharp, N^\star = N/N^\sharp$ are m-integers. We now consider the $(\pi_N)^2$ submatrices of $P \in M$ defined by

$$P_{\alpha\beta} = (P(\alpha + iN, \beta + jN))_{i,j\in\mathbf{Z}^m}$$

where $0 \leq \alpha, \beta < N$. $P \in M$ if and only if $P_{\alpha\beta} \in M$.

Using these submatrices we define an another norm on the space M depending on the two parameters N and s.

We set

$$\|P\|_{N,s} = \Big\|\, (\|P_{\alpha\beta}\|_{M^s})_{0\leq\alpha,\beta<N} \,\Big\|_{N\times N}. \tag{6}$$

Let

$$L^s_{N,N^\sharp} = \Big\{P \in \Lambda_N;\ \|P\|_{N,s} < \infty \text{ and } P_{\alpha\beta} = 0 \text{ if } (\alpha,\beta) \notin \Delta_{N,N^\sharp}\Big\}.$$

By definition the canonical map is a map from $L^s_{N,N^\sharp}$ into $\Lambda^s_{N,N^\sharp}$ defined for $P \in L^s_{N,N^\sharp}$ by

$$\mu(P) = Q = (Q_{\alpha\beta})_{0\leq\alpha,\beta<N}$$

where

$$Q_{\alpha\beta} = (P(\alpha + iN, \beta + jN))_{i,j\in\mathbf{Z}^m}\,.$$

The norm on $L^s_{N,N^\sharp}$ defined by (6) is the norm on $\Lambda^s_{N,N^\sharp}$ induced by μ. Using the notations $P = (P_{\alpha\beta})_{0\leq\alpha,\beta<N}$ where the $P_{\alpha\beta}$'s are the submatrices of P and $Q = (Q_{\alpha\beta})_{0\leq\alpha,\beta<N}$ where the $Q_{\alpha\beta}$'s are matrices coefficients of Q, the map μ is just, in terms of $(P_{\alpha\beta})_{\alpha\beta}$ and $(Q_{\alpha\beta})_{\alpha\beta}$, the identity map.

In other words, with an infinite matrix $P \in L^s_{N,N^\sharp}$, μ associates a finite matrix $Q \in \Lambda \in L^s_{N,N^\sharp}$ with the submatrices of P as coefficients.

We then have

Lemma 1 *The map μ is a unitary linear isomorphism between $L^s_{N,N^\sharp}$ and $\Lambda^s_{N,N^\sharp}$.*

Let us recall that the perturbations in theorem 1 are taken in $\mathcal{A} \cap M^\rho$.

We then have

Lemma 2 *The embedding $\mathcal{A} \cap M^\rho \hookrightarrow L^s_{N,N^\sharp}$ is continuous and μ is continuous map from $\mathcal{A} \cap M^\rho$ into $\Lambda^s_{N,N^\sharp}$.*

Proof of lemma 2: It is easy to check that

$$P \in \mathcal{A} \text{ if and only if } P_{\alpha\beta} = 0,\ \forall\,(\alpha,\beta) \notin \Delta_{N,N^\sharp}.$$

Then it is sufficient to show that

$$\|P\|_{N,s} \leq c\|P\|_{M^\rho} \tag{7}$$

where c depends only on N.

First observe that

$$(P_{\gamma+lN})_{\alpha+kN} = ((P_{\alpha,\alpha+\gamma})_l)_k \tag{8}$$

for all $k, l \in \mathbf{Z}^m$ and all $\alpha, \gamma \in \mathbf{N}^m$ with $0 \leq \alpha, \alpha+\gamma < N$.

Indeed

$$P_{\alpha,\alpha+\gamma}(i,j) = P(\alpha + iN, \alpha+\gamma+jN)$$

thus

$$((P_{\alpha,\alpha+\gamma})_l)_k = (P_{\alpha,\alpha+\gamma}(i, i+l))_{i=k}$$

$$= (P(\alpha+iN, \alpha+\gamma+(i+l)N))_{i=k} = P(\alpha+kN, \alpha+\gamma+(k+l)N)$$

and

$$(P_{\gamma+lN})_{\alpha+kN} = P(\alpha+kN, \alpha+kN+\gamma+lN).$$

Moreover

$$\begin{aligned}
\|P\|_{M^\rho} &= \sup_{l\in\mathbf{Z}^m} \|P_l\|_\infty e^{|l|\rho} \\
&= \sup_{l\in\mathbf{Z}^m} \sup_{k\in\mathbf{Z}^m} |(P_l)_k| e^{|l|\rho} \\
&= \sup_{l\in\mathbf{Z}^m} \sup_{0\leq\beta<N^\sharp} \sup_{k\in\mathbf{Z}^m} |(P_l)_{\alpha+kN}| e^{|l|\rho} \\
&= \sup_{l\in\mathbf{Z}^m} \sup_{(\alpha,\alpha+\gamma)\in\Delta_{N,N^\sharp}} \sup_{k\in\mathbf{Z}^m} |(P_{\gamma+lN})_{\alpha+kN}| e^{|\gamma+lN|\rho}.
\end{aligned} \tag{9}$$

By (8) and (9) we get

$$\begin{aligned}
\|P\|_{M^\rho} &= \sup_{(\alpha,\alpha+\gamma)\in\Delta_{N,N^\sharp}} \sup_{k,l\in\mathbf{Z}^m} |\left((P_{\alpha,\alpha+\gamma})_l\right)_k| e^{|\gamma+lN|\rho} \\
&= \sup_{(\alpha,\alpha+\gamma)\in\Delta_{N,N^\sharp}} \sup_{l\in\mathbf{Z}^m} \|(P_{\alpha,\alpha+\gamma})_l\|_\infty e^{|\gamma+lN|\rho}.
\end{aligned} \tag{10}$$

Remark that $(\alpha, \alpha+\gamma) \in \Delta_{N,N^\sharp}$ gives $0 \leq \gamma < N^\sharp$. Now suppose that the m-integers γ, N and $N^\sharp$ are fixed with $0 \leq \gamma < N^\sharp$.

$$\begin{aligned}|\gamma + lN| &= \sum_{i \text{ with } l_i \geq 0} |l_i N_i + \gamma_i| + \sum_{i \text{ with } l_i < 0} |l_i N_i + \gamma_i| \\ &\geq \inf_{1 \leq i \leq m} (N_i - N_i^\sharp) \sum_{i \text{ with } l_i \geq 0} |l_i| + \sum_{i \text{ with } l_i < 0} |l_i| N_i - \gamma_i. \end{aligned} \tag{11}$$

Using $0 \leq \gamma_i \leq N_i^\sharp - 1$ and $|l_i| \geq 1$ for the second term in the r.h.s. of (11), we obtain

$$|\gamma + lN| \geq |l| \left(1 + \inf_{1 \leq i \leq m} (N_i - N_i^\sharp)\right). \tag{12}$$

Therefore by (10) and (12)

$$\begin{aligned}\|P\|_{M^\rho} &\geq \sup_{(\alpha,\alpha+\gamma)\in\Delta_{N,N^\sharp}} \sup_{l \in \mathbf{Z}^m} \|(P_{\alpha,\alpha+\gamma})_l\|_\infty e^{|l|s} \\ &\geq \sup_{(\alpha,\beta)\in\Delta_{N,N^\sharp}} \sup_{l \in \mathbf{Z}^m} \|(P_{\alpha,\beta})_l\|_\infty e^{|l|s} \\ &\geq \sup_{(\alpha,\beta)\in\Delta_{N,N^\sharp}} \|P_{\alpha,\beta}\|_{M^s} \\ &\geq c_N \|P\|_{N,s} \end{aligned} \tag{13}$$

where $c_N > 0$ depends only on N. This proves (7). □

Remark 5.

It can be proved similarly that the embedding

$$L^s_{N,N^\sharp} \hookrightarrow \mathcal{A} \cap M^{s/\left(1+\inf_{1\leq i\leq m}(N_i+N_i^\sharp)\right)}$$

is continuous.

Lemma 3 *(i)* $\mu(P_1P_2) = \mu(P_1)\mu(P_2)$ *for* $P_1, P_2 \in M$.
(ii) $\mu^{-1}(Q_1Q_2) = \mu^{-1}(Q_1)\mu^{-1}(Q_2)$ *for* $Q_1, Q_2 \in \Lambda_N$.

Proof of lemma 3: We have $(\mu(P))_{\alpha\beta}(i,j) = P(\alpha+iN, \beta+jN)$. Then $\mu(P_1)\mu(P_2)_{\alpha\beta} = \sum_{0\le\gamma<N} (\mu(P_1))_{\alpha\gamma} (\mu(P_2))_{\gamma\beta}$. Thus

$$\left((\mu(P_1)\mu(P_2))_{\alpha\beta}\right)(i,j) = \sum_{0\le\gamma<N} \sum_{k\in\mathbf{Z}^m} (\mu(P_1))_{\alpha\gamma}(i,k)\,(\mu(P_2))_{\gamma\beta}(k,j)$$

$$= \sum_{0\le\gamma<N} \sum_{k\in\mathbf{Z}^m} P_1(\alpha+iN, \gamma+kN) P_2(\gamma+kN, \beta+jN). \tag{14}$$

Moreover

$$\begin{aligned}
(\mu(P_1P_2))_{\alpha\beta}(i,j) &= (P_1P_2)(\alpha+iN, \beta+jN) \\
&= \sum_{k\in\mathbf{Z}^m} P_1(\alpha+iN, k) P_2(k, \beta+jN) \\
&= \sum_{0\le\gamma<N} \sum_{k\in\mathbf{Z}^m} P_1(\alpha+iN, \gamma+kN) P_2(\gamma+kN, \beta+jN).
\end{aligned} \tag{15}$$

Then (i) follows from (14) and (15) and (ii) follows from (i). □

By lemma 2 and lemma 3 it is easy to verify that the proof of theorem 1 follows from the one of theorem 2. If $P \in \mathcal{A} \cap M^\rho$ with $\|P\|_{M^\rho} \le c\Psi(\frac{\sigma}{2})^{-1}$ and $D \in \mathcal{D}_{N,N^\sharp}$ then $\mu(P) \in \Lambda^s_{N,N^\sharp}$ with $\|\mu(P)\|_{\Lambda^s_{N,N^\sharp}} \le c'\Psi(\frac{\sigma}{2})^{-1}$ and $\mu(D) \in \mathcal{E}_{N,N^\sharp}$. From theorem 2 there exist $V, \hat{D}$ such that $V^{-1}(\mu(P)+\hat{D})V = \mu(D)$. Then $\mu^{-1}(V^{-1})(P+\mu^{-1}(\hat{D}))\mu^{-1}(V) = D$.

Thus we restrict ourselves to the proof of theorem 2. For that we use a KAM procedure.

5 Proof of theorem 2.

We first prove several lemmas. In this section $N, N^\sharp$ and $N^\star = N/N^\sharp$ are n-integers with n not necessarily equals to m.

Lemma 4 *$\Lambda^s_{N,N^\sharp}$ is an algebra.*

Proof of lemma 4:

For $Q \in \Lambda_N$ we define the $\pi_{N^\star}$ block matrices of Q by

$$Q^{\eta\xi} = \left(Q_{\alpha+\eta N^\sharp, \beta+\xi N^\sharp}\right)_{0\le\alpha,\beta<N^\sharp}$$

for every $\eta, \xi \in \mathbf{N}^m$ with $0 \le \eta, \xi < N^\star$.

Each $Q^{\eta\xi}$ is then an element of $\Lambda_{N^\sharp}$. For $Q, \tilde{Q} \in \Lambda_N$ we have

$$(Q\tilde{Q})^{\eta\xi} = \sum_{0\le\omega<N^\star} Q^{\eta\omega}\tilde{Q}^{\omega\xi}.$$

Moreover

$$Q \in \Lambda_{N,N^\sharp} \text{ if and only if } Q^{\eta\xi} = 0 \text{ if } \eta \ne \xi.$$

Therefore, when $Q, \tilde{Q} \in \Lambda_{N,N^\sharp}$

$$(Q\tilde{Q})^{\eta\xi} = \begin{cases} Q^{\eta\eta}\tilde{Q}^{\xi\xi} & \text{if } \eta = \xi \\ 0 & \text{if } \eta \ne \xi \end{cases}$$

which completes the proof. $\square$

Lemma 5 *Let $X \in \Lambda^{s-\sigma}_{N,N^\sharp}$ with $0 < \sigma \le s$ and $Y \in \Lambda^s_{N,N^\sharp}$. Then $XY \in \Lambda^{s-\sigma}_{N,N^\sharp}$ and*

$$\|XY\|_{\Lambda^{s-\sigma}_{N,N^\sharp}} \le \frac{b}{\sigma^m}\|X\|_{\Lambda^{s-\sigma}_{N,N^\sharp}}\|Y\|_{\Lambda^s_{N,N^\sharp}} \tag{16}$$

where the constant $b \ge 1$ depends only on m.

Proof of Lemma 5: $XY \in \Lambda_{N,N^\sharp}$ from lemma 4. Moreover

$$\|(XY)_{\alpha\beta}\|_{M^{s-\sigma}} \le \frac{b}{\sigma^m}\sum_{0\le\gamma<N}\|X_{\alpha\gamma}\|_{M^{s-\sigma}}\|Y_{\gamma\beta}\|_{M^s}. \tag{17}$$

as it follows from [P, lemma 4.1].

Define the $N \times N$ real matrices

$$x = (x_{\alpha\beta})_{0\le\alpha,\beta<N} = (\|X_{\alpha\beta}\|_{M^{s-\sigma}})_{0\le\alpha,\beta<N}$$

and

$$y = (y_{\alpha\beta})_{0\le\alpha,\beta<N} = (\|Y_{\alpha\beta}\|_{M^s})_{0\le\alpha,\beta<N}\,.$$

Then

$$0 \le \|(XY)_{\alpha\beta}\|_{M^{s-\sigma}} \le \frac{b}{\sigma^m}(xy)_{\alpha\beta}. \tag{18}$$

The proof is completed by computing the $\|\cdot\|_{N\times N}$ norm of $(\|(XY)_{\alpha\beta}\|_{M^{s-\sigma}})_{0\le\alpha,\beta<N}$ and using (18). □

Lemma 6 *Let $X \in \Lambda^s_{N,N^\sharp}$ such that $\|X-I\|_{\Lambda^s_{N,N^\sharp}} \le \dfrac{\sigma^m}{b}$ for $0<\sigma\le 1$. Then X is invertible in $\Lambda^{s-\sigma}_{N,N^\sharp}$ and*

$$\|X^{-1}-I\|_{\Lambda^{s-\sigma}_{N,N^\sharp}} \le \left(1-\frac{b}{\sigma^m}\|X-I\|_{\Lambda^s_{N,N^\sharp}}\right)^{-1}\|X-I\|_{\Lambda^s_{N,N^\sharp}} \tag{19}$$

Proof of Lemma 6: Remark that if $X\in\Lambda_{N,N^\sharp}$ is invertible in Λ_N, it is invertible in $\Lambda_{N,N^\sharp}$. The rest of proof is routine taking the Neumann series for X^{-1} and using lemma 5 for the estimations. □

For $P\in\Lambda_{N,N^\sharp}$ the diagonal of P is denoted by $[\![P]\!]$

$$([\![P]\!])_{\alpha\beta} = 0 \text{ for } \alpha\ne\beta \text{ and } ([\![P]\!])_{\alpha\alpha} = diagonal(P_{\alpha\alpha}).$$

Lemma 7 *Let $V, V^{-1}\in\Lambda^\sigma_{N,N^\sharp}$ with $\|V-I\|_{\Lambda^\sigma_{N,N^\sharp}}, \|V^{-1}-I\|_{\Lambda^\sigma_{N,N^\sharp}} \le \dfrac{\sigma^m}{2b}$ for $0\le\sigma\le 1$. Then for any $P\in\Lambda_{N,N^\sharp}$ there exists a unique solution $A\in\Lambda^D_N$ to*

$$[\![P+V^{-1}AV]\!] = 0. \tag{20}$$

This solution A satisfies

$$\|A\|_{\Lambda^D_N} \le 2\|[\![P]\!]\|_{\Lambda^D_N}$$

Lemma 7 is a straightforward modification of lemma 4.3 of [P].

Proof of Lemma 7: We have $A - V^{-1}AV = (V^{-1} - I)[V - I, A] + [V - I, A]$. Remark that $[\![[A, B]]\!] = 0$ if $A \in \Lambda_N^D$ and $B \in \Lambda_N$. Therefore $[\![A - V^{-1}AV]\!] = A - [\![V^{-1}AV]\!] = [\![(V^{-1} - I)[V - I, A]]\!]$.

For any $A \in \Lambda_N^D$ we get

$$\|A - [\![V^{-1}AV]\!]\|_{\Lambda_N^D} = \|[\![(V^{-1} - I)[V - I, A]]\!]\|_{\Lambda_N^D}$$

$$\leq \|[\![(V^{-1} - I)[V - I, A]]\!]\|_{\Lambda_{N,N}^0} \leq \frac{1}{2}\|A\|_{\Lambda_{N,N}^0}$$

$$\leq \frac{1}{2}\|A\|_{\Lambda_N^D}$$

The above inequalities follow from lemma 5 and the assumptions on V, V^{-1}. Thus for or any $P \in \Lambda_N$ the map $A \mapsto A - [\![V^{-1}AV]\!]$ is a $\frac{1}{2}$ contraction in Λ_N^D. $\square$

Lemma 8 *Suppose $D \in \mathcal{E}_{N,N^\sharp}$.*

Then for any $P \in \Lambda_{N,N^\sharp}$ the equation $[D, W] + P - [\![P]\!] = 0$ has a unique solution $W \in \Lambda_{N,N^\sharp}$ with $[\![W]\!] = 0$.

Furthermore if $P \in \Lambda_{N,N^\sharp}^s$ and $0 < \sigma \leq s$ then $W \in \Lambda_{N,N^\sharp}^{s-\sigma}$ with

$$\|W\|_{\Lambda_{N,N^\sharp}^{s-\sigma}} \leq \sup_{(\alpha,\beta)\in\Delta_{N,N^\sharp}} \phi_{\alpha\beta}(\sigma)\|P\|_{\Lambda_{N,N^\sharp}^s}$$

where $\phi_{\alpha\beta}(\sigma) = \sup_{r\geq 0} \Omega_{\alpha\beta}(r)e^{-\sigma r} = \sigma^{4m}\Phi_{\alpha\beta}(\sigma)$.

Proof of Lemma 8: The unique solution $W = (W_{\alpha\beta})_{0\leq\alpha,\beta<N}$ with $[\![W]\!] = 0$ is

$$W_{\alpha\beta} = 0 \text{ for } (\alpha, \beta) \notin \Delta_{N,N^\sharp},$$

$$(W_{\alpha\beta})_k = (D_\alpha - T_k D_\beta)^{-1} (P_{\alpha\beta})_k\,,\ \forall\, k \in \mathbf{Z}^m,\ \text{ for } (\alpha, \beta) \in \Delta_{N,N^\sharp} \text{ with } \alpha \neq \beta \tag{21}$$

and

$$(W_{\alpha\alpha})_k = \begin{cases} 0 \ if\ k = 0 \\ (D_\alpha - T_k D_\alpha)^{-1} (P_{\alpha\alpha})_k\,,\ \forall\ 0 \neq k \in \mathbf{Z}^m. \end{cases} \tag{22}$$

Then

$$\begin{aligned}\|\,(W_{\alpha\beta})_k\,\|_\infty &\le \Omega_{\alpha\beta}(|k|)\,e^{-|k|s}\|P_{\alpha\beta}\|_{M^s}\\ &\le \phi_{\alpha\beta}(\sigma)\,e^{-(s-\sigma)|k|}\|P_{\alpha\beta}\|_{M^s},\end{aligned}$$

for every $0 \le \alpha, \beta < N$ and every $k \in \mathbf{Z}^m$.

Consequently

$$\left\|(\|\,(W_{\alpha,\beta})\,\|_{M^s})_{0\le\alpha,\beta<N}\right\|_{N\times N} \le \left\|(\phi_{\alpha\beta}(\sigma)\|P_{\alpha\beta}\|_{M^s})_{0\le\alpha,\beta<N}\right\|_{N\times N}.$$

□

We now write the steps of the scheme as in [P] in order to prove theorem 2.

Suppose $D, \hat{D} \in \Lambda_N^D$. Writing $\hat{D} - D = \sum_{\nu=1}^{\infty} A_\nu$ and $D_n = \sum_{\nu=n}^{\infty} A_\nu$, we will prove that for every positive n there exist $V_n, P_n \in \Lambda_{N,N^\sharp}$ such that

$$V_n^{-1}(\hat{D} + P)V_n = D + P_n + V_n^{-1} D_n V_n$$

with $P_n \to 0$ and $V_n \to V$. Then as n tends to ∞ we get

$$V^{-1}(\hat{D} + P)V = D. \tag{23}$$

For $n = 0$, $P_0 = P$ and $V_0 = I$. To obtain $P_{n+1}, V_{n+1} \in \Lambda_{N,N^\sharp}$ from $P_n, V_n \in \Lambda_{N,N^\sharp}$, first get $A_n \in \Lambda_N^D$ as the solution to

$$[\![P_n + V^{-1} A_n V_n]\!] = 0$$

by using lemma 7. Next obtain $W_n \in \Lambda_{N,N^\sharp}$ as the solution to

$$[D, W_n - I] + P_n + V^{-1} A_n V_n = 0$$

by using lemma 7.

Set $V_{n+1} = V_n W_n$ and $P_{n+1} = W_n^{-1}(P_n + V^{-1} A_n V_n)(W_n - I)$. W_n is invertible in $\Lambda_{N,N^\sharp}$. Then $P_{n+1}, V_{n+1} \in \Lambda_{N,N^\sharp}$ as it follows from lemma 2.

In particular the series $\sum_{\nu=0}^{\infty} A_n$ converges in Λ_N^D and $\hat{D}$ is determined.

Furthermore for every $n \ge 1$ one has

$$\|P_n\|_{\Lambda_{N,N^\sharp}^{s_n}} \le \theta_n^{2^n} \tag{24}$$

$$\|V_n - V_{n-1}\|_{\Lambda^{s_n+\sigma_n}_{N,N^\sharp}}, \|V_n^{-1} - V_{n-1}^{-1}\|_{\Lambda^{s_n}_{N,N^\sharp}} \leq \sigma^{2m}\theta_\infty^{2^{n-1}}, \tag{25}$$

$$\|\hat{D} - D + [\![P]\!]\|_{\Lambda^D_N} \leq 3\theta_\infty^2. \tag{26}$$

Using θ_n and θ_∞ given by [P] we prove that the sequence $(P_n)_{n\geq 0}$ tends to zero in $\Lambda^{s-\sigma}_{N,N^\sharp}$ and the sequence $(V_n)_{n\geq 0}$ tends to an invertible operator V in $\Lambda^{s-\sigma}_{N,N^\sharp}$. Thus (23) is satisfied and this concludes the proof of theorem 2.

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