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Existence of the density of states for multi-dimensional continuum Schrödinger operators with Gaussian random potentials*

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Abstract: A Wegner estimate is proved for quantum systems in multi-dimensional Euclidean space which are characterized by one-particle Schrödinger operators with random potentials that admit a certain one-parameter decomposition. In particular, the Wegner estimate applies to systems with rather general Gaussian random potentials. As a consequence, these systems possess an absolutely continuous integrated density of states, whose derivative, the density of states, is locally bounded. An explicit upper bound is derived.

1. Introduction

The integrated density of states is a quantity of primary interest in the theory and applications of one-particle random Schrödinger operators [SE, BEE+, LGP, CL, PF]. For example, the topological support of the associated measure coincides with the almost-sure spectrum of the infinite-volume operator. Moreover, its knowledge allows to compute the free energy and hence all basic thermostatic quantities of the corresponding non-interacting many-particle system.

An important method to prove the absolute continuity of the integrated density of states, that is, the existence of the density of states, dates back to an idea of Wegner [W]. Nowadays it goes under the name Wegner estimate. Starting with the fundamental work [FS] Wegner-like estimates have also turned out to play a key rôle in rigorous proofs of Anderson localisation in multi-dimensional space, see e.g. [MH, KS, Sp, vDK, AM, CH, Kl, Ki3]. The so-called spectral-averaging technique [Ko1, Ko2, KS, CH, CHM, BCH, Si2] has been very efficient in deriving Wegner estimates for random Schrödinger operators in continuous space. To our knowledge however, if the random potential does not possess an underlying lattice structure the estimates obtained so far are not stringent enough as to deduce the absolute continuity

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of the integrated density of states. Our goal here is to use spectral averaging to prove this property for a truly continuum random potential. More precisely, in building on [CH] we will establish the absolute continuity of the integrated density of states for a Schrödinger operator with a rather general Gaussian random potential in d-dimensional Euclidean space $\mathbb{R}^d$ for all $d \geq 1$. As a by-product we get an explicit upper bound on the density of states. We would also like to remark that the present paper accomplishes one part of the localisation proofs announced in [FLM1] and [FHLM]. The remaining part is deferred to [FLM2].

This paper is organized as follows. In the next section we will formulate our main result. The final section shows that this result is a consequence of a Wegner estimate which is proved to hold for all continuum Schrödinger operators whose random potential admits a certain one-parameter decomposition.

2. Main result

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and let $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $(\omega, x) \mapsto V^{(\omega)}(x)$ be a random field. We list four properties which V may have or not.

(M) V is jointly measurable with respect to the sigma-algebra $\mathcal{A}$ and the Borel sigma-algebra of $\mathbb{R}^d$.

(F) For all bounded open (hyper-) cubes $\Lambda \subset \mathbb{R}^d$ the realisations of V are P-almost surely relatively form-bounded with respect to both $-\frac{1}{2}\Delta_{\Lambda,\mathrm{D}}$ and $-\frac{1}{2}\Delta_{\Lambda,\mathrm{N}}$ with a relative bound strictly smaller than 1. Here $\Delta_{\Lambda,\mathrm{D}}$, respectively $\Delta_{\Lambda,\mathrm{N}}$, denotes the Dirichlet Laplacian, respectively the Neumann Laplacian on the Hilbert space $\mathrm{L}^2(\Lambda)$ of complex-valued functions on Λ which are square-integrable over Λ with respect to the d-dimensional Lebesgue measure $d^d x$.

(E) V is ergodic with respect to translations in $\mathbb{R}^d$.

(G) V is a Gaussian random field which is homogeneous with respect to translations in $\mathbb{R}^d$. It has zero mean, $\int_\Omega dP(\omega)\, V^{(\omega)}(0) = 0$, and its covariance function $x \mapsto C(x) := \int_\Omega dP(\omega)\, V^{(\omega)}(x) V^{(\omega)}(0)$ is continuous at the origin where it obeys $0 < C(0) < \infty$.

Remark 1. Consider a random field V with the Gaussian property (G). Then by Bochner's theorem its covariance function C is bounded and uniformly continuous on $\mathbb{R}^d$. Consequently, [F, Thm. 3.2.2] implies the existence of a separable version of V which has the measurability property (M). From now on it is tacitly asssumed that only this version will be dealt with when we refer to a random field with the property (G). We also remark that (G) implies the form-boundedness property (F). This may be checked with the help of [KM2, Lemma 2.1], since V is P-almost surely in the Lebesgue space $\mathrm{L}^p(\Lambda)$ for any $1 \leq p < \infty$ as follows from

$$\int_\Omega dP(\omega) \int_\Lambda d^d x\, |V^{(\omega)}(x)|^p = \frac{|\Lambda|}{\sqrt{2\pi C(0)}} \int_{\mathbb{R}} dv\, \mathrm{e}^{-v^2/2C(0)}\, |v|^p < \infty\,, \tag{1}$$

where we have used Fubini's theorem and the homogeneity of V. Here $|\Lambda|$ denotes the volume of Λ measured by $d^d x$. A simple sufficient criterion for a Gaussian random field to have the ergodicity property (E) is provided by the mixing condition $\lim_{|x|\to\infty} C(x) = 0$.

Definition 1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded open cube and let V be a random field having the properties* (M) *and* (F). *Then the finite-volume Schrödinger operator*

$$H_{\Lambda,\mathrm{X}}(V) := -\tfrac{1}{2}\,\Delta_{\Lambda,\mathrm{X}} + V\ , \qquad \mathrm{X} = \mathrm{D} \text{ or } \mathrm{X} = \mathrm{N}\ , \tag{2}$$

with random potential V is defined on $\mathrm{L}^2(\Lambda)$ as a sum of quadratic forms. Here V acts as a multiplication operator.

Remark 2. It follows from [KM1] and [KM2] that $H_{\Lambda,\mathrm{X}}(V)$ is measurable and P-almost surely self-adjoint. Its spectrum is P-almost surely discrete and therefore its finite-volume integrated density of states

$$N_{\Lambda,\mathrm{X}}(E) := \mathrm{Tr}\, F_{\Lambda,\mathrm{X}}\big(]-\infty, E[\big) \tag{3}$$

exists P-almost surely. Here Tr denotes the trace over the Hilbert space $\mathrm{L}^2(\Lambda)$ and $F_{\Lambda,\mathrm{X}}$ the spectral family of $H_{\Lambda,\mathrm{X}}(V)$. Consequently, $N^{(\omega)}_{\Lambda,\mathrm{X}}(E)$ is just the number of eigenvalues of $H_{\Lambda,\mathrm{X}}(V^{(\omega)})$, counting multiplicity, which are strictly smaller than E.

After these preparations and writing χ_B for the indicator function of a Borel subset $B \subset \mathbb{R}^d$, we now state a Wegner estimate for $N_{\Lambda,\mathrm{X}}$ in

Theorem 1. *Let $H_{\Lambda,\mathrm{X}}(V)$ be defined as in* (2) *with a random field V having the Gaussian property* (G). *Moreover, assume that there exist a finite signed Borel measure μ on $\mathbb{R}^d$, a cube $\Gamma \subset \mathbb{R}^d$ with $|\Gamma| > 0$ and a constant $\gamma > 0$ such that the covariance function obeys*

$$\int_{\mathbb{R}^d} d\mu(y)\, C(x-y) \geq \gamma\, C(0)\chi_\Gamma(x) \tag{4}$$

for all $x \in \mathbb{R}^d$.

Then for every energy $E \in \mathbb{R}$ there is a constant $0 < W(E) < \infty$, independent of Λ and X, *such that the finite-volume integrated density of states* (3) *obeys*

$$\int_\Omega dP(\omega)\left| N^{(\omega)}_{\Lambda,\mathrm{X}}(E_1) - N^{(\omega)}_{\Lambda,\mathrm{X}}(E_2)\right| \leq |\Lambda|\, |E_1 - E_2|\, W(E) \tag{5}$$

for all $E_1, E_2 \leq E$ and all bounded open cubes $\Lambda \subset \mathbb{R}^d$ with $|\Lambda| \geq |\Gamma|$.

The proof of Theorem 1 will be deferred to the next section.

Remarks 3. (i) In what follows it will be assumed, without loss of generality, that the measure μ is normalized according to

$$\int_{\mathbb{R}^d} d\mu(x) \int_{\mathbb{R}^d} d\mu(y)\; C(x-y) = C(0)\,. \tag{6}$$

(ii) If $C(x) \geq 0$ for all $x \in \mathbb{R}^d$, one may simply choose the signed measure μ to be Dirac's point measure at the origin. Due to the continuity of C and since $C(0) > 0$, Condition (4) is then fulfilled with some sufficiently small cube Γ containing the origin and $\gamma = \inf_{x\in\Gamma} C(x)/C(0)$. With other choices of μ it is also possible to satisfy (4) for certain covariance functions which take on negative values. An example for the case $d = 1$ is given by $C(x) := \int_{\mathbb{R}} dy\, w(x+y)w(y)$ with $w := \chi_{[-3,3]} - \frac{5}{4}\chi_{[-1,1]}$. A suitable choice is $d\mu(x) = m\,\chi_{[-6,6]}(x)dx$, $\Gamma =]-1,1[$ and $\gamma = 8m$, where $m > 0$ is determined by the normalization (6).

(iii) The Lipschitz continuity (5) of the averaged finite-volume integrated density of states implies by the Chebyshev-Markov inequality that the probability of finding a given energy E_0 near the spectrum of $H_{\Lambda,\mathrm{X}}(V)$ is controlled by the inequality

$$P\left(\left\{\omega : \operatorname{dist}\Big(\operatorname{spec}\big(H_{\Lambda,\mathrm{X}}(V^{(\omega)})\big), E_0\Big) < \varepsilon\right\}\right) \leq |\Lambda|\, \varepsilon\, W(E)\,. \tag{7}$$

It is valid for all bounded open cubes $\Lambda \subset \mathbb{R}^d$ with $|\Lambda| \geq |\Gamma|$ and for all energies $E_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $E_0 + \varepsilon \leq E$. Actually it is (7) – or some weakened form of (7), where, for example, $|\Lambda|$ is replaced by $|\Lambda|^\alpha$ with $\alpha > 1$ – that plays a key rôle in proving Anderson localisation for multi-dimensional random Schrödinger operators.

(iv) Theorem 1 can be generalised to situations where the random potential is Gaussian but not $\mathbb{R}^d$-homogeneous, the extension being essentially a matter of notation. This allows in particular to treat correlated Gaussian alloy-type potentials.

Now we assume that the random field V has the ergodicity property (E) in addition to the Gaussian property (G). Due to this and since $\int_\Omega dP(\omega)\ \exp\{-tV^{(\omega)}(0)\} = \exp\{t^2 C(0)/2\} < \infty$ for all $t > 0$, standard techniques show the existence of a non-random left-continuous distribution function N on $\mathbb{R}$, called the integrated density of states (per volume) in the macroscopic limit, such that

$$N(E) = \lim_{\Lambda\uparrow\mathbb{R}^d} \frac{N^{(\omega)}_{\Lambda,\mathrm{X}}(E)}{|\Lambda|}\,. \tag{8}$$

More precisely, there is a set $\Omega_0 \in \mathcal{A}$ of full probability, $P(\Omega_0) = 1$, such that (8) holds for all $\omega \in \Omega_0$, for both boundary conditions X and for all $E \in \mathbb{R}$ except for the at most countably many discontinuity points of N, see [KM2, Ki1] or [Ki2, Sec. 7.3]. The non-randomness of N and Fatou's lemma imply

$$|N(E_1) - N(E_2)| \leq \liminf_{\Lambda\uparrow\mathbb{R}^d} |\Lambda|^{-1} \int_\Omega dP(\omega) \left| N^{(\omega)}_{\Lambda,\mathrm{X}}(E_1) - N^{(\omega)}_{\Lambda,\mathrm{X}}(E_2) \right| \tag{9}$$

for all those $E_1, E_2 \in \mathbb{R}$ for which (8) is true. Accordingly, Theorem 1 and the left-continuity of N yield the Lipschitz continuity of N, namely

$$|N(E_1) - N(E_2)| \leq |E_1 - E_2|\, W\big(\max\{E_1, E_2\}\big) \tag{10}$$

for all $E_1, E_2 \in \mathbb{R}$. These arguments are summarized in

Corollary 1. *Under the assumptions of Theorem* 1 *and supposing property* (E)*, the integrated density of states in the macroscopic limit is absolutely continuous on any bounded interval and its derivative, the density of states, is locally bounded in the sense that*

$$0 \leq \frac{dN(E)}{dE} \leq W(E) \tag{11}$$

for Lebesgue-almost all $E \in \mathbb{R}$.

Remark 4. We infer from the proof of Theorem 1, which is presented in the next section, that the Wegner constant may be taken as

$$W(E) = \frac{\exp\{\beta E + \beta^2 C_E/2\}}{\sqrt{2\pi C(0)}\,\gamma} \left(2\ell_E^{-1} + (2\pi\beta)^{-1/2}\right)^d, \tag{12}$$

where $\beta > 0$ is arbitrary and may be considered as a variational parameter. Furthermore, $C_E := C(0)(1 + B_E^2 - b_E^2)$ and the constants ℓ_E, B_E and b_E are defined below by (16), (21) and (22), respectively. To get (12) we have made use of the normalization (6). For the simple (but not optimal) choice $\beta = (2C_E)^{-1}\big(- E + \sqrt{E^2 + 2C_E/\pi}\big)$ one obtains the leading asymptotic low- and high-energy behaviour

$$\lim_{E\to-\infty} \frac{\ln W(E)}{E^2} = -\frac{1}{2C(0)}, \qquad (13)$$

$$\lim_{E\to\infty} \frac{W(E)}{E^{d/2}} = \frac{3^d\,\mathrm{e}^{1/2\pi}}{\sqrt{2\pi C(0)}\,\gamma}, \qquad (14)$$

which coincides with the known asymptotics [PF, Thms. 5.29, 9.3] of N except for the value of the constant on the right-hand side of (14).

3. A Wegner estimate for continuum Schrödinger operators

In this section we prove Theorem 1 by tracing it back to a Wegner estimate that holds for a wider class of random fields than the ones considered so far. The price one has to pay for the increased generality is a couple of more technical notions. From now on the random field V will only be required to fulfil the measurability property (M) and the form-boundedness property (F).

We follow the line of reasoning laid down in [CH]. The strategy there is based on a one-parameter decomposition of the random field from which we abstract

Definition 2. *A random field $V : \Omega \times \mathbb{R}^d \to \mathbb{R}$ admits a (U, λ, u, ϱ)-decomposition if there exists a random field $U : \Omega \times \mathbb{R}^d \to \mathbb{R}$, a random variable $\lambda : \Omega \to \mathbb{R}$ and a Borel-measurable function $u : \mathbb{R}^d \to \mathbb{R}$ such that*

(i) $V^{(\omega)} = U^{(\omega)} + \lambda^{(\omega)} u$ *for P-almost all ω,*

(ii) the conditional probability measure of λ relative to the sub-sigma-algebra generated by $\{U(x)\}_{x\in\mathbb{R}^d}$ has a jointly measurable density $\varrho : \Omega \times \mathbb{R} \to \mathbb{R}_+$ with respect to the Lebesgue measure on $\mathbb{R}$. Here $\mathbb{R}_+$ is the set of the non-negative reals.

As a variant of the Wegner estimate in [CH] we state

Theorem 2. *Let the cube $\Lambda = \Big(\overline{\bigcup_{j=1}^J \Lambda_j}\Big)^{\mathrm{int}}$ be the interior of the closure of a finite union of pairwise disjoint open cubes Λ_j. For a random field V, which has the properties (M) and (F), define $H_{\Lambda,\mathrm{X}}(V)$ as in (2). Assume that for all $1 \le j \le J$ the random field V admits a $(U_j, \lambda_j, u_j, \varrho_j)$-decomposition subject to the following three conditions: there exist five strictly positive and finite constants ν, v, β, R, Z such that for all $1 \le j \le J$*

(i) $\nu\chi_{\Lambda_j}(x) \le u_j(x)$ *and* $u_j(x)\chi_{\Lambda_j}(x) \le v$ *for all $x \in \mathbb{R}^d$,*

(ii) $\operatorname*{ess\,sup}_{\xi\in\mathbb{R}} \Big(\varrho_j^{(\omega)}(\xi)\, \max\{\mathrm{e}^{-\beta\nu\xi}, \mathrm{e}^{-\beta v\xi}\}\Big) \le R$ *for P-almost all ω,*

(iii) $\displaystyle\int_\Omega dP(\omega)\, \mathrm{Tr}\Big(\chi_{\Lambda_j} \exp\Big\{ -\beta\, H_{\Lambda_j,\mathrm{N}}(U_j^{(\omega)})\Big\}\chi_{\Lambda_j}\Big) \le |\Lambda_j|\, Z.$

Here χ_{Λ_j} is understood as a multiplication operator.

Then the finite-volume integrated density of states (3) *obeys*

$$\int_{\Omega} dP(\omega) \left| N^{(\omega)}_{\Lambda,\mathrm{X}}(E_1) - N^{(\omega)}_{\Lambda,\mathrm{X}}(E_2) \right| \leq |\Lambda| \, |E_1 - E_2| \, \frac{RZ}{\nu} \, \mathrm{e}^{\beta \max\{E_1,E_2\}} \tag{15}$$

for both boundary conditions X *and all* $E_1, E_2 \in \mathbb{R}$.

Before we prove Theorem 2 we show how it can be exploited to deduce Theorem 1.

Proof (of Theorem 1). First we recall that the properties (M) and (F) are implied by the Gaussian property (G), cf. Remark 1. Then, given $E \in \mathbb{R}$ we partition the cube Λ into $J < \infty$ disjoint cubes Λ_j with edges of length obeying $\ell_E/2 \leq |\Lambda_j|^{1/d} \leq \ell_E$, where we have introduced the energy-dependent length

$$\ell_E := \min\{|E|^{-1/2}, |\Gamma|^{1/d}\}\,. \tag{16}$$

It is only because ℓ_E becomes small with increasing $|E|$ that one obtains the low-energy behaviour (13) of the Wegner constant $W(E)$. Next we pick a cube $\Gamma_E \subseteq \Gamma$ with volume $|\Gamma_E| = \ell_E^d$. In what follows, it will be assumed without loss of generality that the edges of Γ_E are parallel to those of the Λ_j's and that the measure μ obeys (6). The main point is that V admits a $(U_j, \lambda_j, u_j, \varrho_j)$-decomposition for all j as can be inferred from the definitions

$$\lambda_j^{(\omega)} \;:=\; (C(0))^{-1/2} \int_{\mathbb{R}^d} d\mu_j(y)\, V^{(\omega)}(y) \tag{17}$$

$$u_j(x) \;:=\; (C(0))^{-1/2} \int_{\mathbb{R}^d} d\mu_j(y)\, C(x-y) \tag{18}$$

$$U_j^{(\omega)}(x) \;:=\; V^{(\omega)}(x) - \lambda_j^{(\omega)} u_j(x) \tag{19}$$

$$\varrho_j^{(\omega)}(\xi) \;:=\; (2\pi)^{-1/2}\, \mathrm{e}^{-\xi^2/2}\,. \tag{20}$$

The signed measures μ_j are defined by suitably translating the signed measure μ such that Condition (i) of Theorem 2 is fulfilled with $v = \sqrt{C(0)}\, B_E$ and $\nu = \sqrt{C(0)}\, b_E$, where

$$B_E := (C(0))^{-1} \sup_{x \in \Gamma_E} \int_{\mathbb{R}^d} d\mu(y)\, C(x-y) \quad \leq \quad |\mu|(\mathbb{R}^d)\,, \tag{21}$$

$$b_E := (C(0))^{-1} \inf_{x \in \Gamma_E} \int_{\mathbb{R}^d} d\mu(y)\, C(x-y) \quad \geq \quad \gamma\,. \tag{22}$$

Note that the integral in (17) is well-defined for P-almost all ω because of $C(0) < \infty$. A decomposition similar to (17) – (20) is used for other purposes in [U, p. 185]. The random field U_j is non-homogeneous and Gaussian with zero mean and covariance function

$$D_j(x,y) := C(x-y) - u_j(x) u_j(y)\,, \tag{23}$$

which obeys $|D_j(x,y)| \leq C(0)(1 - b_E^2)$ for all $x, y \in \Lambda_j$. Moreover, U_j is stochastically independent of the standardized Gaussian random variable λ_j as is consistently taken into account by (20). Hence, Condition (ii) of Theorem 2 is obviously satisfied with $R = (2\pi)^{-1/2} \exp\{\beta^2 C(0) B_E^2/2\}$ for any choice of $\beta > 0$. It remains to check Condition (iii). To do so we claim that

$$\int_\Omega dP(\omega)\,\mathrm{Tr}\left(\chi_{\Lambda_j}\exp\left\{-\beta\, H_{\Lambda_j,N}(U_j^{(\omega)})\right\}\chi_{\Lambda_j}\right)$$
$$\leq \mathrm{Tr}\left(\chi_{\Lambda_j}\exp\left\{-\beta\, H_{\Lambda_j,N}(0)\right\}\chi_{\Lambda_j}\right)\sup_{x\in\Lambda_j}\int_\Omega dP(\omega)\,\mathrm{e}^{-\beta\, U_j^{(\omega)}(x)}\,. \tag{24}$$

If U_j is P-almost surely continuous, and hence bounded on Λ_j, this inequality follows from the Golden-Thompson inequality (see e.g. [RS, p. 320]) and Fubini's theorem. To show (24) under our assumptions on the covariance function, which allow for non-continuous realisations $U_j^{(\omega)}$ with non-zero probability, we adapt the approximation result [Si1, Thm. B.10.1] to the present finite-volume situation with Neumann boundary conditions by using the appropriate Feynman-Kac formula [BR, Thm. 6.3.12] and the fact that P-almost surely $U_j \in \mathrm{L}^p(\Lambda_j)$, $\exp\{-\beta U_j\} \in \mathrm{L}^1(\Lambda_j)$ for arbitrary finite $p \geq 1$ and $\beta > 0$.

Explicit computations show that the supremum in (24) does not exceed $\exp\{\beta^2 C(0)(1-b_E^2)/2\}$ and that there is a constant $z \leq \left(2\ell_E^{-1} + (2\pi\beta)^{-1/2}\right)^d < \infty$ so that the trace on the right-hand side of (24) is bounded by $|\Lambda_j|\, z$. Putting $Z := z\,\exp\{\beta^2 C(0)(1-b_E^2)/2\}$ gives Condition (iii) of Theorem 2 and hence completes the proof of Theorem 1.

The rest of the paper is devoted to a proof of Theorem 2.

Proof (of Theorem 2). Let $E_2 \leq E_1$ and $I := [E_2, E_1[$. By definition of $N_{\Lambda,\mathrm{X}}$ and by the spectral theorem one has

$$\int_\Omega dP(\omega)\left(N_{\Lambda,\mathrm{X}}^{(\omega)}(E_1) - N_{\Lambda,\mathrm{X}}^{(\omega)}(E_2)\right) \leq \mathrm{e}^{\beta E_1}\int_\Omega dP(\omega)\,\mathrm{Tr}\left(F_{\Lambda,\mathrm{X}}^{(\omega)}(I)\,\mathrm{e}^{-\beta\, H_{\Lambda,\mathrm{X}}(V^{(\omega)})}\right). \tag{25}$$

To bound the trace in (25) by

$$\sum_{j=1}^J \mathrm{Tr}\left(\max\{\mathrm{e}^{-\beta\nu\lambda_j^{(\omega)}}, \mathrm{e}^{-\beta v\lambda_j^{(\omega)}}\}\, F_{\Lambda,\mathrm{X}}^{(\omega)}(I)\,\chi_{\Lambda_j}\exp\left\{-\beta\, H_{\Lambda_j,\mathrm{N}}(U_j^{(\omega)})\right\}\chi_{\Lambda_j}\right) \tag{26}$$

we employ the operator inequality

$$H_{\Lambda,\mathrm{X}}(V^{(\omega)}) \geq \sum_{j=1}^J \chi_{\Lambda_j}\left(H_{\Lambda_j,\mathrm{N}}(U_j^{(\omega)}) + \min\{\nu\lambda_j^{(\omega)}, v\lambda_j^{(\omega)}\}\right)\chi_{\Lambda_j} \tag{27}$$

on $\mathrm{L}^2(\Lambda) = \bigoplus_{j=1}^J \mathrm{L}^2(\Lambda_j)$. It is valid for P-almost all ω and follows from Dirichlet-Neumann bracketing, the fact that introducing "Neumann surfaces" lowers eigenvalues (see e.g. [RS, Ch. XIII.15, Prop. 4]), the $(U_j, \lambda_j, u_j, \varrho_j)$-decomposition of V and Condition (i) of Theorem 2. Of course, in the case $\mathrm{X} = \mathrm{N}$ Dirichlet-Neumann bracketing is not needed. Evaluating the trace in (25) in an orthonormal eigenbasis of $H_{\Lambda,\mathrm{X}}(V^{(\omega)})$, using (27) and the Jensen-Peierls inequality [B] one obtains the bound (26).

Let us now estimate the conditional expectation relative to the sub-sigma-algebra generated by $\{U_j(x)\}_{x\in\mathbb{R}^d}$ of the j-th term in the sum (26). Since dominated convergence allows to interchange the trace with the emerging ξ-integration, this conditional expectation may be written as

$$\mathrm{Tr}\left(\int_{\mathbb{R}} d\xi\;\tilde{\varrho}_j^{(\omega)}(\xi)\, F_{j,\xi,\mathrm{X}}^{(\omega)}(I)\,\chi_{\Lambda_j}\exp\left\{-\beta\, H_{\Lambda_j,\mathrm{N}}(U_j^{(\omega)})\right\}\chi_{\Lambda_j}\right). \tag{28}$$

Here we have introduced the abbreviation $\tilde{\varrho}_j^{(\omega)}(\xi) := \varrho_j^{(\omega)}(\xi)\,\max\{\mathrm{e}^{-\beta\nu\xi}, \mathrm{e}^{-\beta v\xi}\}$ and the spectral family $F^{(\omega)}_{j,\xi,\mathrm{X}}$ of the operator $H_{\Lambda,\mathrm{X}}(U_j^{(\omega)} + \xi u_j)$. Cyclic invariance of the trace and the Hölder inequality with the usual operator norm $\|\,.\,\|$ and the trace norm bound (28) from above by

$$\left\| \int_{\mathbb{R}} d\xi\; \tilde{\varrho}_j^{(\omega)}(\xi)\, \chi_{\Lambda_j} F^{(\omega)}_{j,\xi,\mathrm{X}}(I)\, \chi_{\Lambda_j} \right\| \; \mathrm{Tr}\left(\chi_{\Lambda_j} \exp\left\{ -\beta\, H_{\Lambda_j,\mathrm{N}}(U_j^{(\omega)}) \right\} \chi_{\Lambda_j} \right) . \tag{29}$$

Thanks to Conditions (i) and (ii), one can now estimate the operator norm in (29) with the help of the spectral-averaging technique. To do so we apply Corollary 4.2 in [CH] and remark that its validity extends from non-negative bounded functions with compact support to non-negative bounded functions $\tilde{\varrho}_j^{(\omega)}$ by a monotone-convergence argument.

In this way we arrive at the ω- and j-independent upper bound $|I|\, R/\nu$ for the operator norm in (29). Hence, according to Condition (iii) the expectation of (28) with respect to dP is bounded by $|\Lambda_j|\,|I| RZ/\nu$. By virtue of (25) and (26) the proof is complete.

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