ERRATUM TO THE PAPER "INTERNAL LIFSHITS TAILS FOR RANDOM PERTURBATIONS OF PERIODIC SCHRÖDINGER OPERATORS"

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This text is a corrected version of the last section of the paper entitled "Internal Lifshits tails for random perturbations of periodic Schrödinger operators" (preprint 81-97 on the Math. Phys. Archive (`http://rene.math.utexas.edu/mp_arc`).

7. The upper bound

In this section, we will assume that H satisfies assumption (H.1) and that, if we denote its density of states by $n(E)$ then, at the edge 0, we have

$$\limsup_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = d/2$$

Then to prove the upper bound the strategy will be the following: by Theorem 4.1, we know that we just need to upper bound $N_{\mathcal{E}_0}(E)$ (defined in Theorem 4.1). Therefore, we will use (13) and Lemma 3.1 to compare $H^0_\omega = \Pi_0 H_\omega \Pi_0$ with a discrete Anderson model. For this model, the Lifshits tail behaviour hence the upper bound on the integrated density of states has already been proved in [29] or [22].
The idea of the reduction is simple: $N_{\mathcal{E}_0}(E) = N^0(E)$ is the density of states for the reference operator $H^0_\omega = \Pi_0 H_\omega \Pi_0$ where Π_0 is the orthogonal projection on the eigenspace associated to the eigenvalues generating the band at 0. Let $V^0_\omega = \Pi_0 V_\omega \Pi_0$ and $H^0 = \Pi_0 H \Pi_0$. As V^0_ω is non-negative, if a vector minimizes H^0_ω, it has to minimize H^0 hence to be concentrated near the zeros of the $(E_j)_{j\in N}$. But these zeros are non-degenerate and isolated; hence near these zeros, H^0 resembles the usual discrete Laplace operator. On the other hand, if the vector concentrates near a point θ^0 then, in the kernel $V_\omega(\theta,\theta')$, we may freeze $\theta=\theta'=\theta^0$. A close look at the $V_\omega(\theta^0,\theta^0)$ shows that it resembles the usual Anderson potential. A fact that is crucially used is that, as the zeros of the $(E_j)_{j\in N}$ are non-degenerate, it is possible to bound the difference $V_\omega(\theta,\theta')-V_\omega(\theta^0,\theta^0)$ by $(H^0_\omega)^{1/2}$.

The above heuristics tells us that one of the problems we will encounter while proving the upper bound is that we will have to take into account all the points where the Floquet eigenvalues vanish. So we will begin with isolating the contributions of the differents zeros of the Floquet eigenvalues. Let us just shortly recall the structure of the zeros of the Floquet eigenvalues. For $1\le j\le n$, we define $Z_j = \{\theta\in\mathbb{T}^*;\ E_j(\theta)=0\}$; the sequence $(Z_j)_{1\le j\le n}$ is decreasing and $Z_1 = Z$, the set of all points where some Floquet eigenvalue vanishes. Let $m = \sharp Z$. In the sequel, the vectors $(v_j(\cdot,\theta))_{1\le j\le n}$ will be the vector generating the band at 0 defined in Lemma 3.1.

Pick $\theta^0\in Z$, let $j(\theta^0) = \sup N_{\theta^0}$. By the construction made in Lemma 3.1, for θ close to θ^0, the operator H^0 (acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$) is a multiplication by a matrix; the matrix takes the following block diagonal form

$$H^0(\theta) = \begin{pmatrix} B_{j(\theta^0)}(\theta) & 0 & 0 & \cdots & 0 \\ 0 & E_{j(\theta^0)+1}(\theta) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & E_n(\theta) \end{pmatrix}$$

where $B_{j(\theta^0)}(\theta)$ is $j(\theta^0)\times j(\theta^0)$ block given by

$$B_{j(\theta^0)}(\theta) = \begin{pmatrix} \langle v_1(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_1(\cdot,\theta)), Hv_{j(\theta^0)}(\cdot,\theta))\rangle_{L^2(C_0)} \\ \vdots & \ddots & \vdots \\ \langle v_{j(\theta^0)}(\cdot,\theta)), Hv_1(\cdot,\theta))\rangle_{L^2(C_0)} & \cdots & \langle v_{j(\theta^0)}(\cdot,\theta)), Hv_{j(\theta^0)}(\cdot,\theta))\rangle_{L^2(C_0)} \end{pmatrix}.$$

For $\theta^0 \in Z$, define $\varpi_{\theta^0}(\theta) = c\sum_{j=1}^{d}(1-\cos(\theta_j-\theta_j^0))$. Here the constant c is small and will be chosen later on. As the zeros of the Floquet eigenvalues are non-degenerate, there exists $c>0$ and V_{θ^0} an open neighbourhood of θ^0 such that, for $\theta \in V_{\theta^0}$

- for $1 \le j \le j(\theta^0)$, $E_j(\theta) \ge \varpi_{\theta^0}(\theta)$,
- for $j \ge j(\theta^0)$, $E_j(\theta) \ge 2c$.

Hence, if we define $H^b_{\theta^0}$ to be the $n\times n$ diagonal matrix with identical diagonal entries equal to $\varpi_{\theta^0}(\theta)$, for $\theta \in V_{\theta^0}$, we have

$$H^b_{\theta^0}(\theta) \le H^0(\theta)$$

if we choose c (defining ϖ_{θ^0}) small enough
Finally, let us notice that the $(V_{\theta^0})_{\theta^0\in Z}$ may be chosen so that they cover $\mathbb{T}^*$ i.e $\cup_{\theta^0\in Z}V_{\theta^0} = \mathbb{T}^*$ and such that each of them contains exactly one point of Z i.e if $\theta \in Z$, $\theta' \in Z$ and $\theta \neq \theta'$ then $\theta' \notin \overline{V_\theta}$. We order the points in $Z = \{\theta^k;\ 1 \le k \le m\}$. Let $(\chi_k)_{1\le k\le m}$ be a $\mathcal{C}_0^\infty$-partitions of unity on $\mathbb{T}^*$ such that

- for $1 \le k \le m$, $\operatorname{supp}(\chi_k) \subset V_{\theta^k}$,
- for $1 \le k \le m$, $0 \le \chi_k \le 1$,
- for $1 \le k \le m$, $\chi_k \equiv 1$ in some neighbourhood of θ^k.

Then, for all $\theta \in \mathbb{T}^*$, we have $1/m \le \sum_{k=1}^{m}\chi_k^2(\theta) \le 1$ and

$$\sum_{k=1}^{m} H^b_{\theta^k}(\theta)\chi_k^2(\theta) \le H^0(\theta) \tag{51}$$

For $u \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$, we will write $u = ((u_{j,k}))_{\substack{1\le k\le m\\ 1\le j\le n}}$. We will consider u as a system of m columns denoted by $u_{\cdot,k}$ for $1\le k\le m$, each of them belonging to $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$. We endow $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ with the scalar product generating the following euclidean norm

$$\|u\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m} = \sum_{k=1}^{m}\|u_{\cdot,k}\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n} = \sum_{\substack{1\le j\le n\\ 1\le k\le m}}\|u_{j,k}\|^2_{L^2(\mathbb{T}^*)}.$$

We define the mapping $S:\ L^2(\mathbb{T}^*)\otimes\mathbb{C}^n \to L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$ by

$$S(u) = (\chi_k u)_{1\le k\le m} = ((\chi_k u_j))_{\substack{1\le k\le m\\ 1\le j\le n}} \text{ if } u = (u_j)_{1\le j\le n} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n.$$

The adjoint of S, $S^*:\ L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m \to L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$ is then defined by

$$S^*(u) = \left(\sum_{1\le k\le m}\chi_k u_{j,k}\right)_{1\le j\le n} \quad \text{if } u = ((u_{j,k}))_{\substack{1\le k\le m\\ 1\le j\le n}} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m.$$

One easily checks that $\frac{1}{m}I \le S^*\circ S \le I$ (here I is the identity of $L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$) so that S is one-to-one. We then prove the

Lemma 7.1. *There exists $C > 1$ such that, for $u \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$, we have*

$$\langle H^a_\omega Su, Su\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m} \leq C\langle H^0_\omega u, u\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^n}. \tag{52}$$

where the operator H^a acting on $L^2(\mathbb{T}^*) \otimes \mathbb{C}^n \otimes \mathbb{C}^m$ is defined by

$$H^a_\omega u = H^a u + V^a_\omega u = (H^a_{j,k} u_{j,k} + V^a_\omega u_{j,k})_{\substack{1\leq k\leq m\\ 1\leq j\leq n}}$$

where

- $H^a_{j,k}$ *is the multiplication by ϖ_{θ^k} acting as a multiplication on $L^2(\mathbb{T}^*)$.*
- *if V satisfies assumption (H.2bis.2) then* $V^a_\omega = \sum_{\gamma\in\Gamma} \omega_\gamma \Pi_\gamma$,
- *if V satisfies assumption (H.2bis.1) then* $V^a_\omega = \sum_{\gamma\in\Gamma} \omega_\gamma \sum_{\beta\in\Gamma} (1+|\beta|)^{-\nu}\Pi_{\beta+\gamma}$,
- Π_γ *is the orthogonal projector on the vector $e^{i\gamma\theta}$ in $L^2(\mathbb{T}^*)$.*

REMARK 7.1. We immediately see that H^a is unitarily equivalent to the direct sum of $n \cdot m$ random finite difference operators h^a_ω such that

- if V satisfies assumption (H.2bis.2), h^a_ω is the usual Anderson model acting on $\ell^2(\Gamma)$ defined by

$$-\Delta_\Gamma + \sum_{\gamma\in\Gamma} \omega_\gamma \delta_\gamma \otimes \delta_\gamma.$$

- if V satisfies assumption (H.2bis.1), h^a_ω is a long range Anderson model acting on $\ell^2(\Gamma)$ defined by

$$-\Delta_\Gamma + \sum_{\gamma\in\Gamma} \omega_\gamma \sum_{\beta\in\Gamma} (1+|\beta|)^{-\nu} \delta_{\beta+\gamma} \otimes \delta_{\beta+\gamma}.$$

- δ_γ is the vector $(\delta_{\beta\gamma})_{\gamma\in\Gamma}$ and Δ_Γ is the discrete Laplacian on $\ell^2(\Gamma)$ defined by

$$(\Delta_\Gamma u)_\gamma = \sum_{|\beta-\gamma|_\Gamma=1} (u_\beta - 2u_\gamma).$$

Before proving Lemma 7.1, let us end the proof of the upper bound. By Proposition 4.1 and Remark 5.2, we know that the operators H^0_ω and H^a_ω both admit a density of states denoted respectively by $N^0(E)$ and $N^a(E)$. Moreover, for $E \in \mathbb{R}$ such that E is continuity point of N^0 (notice that N^a is continuous (see [25] or [5])), we know that

- $N^0(E)$ is given by

$$N^0(E) = \lim_{q\to+\infty} \frac{1}{\sharp\Lambda_q} \sharp\{\text{eigenvalues of } H^0_{\omega,q} \text{ less than } E\}$$

where

$$H^0_{\omega,q} = \Pi^0_q H^0_\omega \Pi^0_q$$

and $\forall u = (u_j)_{1\le j\le n} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$,

$$\Pi_q^0 u = \left(\sum_{\gamma\in\Lambda_q} e^{i\gamma\theta} \int_{\mathbb{T}^*} e^{-i\gamma\theta} u_j(\theta) d\theta \right)_{1\le j\le n},$$

where $\Lambda_q = \{\gamma \in \Gamma;\ |\gamma|_\Gamma \le q\}$.

- $N^a(E)$ is given by

$$N^a(E) = \lim_{q\to+\infty} \frac{1}{\sharp\Lambda_q} \sharp\{\text{eigenvalues of } H^a_{\omega,q} \text{ less than } E\}$$

where

$$H^a_{\omega,q} = \Pi_q^a H_\omega^a \Pi_q^a$$

and $\forall u = (u_{j,k})_{\substack{1\le j\le n\\ 1\le k\le m}} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^n \otimes \mathbb{C}^m$,

$$\Pi_q^a u = \left(\sum_{\gamma\in\Lambda_q} e^{i\gamma\theta} \int_{\mathbb{T}^*} e^{-i\gamma\theta} u_{j,k}(\theta) d\theta \right)_{\substack{1\le j\le n\\ 1\le k\le m}}.$$

Let us compare N^0 and N^a. Therefore we will need the

Lemma 7.2. *For any $\delta > 0$ and any $l \in \mathbb{N}$, there exists $C_l(\delta) > 0$ such that, for $q \ge 1$,*

$$\left\| \Pi^a_{(1+\delta)q} \circ S \circ \Pi_q^0 - S \circ \Pi_q^0 \right\|_{\mathcal{L}(L^2(\mathbb{T}^*)\otimes\mathbb{C}^n, L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m)} \le C_l(\delta) q^{-l}.$$

Proof. The operator $(I - \Pi^a_{(1+\delta)q}) \circ S \circ \Pi_q^0$ is a $n \times (m \cdot n)$ matrix of operators; each component is acting on $L^2(\mathbb{T}^*)$. The components are of the form $(I - \Pi_{(1+\delta)q})\chi_j \Pi_q$ where $\Pi_q = \sum_{\gamma\in\Lambda_q} e^{i\gamma\theta} \otimes e^{-i\gamma\theta}$ is acting on $L^2(\mathbb{T}^*)$. Then the operator $\Pi_q^0 \chi_j (I - \Pi^a_{(1+\delta)q})\chi_j \Pi_q^0$ has the following kernels

$$[\Pi_q^0 \chi_j (I - \Pi^a_{(1+\delta)q})\chi_j \Pi_q^0](\theta,\theta') = \sum_{\substack{|\beta|_\Gamma \le q\\ |\beta'|_\Gamma\le q}} e^{i\beta\theta - i\beta'\theta'} \sum_{|\gamma|_\Gamma \ge (1+\delta)q} \int_{\mathbb{T}^*} \chi_j e^{i(\beta-\gamma)\theta} d\theta \int_{\mathbb{T}^*} \chi_j e^{i(\gamma-\beta')\theta} d\theta$$

Hence

$$\begin{aligned}
\|(I - \Pi^a_{(1+\delta)q})\chi_j \Pi_q^0\|_{\mathcal{L}(L^2(\mathbb{T}^*))} &\le \sum_{\substack{|\beta|_\Gamma \le q\\ |\beta'|_\Gamma\le q}} \sum_{|\gamma|_\Gamma \ge (1+\delta)q} |\hat{\chi}_j(\beta-\gamma)\hat{\chi}_j(\gamma-\beta')| \\
&\le \sum_{\substack{|\beta|_\Gamma \le q\\ |\beta'|_\Gamma\le q}} \sum_{|\gamma|_\Gamma \ge \delta q} |\hat{\chi}_j(\beta-\beta'-\gamma)\chi_j(\gamma)| \\
&\le C q^{2d} \sum_{|\gamma|_\Gamma \ge \delta q} |\hat{\chi}_j(\gamma)| \\
&\le C_k(\delta) q^{-k} \quad \text{as } \chi_j \in \mathcal{C}_0^\infty(\mathbb{T}^*).
\end{aligned}$$

This ends the proof of Lemma 7.2.

$\square$

Fix $\delta > 0$ and $E > 0$. Pick $u \in \Pi_q^0(L^2(\mathbb{T}^*) \otimes \mathbb{C}^n)$ such that $\langle H_\omega^0 u, u\rangle \leq E\|u\|^2$. Then, by Lemma 7.1, we have, for some $C > 1$

$$\langle H_\omega^a Su, Su\rangle \leq C \cdot E\|u\|^2$$

Hence, by Lemma 7.2,

$$\begin{aligned}\langle H_\omega^a(\Pi^a_{(1+\delta)q}S)u, (\Pi^a_{(1+\delta)q}S)u\rangle &\leq C \cdot E\|u\|^2 + 2|\langle H_\omega^a(1-\Pi^a_{(1+\delta)q}S)u, (\Pi^a_{(1+\delta)q})Su\rangle| + \\ &\quad + |\langle H_\omega^a(1-\Pi^a_{(1+\delta)q})Su, (1-\Pi^a_{(1+\delta)q})Su\rangle| \\ &\leq (C\cdot E + C_\delta q^{-1})\|u\|^2\end{aligned}$$

Moreover, by Lemma 7.2, as $\dfrac{1}{m} I \leq S^* \circ S$,

$$\|u\|^2 \leq \frac{m}{1 - C_\delta q^{-1}} \|(\Pi^a_{(1+\delta)q}S)u\|^2. \tag{53}$$

Hence, for q large enough (depending on δ),

$$\langle H_\omega^a(\Pi^a_{(1+\delta)q}S)u, (\Pi^a_{(1+\delta)q}S)u\rangle \leq C\cdot m(1+\delta)\cdot E\|(\Pi^a_{(1+\delta)q}S)u\|^2$$

By (53), we see that, for large enough q, $\Pi^a_{(1+\delta)q}S\Pi^0_q$ is one-to-one from $\Pi_q^0(L^2(\mathbb{T}^*) \otimes \mathbb{C}^n)$ to $\Pi_q^a(L^2(\mathbb{T}^*) \otimes \mathbb{C}^n \otimes \mathbb{C}^m)$. On the other hand, we know that

$$\begin{aligned}&\sharp\{\text{eigenvalues of } H^0_{\omega,q} \text{ less than } E\} = \\ &\quad = \sup\{\dim(\mathcal{E});\ \mathcal{E} \text{ linear subspace of } \Pi_q^0(L^2(\mathbb{T}^*) \otimes \mathbb{C}^n) \text{ such that} \\ &\qquad\qquad \forall u \in \mathcal{E},\ \langle H^0_\omega u, u\rangle \leq E\|u\|^2\}\end{aligned}$$

and, mutandi mutandis, the same holds for H^a_ω; by what has be done above, for q large enough, we get that

$$\sharp\{\text{eigenvalues of } H^0_{\omega,q} \text{ less than } E\} \leq \sharp\{\text{eigenvalues of } H^a_{\omega,(1+\delta)q} \text{ less than } C(1+\delta)mE\}.$$

Dividing by the volume factor and letting $q \to +\infty$, for E small enough, E a continuity point of $N^0(E)$, we obtain

$$N^0(E) \leq (1+\delta)^d N^a(C\cdot(1+\delta)m\cdot E). \tag{54}$$

Using the right continuity of $N^0(E)$ and its at most enumerable number of discontinuity points, we extend (54) immediately to some interval of the form $[0, E_0]$. Now we can let δ go to 0 and get

$$N^0(E) \leq N^a(C\cdot m\cdot E).$$

To end the proof of the upper bound, we recall that H^a is the direct sum of $n\cdot m$ random (short or long range) Anderson models. One easily deduces the upper bound in Theorem 2.1 from what is known for these usual Anderson models ([29] or [22]). In the case of the long range Anderson model, it is not difficult to extend the results of [29] or [22] to suit our needs. This ends the proof of Theorem 2.1

Remark 7.2. Notice that our assumption on the behaviour of the common probability measure of the $(\omega_\gamma)_{\gamma\in\Gamma}$ is not exactly the same as in [29] or [22]; these can easily be handled by replacing the $(\omega_\gamma)_{\gamma\in\Gamma}$ by a set of new i.i.d random variables (in the Anderson model) defined as follows:

- if $\omega_\gamma \in [0, \text{ess-sup}(\omega_0)/2]$, we set $\tilde{\omega}_\gamma = 0$;
- if $\omega_\gamma \in (\text{ess-sup}(\omega_0)/2, \text{ess-sup}(\omega_0)]$, we set $\tilde{\omega}_\gamma = \text{ess-sup}(\omega_0)/2$.

Then we get that

$$\tilde{A}_\omega = -\Delta_\Gamma + \sum_{\gamma\in\Gamma} \tilde{\omega}_\gamma \delta_\gamma \otimes \delta_\gamma \leq -\Delta_\Gamma + \sum_{\gamma\in\Gamma} \omega_\gamma \delta_\gamma \otimes \delta_\gamma$$

and to $\tilde{A}_\omega$ we may directly apply the results of [29] or [22].
The same idea may be used to remove the assumption on the not too fast decay of the common probability measure of the $(\omega_\gamma)_{\gamma\in\Gamma}$ at the edge of its support. The other assumptions remaining the same, we then get that

$$\limsup_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} \leq \begin{cases} -\dfrac{d}{\nu - d} & \text{if } V \text{ satisfies (H.2bis.1)} \\ -\dfrac{d}{2} & \text{if } V \text{ satisfies (H.2bis.2)} \end{cases}$$

Proof of Lemma 7.1. For $1 \leq j \leq n$ and $1 \leq k \leq m$, we define the function $\tilde{v}_{k,j}(x,\theta) = v_j(x,\theta^k)e^{i(\theta-\theta^k)x}$. Clearly $\tilde{v}_{k,j}$ belongs to $L^2(\mathbb{T}^*, \mathcal{H}^2_\theta)$ and $\|\tilde{v}_{k,j}\|_{1,H,\infty} < +\infty$. We define

$$\delta v_{k,j}(x,\theta) = \frac{1}{\sqrt{\varpi(\theta)}} \left(\tilde{v}_j(x,\theta) - v_{k,j}(x,\theta)\right).$$

As $v_k(x,\theta)$ is analytic in θ for θ close to θ^0, we see that $\theta \mapsto \|\delta v_{k,j}(x,\theta)\|_{L^2(C_0)}$ satisfies the assumption of Lemma 5.5. As the family $(V_\omega)_\omega$ is bounded in $L^p_{\text{loc,unif}}$ (for some p chosen as in (H.1)), Lemma 5.5 tells us that the family of operators $(\mathrm{Op}_{1,\delta v_{k,j}(\cdot,\theta),V_\omega})_\omega$ is bounded. Now we compute for $u = (u_j)_{1\leq j\leq n} \in L^2(\mathbb{T}^*) \otimes C^n$,

$$\begin{aligned} \langle V^0_\omega u, u\rangle &= \sum_{1\leq k,k'\leq m} \sum_{1\leq j,j'\leq n} \langle v_{\omega,jj'}\chi_k u_j, \chi_{k'} u_{j'}\rangle \\ &= \left\| \sum_{\substack{1\leq j\leq n\\ 1\leq k\leq m}} \mathrm{Op}_{1,v_j(\cdot,\theta),V_\omega}(\chi_k u_j)\right\|^2_{L^2(\mathbb{R}^d)} \\ &\geq \frac{1}{2}\left\| \sum_{\substack{1\leq j\leq n\\ 1\leq k\leq m}} \mathrm{Op}_{1,\tilde{v}_{k,j}(\cdot,\theta),V_\omega}(\chi_k u_j)\right\|^2_{L^2(\mathbb{R}^d)} \\ &\qquad - \left\| \sum_{\substack{1\leq j\leq n\\ 1\leq k\leq m}} \mathrm{Op}_{1,\delta v_{k,j}(\cdot,\theta),V_\omega}(\sqrt{\varpi_k}\chi_k u_j)\right\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

where $\varpi_k = \varpi_{\theta^k}$.

Hence, for some $C>0$ we get, uniformly in ω

$$\left\|\sum_{\substack{1\leq j\leq n\\ 1\leq k\leq m}} \mathrm{Op}_{1,\tilde v_{k,j}(\cdot,\theta),V_\omega}(\chi_k u_j)\right\|^2_{L^2(\mathbb{R}^d)} \leq C\left(\langle V^0_\omega u,u\rangle + \sum_{1\leq j\leq n}\|\sqrt{\varpi_k}\chi_k u_j\|^2\right)$$
$$= C\langle H^0_\omega u,u\rangle$$

by (51).
This immediately tell us that, for some $C>0$

$$\langle H^b_\omega Su, Su\rangle \leq C\langle H^0_\omega u,u\rangle \tag{55}$$

where

- $H^b_\omega = H^a + V^b_\omega$,
- H^a is defined in Lemma 7.1,
- V^b_ω is the operator matrix with entries the operators with the following kernels

$$v^b_{\omega,jj'kk'}(\theta,\theta') = \int_{\mathbb{R}^d} V_\omega(x) v_{k,j}(x,\theta^k)\overline{v_{k',j'}(x,\theta^{k'})}e^{i((\theta-\theta^k)-(\theta'-\theta^{k'}))x}dx.$$

Let us study these kernels more precisely; one has

$$\int_{\mathbb{R}^d} V_\omega(x) v_{k,j}(x,\theta^k)\overline{v_{k',j'}(x,\theta^{k'})}e^{i((\theta-\theta^k)-(\theta'-\theta^{k'}))x}dx = \sum_{\gamma\in\Gamma}\omega_\gamma e^{i(\theta-\theta')\gamma} v^b_{jj'kk'}(\theta,\theta') \tag{56}$$

where

$$v^b_{jj'kk'}(\theta,\theta') = \int_{\mathbb{R}^d} V(x) v_j(x,\theta^k)\overline{v_{j'}(x,\theta^{k'})}e^{i((\theta-\theta^k)-(\theta'-\theta^{k'}))x}dx.$$

Remark 7.3. In (56), the sum converges in the strong operator topology.

So we get that

$$V^b_\omega = \sum_{\gamma\in\Gamma}\omega_\gamma V^b_\gamma$$

where V^b_γ is an operator matrix with entries the $((v^b_{jj'kk'}))_{1\leq j,j'\leq n,\ 1\leq k,k'\leq m}$. By definition, for $u=(u_{k,j})_{1\leq j\leq n,\ 1\leq k\leq m}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$, we have

$$\langle V^b_\gamma u,u\rangle = \int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\leq j\leq n\\ 1\leq k\leq m}} v_j(x,\theta^k)\int_{C^*_0} e^{i(\theta-\theta_k)x}e^{-i\theta\gamma}u_{k,j}(\theta)d\theta\right|^2 dx. \tag{57}$$

To lower bound the operator V^b_ω we will need to distinguish the two cases in assumption (H.2.bis).

7.0.1. *In case 2) of assumption (H.2bis).* In this case, $\mid x \mid^2 V(x)$ is in $\ell^1(L^p(\mathbb{R}^d))$ hence for $\gamma \in \Gamma$ and $u = (u_{k,j})_{1\leq j\leq n,\ 1\leq k\leq m} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^n \otimes \mathbb{C}^m$, expanding (57), we get

(58)

$$\begin{aligned}\langle V^b_\gamma u, u\rangle = &\int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right|^2 dx \\ &+ 2\mathrm{Re}\left(\int_{\mathbb{R}^d} V(x) \left(\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} g_k(x,\theta) e^{-i\theta\gamma} \sqrt{\varpi_k(\theta)} u_{k,j}(\theta) d\theta \right) \cdot \right. \\ &\left. \cdot \overline{\left(\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right)} dx \right) \\ &+ \int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} g_k(x,\theta) e^{-i\theta\gamma} \sqrt{\varpi_k(\theta)} u_{k,j}(\theta) d\theta \right|^2 dx\end{aligned}$$

where, for $1 \leq k \leq m$, we defined

$$g_k(x,\theta) = \frac{e^{i(\theta-\theta^k)x} - 1}{\sqrt{\varpi_k(\theta)}}$$

Then, as θ_k is the only zero of $\varpi_k(\theta)$ and it is non-degenerate, there exists $C > 0$ such that, for $x \in \mathbb{R}^d$, $\theta \in \mathbb{T}^*$ and $1 \leq k \leq m$, we have

$$|g_k(x,\theta)| \leq C(1 + |x|). \tag{59}$$

Define

$$A = \int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right|^2 dx$$

$$B = \int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} g_k(x,\theta) e^{-i\theta\gamma} \sqrt{\varpi_k(\theta)} u_{k,j}(\theta) d\theta \right|^2 dx$$

By (58), using the Cauchy-Schwartz inequality, we get

$$\langle V^b_\gamma u, u\rangle \geq (\sqrt{A} - \sqrt{B})^2 \geq (1-\alpha)A + (1 - 1/\alpha)B.$$

where $0 < \alpha < 1$.

Choosing α small, we get that, for some $C>0$, we have

(60)

$$\int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_{k,j}(\theta)d\theta\right|^2 dx$$
$$\leq C\left(\int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi_k(\theta)}u_{k,j}(\theta)d\theta\right|^2 dx + \langle V^b_\gamma u,u\rangle\right).$$

Expanding the square in the left hand side term of (60), we see that

$$\int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{-i\theta\gamma}u_{k,j}(\theta)d\theta\right|^2 dx = \sum_{\substack{1\leq j,j'\leq n\\1\leq k,k'\leq m}} v_{jkj'k'}\hat{u}_{k,j}(\gamma)\overline{\hat{u}_{k',j'}(\gamma)}$$

where

- $\hat{u}_{k,j}(\gamma)$ is the γ-th Fourier coefficient of $u_{k,j}$ (up to a normalization constant).
- $v_{jkj'k'} = \displaystyle\int_{\mathbb{R}^d} V(x)v_j(x,\theta^k)\overline{v_{j'}(x,\theta^{k'})}dx.$

Now we have the

Lemma 7.3. *The matrix* $((v_{jkj'k'}))_{1\leq j,j'\leq n,\ 1\leq k,k'\leq m}$ *is positive definite.*

Proof The positivity is obvious. Assume it is not definite; then it has 0 as an eigenvalue. In other words there exists $(\lambda_{k,j})_{1\leq j\leq n,\ 1\leq k\leq m}$ such that

$$\int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} \lambda_{k,j}v_j(x,\theta^k)\right|^2 dx = 0.$$

Hence as $V(x)$ is non-negative and does not vanish almost everywhere, we know that, for x in some open set, we have

$$v(x) = \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} \lambda_{k,j}v_j(x,\theta^k) = 0.$$

By construction (see Lemma 3.1) all the $(v_j(\cdot,\theta^k))_{1\leq j\leq n,\ 1\leq k\leq m}$ are solutions of the elliptic equation $Hu=0$ (here $H=-\Delta+W$ is the background periodic Schrödinger operator); hence v is also a solution of $Hu=0$. By the unique continuation principle (e.g. see [10]), v vanishes everywhere. So we get that, for $x\in\mathbb{R}^d$,

$$\sum_{1\leq k\leq m} w_k(x) = \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} \lambda_{k,j}v_j(x,\theta^k) = 0$$

where $w_k(x) = \sum_{1\leq j\leq n} \lambda_{k,j} v_j(x,\theta^k)$.

Hence, for $1 \leq k \leq m$, we have $w_k \equiv 0$ on $\mathbb{R}^d$. Indeed the functions $(w_k)_{1\leq k\leq m}$ are Γ-quasi-periodic with periodicity module respectively $e^{i\theta^k}$. These modules are two by two distinct. We recall that C_0 denotes the fundamental domain of the lattice Γ and set $\Lambda_l = \{x \in \mathbb{R}^d;\ |x|_\gamma \leq l\}$. Then, for $1 \leq k \leq m$, we compute

$$\begin{aligned}
0 &= \frac{1}{\mathrm{Vol}(\Lambda_l)} \int_{\Lambda_l} v(x)\overline{w_k(x)}dx \\
&= \frac{1}{\mathrm{Vol}(C_0)} \int_{C_0} |w_k(x)|^2 dx + \sum_{k'\neq k} \sum_{\gamma\in\Gamma\cap\Lambda_l} \frac{1}{\mathrm{Vol}(\Lambda_l)} \int_{\gamma+C_0} w_{k'}(x)\overline{w_k(x)}dx \\
&= \frac{1}{\mathrm{Vol}(C_0)} \int_{C_0} |w_k(x)|^2 dx + \sum_{k'\neq k} \int_{C_0} w_{k'}(x)\overline{w_k(x)}dx \frac{1}{\mathrm{Vol}(\Lambda_l)} \left(\sum_{\gamma\in\Gamma\cap\Lambda_l} e^{i(\theta^{k'}-\theta^k)\gamma} \right)
\end{aligned}$$

Now as the $(\theta^k)_{1\leq k\leq m}$ are two by two distinct, for $k' \neq k$, computing the geometric series gives

$$\lim_{l\to+\infty} \frac{1}{\mathrm{Vol}(\Lambda_l)} \left(\sum_{\gamma\in\Gamma\cap\Lambda_l} e^{i(\theta^{k'}-\theta^k)\gamma} \right) = 0$$

Hence $\int_{C_0} |w_k(x)|^2 dx = 0$ so that $\sum_{1\leq j\leq n} \lambda_{k,j} v_j(\cdot,\theta^k) \equiv 0$ on $\mathbb{R}^d$, this for any $1 \leq k \leq m$. This contradicts the fact that, for every $1 \leq k \leq m$, the $(v_j(\cdot,\theta^k))_{1\leq j\leq n}$ are linearly independent. This ends the proof of Lemma 7.3.

□

Lemma 7.3 tells us that, for some $C > 0$, we have

$$\sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} \left| \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right|^2 \leq C \cdot \int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right|^2 dx \tag{61}$$

If we sum (60) over γ, using (61) and the definition of V^b_ω, we get

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \omega_\gamma \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} \left| \int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta) d\theta \right|^2 \\
&\leq C \left(\sum_{\gamma\in\Gamma} \omega_\gamma \int_{\mathbb{R}^d} V(x) \left| \sum_{\substack{1\leq j\leq n\\1\leq k\leq m}} v_j(x,\theta^k) \int_{\mathbb{T}^*} g_k(x,\theta) e^{-i\theta\gamma} \sqrt{\varpi_k(\theta)} u_{k,j}(\theta) d\theta \right|^2 dx + \langle V^b_\omega u, u \rangle \right)
\end{aligned}$$

As the random variables $(\omega_\gamma)_\gamma$ are bounded, using Parseval's identity, (59) and assumption (H.2bis.1), we obtain, for some $C > 0$,

(62)

$$\sum_{\gamma\in\Gamma}\omega_\gamma \sum_{\substack{1\le j\le n\\1\le k\le m}} \left|\int_{\mathbb{T}^*} e^{-i\theta\gamma} u_{k,j}(\theta)d\theta\right|^2$$

$$\le \sqrt{C}\left(\sum_{\gamma\in\Gamma}\int_{\mathbb{R}^d} V(x)\left|\sum_{\substack{1\le j\le n\\1\le k\le m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi_k(\theta)}u_{k,j}(\theta)d\theta\right|^2 dx + \langle V^b_\omega u, u\rangle\right)$$

$$\le \sqrt{C}\left(\int_{\mathbb{R}^d} V(x)\sum_{\substack{1\le j\le n\\1\le k\le m}}\int_{\mathbb{T}^*}\left|g_k(x,\theta)v_j(x,\theta^k)\sqrt{\varpi_k(\theta)}u_{k,j}(\theta)\right|^2 d\theta dx + \langle V^b_\omega u, u\rangle\right)$$

$$\le C\left(\sum_{\substack{1\le j\le n\\1\le k\le m}}\int_{\mathbb{R}^d} V(x)|v_j(x,\theta^k)|^2(1+|x|)^2dx\int_{\mathbb{T}^*}\varpi_k(\theta)|u_{k,j}(\theta)|^2d\theta + \langle V^b_\omega u, u\rangle\right)$$

$$\le C\left(\sum_{\substack{1\le j\le n\\1\le k\le m}}\int_{\mathbb{T}^*}\varpi_k(\theta)|u_{k,j}(\theta)|^2d\theta + \langle V^b_\omega u, u\rangle\right),$$

This tells us that there exists $C > 0$ such that for $u \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n\otimes\mathbb{C}^m$,

$$\langle H^a_\omega u, u\rangle \le C\cdot\langle (H^a + V^b_\omega)u, u\rangle,$$

where H^a_ω is defined in Lemma 7.1. If we now use (55), we get that that there exists $C > 0$ such that for $u \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$

$$\langle H^a_\omega Su, Su\rangle \le C\cdot\langle H^0_\omega u, u\rangle.$$

This ends the proof of Lemma 7.1 if V satisfies assumption (H.2bis) 2).

7.0.2. *In case 1) of assumption (H.2bis).* As $\mid x \mid^2 V(x)$ is not in $\ell^1(L^p(\mathbb{R}^d))$ in this case, using (H.2bis.1), for some $C>0$, we get

$$
\begin{aligned}
\langle V_\gamma^b u, u\rangle &= \sum_{\beta\in\Gamma}\int_{C_0} V(x-\beta)\left|\sum_{\substack{1\le j\le n\\ 1\le k\le m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{i\theta x}e^{-i\theta(\gamma+\beta)}u_{k,j}(\theta)d\theta\right|^2 dx\\
&\ge \frac{1}{C}\sum_{\beta\in\Gamma}(1+|\beta|)^{-\nu}\int_{C_0} g_-(x)\left|\sum_{\substack{1\le j\le n\\ 1\le k\le m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{i\theta x}e^{-i\theta(\gamma+\beta)}u_{k,j}(\theta)d\theta\right|^2 dx\\
&= \frac{1}{C}\sum_{\beta\in\Gamma}(1+|\beta|)^{-\nu}s_{\gamma,\beta}
\end{aligned}
$$

We can then analyse each of the $s_{\gamma,\beta}$ in the same way as in in the previous case to get, for some $C>0$

$$
\begin{aligned}
&\sum_{\substack{1\le j\le n\\ 1\le k\le m}}\left|\int_{\mathbb{T}^*} e^{-i\theta(\gamma+\beta)}u_{k,j}(\theta)d\theta\right|^2\\
&\le C\left(s_{\gamma,\beta}+\int_{C_0} g_-(x)\left|\sum_{\substack{1\le j\le n\\ 1\le k\le m}} v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta(\gamma+\beta)}\sqrt{\varpi_k(\theta)}u_{k,j}(\theta)d\theta\right|^2 dx\right).
\end{aligned}
$$

If we now multiply this by $\omega_\gamma\cdot(1+|\beta|)^{-\nu}$ and sum it over γ and β, using the same arguments as before, we get, for some $C>0$,

$$
\begin{aligned}
&\sum_{\substack{1\le j\le n\\ 1\le k\le m}}\sum_{\substack{\gamma\in\Gamma\\ \beta\in\Gamma}}(1+|\beta|)^{-\nu}\omega_\gamma\left|\int_{\mathbb{T}^*} e^{-i\theta(\gamma+\beta)}u_{k,j}(\theta)d\theta\right|^2\\
&\qquad\qquad\le C\left(\sum_{\substack{1\le j\le n\\ 1\le k\le m}}\int_{\mathbb{T}^*}\varpi_k(\theta)|u_{k,j}(\theta)|^2 d\theta+\langle V_\omega^b u, u\rangle\right).
\end{aligned}
$$

This immediately implies that there exists $C>0$ such that for $u\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^n$,

$$\langle H_\omega^a Su, Su\rangle \le C\langle H_\omega^0 u, u\rangle$$

where H_ω^a is defined in Lemma 7.1. This ends the proof of Lemma 7.1.

□

References

[1] S. Agmon. On positive solutions of elliptic operators with periodic coefficients in $L^2(\mathbf{R}^n)$, spectral results and extension to elliptic operators on Riemannian manifolds. In I. Knowles and R. Lewis, editors, *Differential equations*, pages 7–17, Birmingham, 1984. North-Holland.

[2] J.-M. Barbaroux, J.-M. Combes, and P. Hislop. Localization at internal band edges. Preprint, Centre de Physique Théorique, Marseille, 1996.

[3] M. Sh Birman. Perturbations of operators with periodic coefficients. In *Schrödinger Operators: standard and non-standard*, Dubna, 1988.

[4] R. Carmona. Random Schrödinger operators. In *Ecole d'été de Probabilité de Saint-Flour, XIV*, volume 1180 of *Lecture Notes in Mathematics*, pages 1–121, 1984.

[5] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators*. Springer Verlag, Berlin, 1987.

[6] F. Daumer. Equation de Schrödinger avec champ électrique périodique et champ magnétique constant dans l'approximation de "tight-binding". *Communications in Partial Differential Equations*, 18(5-6):1021–1041, 1993.

[7] Y Colin de Verdière. Sur les singularités de Van Hove génériques. In *Analyse globale et physique mathematique (Lyon, 1989)*, volume 46 of *Mémoire de la Société Mathématique de France*, pages 99–110, 1991. Colloquium en l'honneur d'E. Combet.

[8] M. Eastham. *The spectral theory of periodic differential operators*. Scottish Academic Press, Edinburgh, 1973.

[9] B. Helffer and J. Sjöstrand. On diamagnetism and the De Haas-Van Alphen effect. *Annales de l'Institut Henri Poincaré, série Physique Théorique*, 52:303, 1990.

[10] D. Jerison and C. Kenig. Unique continuation and absence of positive eigenvalues for Schrödinger operators. *Annals of Mathematics*, 121:463–494, 1985.

[11] T. Kato. *Perturbation Theory for Linear Operators*. Springer Verlag, Berlin, 1980.

[12] W. Kirsch. Random Schrödinger operators. In A. Jensen H. Holden, editor, *Schrödinger Operators*, number 345 in Lecture Notes in Physics, Berlin, 1989. Springer Verlag. Proceedings, Sonderborg, Denmark 1988.

[13] W. Kirsch and F. Martinelli. On the density of states of Schrödinger operators with a random potential. *Journal of Physics A*, 15:2139–2156, 1982.

[14] W. Kirsch and F. Martinelli. On the spectrum of Schrödinger operators with a random potential. *Communications in Mathematical Physics*, 85:329–350, 1982.

[15] W. Kirsch and B. Simon. Lifshits tails for the Anderson model. *Journal of Statistical Physics*, 38:65–76, 1985.

[16] W. Kirsch, P. Stollman, and G. Stolz. Localization for random perturbations of periodic Schrödinger operators. *Preprint*, 1996.

[17] F. Klopp. Étude semi-classique d'une perturbation d'un opérateur de Schrödinger périodique. *Annales de l'Institut Henri Poincaré, série Physique Théorique*, 55:459–509, 1991.

[18] F. Klopp. An asymptotic expansion for the density of states of a random Schrödinger operator with Bernoulli disorder. *Random Operators and Stochastic Equations*, 3(4):315–332, 1995.

[19] F. Klopp. Localization for some continuous random Schrödinger operators. *Communications in Mathematical Physics*, 167:553–570, 1995.

[20] F. Klopp and J. Ralston. Generic behaviour of open band edges for periodic Schrödinger operators and applications to Lifshits tail asymptotics. Work in progress.

[21] P. Kuchment. *Floquet theory for partial differential equations*, volume 60 of *Operator Theory: Advances and Applications*. Birkhäuser, Basel, 1993.

[22] G. Mezincescu. Internal Lifshits singularities of disordered finite-difference Schrödinger operators. *Communications in Mathematical Physics*, 103:107–116, 1986.

[23] G. Mezincescu. Lifshits singularities for periodic operators plus random potentials. *Journal of Statistical Physics*, 49:1081–1090, 1987.

[24] G. Mezincescu. Internal Lifshits singularities for one dimensional Schrödinger operators. *Communications in Mathematical Physics*, 158:315–325, 1993.
[25] L. Pastur and A. Figotin. *Spectra of Random and Almost-Periodic Operators*. Springer Verlag, Berlin, 1992.
[26] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Vol IV: Analysis of Operators*. Academic Press, New-York, 1978.
[27] M.A. Shubin. Spectral theory and index of elliptic operators with almost periodic coefficients. *Russian Mathematical Surveys*, 34:109–157, 1979.
[28] B. Simon. Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7:447–526, 1982.
[29] B. Simon. Lifshits tails for the Anderson model. *Journal of Statistical Physics*, 38:65–76, 1985.
[30] B. Simon. Internal Lifshits tails. *Journal of Statistical Physics*, 46:911–918, 1987.
[31] J. Sjöstrand. Microlocal analysis for periodic magnetic Schrödinger equation and related questions. In *Microlocal analysis and applications*, volume 1495 of *Lecture Notes in Mathematics*, Berlin, 1991. Springer Verlag.
[32] M. Skriganov. Geometric and arithmetic methods in the spectral theory of multidimensional periodic operators. *Proceedings of the Steklov Institute of Mathematics*, 171:1–121, 1987.

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