On the evaluation of the norm of an integral operator associated with the stability of one-electron atoms

V.I.Burenkov and W.D.Evans

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Abstract

The norm of an integral operator occurring in the partial wave decomposition of an operator B introduced by Brown and Ravenhall in a model for relativistic one-electron atoms is determined. The result implies that B is non-negative and has no eigenvalue at 0 when the nuclear charge does not exceed a specified critical value.

1 Introduction

The operator referred to in the title is defined on $L^2(0,\infty)$ by

$$(T\phi)(x) := \int_0^\infty t(x,y)\phi(y)dy, \quad 0 < x < \infty, \tag{1}$$

where,

$$\begin{aligned} t(x,y) = \ & \tfrac{1}{2} \left\{ \sqrt{\frac{\sqrt{x^2+1}+1}{x^2+1}} g_0(x/y) \sqrt{\frac{\sqrt{y^2+1}+1}{y^2+1}} \right. \\ & + \left. \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}} g_1(x/y) \sqrt{\frac{\sqrt{y^2+1}-1}{y^2+1}} \right\} \end{aligned} \tag{2}$$

with

$$g_0(u) = \log\left|\frac{u+1}{u-1}\right|, g_1(u) = \frac{1}{2}\left(u+\frac{1}{u}\right)\log\left|\frac{u+1}{u-1}\right| - 1, \quad u > 0.$$

To describe its role in relativistic stability, we require some background information. It is well-known that the Dirac operator describing relativistic one-particle systems is unbounded below, and that problems occur when it is extended as a model for multi-particle systems. The root of the problem is that the Dirac operator describes two different particles, namely electrons and positrons. In the paper [2] Brown and Ravenhall overcame this difficulty by projecting onto the electron subspaces only. Specifically, for a relativistic electron in the field of its nucleus, their operator is

$$B := \Lambda_+(D_0 - \frac{e^2 Z}{|\cdot|})\Lambda_+. \tag{3}$$

The notation in (1.3) is as follows:

- D_0 is the free Dirac operator

  $$D_0 = c\alpha \cdot \frac{\hbar}{i}\nabla + mc^2\beta \equiv \sum_{j=1}^{3} c\frac{\hbar}{i}\alpha_j \frac{\partial}{\partial x_j} + mc^2\beta$$

  where $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and β are the Dirac matrices given by

  $$\alpha_j = \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_j & 0_2 \end{pmatrix}, \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}$$

  with $0_2, 1_2$ the zero and unit 2×2 matrices respectively, and σ_j the Pauli matrices

  $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

- Λ_+ denotes the projection of $L^2(R^3) \otimes C^4$ onto the positive spectral subspace of D_0, that is $\chi_{(0,\infty)} D_0$. If we set

  $$\hat{f}(\mathbf{p}) \equiv \mathcal{F}(\mathbf{p}) := (\frac{1}{2\pi\hbar})^{3/2} \int_{R^3} e^{-i\mathbf{p}.\mathbf{x}/\hbar} f(\mathbf{x}) d\mathbf{x}$$

  for the Fourier transform of f, then it follows that

  $$(\widehat{\Lambda_+ f})(\mathbf{p}) = \Lambda_+(\mathbf{p})\hat{f}(\mathbf{p})$$

  where

  $$\Lambda_+(\mathbf{p}) = \frac{1}{2} + \frac{c\alpha \cdot \mathbf{p} + mc^2\beta}{2e(p)}, e(p) = \sqrt{c^2p^2 + m^2c^4}, \tag{4}$$

  with $p = |\mathbf{p}|$;

- $2\pi\hbar$ is Planck's constant, c the velocity of light, m the electron mass, $-e$ the electron charge and Z the nuclear charge.

The underlying Hilbert space in which B acts is

$$\mathcal{H} = \Lambda_+(L^2(R^3) \otimes C^4) \tag{5}$$

and when it is bounded below,B generates a self-adjoint operator (also denoted by B) defined on $\Lambda_+(H^1(R^3) \otimes C^4)$; it is the Friedrichs extension of the restriction of B to $\Lambda_+(C_0^\infty(R^3) \otimes C^4)$.

The operator B was later used by Bethe-Salpeter (see [1]) and is referred to with their name in [3]. In [3] it is proved that B is bounded below if and only if the nuclear charge Z does not exceed the critical value

$$Z_c = 2/[(\frac{\pi}{2} + \frac{2}{\pi})\alpha], \quad \alpha = e^2/\hbar c, \tag{6}$$

where α is Sommerfeld's fine structure constant; this range of Z covers all natural elements. When $Z = Z_c$, it is proved in [3] that

$$B \geq -(\frac{\pi^2 - 4}{\pi^2 + 4})mc^2.$$

However, in [4] Hardekopf and Sucher had investigated B numerically and predicted that B is in fact non-negative, and that, as for the Dirac operator, the ground state energy vanishes for $Z = Z_c$, i.e. 0 is an eigenvalue of B. The first part of this prediction of Hardekopf and Sucher has recently been confirmed, but the second part contradicted, by Tix in [5]. Following the basic strategy in [3], but with a better choice of trial functions, Tix obtains a lower bound for B which is shown to be positive for $Z \leq Z_c$, specifically

$$B \geq mc^2(1 - \alpha Z - 0.002\frac{Z}{Z_c}) > 0; \tag{7}$$

the numerical factor is roughly 0.09 for $Z = Z_c$.

¿From the partial wave analysis of B, it is shown in [3] that for all $\psi \in \Lambda_+(H^1(R^3) \otimes C^4)$ and with $(\cdot, \cdot)$ the standard inner product on $L^2(R^3) \otimes C^4$,

$$\begin{aligned}(B\psi, \psi) &= \sum_{(l,m,s)\in I} \left\{ \int_0^\infty e(p)|a_{l,m,s}(p)|^2 dp \right. \\ &- \left. \frac{\alpha c Z}{\pi} \int_0^\infty \int_0^\infty \overline{a}_{l,m,s}(p')k_{l,s}(p',p)a_{l,m,s}(p)dp dp' \right\}\end{aligned} \tag{8}$$

where I is the index set

$$\begin{aligned} I = \{(l,m,s) : l \in N_0, \quad m &= -l - 1/2, \cdots, l + 1/2, s = 1/2, -1/2, \\ |m| &\neq l + 1/2 \quad \text{when } s = -1/2\}, \end{aligned} \tag{9}$$

the kernels $k_{l,s}(p',p)$ are given by

$$k_{l,s}(p',p) = \frac{[e(p')+e(0)]Q_l(\frac{1}{2}[\frac{p}{p'}+\frac{p'}{p}])[e(p)+e(0)] + c^2p'Q_{l+2s}(\frac{1}{2}[\frac{p}{p'}+\frac{p'}{p}])p}{(2e(p)[e(p)+e(0)])^{1/2}(2e(p')[e(p')+e(0)])^{1/2}} \tag{10}$$

and

$$\sum_{(l,m,s)\in I}\int_0^\infty |a_{l,m,s}(p)|^2dp = \|\psi\|^2 := \sum_{j=1}^4\int_{R^3}|\psi_j|^2d\mathbf{x}. \tag{11}$$

In (1.10) the Q_l are the Legendre functions of the second kind. The strategy in [3] was based on this decomposition of B and the observation that

$$0 \le k_{l,s}(p',p) \le k_{0,1/2}(p',p), \quad l \in N_0, s = 1/2, -1/2. \tag{12}$$

It would follow that $B \ge 0$ for $Z \le Z_c$ if and only if

$$\int_0^\infty\int_0^\infty a(p')k_{0,1/2}(p',p)a(p)dp'dp \le \frac{\pi}{\alpha c Z_c}\int_0^\infty e(p)a(p)^2dp \tag{13}$$

for all non-negative measurable functions a. On setting

$$g_l(u) = Q_l(\frac{1}{2}[u+\frac{1}{u}]), \quad l \in N_0 \tag{14}$$

$$p = mcx, p' = mcy, \phi(x) = \sqrt{e(mcx)}a(mcx),$$

(1.13) becomes

$$\int_0^\infty\int_0^\infty t(x,y)\phi(x)\phi(y)dxdy \le (\frac{\pi^2}{4}+1)\int_0^\infty \phi(x)^2dx, \tag{15}$$

where $t(\cdot,\cdot)$ is defined in (1.2). What we prove in this paper is that the constant $\frac{\pi^2}{4}+1$ in the inequality (1.15) is sharp, and there are no extremal functions. Furthermore, we show that these results imply that $B \ge (1-\frac{Z}{Z_c})mc^2$ for $Z \le Z_c$ and 0 is not an eigenvalue of B when $Z = Z_c$. Much of the analysis continues to be valid for analogous inequalities defined by general kernel functions $t_{l,s}$ derived from the $k_{l,s}$.

2 The main results

The operator T defined on $L^2(0,\infty)$ by (1.1)is readily seen to be a bounded symmetric operator and so

$$\sup\left\{\frac{|(T\phi,\phi)|}{\|\phi\|^2} : \phi \in L^2(0,\infty), \phi \ne 0\right\} = \|T|L^2(0,\infty) \to L^2(0,\infty)\|. \tag{16}$$

Our main result is

Theorem 2.1 *Let T be defined by (1.1). Then*

1. $\|T|L^2(0,\infty)\to L^2(0,\infty)\| = \frac{\pi^2}{4}+1$;
2. *the operator T has no extremal functions.*

Remark 2.2 *We recall that ϕ is an extremal function of a bounded symmetric operator T if $\phi\in L^2(0,\infty), \phi\neq 0$ a.e. and $\|T|L^2(0,\infty)\to L^2(0,\infty)\| = |(T\phi,\phi)|/\|\phi\|^2$. Hence Theorem 2.1 means the following: for all non-negative measurable functions $\phi\in L^2(0,\infty)$ with $\phi\neq 0$ a.e.*

$$\int_0^\infty\int_0^\infty t(x,y)\phi(x)\phi(y)dxdy < (\frac{\pi^2}{4}+1)\int_0^\infty \phi^2(x)dx \tag{17}$$

and the constant $\frac{\pi^2}{4}+1$ is sharp. In turn, this implies that the inequality (1.13) is valid, the constant $\pi/\alpha c Z_c$ is sharp and there is no function $a\in L^2(0,\infty;e(p)dp)$ which is not null and for which there is equality in (1.13). A consequence of Theorem 2.1 is

Theorem 2.3 *Let B be the self-adjoint operator generated in $\mathcal{H}$ by (1.3) and let Z_c be given by (1.6). Then*

1. *if $Z\leq Z_c, B\geq (1-\frac{Z}{Z_c})mc^2$;*
2. *if $Z=Z_c, 0$ is not an eigenvalue of B;*
3. *if $Z>Z_c, B$ is unbounded below.*

Proof. Part 3 is proved in [3]. From (1.8),(1.10),(1.11) and (1.12) it follows that

$$\begin{aligned}
(B\psi,\psi) &\geq \sum_{(l,m,s)\in I}\Bigg\{\int_0^\infty e(p)|a_{l,m,s}(p)|^2dp \\
&- \frac{\alpha c Z}{\pi}\int_0^\infty\int_0^\infty |a_{l,m,s}(p')|k_{0,1/2}(p',p)|a_{l,m,s}(p)|dpdp'\Bigg\} \\
&\geq \sum_{(l,m,s)\in I}\int_0^\infty \left(1-\frac{Z}{Z_c}\right)e(p)|a_{l,m,s}(p)|^2dp \\
&\geq \left(1-\frac{Z}{Z_c}\right)mc^2\|\psi\|^2
\end{aligned}$$

which establishes part 1. To prove part 2, suppose 0 is an eigenvalue of B with corresponding eigenfunction ψ. By (1.12) and (1.13), all the summands on the right-hand side of (1.8) are non-negative and consequently are zero as now $B\psi=0$. Also (1.11) implies that at least one of the functions $a_{l,m,s}, a_{l_0,m_0,s_0}$ say, is not null. But this would imply that there is equality in (1.13) with the function $a=|a_{l_0,m_0,s_0}|$, contrary to Remark 2.2. Hence the proof is complete.

Remark 2.4 *When the mass $m = 0$, a proof of Theorem 2.3 is given in [3]. On setting $p = x, p' = y, \phi(x) = \sqrt{cx}a(x)$ in (1.13) when $m = 0$ we obtain*

$$\int_0^\infty \int_0^\infty t_0(x,y)\phi(x)\phi(y)dxdy \leq \left(\frac{\pi^2}{4}+1\right)\int_0^\infty \phi(x)^2 dx \tag{18}$$

where

$$t_0(x,y) := \frac{1}{2\sqrt{xy}}\left\{g_0\left(\frac{x}{y}\right)+g_1\left(\frac{x}{y}\right)\right\}. \tag{19}$$

We shall prove in Section 3 that the integral operator T_0 with kernel t_0 satisfies Theorem 2.1, and thus yields the analogue of Theorem 2.3 in the case $m = 0$.

3 Proof of Theorem 2.1

The starting point is the following simple result. We shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner-product and norm respectively in $L^2(0,\infty)$. It is sufficient to consider only real-valued functions in $L^2(0,\infty)$ throughout this section.

Lemma 3.1 *Let f, g, h be real-valued, measurable functions on $(0,\infty)$. Moreover, let g and h be positive and*

$$g(1/u) = g(u), \quad 0 < x < \infty. \tag{20}$$

Then

$$\int_0^\infty \int_0^\infty f(x)g\left(\frac{x}{y}\right)f(y)dxdy \leq \int_0^\infty f(x)^2\left\{\int_0^\infty \frac{h(y)}{h(x)}g\left(\frac{y}{x}\right)dy\right\}dx. \tag{21}$$

Equality holds if and only if $f(x) = Ah(x)$ a.e. on $(0,\infty)$, where A is a constant.

Proof. By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty f(x)g(x/y)f(y)dxdy\\
&= \int_0^\infty\int_0^\infty f(x)\sqrt{g(x/y)\frac{h(y)}{h(x)}}f(y)\sqrt{g(y/x)\frac{h(x)}{h(y)}}dxdy\\
&\leq \left(\int_0^\infty\int_0^\infty f(x)^2 g(x/y)\frac{h(y)}{h(x)}dxdy\right)^{1/2}\left(\int_0^\infty\int_0^\infty f(y)^2 g(y/x)\frac{h(x)}{h(y)}dxdy\right)^{1/2}\\
&= \int_0^\infty f(x)^2\left(\int_0^\infty g(y/x)\frac{h(y)}{h(x)}dy\right)dx
\end{aligned}$$

Equality holds if and only if, for some constants μ and λ

$$\mu f(x)\sqrt{g(x/y)\frac{h(y)}{h(x)}} = \lambda f(y)\sqrt{g(y/x)\frac{h(x)}{h(y)}}$$

a.e. on $(0,\infty)\times(0,\infty)$. This is equivalent to $f(x)=Ah(x)$ a.e. on $(0,\infty)$, where A is a constant.

Lemma 3.2 *Let G be the symmetric operator defined on $L^2(0,\infty)$ by*

$$Gf(x) := \int_0^\infty \frac{g(x/y)}{\sqrt{xy}} f(y)dy, \quad 0<x<\infty, \tag{22}$$

where g is a positive measurable function satisfying (3.1). Then

$$\|G|L^2(0,\infty)\to L^2(0,\infty)\| = \int_0^\infty g(u)\frac{du}{u}. \tag{23}$$

Moreover, there are no extremal functions.

Proof. By Lemma 3.1 with $h(u)=1/u$ we get

$$\begin{aligned} |(Gf,f)| &= |\int_0^\infty\int_0^\infty \frac{f(x)}{\sqrt{x}} g(x/y)\frac{f(y)}{\sqrt{y}}dxdy| \\ &\le \int_0^\infty \left(\frac{f(x)}{\sqrt{x}}\right)^2 \left(\int_0^\infty \frac{x}{y} g(y/x)dy\right)dx \\ &= \int_0^\infty g(u)\frac{du}{u}\int_0^\infty f(x)^2dx. \end{aligned}$$

Hence,

$$\|G|L^2(0,\infty)\to L^2(0,\infty)\| \le \int_0^\infty g(u)\frac{du}{u}. \tag{24}$$

Furthermore, equality in (3.5) can hold if and only if $f(x)=A/\sqrt{x}$ a.e. on $(0,\infty)$. Since $A/\sqrt{x}\notin L^2(0,\infty)$ unless $A=0$, it follows that for all $f\in L^2(0,\infty), f\ne 0$ *a.e.*,

$$|(Gf,f)| < \left(\int_0^\infty g(u)\frac{du}{u}\right)\|f\|^2. \tag{25}$$

In order to establish the inequality converse to (3.5) we take $f_\delta(x)=\frac{\chi_{(1,\delta)}(x)}{\sqrt{x}}$ as a test function , where $\chi_{(1,\delta)}$ denotes the characteristic function of $(1,\delta), 1<\delta<\infty$. By l'Hospital's Rule, we have as $\delta\to\infty$,

$$\begin{aligned} \|G|L^2(0,\infty)\to L^2(0,\infty)\| &\ge \lim_{\delta\to\infty}\frac{(Gf_\delta,f_\delta)}{\|f_\delta\|^2} \\ &= \lim_{\delta\to\infty}\left\{(\ln\delta)^{-1}\int_1^\delta\left(\int_1^\delta g(y/x)\frac{dy}{y}\right)\frac{dx}{x}\right\} \\ &= \lim_{\delta\to\infty}\left\{\int_1^\delta g(y/\delta)\frac{dy}{y}+\int_1^\delta g(\delta/x)\frac{dx}{x}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\delta\to\infty}\int_{1/\delta}^{\delta} g(u)\frac{du}{u} \\
&= \int_0^\infty g(u)\frac{du}{u}.
\end{aligned}
$$

The equality (3.4) follows from (3.5). From (3.6) it follows that there is no extremal function.

Lemma 3.3 *Let T_0 be the symmetric operator in $L^2(0,\infty)$ defined by*

$$T_0 f(x) := \int_0^\infty t_0(x,y)f(y)dy,$$

where t_0 is given by (2.4). Then

1. $\|T_0|L^2(0,\infty)\to L^2(0,\infty)\| = \frac{\pi^2}{4}+1$;
2. *there are no extremal functions.*

Proof. The results follow from Lemma 3.2 since

$$
\begin{aligned}
\int_0^\infty g_0(u)\frac{du}{u} &= 2\int_0^1 g_0(u)\frac{du}{u} \\
&= 2\int_0^1 \log\left|\frac{u+1}{u-1}\right|\frac{du}{u} \\
&= 4\int_0^1\left(\sum_{k=0}^\infty \frac{u^{2k}}{2k+1}\right)du \\
&= 4\sum_{k=0}^\infty \frac{1}{(2k+1)^2} \\
&= \frac{\pi^2}{2}
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\int_0^\infty g_1(u)\frac{du}{u} &= 2\int_0^1 g_1(u)\frac{du}{u} \\
&= 2\int_0^1\left(\frac{1}{2}\left[u+\frac{1}{u}\right]\log\left|\frac{u+1}{u-1}\right|-1\right)\frac{du}{u} \\
&= 2\lim_{\epsilon\to 0+,\delta\to 1-}\left[\frac{1}{2}\left(u-\frac{1}{u}\right)\log\left|\frac{u+1}{u-1}\right|\right]_\epsilon^\delta \\
&= 2.
\end{aligned}
\tag{27}
$$

Lemma 3.4 *The operator T defined in (1.1) satisfies*

$$\|T|L^2(0,\infty)\to L^2(0,\infty)\| \ge \frac{\pi^2}{4}+1.$$

Proof. As in Lemma 3.2, we take $f_\delta(x) = \frac{\chi_{(1,\delta)}(x)}{\sqrt{x}}, 1 < \delta < \infty$, as a test function. By l'Hospital's rule we obtain

$$\begin{aligned}
\|T|L^2(0,\infty) \to L^2(0,\infty)\| &\geq \lim_{\delta\to\infty} \frac{(Tf_\delta, f_\delta)}{\|f_\delta\|^2} \\
&= \lim_{\delta\to\infty} \left\{ (\ln\delta)^{-1} \int_1^\delta \left(\int_1^\delta t(x,y) \frac{dy}{\sqrt{y}} \right) \frac{dx}{\sqrt{x}} \right\} \\
&= \lim_{\delta\to\infty} \left\{ \sqrt{\delta} \int_1^\delta t(\delta,y) \frac{dy}{\sqrt{y}} \right. \\
&+ \left. \sqrt{\delta} \int_1^\delta t(x,\delta) \frac{dx}{\sqrt{x}}) \right\} \\
&= 2 \lim_{\delta\to\infty} \int_{1/\delta}^1 \delta t(\delta, \delta u) \frac{du}{\sqrt{u}}.
\end{aligned}$$

It is readily seen from (1.2) that for $1 < \delta < \infty$,

$$\frac{\delta t(\delta,\delta u)}{\sqrt{u}} \leq \frac{g_0(u) + g_1(u)}{u} \in L(0,1)$$

and

$$\lim_{\delta\to\infty} \frac{\delta t(\delta,\delta u)}{\sqrt{u}} = \frac{g_0(u) + g_1(u)}{2u}.$$

Hence, by the Dominated Convergence Theorem, (3.7) and (3.8), we have

$$\|T|L^2(0,\infty) \to L^2(0,\infty)\| \geq \int_0^1 \frac{g_0(u) + g_1(u)}{u} du = \frac{\pi^2}{4} + 1.$$

Lemma 3.5 *For all functions h_0, h_1 which are positive and measurable on $(0,\infty)$*

$$\|T|L^2(0,\infty) \to L^2(0,\infty)\| \leq A(h_0,h_1), \tag{28}$$

where

$$\begin{aligned}
A(h_0,h_1) &= \frac{1}{2} \sup_{0<x<\infty} \left(\frac{\sqrt{x^2+1}+1}{x^2+1} \int_0^\infty \frac{h_0(y)}{h_0(x)} g_0(y/x) dy \right. \\
&+ \left. \frac{\sqrt{x^2+1}-1}{x^2+1} \int_0^\infty \frac{h_1(y)}{h_1(x)} g_1(y/x) dy \right).
\end{aligned}$$

The operator T has an extremal function ϕ if and only if

$$\phi(x) = A_0 h_0(x) \sqrt{\frac{x^2+1}{\sqrt{x^2+1}+1}} = A_1 h_1(x) \sqrt{\frac{x^2+1}{\sqrt{x^2+1}-1}} \tag{29}$$

and

$$\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty \frac{h_0(y)}{h_0(x)}g_0(y/x)dy + \frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty \frac{h_1(y)}{h_1(x)}g_1(y/x)dy = A_3, \tag{30}$$

for some non-zero constants A_1, A_2, A_3.

Proof. By Lemma 3.1,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty t(x,y)\phi(x)\phi(y)dxdy \\
&= \frac{1}{2}\left\{\int_0^\infty\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}+1}{x^2+1}}\phi(x)g_0(x/y)\sqrt{\frac{\sqrt{y^2+1}+1}{y^2+1}}\phi(y)dxdy\right. \\
&+ \left.\int_0^\infty\int_0^\infty \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}\phi(x)g_1(x/y)\sqrt{\frac{\sqrt{y^2+1}-1}{y^2+1}}\phi(y)dxdy\right\} \\
&\leq \frac{1}{2}\int_0^\infty\left(\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty \frac{h_0(y)}{h_0(x)}g_0(y/x)dy\right. \\
&+ \left.\frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty \frac{h_1(y)}{h_1(x)}g_1(y/x)dy\right)\phi(x)^2dx \\
&\leq A(h_0,h_1)\|\phi\|^2.
\end{aligned}$$

Moreover, the first inequality becomes an equality if and only if

$$\sqrt{\frac{\sqrt{x^2+1}+1}{x^2+1}}\phi(x) = A_0h_0(x), \sqrt{\frac{\sqrt{x^2+1}-1}{x^2+1}}\phi(x) = A_1h_1(x)$$

a.e. on $(0,\infty)$, for some constants A_0, A_1. The second inequality becomes an equality if and only if $\phi(x) = 0$ a.e. on the set of all $x \in (0,\infty)$ for which

$$\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty \frac{h_0(y)}{h_0(x)}g_0(x/y)dy + \frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty \frac{h_1(y)}{h_1(x)}g_1(y/x)dy < A(h_0,h_1).$$

Since an extremal function ϕ is not null, this inequality can only be satisfied on a set of zero measure. Consequently, (3.11) holds a.e. for some constant A_3.

Remark 3.6 *We note that for all functions* h_0, h_1 *which are positive and measurable on* $(0,\infty)$

$$\liminf_{x\to\infty}\frac{1}{2}\left(\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty \frac{h_0(y)}{h_0(x)}g_0(y/x)dy\right.$$

$$+\frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty \frac{h_1(y)}{h_1(x)}g_1(y/x)dy\Bigg)$$
$$\geq \frac{\pi^2}{4}+1.$$

Indeed, let

$$\hat{h}_j(\xi) := \liminf_{x\to\infty}\frac{h_j(\xi x)}{h_j(x)}, \quad 0<\xi<\infty, j=0,1,$$

where the $\liminf$ *can be finite or infinite. Then*

$$\begin{aligned}
\hat{h}_j(1/\xi) &= \liminf_{x\to\infty}\frac{h_j(x/\xi)}{h_j(x)} \\
&= \liminf_{y\to\infty}\frac{h_j(y)}{h_j(\xi y)} \\
&= \frac{1}{\hat{h}_j(\xi)}.
\end{aligned}$$

By Fatou's Theorem

$$\begin{aligned}
&\liminf_{x\to\infty}\frac{1}{2}\Bigg(\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty \frac{h_0(y)}{h_0(x)}g_0(y/x)dy \\
&\quad+\frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty \frac{h_1(y)}{h_1(x)}g_1(y/x)dy\Bigg) \\
&\quad=\liminf_{x\to\infty}\frac{1}{2}\Bigg(\frac{\sqrt{x^2+1}+1}{x^2+1}\int_0^\infty x\frac{h_0(ux)}{h_0(x)}g_0(u)du \\
&\quad+\frac{\sqrt{x^2+1}-1}{x^2+1}\int_0^\infty x\frac{h_1(ux)}{h_1(x)}g_1(u)du\Bigg) \\
&\quad\geq \frac{1}{2}\left(\int_0^\infty \liminf_{x\to\infty}\left[\frac{h_0(ux)}{h_0(x)}\right]g_0(u)du+\int_0^\infty \liminf_{x\to\infty}\left[\frac{h_1(ux)}{h_1(x)}\right]g_1(u)du\right) \\
&\quad=\frac{1}{2}\left(\int_0^\infty \hat{h}_0(u)g_0(u)du+\int_0^\infty \hat{h}_1(u)g_1(u)du\right).
\end{aligned}$$

Furthermore, on substituting $u=v-\sqrt{v^2-1}$ *when* $0<u<1$ *and* $u=v+\sqrt{v^2-1}$ *when* $u>1$ *we have*

$$\begin{aligned}
\int_0^\infty \hat{h}_j(u)g_j(u)du &= \int_0^\infty \hat{h}_j(u)Q_j\left(\frac{1}{2}\left[u+\frac{1}{u}\right]\right)du \\
&= \int_1^\infty \{\hat{h}_j(v-\sqrt{v^2-1})(v-\sqrt{v^2-1}) \\
&+\ \hat{h}_j(v+\sqrt{v^2+1})(v+\sqrt{v^2+1})\}Q_j(v)\frac{dv}{\sqrt{v^2-1}}
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty \left\{ \frac{1}{\hat{h}_j(v+\sqrt{v^2-1})(v+\sqrt{v^2-1})} \right.\\
&+ \left. \hat{h}_j(v+\sqrt{v^2-1})(v+\sqrt{v^2-1}) \right\} Q_j(v)\frac{dv}{\sqrt{v^2-1}}\\
&\geq 2\int_1^\infty Q_j(v)\frac{dv}{\sqrt{v^2-1}}\\
&= \int_0^\infty Q_j(\frac{1}{2}[u+\frac{1}{u}])\frac{du}{u}\\
&= \int_0^\infty g_j(u)\frac{du}{u}.
\end{aligned}
$$

This verifies the assertion. We also note that equality holds if and only if $\hat{h}_j(u) = 1/u$ *a.e. on* $(0,\infty)$. *Thus to prove that* $A(h_0,h_1) \leq \frac{\pi^2}{4}+1$, *and hence complete the proof of Theorem 2.1, we must choose* h_0 *and* h_1 *in such a way that* $\hat{h}_0(u) = \hat{h}_1(u) = 1/u$ *a.e. on* $(0,\infty)$.

Lemma 3.7 *For all* $\phi \in L^2(0,\infty), \phi(x) \neq 0$ *a.e., we have*

$$\int_0^\infty \int_0^\infty t(x,y)\phi(x)\phi(y)dxdy < C\int_0^\infty \phi(x)^2 dx, \tag{31}$$

where

$$C = \sup_{0<x<\infty} F(x) \tag{32}$$

and

$$F(x) = \frac{\pi}{2}(\sqrt{x^2+1}+1)\frac{\arctan x}{x} + \frac{(\sqrt{x^2+1}-1)x}{x^2+1}. \tag{33}$$

Proof. We apply Lemma 3.5 with the choice

$$h_0(x) = \frac{x}{x^2+1}, \quad h_1(x) = \frac{1}{x}. \tag{34}$$

¿From (3.8),

$$\int_0^\infty h_1(y)g_1(y/x)dy = 2.$$

Also, on using Cauchy's Residue Theorem, we obtain

$$
\begin{aligned}
\int_0^\infty h_0(y)g_0(y/x)dy &= \int_0^\infty \frac{y}{y^2+1}\log\left|\frac{x+y}{x-y}\right| dy\\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{y}{y^2+1}\log\left|\frac{x+y}{x-y}\right| dy\\
&= \frac{x^2}{2}\int_{-\infty}^\infty \frac{u}{(xu)^2+1}\log\left|\frac{1+u}{1-u}\right| du
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^2}{2}\Re\left[\int_{-\infty}^{\infty}\frac{u}{(xu)^2+1}\log\left(\frac{1+u}{1-u}\right)du\right] \\
&= \pi\Re\left[\frac{i}{2}\log\left(\frac{x+i}{x-i}\right)\right] \\
&= \pi\arctan x.
\end{aligned}
$$

Thus (3.13) is confirmed. Since the equality (3.10) is not satisfied by the choice of h_0, h_1 in (3.15) for any constants A_0, A_1, it follows from Lemma 3.5 that there is strict inequality in (3.12).

The final link in the chain of arguments is

Lemma 3.8 *The constant C in (3.13) is given by*

$$
C = \frac{\pi^2}{4} + 1. \tag{35}
$$

Proof. Since $\lim_{x\to\infty} F(x) = \frac{\pi^2}{4} + 1$, we have that $C \geq \frac{\pi^2}{4} + 1$. To prove the reverse inequality we start by substituting $x = \tan 2v$ in $F(x)$ to obtain

$$
F(\tan 2v) = \frac{\pi v + 4\sin^4 v}{\tan v}, \quad 0 \leq v \leq \pi/4.
$$

We therefore need to prove that

$$
f(v) := \pi v + 4\sin^4 v - (\frac{\pi^2}{4} + 1)\tan v \leq 0, \quad 0 \leq v \leq \pi/4.
$$

The following identities for the derivatives are easily verified :

$$
\begin{aligned}
f^{(1)}(v) &= \pi + 16\sin^3 v\cos v - (\frac{\pi^2}{4} + 1)\sec^2 v, \\
f^{(2)}(v) &= 2\sin v\sec^3 v g(v),
\end{aligned}
$$

where

$$
g(v) = 3\sin 2v + 3\sin 4v + \sin 6v - (\frac{\pi^2}{4} + 1)
$$

and

$$
g^{(1)}(v) = 12\cos 4v(1 + \cos 2v).
$$

Since $g(0) < 0, g(\pi/8) > 0, g(\pi/4) < 0, g^{(1)}(v) > 0$ on $[0, \pi/8)$ and $g^{(1)}(v) < 0$ on $(\pi/8, \pi/4]$ there exist v_1, v_2 such that $0 < v_1 < v_2 < \pi/4, g(v_1) = g(v_2) = 0, g(v) < 0$ on $[0, v_1)$ and $(v_2, \pi/4]$, and $g(v) > 0$ on (v_1, v_2). Thus $f(0) = 0, f^{(1)}(0) < 0, f^{(2)}(0) = 0, f(\pi/4) = 0, f^{(1)}(\pi/4) > 0$, and $f^{(2)}(\pi/4) < 0$. Moreover, $f^{(2)}$ vanishes at v_1 and v_2, is negative on $(0, v_1)$ and (v_2, π), and positive on (v_1, v_2). In particular, it follows that $f^{(1)}$ is negative on $[0, v_1]$ and positive on $[v_2, \pi/4]$.

Suppose that $f(\xi) = 0$ for some $\xi \in (0, \pi/4)$. From $f(0) = f(\xi) = f(\pi/4) = 0$, and the last sentence of the previous paragraph, it follows that there exist η_1, η_2 such that $f^{(1)}(\eta_1) = f^{(1)}(\eta_2) = 0$ and $v_1 < \eta_1 < \eta_2 < v_2$. Consequently, there exists $v_3 \in (v_1, v_2)$ such that $f^{(2)}(v_3) = 0$ which is contrary to what was established in the previous paragraph. Thus $f(v) \neq 0$ on $(0, \pi/4)$. Since $f(0) = f(\pi/4) = 0$ and $f^{(1)}(0) < 0, f^{(1)}(\pi/4) > 0$, it follows that $f(v) \leq 0$ on $[0, \pi/4]$. The proof is therefore complete.

Proof of Theorem 2.1. Part 1 follows from Lemmas 3.4,3.7 and 3.8, and part 2 from (3.12).

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School of Mathematics,
University of Wales, Cardiff,
Senghennydd Road,
Cardiff CF2 4YH.