A NOTE ABOUT A BURTON KEANE'S THEOREM

BY

H. Guiol*

Instituto de Matemática e Estatística

Universidade de São Paulo

BP 66281, 05389-970 São Paulo SP, Brasil

e-mail: herve@ime.usp.br

march 1997

Abstract

In BURTON-KEANE'S structural paper [1] a situation has not been lightened, such that the proof of their result about the topological structure of ribbons in the two-dimensional site percolation has a gap. We introduce the confluent zone notion and prove, using Burton-Keane's arguments, the almost surely absence of these zones (This completes the proof of the topological structure of ribbons announced by the authors). In the end we give a alternative proof of the corollary of Theorem 2 in [1].

1 Definitions and recalls :

We begin by following line by line what BURTON and KEANE have stated in [1].

1.1 Basic Clustering

We will use the classic Euclidean distance d between two points x and y of $\mathbf{Z}^2$ given by the quadratic norm $\|.\|_2$. We say that two points of $\mathbf{Z}^2$ are **neighbors** if they are at distance 1 from each other.

We call **path** a sequence (finite or not) $(x_1, ..., x_n)$ such that $d(x_i, x_{i+1}) = 1$ for all $i = 1, ..., n-1$ and $x_i \neq x_j$ if $i \neq j$ *i.e.* to us a path is a chain of neighboring points with no loops.

*Partially supported by FAPESP grant 96/04859-9

Definition 1.1 *Let S be a subset of $\mathbf{Z}^2$, and define the relation $\mathcal{R}$ by*

$$x\,\mathcal{R}\,y \quad \Leftrightarrow \text{ There exists a path from } x \text{ to } y \text{ in } S.$$

$\mathcal{R}$ is of course an equivalent relation over S. We call **components** *of S the equivalent classes of $\mathcal{R}$.*

We are now going to distinguish finite and infinite components.

Definition 1.2 *We set*
$int(S) := \bigcup\{K\ :\ K$ *finite component of* $\mathbf{Z}^2 \setminus S\}$ *the* **interior** *of S;*
$ext(S) := \bigcup\{K\ :\ K$ *infinite component of* $\mathbf{Z}^2 \setminus S\}$ *the* **exterior** *of S;*
$cl(S) :=\ S \cup int(S)$ *the* **closure** *of S.*

Now for a given realization of site percolation in $\mathbf{Z}^2$ *i.e.* for any x in $\{0,1\}^{\mathbf{Z}^2}$ and for any i in $\{0,1\}$, we set

$$G_i := \{z \in \mathbf{Z}^2\ :\ x(z) = i\}.$$

Definition 1.3 *The components of G_i are called i-***clusters**.

Remark 1.4 *-A cluster in general will be either a 0-cluster or an 1-cluster.*
-The collection of clusters for a given $x \in \{0,1\}^{\mathbf{Z}^2}$ partitions $\mathbf{Z}^2$.

Definition 1.5 *Let C and C' be two clusters, and suppose $C \subseteq cl(C')$. Then, we write $C \prec C'$ and*

say that C' **encloses** *C, or C is enclosed by C'.*

Notice that the enclosure relation defines a partial ordering on the set of clusters.

Definition 1.6 *Maximal elements of this ordering are called* **essentials clusters***, and the others are called inessentials clusters.*

Remark 1.7 *Any inessential cluster is finite.*

We can now distinguish two sorts of realizations :

Definition 1.8 *A realization $x \in \{0,1\}^{\mathbf{Z}^2}$ is called*
(i) **essential** *if each cluster is enclosed by an (unique) essential cluster.*
(ii)**infinite cascade** *if each cluster is inessential (and finite).*

The following proposition, proved in [1], shows that there exists only two possibilities :

Proposition 1.9 *A realization is either an essential or an infinite cascade.*

Remark 1.10 *-In [1] it is proved, with an ergodic assumption, that these events have probability zero or one.*

-We will focus on essentials realizations which is in general the most interesting and the most frequent case.

When a cluster is essential it can be finite or not. Therefore, if we are interested in finite essential clusters, it is natural to characterize their positions from each other. In order to do this, R. Burton and M. Keane changed the previous definition of neighborhood and introduced the following $*$-neighborhood notion.

Two points in $\mathbf{Z}^2$ are said to be **$*$-neighbors** if their Euclidean distance is either 1 or $\sqrt{2}$.

As before a finite or infinite sequence $(x_1, ..., x_n, ...)$ is said to be a **$*$-path** if $d(x_i, x_{i+1}) \leq \sqrt{2}$ and $x_i \neq x_j$ for all indices i, j, $i \neq j$.

For a given (essential) realization x we set

$$H := \bigcup \{cl(C) \,:\, C \text{ is an essential finite cluster (of } x)\}.$$

We also introduce the equivalence relation $\mathcal{R}^*$ on H by

$$x \, \mathcal{R}^* \, y \Leftrightarrow \text{ there exists a } *\text{-path in } H \text{ from } x \text{ to } y.$$

Definition 1.11 *We call*

rocks *equivalence classes of $\mathcal{R}^*$ which are of finite cardinality;*

$*$-ribbons *the infinite ones;*

On H^c we define the

0-ribbon *: is the union of all clusters enclosed in a fix infinite (essential) 0-cluster.*

1-ribbon *: is the union of all clusters enclosed in a fix infinite (essential) 1-cluster.*

We have now the following proposition ([1]) :

Proposition 1.12 *If x is an essential realization, then $\mathbf{Z}^2$ is partitioned into 0-ribbons, 1-ribbons, $*$-ribbons and rocks such that :*

(i) Ribbons of the same type are never neighbors;

(ii) $*$-ribbons and rocks are never $*$-neighbors;

(iii) Ribbons are infinite, and rocks are finite;

(iv) The closure of any cluster is contained in a ribbon or a rock.

Remark 1.13 *R. Burton and M. Keane noted that a similar procedure to the above one should be possible for realizations of site percolation in higher dimensions in $\mathbf{Z}^d$, $d \geq 3$.*

1.2 Stationary Clustering

We are going to introduce a dynamic on $\{0,1\}^{\mathbf{Z}^2}$ and this will provide us more things about the "topological structure" of rocks and ribbons.

We denote $X := \{0,1\}^{\mathbf{Z}^2}$ and we consider the probability space $(X, \mathcal{B}_X, \mu)$. We suppose that the measure μ is stationary and ergodic.

Remark 1.14 *¿From now on it is clear that the event $\{x \in X\ ;\ x$ is infinite cascade $\}$ is $\mathbf{Z}^2$-invariant and so by ergodicity have probability 0 or 1. Of course, it is the same for its complementary event $\{x \in X\ ;\ x$ is essential $\}$.*

Definition 1.15 *We say that μ is* **essential** *if and only if μ-almost all realization is essential. Otherwise μ is said to be an infinite cascade.*

2 Topological structure of ribbons.

Up to now we assume that μ is essential.

2.1 Clustering in an Essential Process

R. Burton and M. Keane have given two theorems. In the first one, they show that rocks could be eliminated (cf page 6-7 of [1]). The second one says

Theorem 2.1 *Let Q be the event "x contains a ribbon whose complement in $\mathbf{Z}^2$ has at least 3 components". Then $\mu(Q) = 0$.*

But this theorem is not sufficient to conclude that ribbons are topologically strips as is suggested by the following counter example :

Definition 2.2 *Let $x \in \{0,1\}^{\mathbf{Z}^2}$. We call* **confluent zone** *for the realization x a square of four points $z_1 = (u,v), z_2 = (u+1,v), z_3 = (u,v+1), z_4 = (u+1,v+1)$ in $\mathbf{Z}^2$ such that $x(z_1) = x(z_4) \neq x(z_2) = x(z_3)$ and such that this zone has at least three (distinct) ribbons as neighbors.*

If such a zone exists then it is very easy to see that some ribbon may have two different neighbors on the same side. So that the main result of [1] should fail. Fortunately for a stationary ergodic measure this counter-example has measure zero.

Theorem 2.3 *Let $T := \{x \in \{0,1\}^{\mathbf{Z}^2}\ :\ x$ has a confluent zone $\}$. Then $\mu(T) = 0$.*

Proof :

Clearly, T is $\mathbf{Z}^2$-invariant. Recall that μ is ergodic, and assume that $\mu(T) = 1$.

So there exists an $\alpha > 0$ such that $\mu\{ x \in T\, , 0$ *is the (lexicographic) smallest point in a confluent zone of* $x\} = \alpha$.

Scheme of the proof is similar to the one given in [1] for theorem 1.

(Let $A_z := \{x \in T\, , z$ *is the smallest point in a confluent zone of* $x\}$, we have $T = \cup_{z \in \mathbf{Z}^2} A_z$. Therefore there exists z_0 such that $\mu(A_{z_0}) > 0$ and by stationarity $\mu(A_0) > 0$).

Now, apply the Tempel'man's ergodic theorem to the event

$$A := \{\, 0 \textit{ is the smallest point in a confluent zone } \}$$

and deduce that for some n (large enough), one can find εn^2 confluent zones in the box $K_n = [0, n] \times [0, n]$.

¿From three neighbors of each of these zones we construct three (infinite) distinct paths in three (distinct) ribbons. For each confluent zones their three paths intersect with the boundary of the box K_n.

If we now start from $(0, 0)$ on the boundary of K_n and go throw the boundary in a clockwise way, we call the second of these three points we meet the **central** point. An elementary application of the Jordan's curve theorem shows that the central points of different confluent zones must be distinct. Hence, the boundary of K_n contains at least εn^2 points which is absurd for large n. ■

Remark 2.4 *$\mu(Q) = 0$ and $\mu(T) = 0$ mean that for μ-almost every realization $x \in \{0,1\}^{\mathbf{Z}^2}$, a ribbon of x is topologically a "strip", i.e. it has either*

-one neighbor on each "side" (a real strip);

-only one neighbor (a half plane);

-no neighbor (the whole plane).

3 Finite energy measure

Definition 3.1 *For $z \in \mathbf{Z}^2$ and $i \in \{0, 1\}$ define $\tau : X \to X$ by*

$$\tau(x)(z') := \begin{cases} x(z') \text{ if } z' \neq z \\ i \text{ if } z' = z. \end{cases}$$

The measure μ is said to be of **finite energy** *if for any event A with $\mu(A) > 0$, any $z \in \mathbf{Z}^2$ and any $i \in \{0, 1\}$ one has*

$$\mu(\tau(A)) > 0.$$

Corollary 3.2 *(Burton-Keane [1])*

If μ has finite energy, then almost every $x \in X = \{0,1\}^{\mathbf{Z}^2}$ contains at most two ribbons.

The proof in [1] does not seem to be clear. Here we give another proof using a result in [2] :

Theorem 3.3 *(Burton-Keane [2])*

Let μ be of finite energy and choose $i \in \{0,1\}$ then μ-almost every $x \in X$ has at most one infinite i-cluster.

Proof of corollary 3.2

If it is not $\mu\{$ *x has a ribbon with two neighbors* $\} > 0$, and thus we have only the three following possibilities :

(Recall that, from theorem 2.1 and theorem 2.3, ribbons are topologically strips.)

First case :

We have three different type of ribbons, then changing the realization on large enough finite box we obtain a ribbon whose complement in $\mathbf{Z}^2$ has 3 components. So, by theorem 2.1, this event has measure zero.

Second case :

Two ribbons are of the same type but are not $*$-ribbons, then this event has measure zero, by theorem 3.3 on the uniqueness of i-clusters.

Third case :

We have two $*$-ribbons, then changing the realization on a large enough finite box we go back to the preceding case. ■

Acknowledgments

The author wants to thank M. Decking, P. Liardet and E. Andjel to introduce him in this percolation problematic. Special thanks also to R. Burton, E. Saada, C. Bezuidenhout, G. Grimmett, V. Belitsky, P. Ferrari, L.R. Fontes, J. Neves, S. Martinez and A. Nogueira for interesting discussions.

References

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