Centre de Physique Théorique*, **CNRS Luminy, Case 907**
F-13288 Marseille – Cedex 9

STABILITY AND INSTABILITY OF THE WAVE EQUATION SOLUTIONS IN A PULSATING DOMAIN

J. DITTRICH[1], P. DUCLOS[2] and N. GONZALEZ[2,3]

Abstract

The behavior of energy is studied for the real scalar field satisfying d'Alembert equation in a finite space interval $0 < x < a(t)$; the endpoint $a(t)$ is assumed to move slower than the light and periodically in most parts of the paper. The boundary conditions are of Dirichlet and Neumann type. We give sufficient conditions for the unlimited growth, the boundedness and the periodicity of the energy E. The case of unbounded energy without infinite limit ($0 < \liminf_{t\to+\infty} E(t) < \limsup_{t\to+\infty} E(t) = +\infty$) is also possible. For the Neumann boundary condition, E may decay to zero as the time tends to infinity. If a is periodic, the solution is determined by a homeomorphism $\overline{F}$ of the circle related to a. The behavior of E depends essentially on the number theoretical characteristics of the rotation number of $\overline{F}$.

Key-Words: wave equation, time-dependent domain, homeomorphisms of the circle.

May 1997
CPT-97/P.3484

anonymous ftp or gopher: cpt.univ-mrs.fr

*Unité Propre de Recherche 7061

[1]Nuclear Physics Institute, Academy of Sciences of the Czech Republic, CZ-250 68 Řež, and Doppler Institute of Mathematical Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Prague, Czech Republic
e-mail address: dittrich@ujf.cas.cz

[2]and PhyMat, Université de Toulon et du Var, La Garde, France
e-mail address: duclos@naxos.unice.fr

[3]and Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Prague, Czech Republic
e-mail address: gonzalez@cptsg2.univ-mrs.fr

Contents

1 Introduction

The study of the so-called Fermi accelerators becomes more and more extensive. The name comes from Fermi's considerations on the possible mechanism of cosmic rays acceleration [1]. In the later studies up to contemporary ones, they serve as simple prototypes of the externally driven dynamical systems, mainly in the connection with the deterministic and chaotic behavior of the classical and quantum systems. The first mechanical models were proposed by Ułam [2], the rigorous results in Newtonian mechanics (Pustyl'nikov [3, 4]) and in special-relativistic classical mechanics (Pustyl'nikov [5, 6, 4]) were obtained much later. Only as a sample of papers in nonrelativistic quantum mechanics let us mention Karner [7], Šeba [8], Dembiński, Makowski and Pepłowski [9], Dodonov, Klimov and Nikonov [10], Duclos and Šťovíček [11]. Similar problems for classical wave equation (Balazs [12], Cooper [13], Dittrich, Duclos and Šeba [14], Cooper and Koch [15], Gignoux and Meplan [16]) and Maxwell equations (Cooper [17]) were also considered. An analogous model in quantum field theory was treated e.g. by Moore [18], Calucci [19], Dodonov, Klimov and Nikonov [20], Johnston and Sarkar [21]. In the present paper, we continue and extend the study for the classical d'Alembert equation.

Let us consider the one-dimensional wave equation in a domain with one spatial boundary fixed and the second one moving slower than the wave velocity. Let us assume that the boundary motion is described by a Lipschitz continuous function a and assume that the field satisfies either Dirichlet or Neumann boundary conditions. We describe the behavior of the energy E of the field in more details and for a wider class of functions a than the papers [13, 14, 15, 16] which treat only the case $a \in \mathcal{C}^k(\mathbb{R})$ $(k \geq 2)$ and a periodic.

The key to the results is that the orbits of the characteristics of the wave equation are given by a Lipschitz homeomorphism F of $\mathbb{R}$ which depends only on a and becomes the lift of a homeomorphism of the circle when a is periodic. According to the arithmetic properties of the rotation number of F and to the regularity of a, E will behave differently. It is worth remarking that our results are nonperturbative (in other words there are no assumptions on the smallness of time variations of a). Section 2 is devoted to the formulation of the problems (Dirichlet and Neumann), the link between a and F, the existence and unicity of a weak solution in the space of finite energy fields. Section 3 is divided into two subsections: Stability (E is bounded from below and above) and Instability (E is unbounded), both for the Dirichlet problem. In Subsection 3.1, we give a universal lower bound (a needs just be bounded), a necessary and sufficient condition of periodicity and sufficient conditions for the existence of an upper bound. In Subsection 3.2, we distinguish two types of instability: a weaker one (the limit superior is infinite) and a stronger one (the limit inferior is infinite). We first give a sufficient condition of instability and we state a sufficient condition of strong instability. The proof introduces the condition for occurence of "wandering characteristics" and therefore generalizes the condition of "periodic characteristics" given by Cooper [13]. We compute an asymptotics which shows the exponential increase of E due to the presence of periodic characteristics; in this case, under any small periodic perturbation of the boundary motion a, E keeps increasing exponentially. We also show on examples that E may have an infinite limit superior but a finite limit inferior in the case where there are no periodic characteristics and the boundary a is smooth; this answers negatively to a conjecture of Cooper [17]. Cooper and Koch [15] already used diffeomorphisms of the circle in the Dirichlet problem and they show that the spectrum of the evolution operator on one period depends on the rotation number. Section 4 is devoted to the Neumann problem. To our knowledge, this is the first treatment in the literature. As we shall see, E behaves in a completely different manner. In Subsection 4.1, we prove that E is universally bounded from above except if the initial conditions are singular (in which case we show that E may diverge exponentially). The periodicity and lower bounds are proven like for the Dirichlet problem. We also propose conditions for an asymptotically periodic energy and give explicit asymptotics for this case. In Subsection 4.2, we show that for appropriate initial conditions E decays (and even exponentially fast) to 0.

Let a be a stricly positive real function to be precised later. The problems we consider are the *Dirichlet problem*:

$$\begin{aligned}
\varphi_{tt} - \varphi_{xx} &= 0\,, \quad t \in \mathbb{R}, \quad 0 < x < a(t)\,, && (1)\\
\varphi(x,0) &= \varphi_0(x)\,, \quad 0 < x < a(0)\,, && (2)\\
\varphi_t(x,0) &= \varphi_1(x)\,, \quad 0 < x < a(0)\,, && (3)\\
\varphi(0,t) &= 0\,, \quad t \in \mathbb{R}\,, && (4)\\
\varphi(a(t),t) &= 0\,, \quad t \in \mathbb{R}\,, && (5)
\end{aligned}$$

and the *Neumann problem* for which

$$\begin{aligned}
\varphi_x(0,t) &= 0\,, \quad t \in \mathbb{R}\,, && (6)\\
\varphi_x(a(t),t) &= 0\,, \quad t \in \mathbb{R} && (7)
\end{aligned}$$

are required instead of (4)-(5). Since it will not play a rôle in the mathematical analysis, the wave velocity of the field φ is normalized to 1. In addition, if a is periodic, then by a rescaling in the parameters, one can also take the period equal to 1; this will simplify our notations. The energy of the field φ is given by the standard expression

$$E(t) := \frac{1}{2}\int_0^{a(t)} \left(|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2\right)\, dx\,. \tag{8}$$

2 Preliminaries

2.1 Notations, definitions and preliminary results

We start with some notations and known results. Not to be very formal in trivialities, some equivalent spaces of functions (like functions on a circle and their liftings on a line) are indentified in our notation explained below.

Let X be either the set $\mathbb{Z}$ (the integers) or $\mathbb{N}$ (the nonnegative integers) or $\mathbb{Q}$ (the rational numbers) or $\mathbb{R}$ (the real numbers). Then $X^* := X \setminus \{0\}$, $X_+ := \{x \in X;\ x \geq 0\}$ and $X_+^* := X^* \cap X_+$. For any set X, we denote by $\dot{X}$ or Int X its interior. Denote by $\mathbb{T}$ the 1-dimensional torus (the circle of unit length) and by X either $\mathbb{T}$ or $\mathbb{R}$. Let $\mathcal{C}^0(X)$ be the space of the continuous functions and let $\mathcal{C}^k(X)$, $k \in \mathbb{N}^*$, be the space of the k-times continuously differentiable functions; the k-th derivative of a function F is denoted by $F^{(k)}$ or $D^k F$. Denote by $\mathcal{C}^k(\mathbb{T})$ the set of 1-periodic and k-times continuously differentiable functions on $\mathbb{R}$. One defines the norms $\|F\|_0 := \sup_{x\in\mathbb{T}} |F(x)|$ on $\mathcal{C}^0(\mathbb{T})$ and $\|F\|_k := \max_{0\leq i\leq k} \|D^i F\|_0$ on $\mathcal{C}^k(\mathbb{T})$ for finite $k \in \mathbb{N}$. By $\mathcal{C}^\infty(X)$ (resp. $\mathcal{C}^\omega(X)$) one denotes the space of infinitely differentiable (resp. $\mathbb{R}$-analytic) functions on X. For a measurable function $F : X \to \mathbb{R}$, we shall denote by $F_{\min}$ and $F_{\max}$ its essential infimum and its essential supremum respectively. Let $\mathrm{Lip}_\beta(X)$ be the space of Hölder continuous functions with exponent $\beta \in (0,1]$. By definition, if $\beta \in (0,1)$, $\mathcal{C}^\beta(X) := \mathrm{Lip}_\beta(X)$. If $\beta = 1$, $\mathrm{Lip}(X) := \mathrm{Lip}_1(X)$ is the set of Lipschitz functions. We shall denote the Lipschitz constant of a function F by $L(F) := \sup_{x,y\in X, x\neq y} \left|\frac{F(x)-F(y)}{x-y}\right|$. Let $\pi : \mathbb{R} \to \mathbb{T}$, $x \mapsto x + \mathbb{Z}$ be the canonical projection. For any continuous map $\overline{F} : \mathbb{T} \to \mathbb{T}$, the function F satisfying $\overline{F} \circ \pi = \pi \circ F$ is called a lift of $\overline{F}$ to $\mathbb{R}$. Denote by $\mathrm{Diff}^k(\mathbb{R})$ the $\mathcal{C}^k$-diffeomorphisms on $\mathbb{R}$. Let $[x]$ be the integer part of a real number x. One calls $\mathcal{D}^k(\mathbb{T})$, $k \in \mathbb{R}_+ \cup \{+\infty, \omega\}$, the set of lifts of the orientation-preserving $\mathcal{C}^k$-diffeomorphisms of $\mathbb{T}$, i.e. $\mathcal{D}^k(\mathbb{T}) := \left\{F \in \mathrm{Diff}^{[k]}(\mathbb{R});\ D^{[k]}(F - \mathrm{Id}) \in \mathcal{C}^{k-[k]}(\mathbb{T})\right\}$. One calls $F \in \mathcal{D}^0(\mathbb{T})$ a Lipschitz homeomorphism if F and F^{-1} are Lipschitz continuous. Recall that a 1-periodic function $F : \mathbb{R} \to \mathbb{R}$ is of bounded variation on $\mathbb{T}$ if F is of bounded variation on $[0,1]$ and one denotes by $\mathcal{BV}(\mathbb{T})$ the set of functions of bounded variation on $\mathbb{T}$.

For any $F \in \mathcal{D}^0(\mathbb{T})$, the rotation number $\rho(F)$ is defined by:

$$\rho(F) := \lim_{n\to+\infty} \frac{F^n(x) - x}{n}\ , \quad x \in \mathbb{R}\ , \tag{9}$$

where $F^n := F \circ F \circ \ldots \circ F$ is the n-th iterate of F. In Herman [22, prop. II.2.3, p. 20], the limit (9) is proven to exist (it is a real number independent of x) and to be uniform w.r.t. x. If in the sequel $\rho(F) = \frac{p}{q}$ for $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$, it is always assumed that p and q are relatively primes.

Definition 2.1 *A point $x_0 \in \mathbb{R}$ is said to be a periodic point of period $q \in \mathbb{N}^*$ of $F \in \mathcal{D}^0(\mathbb{T})$ if there exists $p \in \mathbb{N}$ such that $F^q(x_0) = x_0 + p$.*

The point x_0 is said to be attracting if there exists a neighborhood U of x_0 such that for all $x \in U$, $F^{nq}(x) - np$ tends to x_0 as n tends to $+\infty$. If x_0 is an attracting periodic point of F^{-1}, then x_0 is called a repelling periodic point of F.

$F \in \mathcal{D}^1(\mathbb{T})$ is said to be a Morse-Smale diffeomorphism if F has a finite nonzero number of periodic points a_k (with period $q \in \mathbb{N}^*$), all of them hyperbolic (i.e. either $DF^q(a_k) < 1$ and in this case the point is attracting or $DF^q(a_k) > 1$ and in this case the point is repelling.).

One can show (e.g. Herman [22, prop. II.5.3, p. 24]) that the existence of a periodic point x_0 for $F \in \mathcal{D}^0(\mathbb{T})$, $F^q(x_0) = x_0 + p$, is equivalent to $\rho(F) = \frac{p}{q} \in \mathbb{Q}$. It means that if the rotation number is irrational then there are no periodic points.

The minimal assumptions that we shall make on the moving boundary described by a are the following:

Assumption 2.2 *The function a is stricly positive, Lipschitz continuous on $\mathbb{R}$ with $L(a) \in [0, 1)$.*

Other assumptions will be always given explicitly. In particular if a is 1-periodic, then $a \in \mathrm{Lip}(\mathbb{T})$.

Let Id be the identity on $\mathbb{R}$. Define $h := \mathrm{Id} - a$, $k := \mathrm{Id} + a$ on $\mathbb{R}$. Under Assumption 2.2, it is easy to see that h, k, h^{-1}, k^{-1} are Lipschitz homeomorphisms on $\mathbb{R}$ and $L(k), L(h) \leq 1 + L(a)$, $L(h^{-1}), L(k^{-1}) \leq \frac{1}{1-L(a)}$. Hence $k \circ h^{-1}, h \circ k^{-1}$ are also Lipschitz homeomorphisms on $\mathbb{R}$ and $L(k \circ h^{-1}), L(h \circ k^{-1}) \leq \frac{1+L(a)}{1-L(a)}$. If moreover a is 1-periodic, then $h, k, k \circ h^{-1}, h \circ k^{-1} \in \mathcal{D}^0(\mathbb{T})$. The proof of the following lemma is left to the reader.

Lemma 2.3 *The sets $\mathcal{A}_n := \{a \in \mathcal{C}^n(\mathbb{T});\ a > 0,\ a \in \mathrm{Lip}(\mathbb{T}),\ L(a) \in [0, 1)\}$ and $\mathcal{F}_n := \{F \in \mathcal{D}^n(\mathbb{T});\ F > \mathrm{Id},\ F \in \mathrm{Lip}(\mathbb{R}),\ F^{-1} \in \mathrm{Lip}(\mathbb{R})\}$ equipped with $\mathcal{C}^n$-topologies are homeomorphic for $n \in \mathbb{N} \cup \{+\infty, \omega\}$. If $n \geq 1$, $\mathcal{A}_n$ and $\mathcal{F}_n$ are open subsets of $\mathcal{C}^n(\mathbb{T})$ and $\mathcal{D}^n(\mathbb{T})$ respectively.*

More generally there exists a bijection between the set of functions a satisfying Assumption 2.2 and the set of Lipschitz homeomorphisms F on $\mathbb{R}$ with $F > \mathrm{Id}$. We will need the relations:

$$F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1} = \mathrm{Id} + 2a \circ (\mathrm{Id} - a)^{-1}, \tag{10}$$

$$a = \left(\frac{F - \mathrm{Id}}{2}\right) \circ \left(\frac{F + \mathrm{Id}}{2}\right)^{-1}. \tag{11}$$

In the sequel, there is a deep interplay between the invariant measures of diffeomorphisms of the circle and the physics of the problems. Let us remind some definitions. Let X be a compact metric space (for instance, $X = \mathbb{T}$) and $F : X \to X$ be a continuous map. The measure μ is said to be an invariant measure of F iff μ belongs to the set of probability measures on X (i.e. $\mu \in (\mathcal{C}^0(X))'$ the dual space of $\mathcal{C}^0(X)$, $\mu \geq 0$ and $\mu(X) = 1$) and for every μ-measurable set A, $\mu(F^{-1}(A)) = \mu(A)$. According to Krylov and Bogolyubov's theorem (e.g. Katok and Hasselblatt [23, th. 4.1.1, p. 135]), for any continuous map of X, there exists at least one invariant measure. For the particular case of $F \in \mathcal{D}^0(\mathbb{T})$: if $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, the invariant measure μ is unique (F is said uniquely ergodic, see Herman [22, prop. II.8.5, p. 28]). If $\rho(F) \in \mathbb{Q}$, the invariant measure is in general not unique and it may be atomic. As an illustration we prove the lemma:

Lemma 2.4 *Let $a \in \mathcal{A}_0$ and F defined by (10). Let p, $q \in \mathbb{N}^*$, $x_0 \in \mathbb{R}$. Then:*

$$F^q(x_0) = x_0 + p \Longleftrightarrow \sum_{k=0}^{q-1} a \circ h^{-1} \circ F^k(x_0) = \frac{p}{2}\ . \tag{12}$$

Proof: Assume: $F^q(x_0) = x_0 + p$. Thus $\rho(F) = \frac{p}{q}$. Clearly

$$\mu := \frac{1}{q} \sum_{k=0}^{q-1} \delta_{\overline{F}^k(x_0)}$$

is an invariant probability measure for $\overline{F}$ ($\delta_{\overline{F}^k(x_0)}$ is the Dirac measure at $\overline{F}^k(x_0)$). Let $\psi := F - \mathrm{Id} \in \mathcal{C}^0(\mathbb{T})$. By formula (10), $\psi = 2a \circ h^{-1}$. According to Herman [22, prop. II.2.3, p. 20],

$$\rho(F) = \int_{\mathbb{T}} \psi \, d\mu\ ,$$

so that

$$\frac{p}{q} = 2 \int_{\mathbb{T}} a \circ h^{-1} \, d\mu\ . \tag{13}$$

Inserting μ into (13), the equality $\sum_{k=0}^{q-1} a \circ h^{-1} \circ F^k(x_0) = \frac{p}{2}$ is proven.

Conversely, assume the formula on the right-hand side of the equivalence holds and let $\psi := F - \mathrm{Id} \in \mathcal{C}^0(\mathbb{T})$. Then $F^n = \mathrm{Id} + \sum_{k=0}^{n-1} \psi \circ F^k$, so $F^q(x_0) = x_0 + 2 \sum_{k=0}^{q-1} a \circ h^{-1} \circ F^k(x_0) = x_0 + p$. ∎

Remark 2.5 Let $x_0 \in I_0$. Then $\{(a \circ h^{-1} \circ F^n(x_0), h^{-1} \circ F^n(x_0));\ n \in \mathbb{N}\}$ is the set of intersections between the characteristic $t + x = x_0$, reflecting against the boundaries, and the moving boundary in the (x, t) plane. The physical explanation of Lemma 2.4 is the following: the relation (12) expresses the existence of a periodic characteristic of period p after q bounces.

In order to apply Denjoy's theorem we will use the functions of class P (Herman [22, def. VI.4.1, p. 74]).

Definition 2.6 *Let $F \in \mathcal{D}^0(\mathbb{T})$. Then F is of class P if F is differentiable up to a countable set (u.c.s.) and if its derivative is equal u.c.s. to a (1-periodic) function which is bounded from below by a positive number and is of bounded variation on $\mathbb{T}$.*

Let us give a slight generalisation of the variation of a function and define a suitable class of functions a.

Definition 2.7 *Let f be a function defined u.c.s. in $\mathbb{T}$. One denotes:*

$$\widehat{\mathrm{Var}}(f) := \inf\left\{\mathrm{Var}(b);\ b : \mathbb{T} \to \mathbb{R},\ b(x) = f(x) \text{ u.c.s. in } \mathbb{T}\right\}.$$

A function a is said to be of class Q if $a \in \mathcal{A}_0$, a is differentiable u.c.s. in $\mathbb{T}$ and $\widehat{\mathrm{Var}}(a') < +\infty$.

A link between the classes P and Q is established in the following lemma.

Lemma 2.8 *If a is of class Q, then $F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1}$ is of class P and*

$$\widehat{\mathrm{Var}}(F') \leq \frac{2}{(1 - L(a))^2}\widehat{\mathrm{Var}}(a') .$$

Proof: For any $\epsilon > 0$ there exists $b_1 : \mathbb{T} \to \mathbb{R}$ such that $b_1 = a'$ u.c.s. in $\mathbb{T}$ and $\mathrm{Var}(b_1) < \widehat{\mathrm{Var}}(a') + \epsilon$. Defining:

$$\forall x \in \mathbb{T}, \quad b(x) := \begin{cases} L(a) & \text{if } \ b_1(x) > L(a) \\ b_1(x) & \text{if } \ -L(a) \leq b_1(x) \leq L(a) \\ -L(a) & \text{if } \ b_1(x) < -L(a)\ , \end{cases}$$

then $b = a'$ u.c.s. in $\mathbb{T}$ and $\mathrm{Var}(b) \leq \mathrm{Var}(b_1) < \widehat{\mathrm{Var}}(a') + \epsilon$. We have: $F' = \frac{1+a' \circ h^{-1}}{1-a' \circ h^{-1}}$ exists u.c.s. and is bounded in the L^∞-norm. Since $h^{-1} \in \mathcal{D}^0(\mathbb{T})$, $b \circ h^{-1} \in \mathcal{BV}(\mathbb{T})$ and $\mathrm{Var}(b \circ h^{-1}) = \mathrm{Var}(b)$. Let $G := \frac{1+b \circ h^{-1}}{1-b\circ h^{-1}}$. Then $G = F'$ u.c.s. and

$$\begin{aligned} \mathrm{Var}(G) &= \sup \sum_{k=1}^{n} \left| \frac{1 + b \circ h^{-1}(t_k)}{1 - b \circ h^{-1}(t_k)} - \frac{1 + b \circ h^{-1}(t_{k-1})}{1 - b \circ h^{-1}(t_{k-1})} \right| \\ &= \sup \sum_{k=1}^{n} \frac{2|b \circ h^{-1}(t_k) - b \circ h^{-1}(t_{k-1})|}{|1 - b \circ h^{-1}(t_k)| \cdot |1 - b \circ h^{-1}(t_{k-1})|} \\ &\leq \frac{2}{(1 - L(a))^2}\mathrm{Var}(b \circ h^{-1}) \\ &= \frac{2}{(1 - L(a))^2}\mathrm{Var}(b) < \frac{2}{(1 - L(a))^2}\left(\widehat{\mathrm{Var}}(a') + \epsilon\right) \end{aligned}$$

where the supremum is taken over all $0 \le t_0 \le t_1 \le \ldots \le t_n \le 1$. Since this inequality holds for any $\epsilon > 0$,

$$\widehat{\mathrm{Var}}(F') \le \mathrm{Var}(G) \le \frac{2}{(1-L(a))^2}\widehat{\mathrm{Var}}(a') \; . \quad \blacksquare$$

We recall that $F := k \circ h^{-1}$ and we define:

$$\forall n \in \mathbb{Z}, \quad I_n := [F^n(-a(0)), F^n(a(0)))\,. \tag{14}$$

For convenience, denote $x_n := F^n(a(0))$, for all $n \in \mathbb{Z}$. Since $F(-a(0)) = a(0)$, $I_n = [x_{n-1}, x_n)$. If a is bounded from below and from above, then

$$\forall n \in \mathbb{Z}, \quad 2a_{\min} \le x_{n+1} - x_n \le 2a_{\max}\,. \tag{15}$$

It can be easily seen that

$$\bigcup_{n\in\mathbb{Z}} I_n = \mathbb{R} \quad \text{and} \quad \forall n \neq m,\ I_m \cap I_n = \emptyset\,. \tag{16}$$

Clearly $\{I_n\}_{n\in\mathbb{Z}}$ is a partition of $\mathbb{R}$.

The aim of the present paper is to study the energy E of the field:

$$E(t) = \frac{1}{2}\left(\|\varphi_x(\cdot,t)\|^2_{(0,a(t))} + \|\varphi_t(\cdot,t)\|^2_{(0,a(t))}\right)\,,$$

where we denote by $\|\cdot\|_X$ the L^2-norm on the measurable set X with respect to the Lebesgue measure m. It will appear useful to give the following definition.

Definition 2.9 *We shall say that the model is stable if E is bounded from below and from above by strictly positive constants. Equivalently the model is unstable if the limit superior (resp. limit inferior) of E is infinite (resp. zero). Moreover if the limit of E exists and is infinite or zero, then the model is said strongly unstable.*

Let $\Sigma \subset \mathbb{R}^n$, $n \ge 1$, be an open set. Let $\mathcal{D}(\Sigma) := \mathcal{C}_0^\infty(\Sigma)$ and $\mathcal{D}'(\Sigma)$ be its dual. We will use also the standard Sobolev spaces $H^1(\Sigma)$, $H_0^1(\Sigma)$, $H^1_{\mathrm{loc}}(\Sigma)$ and $H^1_{0,\mathrm{loc}}(\Sigma)$ defined as the space of distributions φ such that for all $\psi \in \mathcal{D}(\mathbb{R}^n)$, $\psi\phi \in H_0^1(\Sigma)$.

Definition 2.10 *Let $\Omega := \{(x,t) \in \mathbb{R}^2;\ 0 < x < a(t)\}$. Then $\varphi \in H^1_{\mathrm{loc}}(\Omega)$ is called a weak solution of (1), (4)-(5) or of (1), (6)-(7) if*

$$\forall \psi \in \mathcal{D}(\Omega), \quad (\varphi_t, \psi_t)_\Omega - (\varphi_x, \psi_x)_\Omega = 0 \tag{17}$$

(or simply $\varphi_{tt} - \varphi_{xx} = 0$ in $\mathcal{D}'(\Omega)$) and the boundary conditions (4)-(5) or (6)-(7) are satisfied.

Remark 2.11 The traces needed for the initial and boundary conditions do not exist for general φ in $H^1_{\mathrm{loc}}(\Omega)$. However they exist if φ satisfies (17); see Theorems 2.12 and 2.13 below.

2.2 Existence and unicity of the solution

We were not able to find in the literature a proof of the existence and unicity of a solution with finite energy to our problems when the boundary a is not smooth. Indeed the standard technique consists in transforming the problems into new ones with fixed boundaries and then to apply the general theory of hyperbolic partial differential equations with variable coefficients (cf. Ladyzhenskaya [25], Lions and Magenes [26], etc.). But this requires roughly that a is $\mathcal{C}^2$. That is why we give in details the proof for more general a.

Theorem 2.12 *(Dirichlet problem)*
If $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$ and

$$(\varphi_0, \varphi_1) \in H_0^1((0, a(0))) \times L^2((0, a(0))) \ ,$$

then there exists a unique weak solution φ of the Dirichlet problem satisfying the initial conditions (2)-(3). Moreover there exists $f \in H^1_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\varphi(x,t) = f(t+x) - f(t-x) \quad a.e.\ in\ \Omega \tag{18}$$

and $\varphi \in L^\infty(\Omega) \cap H^1_{0,\mathrm{loc}}(\Omega)$.

Proof: *First step: the form of general solution and the regularity of traces.* By the assumptions on Ω, if a weak solution φ exists, then there exist f, $g \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that $\varphi(x,t) = f(t+x) + g(t-x)$ a.e. in Ω. Moreover φ can be extended continuously to $\overline{\Omega}$ (so that the last relation holds in $\overline{\Omega}$). The proof is given in appendix (Lemma B). Functions of such a form have traces on the boundaries of Ω and on the intersections of Ω with lines $t = constant$ which are given by continuous extension; moreover these traces are in $H^1_{\mathrm{loc}}(\partial\Omega)$ and in $H^1((0, a(t)))$ respectively.
Second step: the fixed boundary. Since φ has a trace $\varphi(0,t) = f(t) + g(-t)$ on $x = 0$, $\varphi(0,t) = 0$ implies that $g = -f$. Then $\varphi(x,t) = f(t+x) - f(t-x)$ a.e. in Ω.
Third step: the initial conditions. Since φ has a trace on $t = 0$ given by $\varphi(x,0) = f(x) - f(-x)$ and $\varphi_t(x,0) = f'(x) - f'(-x)$ a.e. in $(0, a(0))$, the initial conditions (2)-(3) give:

$$\forall x \in [0, a(0)], \quad f(x) := f(0) + \frac{1}{2}\left(\varphi_0(x) + \int_0^x \varphi_1(y)\,dy\right) , \tag{19}$$

$$f(-x) := f(0) + \frac{1}{2}\left(-\varphi_0(x) + \int_0^x \varphi_1(y)\,dy\right) . \tag{20}$$

The constant $f(0)$ is arbitrary. Clearly $f \in H^1(I_0)$, since $\varphi_0(0) = 0$ by assumption.
Fourth step: the moving boundary. The function φ has a trace on $x = a(t)$ and $\varphi(a(t), t) = 0$; this implies that

$$f \circ F = f \quad \text{on } \mathbb{R} \quad \text{with } F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1} \ . \tag{21}$$

Fifth step: construction of a solution. By iteration of the formula (21): $\forall n \in \mathbb{Z},\ f \circ F^n = f$. Thus the function f is known on I_n for all $n \in \mathbb{Z}$, and $f \in \mathcal{C}^0(\dot{I}_n)$ for all $n \in \mathbb{Z}$. Since $\varphi_0(a(0)) = 0$ and $F(-a(0)) = a(0)$, f is continuous at $\pm a(0)$, hence $f \in \mathcal{C}^0(\mathbb{R})$ by iterations. Since F is Lipschitz continuous, $f \in H^1(I_n)$ for all $n \in \mathbb{Z}$, thus $f \in H^1_{\text{loc}}(\mathbb{R})$ and $\varphi \in H^1_{0,\text{loc}}(\Omega)$. Note that $\|f\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(I_0)} < +\infty$. Obviously $\varphi \in L^\infty(\Omega)$, since $\|\varphi\|_{L^\infty(\Omega)} \leq 2\|f\|_{L^\infty(I_0)} < +\infty$.
Sixth step: unicity. Up to a constant, f is uniquely determined on I_0 and by (21), f is then unique on $\mathbb{R}$. Thus φ is unique, since the difference $f(t+x) - f(t-x)$ is independent of the constant $f(0)$. ∎

Theorem 2.13 *(Neumann problem)*
If $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$ and

$$(\varphi_0, \varphi_1) \in H^1((0, a(0))) \times L^2((0, a(0))) \ ,$$

then there exists a unique weak solution φ of the Neumann problem satisfying the initial conditions (2)-(3). Moreover there exists $f \in H^1_{\text{loc}}(\mathbb{R})$ such that

$$\varphi(x,t) = f(t+x) + f(t-x) \quad a.e.\ in\ \Omega \ . \tag{22}$$

The proof, very similar to that for the Dirichlet problem, is given in appendix. One just needs to notice that the moving boundary condition yields the essential functional equation

$$f' \circ F = f' \quad \text{a.e. in } \mathbb{R} \ . \tag{23}$$

In the sequel the initial conditions are assumed to satisfy the requirements for the existence and unicity of the weak solution if other assumptions are not mentioned. Let us give also conditions for the existence of classical $\mathcal{C}^2$-solutions. The proof is left to the reader.

Theorem 2.14 *Let $a \in \mathcal{C}^2(\mathbb{R})$, $|a'| < 1$, $a > 0$ and $(\varphi_0, \varphi_1) \in \mathcal{C}^2([0, a(0)]) \times \mathcal{C}^1([0, a(0)])$. If*

$$\begin{gathered} \varphi_0(0) = 0 \ , \quad \varphi_0(a(0)) = 0 \ , \quad \varphi_0''(0) = 0 \ , \\ \varphi_1(0) = 0 \ , \quad \varphi_1(a(0)) + a'(0)\varphi_0'(a(0)) = 0 \ , \\ (1 + a'^2(0))\varphi_0''(a(0)) + 2a'(0)\varphi_1'(a(0)) + a''(0)\varphi_0'(a(0)) = 0 \ , \end{gathered}$$

then there exists a unique classical $\mathcal{C}^2$-solution of the equations (1)-(5). If

$$\varphi_0'(0) = 0 \ , \quad \varphi_0'(a(0)) = 0 \ , \quad \varphi_1'(0) = 0 \ , \quad \varphi_1'(a(0)) + a'(0)\varphi_0''(a(0)) = 0 \ ,$$

then there exists a unique classical $\mathcal{C}^2$-solution of the equations (1)-(3), (6)-(7).

2.3 Some useful lemmas

We start by some considerations on the energy E.

Lemma 2.15 *Under the assumptions of Theorem 2.12 (Dirichlet problem) or Theorem 2.13 (Neumann problem) the following assertions hold:*

1) $\forall t \in \mathbb{R},\ E(t) = \|f'\|^2_{(h(t),k(t))} < +\infty$.

2) If $E(0) = 0$, *then for all* $t \in \mathbb{R}$, $E(t) = 0$.

3) E *is absolutely continuous on* $\mathbb{R}$.

4) Finally:

$$\begin{aligned} E' &= -2a'\frac{k'}{h'}|f' \circ k|^2 \quad \textit{a.e. in } \mathbb{R} \quad \textit{(Dirichlet problem)}, && (24)\\ E' &= 2a'|f' \circ k|^2 \quad \textit{a.e. in } \mathbb{R} \quad \textit{(Neumann problem)}. && (25) \end{aligned}$$

Proof: Let us show 1) for the Dirichlet problem (the proof is similar for the Neumann problem). Since $\varphi(x,t) = f(t+x) - f(t-x)$ a.e. in Ω,

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|\varphi_x(\cdot,t)\|^2_{(0,a(t))} + \|\varphi_t(\cdot,t)\|^2_{(0,a(t))}\right)\\ &= \frac{1}{2}\left(\|f'(t+\cdot) + f'(t-\cdot)\|^2_{(0,a(t))} + \|f'(t+\cdot) - f'(t-\cdot)\|^2_{(0,a(t))}\right)\\ &= \|f'\|^2_{(h(t),k(t))}\ . \end{aligned}$$

For all $t \in \mathbb{R}$, $k(t) - h(t) = 2a(t)$ is finite, since a is continuous. The function f' is in $L^2_{\text{loc}}(\mathbb{R})$, therefore $E(t) < +\infty$ for all $t \in \mathbb{R}$. This proves 1).

If $E(0) = 0$, then $f'(x) = 0$ a.e. on I_0. By the relations (21) or (23), $f'(x) = 0$ a.e. in $\mathbb{R}$ and 2) is proved.

Part 3) follows from 1), the Lipschitz property of h and k and the fact that $f' \in L^2_{\text{loc}}(\mathbb{R})$.

Then the differentiation of E (the derivatives exist a.e. by 3)) and the use of the relations (21) (Dirichlet problem) or (23) (Neumann problem) give 4). ∎

Remark 2.16 Statement 4) of Lemma 2.15 says that E and a vary in opposite sense in the Dirichlet problem and in the same sense in the Neumann problem. This is a manisfestation of the Doppler effect.

In Lemma 2.17 we give a useful formula for E. We recall that the intervals $\{I_n\}_{n\in\mathbb{Z}}$ are defined by (14).

Lemma 2.17 *Let* $a \in \text{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$. *Then for all* t *in* $\mathbb{R}$, *there exists a unique* $n(t)$ *in* $\mathbb{Z}$ *such that* $h(t) \in I_{n(t)}$. *This defines a function*

$n : \mathbb{R} \to \mathbb{Z}$ with $n(0) = 0$. For the Dirichlet problem there exists a function α_D such that

$$\forall t \in \mathbb{R}, \quad E(t) = \alpha_D(t) \left\| \frac{f'}{\sqrt{DF^{n(t)}}} \right\|_{I_0}^2 \tag{26}$$

and for the Neumann problem there exists a function α_N such that

$$\forall t \in \mathbb{R}, \quad E(t) = \alpha_N(t) \left\| f' \sqrt{DF^{n(t)}} \right\|_{I_0}^2 . \tag{27}$$

Moreover $\alpha_D(0) = \alpha_N(0) = 1$ and

$$\forall t \in \mathbb{R}, \quad \frac{1}{F'_{\max}} \leq \alpha_D(t) \leq \frac{1}{F'_{\min}} , \quad F'_{\min} \leq \alpha_N(t) \leq F'_{\max} .$$

If in addition a is 1-periodic, then

$$\forall t \in \mathbb{R}^*, \quad \frac{1}{\rho(F)} - \frac{a(t)}{\rho(F)t} - \frac{1 + a(0)}{\rho(F)|t|} < \frac{n(t)}{t} < \frac{1}{\rho(F)} - \frac{a(t)}{\rho(F)t} + \frac{1 + a(0)}{\rho(F)|t|} . \tag{28}$$

Proof: The first statement about the existence of the function n is true since $\{I_n\}_{n \in \mathbb{Z}}$ is a partition of $\mathbb{R}$.

Assume $E(0) > 0$ (the opposite case is trivial), then $f' \neq 0$ (as a function in $L^2(I_0)$) and for all $n \in \mathbb{Z}$, $\|f'\|_{I_n}^2 = \left\| \frac{f'}{\sqrt{DF^n}} \right\|_{I_0}^2$, since I_n is mapped by F^{-n} onto I_0 and $DF^n \cdot f' \circ F^n = f'$ a.e. on $\mathbb{R}$. Therefore by Lemma 2.15, $\|f'\|_{I_n} > 0$. It follows that $\alpha_D(t) := \frac{E(t)}{\|f'\|_{I_{n(t)}}^2}$ is well defined. To find the bounds on α_D we use again Lemma 2.15 and the relation: $\forall t \in \mathbb{R}$,

$$\begin{aligned} \|f'\|_{(h(t),k(t))}^2 &= \|f'\|_{(h(t),x_{n(t)})}^2 + \|f'\|_{(x_{n(t)},k(t))}^2 \\ &= \|f'\|_{(h(t),x_{n(t)})}^2 + \left\| \frac{f'}{\sqrt{F'}} \right\|_{(x_{n(t)-1},h(t))}^2 . \end{aligned}$$

Similar considerations hold for the Neumann case.

If a is 1-periodic, according to Herman [22, prop. II.2.3, p. 20],

$$\forall n \in \mathbb{Z}^*,\ \forall x \in \mathbb{R}, \quad -\frac{1}{|n|} < \frac{F^n(x) - x}{n} - \rho(F) < \frac{1}{|n|} .$$

Then, choosing $x = \pm a(0)$ and taking into account that $F^n(-a(0)) \leq h(t) < F^n(a(0))$, we have

$$\forall n \in \mathbb{Z}, \quad \forall t \in h^{-1}(I_n), \quad n\rho(F) - 1 - a(0) < h(t) < n\rho(F) + 1 + a(0) .$$

Since $\rho(F) > 0$ under our assumptions, (28) is proven. ∎

3 Dirichlet problem

3.1 Stability

3.1.1 A universal lower bound

We prove that the energy cannot become arbitrarily small. Note that a is not necessarily periodic. We recall that $f'(x) = \frac{1}{2}(\varphi_0'(x) + \varphi_1(x))$ a.e. in $(0, a(0))$ and $f'(x) = \frac{1}{2}(\varphi_0'(-x) - \varphi_1(-x))$ a.e. in $(-a(0), 0)$. Let us denote $M_c := \{x \in I_0;\ 0 < |f'(x)| \le c\}$ for $c \in \mathbb{R}_+^*$ and $\hat{E}_0 := \sup_{c \in \mathbb{R}_+^*} \frac{\|f'\|_{M_c}^4}{2a(0)c^2}$. It is easy to see that $0 < \hat{E}_0 \le E(0)$ if $E(0) > 0$.

Theorem 3.1 *Assume that $E(0) > 0$, $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$ and $a_{\max} < +\infty$. Then*

$$\forall t \in \mathbb{R}, \quad E(t) \ge \frac{a(0)}{a_{\max} F'_{\max}} \hat{E}_0 \ . \tag{29}$$

If in addition φ_0', $\varphi_1 \in L^\infty((0, a(0)))$, then

$$\forall t \in \mathbb{R}, \quad E(t) \ge \frac{1}{F'_{\max}} \frac{E(0)^2}{2a_{\max} \|f'\|_{L^\infty(I_0)}^2} \ . \tag{30}$$

Proof: By Cauchy-Schwarz inequality, for any $c \in \mathbb{R}_+^*$:

$$\forall n \in \mathbb{Z},\ \|f'\|_{M_c}^2 \le \left\| \frac{f'}{\sqrt{DF^n}} \right\|_{M_c} \left\| f' \sqrt{DF^n} \right\|_{M_c} \le \left\| \frac{f'}{\sqrt{DF^n}} \right\|_{M_c} \sqrt{2a_{\max} c^2} \ ,$$

since $\left\| \sqrt{DF^n} \right\|_{M_c}^2 \le \left\| \sqrt{DF^n} \right\|_{I_0}^2 = F^n(a(0)) - F^n(-a(0)) \le 2a_{\max}$ (see the formula (15)). By Lemma 2.17,

$$\forall t \in \mathbb{R}, \quad E(t) \ge \frac{1}{F'_{\max}} \left\| \frac{f'}{\sqrt{DF^{n(t)}}} \right\|_{M_c}^2 \ge \frac{1}{F'_{\max}} \frac{\|f'\|_{M_c}^4}{2a_{\max} c^2} \ .$$

This proves (29). If moreover φ_0', $\varphi_1 \in L^\infty((0, a(0)))$, then $f' \in L^\infty(I_0)$ and choosing $c := \|f'\|_{L^\infty(I_0)}$, we have: $\|f'\|_{M_c}^2 = E(0)$ and (30) follows. ∎

Remark 3.2 The condition $a_{\max} < +\infty$ is necessary for Theorem 3.1, since the energy may vanish asymptotically otherwise. To see that, assume that $a \in \mathcal{C}^1(\mathbb{R})$ is stricly increasing and $a' \ge \gamma > 0$ on $\mathbb{R}_+$. Then $a_{\max} = +\infty$ and $\lim_{t \to +\infty} E(t) = 0$. In fact, $F' \ge \frac{1+\gamma}{1-\gamma}$ on $[-a(0), +\infty)$. Then, by Lemma 2.17, $E(t) \le \left(\frac{1-\gamma}{1+\gamma}\right)^{n(t)} \int_{I_0} |f'(x)|^2 \, dx = E(0) \left(\frac{1-\gamma}{1+\gamma}\right)^{n(t)}$ for $t \ge 0$. Since $h(t)$ tends to infinity as t tends to infinity and $h(t) \le F^{n(t)}(a(0))$, $n(t)$ tends to infinity as t tends to infinity. Thus $\lim_{t \to +\infty} E(t) = 0$.

3.1.2 Periodicity

Special case of bounded energy is the energy periodic in time. We find the necessary and sufficient condition for the energy to be periodic for all initial data.

Theorem 3.3 *1) Let $a \in \mathcal{A}_0$. If $F^q = \mathrm{Id} + p$ for some $q \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$, then*

$$\forall (x,t) \in \Omega, \quad \varphi(x, t+p) = \varphi(x,t) \quad \textit{and} \quad \forall t \in \mathbb{R}, \quad E(t+p) = E(t)\ .$$

2) Let $a \in \mathrm{Lip}(\mathbb{R})$ such that $L(a) \in [0,1)$, $a > 0$. If there exists $p \in \mathbb{R}_+^*$ such that for all $(\varphi_0, \varphi_1) \in H_0^1((0,a(0))) \times L^2((0,a(0)))$, the relation

$$E(t+p) = E(t) \tag{31}$$

holds for all $t \in \mathbb{R}$, then a is periodic of period p and if a is not constant in $\mathbb{R}$, there exists $q \in \mathbb{N}$, $q \geq 2$, such that $F^q = \mathrm{Id} + p$.

Proof: 1) Let $a \in \mathcal{A}_0$, $F^q = \mathrm{Id} + p$ with $q \in \mathbb{N}^*$, $p \in \mathbb{N}^*$. Then the relation $f \circ F = f$ implies $f \circ (\mathrm{Id}+p) = f$ and the first statement follows from Lemma 2.15 1).

2) Let the assumptions of the second part hold now.
First step. Using the relations (19), (20) and (21), it is not difficult to see that for any $t \in \mathbb{R}$ and any $\psi \in L^2(I(t))$, $I(t) := (h(t), k(t))$, such that $\int_{I(t)} \psi\, dm = 0$, there exist initial conditions $(\varphi_0, \varphi_1) \in H_0^1((0,a(0))) \times L^2((0,a(0)))$ such that the restriction of f' to $I(t)$ is ψ. It is also easy to prove that if $\int_{I(t)} |\psi|^2 g\, dm = 0$ for some $g \in L^\infty(I(t))$ and all ψ of the above properties, then $g = 0$ a.e. in $I(t)$. To see that, choose e.g. $\psi(x) := \pm 1$ at the points where $g(x) \geq 0$ and 0 elsewhere; the signs can be chosen in such a way that $\int_{I(t)} \psi\, dm = 0$. Then we see that $g(x) > 0$ only on a set of zero measure. The same holds for $g(x) < 0$.
Second step. We recall the relation equivalent to (24):

$$E'(t) = -2\frac{h'(t)}{k'(t)} |f'(h(t))|^2 a'(t) \quad \text{a.e. on } \mathbb{R}\ .$$

Substituting this expression into the derivative of equation (31) we obtain

$$\frac{h'(t+p)}{k'(t+p)} |f'(h(t+p))|^2 a'(t+p) = \frac{h'(t)}{k'(t)} |f'(h(t))|^2 a'(t) \quad \text{a.e. on } \mathbb{R}\ . \tag{32}$$

For any $t \in \mathbb{R}$ there exist $t' \in \mathbb{R}$ and $q \in \mathbb{N}$ such that $h(t) \in I(t')$ and $h(t+p) \in F^q(I(t'))$. Here the integer q depends on t and t' but can be chosen as constant in a neighborhood of a given t. Using the relation $f' \circ F^{-q} \cdot DF^{-q} = f'$ a.e. on $\mathbb{R}$, the equation (32) reads

$$\frac{h'(t+p)}{k'(t+p)} DF^{-q}(h(t+p))^2 |f'(F^{-q}(h(t+p)))|^2 a'(t+p) =$$
$$= \frac{h'(t)}{k'(t)} |f'(h(t))|^2 a'(t) \quad \text{a.e. on } \mathbb{R} \,. \tag{33}$$

Here all the factors are nonzero except of a' values and f' values corresponding to the arguments in the same interval $I(t')$. Let $M_0 := \{x \in \mathbb{R};\, a'(x) = 0\}$, $M_1 := \{x \in \mathbb{R};\, a'(x)$ exists, $a'(x) \neq 0$ and (33) holds$\}$. The relation (33) is valid by assumption for all initial conditions (φ_0, φ_1) from the required set and consequently $a'(t) = a'(t+p) = 0$ for almost every $t \in M_0$ and

$$F^{-q}(h(t+p)) = h(t) \quad \text{for } t \in M_1 \,, \tag{34}$$

since otherwise we could simply construct (φ_0, φ_1) such that one side of the equation (33) is zero while the other is nonzero. At almost every points of M_1 which are accumulation points of M_1 we can differentiate equation (34) and substitute the result into equation (33). Choosing the initial conditions such that $f'(h(t)) \neq 0$ and realizing that $|a'| < 1$, a simple calculation gives: $a'(t+p) = a'(t)$ at almost every considered accumulation points. Since the isolated points of M_1 form a countable and therefore zero-measure set, the last relation holds a.e. in M_1 and M_0, i.e. a.e. in $\mathbb{R}$. So $a(t+p) = a(t) + c$ with a constant $c \in \mathbb{R}$. However, a is positive by assumption. Therefore $c = 0$ and p is a period of a, and also the relations $h(t+p) = h(t) + p$, $k(t+p) = k(t) + p$ and $F(t+p) = F(t) + p$ hold.

If a is not constant in $\mathbb{R}$, then $m(M_1) > 0$ and there exists a point t such that (34) holds. The relation $E(t+p) = E(t)$ expressed with the help of Lemma 2.15 1) leads to

$$\int_{I(t)} \left(1 - \frac{1}{DF^q}\right) |f'|^2 \, dm = 0 \,.$$

According to the first step, the relation $DF^q(x) = 1$ follows for almost every $x \in I(t)$ and then

$$F^q(x) = x + p \,,$$

taking into account (34). By application of F^n we now see that the last relation holds in every interval $F^n(I(t))$ with $n \in \mathbb{Z}$ and therefore in $\mathbb{R}$. ∎

Example 3.4 When $q = 1$ the condition of the part 1) of the above theorem is satisfied iff a is constant and equal to $a = \frac{p}{2}$. For $q = 2$ we are able to prove that

$$F^2 = \mathrm{Id} + p \Longleftrightarrow \forall t \in \mathbb{R}, \quad a(t) + a(t + \frac{p}{2}) = \frac{p}{2} \,.$$

An example of such function is: $a(t) := \frac{p}{4} + \alpha \sin(2\pi t)$ for some $p \in \mathbb{N}^*$, p odd, and $|\alpha| < \frac{1}{2\pi}$. It would be interesting to find such an explicit characterization for any q.

3.1.3 Absence of strong instability

Proposition 3.5 *Assume that a is of class Q and $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$. Let $V := \widehat{\mathrm{Var}}(\ln F')$. Then there exists a subsequence $\{I_{\pm q_n}\}_{n\in\mathbb{N}}$ of the sequence $\{I_n\}_{n\in\mathbb{Z}}$ defined by (14) such that q_n tends to infinity as n tends to infinity and*

$$\forall n \in \mathbb{N}, \quad \forall t \in h^{-1}(I_{\pm q_n}), \quad \frac{e^{-V}E(0)}{F'_{\max}} \le E(t) \le \frac{e^{V}E(0)}{F'_{\min}} . \tag{35}$$

Proof: By the conditions assumed on a, F is of class P (see Definition 2.6 and Lemma 2.8). Then the Denjoy's inequality holds (Herman [22, prop. VI.4.4, p. 75]): $\forall n \in \mathbb{N}$, $e^{-V} \le DF^{\pm q_n} \le e^{V}$ u.c.s.. The rationals $\frac{p_n}{q_n} \in \mathbb{Q}$ are the convergents of $\rho(F)$. The integers $q_n \in \mathbb{N}$ satisfy the following properties: $q_{n+1} > q_n$ $(n \ge 1)$, $q_n \ge 2^{\frac{n}{2}}$ $(n \ge 2)$. For further details see Herman [22, section V, p. 57] or Lang [27]. It follows from (26) that: $\forall n \in \mathbb{N}$, $\forall t \in h^{-1}(I_{\pm q_n})$, $e^{-V}E(0)\alpha_D(t) \le E(t) \le e^{V}E(0)\alpha_D(t)$. ∎

Remark 3.6 The behavior described by Proposition 3.5 is in fact more general. Let $m, n \in \mathbb{N}$ and define the sequence of intervals $J_{n,m} := \cup_{k=-m}^{m} I_{q_n+k}$. Then we have for all $t \in h^{-1}(J_{n,m})$,

$$\left(\min\left\{F'_{\min}, \frac{1}{F'_{\max}}\right\}\right)^{m+1} e^{-V}E(0) \le E(t) \le$$
$$\le \left(\max\left\{F'_{\max}, \frac{1}{F'_{\min}}\right\}\right)^{m+1} e^{V}E(0) .$$

Notice that under the condition of Proposition 3.5, the case of strong instability is excluded. The sequence $\{q_n\}_{n\in\mathbb{N}}$ used in the proof diverges at least exponentially.

3.1.4 A sufficient condition of stability

Let us first give some definitions. If $F \in \mathcal{D}^l(\mathbb{T})$, $0 \le l \le \omega$, then F is said to be $\mathcal{C}^k$-conjugate to $R_{\rho(F)} := \mathrm{Id} + \rho(F)$, $0 \le k \le l$, if there exists $g \in \mathcal{D}^k(\mathbb{T})$ such that $g \circ F = R_{\rho(F)} \circ g$. The function g is called the conjugacy of F. We start with a rather abstract but general theorem.

Theorem 3.7 *Assume $a \in \mathcal{A}_0$, $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$ and denote by μ the unique invariant measure of $\overline{F}$. Assume that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that the Radon-Nikodym derivative of μ w.r.t. m satisfies: $0 < \lambda_1 \le \frac{d\mu}{dm} \le \lambda_2 < +\infty$. Then*

$$\forall t \in \mathbb{R}, \quad \frac{\lambda_1}{\lambda_2}\frac{E(0)}{F'_{\max}} \le E(t) \le \frac{\lambda_2}{\lambda_1}\frac{E(0)}{F'_{\min}} . \tag{36}$$

Conversely, if $a \in \mathcal{A}_0$, $E(0) > 0$ and E is bounded from above by a strictly positive constant, then there exists an invariant measure of $\overline{F}$ which is not singular w.r.t. m.

Proof: Let $\phi := \frac{d\mu}{dm}$ be the Radon-Nikodym derivative and set $g(x) := \int_0^x d\mu$. By Herman [22, section II, p. 19], $g \circ F = g + \rho(F)$. Obviously g is absolutely continuous and $\lambda_1 \leq g' = \phi \leq \lambda_2$ a.e.. It follows that $g \in \mathcal{D}^0(\mathbb{T})$ and $\lambda_2^{-1} \leq Dg^{-1} \leq \lambda_1^{-1}$ a.e.. Now $F = g^{-1} \circ R_{\rho(F)} \circ g$, hence $DF^n = Dg^{-1} \circ R_{n\rho(F)} \circ g \cdot g'$ a.e.. Using Lemma 2.17, we have:

$$\forall t \in \mathbb{R}, \quad \frac{1}{F'_{\max}} \int_{I_0} \frac{|f'(x)|^2}{DF^{n(t)}(x)}\, dx \leq E(t) \leq \frac{1}{F'_{\min}} \int_{I_0} \frac{|f'(x)|^2}{DF^{n(t)}(x)}\, dx$$

and (36) follows immediately.

The statement of the second part of the theorem is equivalent to: if all the invariant measures of $\overline{F}$ are singular w.r.t. m, then $\sup_{t\in\mathbb{R}} E(t) = +\infty$. Since by Corollary 3.15 (which is proven independently), the supremum of E is infinite under this condition, the proof is complete. ∎

Remark 3.8 From the proof of Theorem 3.7, it can be seen that if $F \in \mathcal{F}_0$ is Lipschitz conjugate to the translation $R_{\rho(F)}$ (i.e. the conjugacy and its inverse are Lipschitz continuous), then the model is stable.

Let us give an explicit example of stability.

Example 3.9 Assume that $a \in \mathcal{A}_0$ is defined on one period by

$$a(t) := \begin{cases} \frac{\lambda-1}{\lambda+1}t + \frac{b}{\lambda+1} & \text{if } \frac{b}{2} \leq t \leq \frac{(\lambda+1)t_0+b}{2} \\ \frac{\lambda^{-\beta}-1}{\lambda^{-\beta}+1}t + \frac{1+b-\lambda^{-\beta}}{1+\lambda^{-\beta}} & \text{if } \frac{(\lambda+1)t_0+b}{2} \leq t \leq \frac{b}{2}+1 \end{cases}$$

for $b, \beta \in \mathbb{R}_+^*$, $\lambda > 1$ and $t_0 := \frac{1-\lambda^{-\beta}}{\lambda-\lambda^{-\beta}} \in (0,1)$. Computing $F := (\mathrm{Id}+a) \circ (\mathrm{Id}-a)^{-1}$, it can be shown that F satisfies assumptions of Herman [22, prop. VI.7.7.1, p. 82] and we have: if $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$ (by a suitable choice of b) and $\frac{\beta}{1+\beta} \in \mathbb{Z}\rho(F) \pmod 1$, then this model is stable.

By adding some conditions, we can prove a general result of stability. First we recall the definition:

Definition 3.10 *Let $\beta \in \mathbb{R}_+$. Then $\alpha \in \mathbb{R}$ is a β-Diophantine number if there exists $C > 0$ such that*

$$\forall \frac{p}{q} \in \mathbb{Q}\,, \quad \left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{2+\beta}}\,.$$

The set of Diophantine numbers is of full measure and meager in the real numbers. For definitions and details see Herman [22, section I, p. 11 and section V, p. 57] or Lang [27].

Theorem 3.11 *Assume: (i) $a \in \mathcal{C}^k(\mathbb{T})$, $k \in \mathbb{R}$, $k > 2$, $|a'| < 1$ and $a > 0$ and (ii) $\rho(F)$ is β-Diophantine with $\beta < k-2$. Then (36) holds.*

Proof: The assumption (i) implies that $F \in \mathcal{D}^k(\mathbb{T})$. A result of Katznelson and Ornstein [28] gives: let $F \in \mathcal{D}^k(\mathbb{T})$ with $\rho(F)$ a β-Diophantine number and $k > 2+\beta$. Then F is $\mathcal{C}^1$-conjugate to $\mathrm{Id}+\rho(F)$. By Remark 3.8, the proof is complete. ∎

3.2 Instability

3.2.1 A universal upper bound

It is not assumed here that a is periodic. We recall that $n(t)$ is defined in Lemma 2.17.

Proposition 3.12 *If $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$, then*

$$\forall t \in \mathbb{R}_+, \quad E(t) \leq E(0) \left(\frac{1}{F'_{\min}} \right)^{n(t)+1} . \tag{37}$$

If moreover $a_{\min} > 0$ and $E(0) > 0$, then the maximal rate of exponential increase of E defined by $r := \limsup_{t \to +\infty} \frac{\ln E(t)}{t}$ satisfies

$$r \leq \frac{1}{2a_{\min}} \ln \frac{1}{F'_{\min}} . \tag{38}$$

Proof: The first statement follows from Lemma 2.17:

$$\forall t \in \mathbb{R}_+, \quad E(t) = \alpha_D(t) \left\| \frac{f'}{\sqrt{DF^{n(t)}}} \right\|^2_{I_0} \leq \left(\frac{1}{F'_{\min}} \right)^{n(t)+1} \|f'\|^2_{I_0} .$$

Since $a_{\min} \leq a \leq a_{\max}$, then $\mathrm{Id} + 2a_{\min} \leq F \leq \mathrm{Id} + 2a_{\max}$ (since $F = \mathrm{Id} + 2a \circ h^{-1}$), hence $\mathrm{Id} + 2na_{\min} \leq F^n \leq \mathrm{Id} + 2na_{\max}$. Combining these inequalities and the fact that by definition $F^{n(t)}(-a(0)) \leq t - a(t) < F^{n(t)}(a(0))$, it is easy to prove that for all $t \in \mathbb{R}_+$,

$$\frac{h(t) - a(0)}{2a_{\max}} \leq n(t) \leq \frac{h(t) + a(0)}{2a_{\min}}$$

(with the convention that if $a_{\max} = +\infty$, then $a_{\max}^{-1} = 0$). Using (37), (38) follows. ∎

Remark 3.13 We will see in Remark 3.26 f) that this bound is almost optimal. Notice that in the case where a is periodic, one can deduce from the spectral analysis in Cooper and Koch [15] an other estimate, namely $r \leq 2 \ln r_U$ with r_U the spectral radius of the evolution operator on one period. This estimate is more accurate if there exists no attracting hyperbolic periodic point, since in this case $r_U = 1$ and therefore $r = 0$. It then follows that the rate of increase of E is subexponential.

3.2.2 A sufficient condition of instability

We recall that if $F \in \mathcal{D}^0(\mathbb{T})$, then $\overline{F}$ denotes its projection on the circle. Let us identify $[-a(0), -a(0) + 1] =: J \subset \mathbb{R}$ with $\mathbb{T}$ and denote for any $X \subset J$, $\tilde{X} := (\bigcup_{m,n \in \mathbb{Z}} (F^n(X) + m)) \cap I_0$.

Theorem 3.14 *Let $a \in \mathcal{A}_0$. Assume that for every invariant measure μ of $\overline{F}$, there exists a Borel set B in $\mathbb{T}$ such that $\mu(B) = 0$ and*

$$\|\varphi_0' + \varphi_1\|_{(\tilde{B} \cap (0,a(0)))} > 0 \quad or \quad \|\varphi_0' - \varphi_1\|_{((-\tilde{B}) \cap (0,a(0)))} > 0 \,, \tag{39}$$

then $m(B) > 0$ (i.e. m is not absolutely continuous w.r.t. μ) and

$$\limsup_{t \to +\infty} E(t) = +\infty \,.$$

Proof: The relation $m(B) > 0$ follows from (39) and from the fact that F is a Lipschitz homeomorphism. It is also clear that there exist two integers m_0, $n_0 \in \mathbb{Z}$ such that $\|f'\|_{L^1(M)} > 0$ for the set $M := (F^{n_0}(B) + m_0) \cap I_0$. Let us introduce a sequence of probability measures on $\mathbb{T}$ defined for all m-measurable set $X \subset \mathbb{T}$ by

$$m_n(X) := \int_X \frac{1}{n} \sum_{k=0}^{n-1} D\overline{F}^k \, dm \,. \tag{40}$$

Since the probability measures form a weak* compact subset of $\mathcal{C}^0(\mathbb{T})'$, there exists a subsequence $\{m_{n_i}\}_{i \in \mathbb{N}}$ which converges in the weak* topology to a Borel measure μ. It is easily seen that μ is an invariant measure of $\overline{F}$. By assumption, there exists a Borel set B in $\mathbb{T}$ such that $\mu(B) = 0$ and $m(B) > 0$. Due to the regularity of μ and m, for every $\epsilon > 0$, there exists an open set V_ϵ in J such that V_ϵ contains B and $\mu(V_\epsilon) < \epsilon$; there also exists a compact set K_ϵ in J such that B contains K_ϵ and $m(B \setminus K_\epsilon) < \epsilon$. According to Urysohn's lemma, there exists a function $g_\epsilon \in \mathcal{C}^0(\mathbb{T})$ such that the support of g_ϵ is contained in V_ϵ, the range of g_ϵ is included in $[0,1]$ and g_ϵ is equal to 1 on K_ϵ. With the help of the function g_ϵ it is easily seen that there exists $i_\epsilon \in \mathbb{N}$ such that for all $i \geq i_\epsilon$,

$$m_{n_i}(K_\epsilon) = \int_{K_\epsilon} \frac{1}{n_i} \sum_{k=0}^{n_i-1} D\overline{F}^k \, dm < \epsilon \,.$$

Let us denote $N_\epsilon := F^{n_0}(K_\epsilon) + m_0$ and $M_\epsilon := N_\epsilon \cap I_0$. We will extend the measure m_n to the Borel sets in $\mathbb{R}$ by formula (40) with $\overline{F}$ replaced by F. A straightforward calculation shows that

$$m_{n_i}(N_\epsilon) = \frac{1}{n_i} \int_{K_\epsilon} \left(\sum_{k=0}^{n_i-1} D\overline{F}^k - \sum_{k=0}^{n_0-1} D\overline{F}^k + \sum_{k=n_i}^{n_0+n_i-1} D\overline{F}^k \right) dm \,.$$

The integral of the last two sums is bounded by n_0. So there exists an integer i_ϵ' such that for all $i \geq i_\epsilon'$, $m_{n_i}(N_\epsilon) < 2\epsilon$. Since $m(M \setminus M_\epsilon) \leq DF_{\max}^{n_0}\epsilon$, we have: $\|f'\|_{L^1(M_\epsilon)} > \frac{1}{2}\|f'\|_{L^1(M)} > 0$ for sufficiently small ϵ. Now we will establish a useful inequality using Cauchy-Schwarz inequality. Let $A \subset I_0$ be an m-measurable arbitrary set. Then:

$$\begin{aligned}\|f'\|_{L^1(A)} &= \frac{1}{n}\sum_{k=0}^{n-1}\|f'\|_{L^1(A)} \leq \frac{1}{n}\sum_{k=0}^{n-1}\left\|\frac{f'}{\sqrt{DF^k}}\right\|_A \left\|\sqrt{DF^k}\right\|_A \\ &\leq \left(\frac{1}{n}\sum_{k=0}^{n-1}\left\|\frac{f'}{\sqrt{DF^k}}\right\|_A^2\right)^{\frac{1}{2}} \left(\frac{1}{n}\sum_{k=0}^{n-1}\left\|\sqrt{DF^k}\right\|_A^2\right)^{\frac{1}{2}} .\end{aligned}$$

Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers such that $t_k \in h^{-1}(I_k)$ for all $k \in \mathbb{N}$ and choose $A := M_\epsilon$:

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1}E(t_k) &= \frac{1}{n}\sum_{k=0}^{n-1}\alpha_D(t_k)\left\|\frac{f'}{\sqrt{DF^k}}\right\|_{I_0}^2 \geq \frac{1}{F'_{\max}}\frac{1}{n}\sum_{k=0}^{n-1}\left\|\frac{f'}{\sqrt{DF^k}}\right\|_{M_\epsilon}^2 \\ &\geq \frac{1}{F'_{\max}}\frac{\|f'\|_{L^1(M_\epsilon)}^2}{\frac{1}{n}\sum_{k=0}^{n-1}\left\|\sqrt{DF^k}\right\|_{M_\epsilon}^2} \geq \frac{1}{F'_{\max}}\frac{\|f'\|_{L^1(M)}^2}{4m_n(N_\epsilon)}\end{aligned}$$

and then for all $i \geq i'_\epsilon$,

$$\frac{1}{n_i}\sum_{k=0}^{n_i-1}E(t_k) > \frac{\|f'\|_{L^1(M)}^2}{8F'_{\max}}\frac{1}{\epsilon}$$

and since ϵ can take arbitrary small values,

$$\lim_{i\to+\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}E(t_k) = +\infty .$$

It follows that the limit superior of E is infinite as t tends to infinity. ∎

Corollary 3.15 *Let $a \in \mathcal{A}_0$ and assume that $E(0) > 0$.*

1) If every invariant measure of $\overline{F}$ is singular w.r.t. m, then

$$\limsup_{t\to+\infty}E(t) = +\infty .$$

2) If $\rho(F) \in \mathbb{R}\setminus\mathbb{Q}$ and the (unique) invariant measure of $\overline{F}$ is not absolutely continuous w.r.t. m, then

$$\limsup_{t\to+\infty}E(t) = +\infty .$$

Proof: Assume that the assumptions of 1) hold: then the Borel set B of Theorem 3.14 can be chosen such that $m(B) = 1$ and $\mu(B) = 0$, so $m(I_0 \setminus \tilde{B}) = 0$. Since $E(0) > 0$, $\|f'\|_{\tilde{B}} > 0$ and the conclusion of Theorem 3.14 holds.

Assume that the assumptions of 2) hold: then there exists a Borel set B in $\mathbb{T}$ such that $m(B) = 0$ and $\mu(B) > 0$. Denote $C := \cup_{n\in\mathbb{Z}}\overline{F}^n(B)$. Since

$F \in \mathcal{F}_0$, $\overline{F}(C) = C$ (i.e. C is $\overline{F}$-invariant) and $m(C) = 0$. Now we will prove that $\mu(C) = 1$ which will show that μ and m are singular. Then the first part of this corollary ends the proof. Let g be the conjugacy of F; g is continuous and nondecreasing (see Herman [22, prop. II.7.1, p. 25]) and $\overline{g} \circ \overline{F}(C) = \overline{g}(C) + \rho(F)$, hence $\overline{g}(C)$ is invariant by the irrational rotation $\mathrm{Id} + \rho(F)$ and since m is its unique invariant measure, it follows that $m(\overline{g}(C)) = 0$ or $m(\overline{g}(C)) = 1$. But $m(\overline{g}(B)) = \mu(B) > 0$, thus $\mu(C) = m(\overline{g}(C)) = 1$. ∎

Example 3.16 If in Example 3.9, $\beta = 1$ and $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, then the limit superior of the energy as the time tends to infinity is infinite, whenever $E(0) > 0$ (see Herman [22, th. VI.7.4, p. 79] and Corollary 3.15 2)).

Remark 3.17 In view of Theorems 3.7, 3.14 and Corollary 3.15, we put forward the question: whether there exists an example of $F \in \mathcal{F}_0$ with an invariant measure equivalent to m and leading to an infinite limit superior of the energy. We do not know presently the answer.

3.2.3 A sufficient condition of strong instability

We give sufficient condition for the unlimited growth of energy in this section. The main results are contained in Theorem 3.20 and its corollaries.

Definition 3.18 *A point $x \in \mathbb{R}$ is called a wandering point for $F \in \mathcal{F}_0$ if there exists an open neighborhood U of x in $\mathbb{R}$ such that*

$$\forall n \in \mathbb{N}^*, \quad \forall m \in \mathbb{Z}, \quad (F^n(U) + m) \cap U = \emptyset \, . \tag{41}$$

The set of wandering points in $\dot{I}_0$ will be denoted by

$$W(F) := \{x \in \dot{I}_0;\ x \text{ is a wandering point for } F\} \, .$$

Clearly, $W(F)$ is an open subset of $\dot{I}_0$ and $W(F) = W(F^{-1})$.

Lemma 3.19 *Let $a \in \mathcal{A}_0$. Then for all $x \in W(F)$ there exists an open set U such that $x \in U \subset W(F)$ and*

$$\lim_{n \to \pm\infty} \|DF^n\|_{L^1(U)} = 0 \, , \qquad \sum_{n=-\infty}^{+\infty} \|DF^n\|_{L^1(U)} \leq 2a(0) \, . \tag{42}$$

Proof: Since $x \in W(F)$ is a wandering point, there exists an open set $U \ni x$ in $\dot{I}_0$ such that $U \subset W(F)$ and $(F^n(U)+m) \cap U = \emptyset$ for all $n \in \mathbb{N}^*$ and $m \in \mathbb{Z}$. It follows easily that $\sum_{n=-\infty}^{+\infty} \|DF^n\|_{L^1(U)} = \sum_{n=-\infty}^{+\infty} m(F^n(U)) \leq 2a(0)$. ∎

Theorem 3.20 *If $a \in \mathcal{A}_0$ and:*

$$\|\varphi_0' + \varphi_1\|_{(W(F)\cap(0,a(0)))} > 0 \quad \textit{or} \quad \|\varphi_0' - \varphi_1\|_{(-W(F)\cap(0,a(0)))} > 0 \, ,$$

then

$$\lim_{t\to\pm\infty} E(t) = +\infty \ . \tag{43}$$

Moreover, for all $0 < c < +\infty$ *there exists an increasing sequence of integer numbers* $\{n_k\}_{k=-\infty}^{+\infty}$ *such that*

$$\forall t \in h^{-1}(I_{n_k}), \quad E(t) \geq c\frac{|t|}{\rho(F)} \ln \frac{|t|}{\rho(F)} \ . \tag{44}$$

Proof: Since $\|f'\|_{W(F)} > 0$ by assumption, $W(F)$ is nonempty, so by Lemma 3.19, there exists a subset $U \subset W(F)$ such that

$$\lim_{n\to+\infty} \|DF^n\|_{L^1(U)} = 0 \tag{45}$$

and $\|f'\|_U > 0$. Since $f' \in L^2_{\text{loc}}(\mathbb{R})$ and U is a bounded set,

$$0 < \|f'\|_{L^1(U)} < +\infty \ . \tag{46}$$

By Cauchy-Schwarz inequality,

$$\|f'\|_{L^1(U)} \leq \left\| \frac{f'}{\sqrt{DF^n}} \right\|_U \left\| \sqrt{DF^n} \right\|_U \ . \tag{47}$$

Then by Lemma 2.17,

$$\forall t \in \mathbb{R}, \quad E(t) \geq \frac{1}{F'_{\max}} \left\| \frac{f'}{\sqrt{DF^{n(t)}}} \right\|_U^2 \geq \frac{1}{F'_{\max}} \frac{\|f'\|^2_{L^1(U)}}{\|DF^{n(t)}\|^2_{L^1(U)}}$$

according to (47). Now, (45) and (46) imply (43).

Due to Lemma 3.19, for any $d > 0$ there exists an increasing sequence of integer numbers $\{n_k\}_{k=-\infty}^{+\infty}$ such that $\|DF^{n_k}\|_{L^1(U)} \leq \dfrac{d}{|n_k| \ln |n_k|}$ (otherwise the series of integrals in (42) would diverge). If $t \in h^{-1}(I_{n_k})$, then $h(t) \leq F^{n_k}(a(0))$ and $\left| \dfrac{F^{n_k}(a(0)) - a(0)}{n_k} - \rho(F) \right| < \dfrac{1}{|n_k|}$ by relation (28). Assume that $t \geq h^{-1}(a(0))$, then $n_k > 0$ and $\frac{h(t)-a(0)-1}{\rho(F)} < n_k$. Now for all $t \in h^{-1}(I_{n_k})$, $t > h^{-1}(a(0)) + 1$,

$$E(t) \geq \frac{1}{F'_{\max}} \frac{\|f'\|^2_{L^1(U)}}{d} \frac{|t - a(t) - a(0) - 1|}{\rho(F)} \ln \frac{|t - a(t) - a(0) - 1|}{\rho(F)} \ .$$

The same arguments can be repeated for $t < 0$. The required relation (44) follows. ∎

Corollary 3.21 *Assume* $a \in \mathcal{A}_0$, $\rho(F) = \frac{p}{q} \in \mathbb{Q}^*$ *and* $F^q \neq \text{Id} + p$. *Denote by* $A := \{x \in I_0;\ F^q(x) \neq x + p\}$ *the set of nonperiodic points. If* $\|\varphi_0' + \varphi_1\|_{(A\cap(0,a(0)))} \neq 0$ *or* $\|\varphi_0' - \varphi_1\|_{(-A\cap(0,a(0)))} \neq 0$, *then* $\lim_{t\to\pm\infty} E(t) = +\infty$.

Proof: We shall verify that $W(F)$ consists of the complement of the set of periodic points (thus $W(F) = A$) and $m(W(F)) > 0$. Since the set of wandering points is open it is sufficient to prove that at least one wandering point exists. By assumption, there exists a point $x_1 \in I_0$ such that $F^q(x_1) \neq x_1 + p$. Evidently, $F^q(x_1) \neq x_1 + n$ for any $n \in \mathbb{Z}$ (otherwise $\rho(F)$ would have a value different from $\frac{p}{q}$). There also exists a point $x_0 \in I_0$ such that $F^q(x_0) = x_0 + p$ as $\rho(F) = \frac{p}{q}$. Let us pass from I_0 and F to $\mathbb{T}$ and the projected map $\overline{F} : \mathbb{T} \to \mathbb{T}$. Let us call $\overline{x}_1 := \pi(x_1)$ and $\overline{x}_0 := \pi(x_0)$. There exists a maximal (connected) open interval U in $\mathbb{T}$ containing $\overline{x}_1$ such that $\overline{F}^q(x) \neq x$ for all $x \in U$. The interval U does not cover the whole $\mathbb{T}$ (e.g. $\overline{x}_0 \notin U$). The endpoints of U are fixed points of $\overline{F}^q$, one is attracting and the second is repelling. Since $\overline{F}^q$ is orientation-preserving and continuous, $\overline{F}^q(U) = U$. It is easily seen that all the points $\overline{x}_1, \overline{F}(\overline{x}_1), \ldots, \overline{F}^q(\overline{x}_1)$ are different. So there exists an open interval V such that $\overline{x}_1 \in V \subset U$ and the sets $V, \overline{F}(V), \ldots, \overline{F}^q(V)$ are pairwise disjoint. Looking for the ordering of endpoints of the intervals $\overline{F}^{nq}(V)$ in U it is seen that $V \cap \overline{F}^{nq}(V) = \emptyset$ for $n \in \mathbb{Z}^*$ (since $\overline{F}^q$ is monotone in U).

Now we prove that $V \cap \overline{F}^{nq+l}(V) = \emptyset$ for $n \in \mathbb{Z}^*$, $l \in \{0, 1, \ldots, q-1\}$. Let us assume the opposite: there exist $y \in V$ and $z \in V$ such that $y = \overline{F}^{nq+l}(z)$ for some $n \in \mathbb{Z}^*$ and $l \in \{0, 1, \ldots, q-1\}$. Then $\overline{F}^{mq}(y) = \overline{F}^l(\overline{F}^{(n+m)q}(z))$ for any $m \in \mathbb{Z}$. In the limit $m \to +\infty$, we have $\overline{F}^{mq}(y) \to b$, $\overline{F}^{(n+m)q}(z) \to b$ where b is the attracting endpoint of U. So $\overline{F}^l(b) = b$ which contradicts the value of $\rho(F) = \frac{p}{q}$. As a result, we have proved that $V \cap \overline{F}^m(V) = \emptyset$ for all $m \in \mathbb{Z}^*$ and x_1 is a wandering point. Since x_1 is an arbitrary nonperiodic point, then $A \subset W(F)$ (and even $A = W(F)$). Thus assumptions of Theorem 3.20 are valid. ∎

Corollary 3.22 *If $a \in \mathcal{A}_0$, F has a finite nonzero number of periodic points and $E(0)$ is strictly positive, then $\lim_{t \to \pm\infty} E(t) = +\infty$.*

Proof: By assumption, there exist p, $q \in \mathbb{N}^*$ such that $\rho(F) = \frac{p}{q}$. Moreover $[-a(0), a(0)] \setminus W(F)$ consists of a finite number of points; this can be seen in the proof of Corollary 3.21. Thus $m(W(F)) = m(I_0) = 2a(0) > 0$, which implies that $\|f'\|_{W(F)} > 0$. Since the assumptions of Theorem 3.20 are satisfied, E tends to infinity as t tends to infinity. ∎

Corollary 3.23 *Assume $a \in \mathcal{A}_0$, $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, F is not $\mathcal{C}^0$-conjugate to $\mathrm{Id} + \rho(F)$ and $\varphi_0' + \varphi_1 \neq 0$ a.e. in $W(F) \cap (0, a(0))$ or $\varphi_0' - \varphi_1 \neq 0$ a.e. in $(-W(F)) \cap (0, a(0))$. Then $\lim_{t \to \pm\infty} E(t) = +\infty$.*

Proof: To apply Theorem 3.20, let us prove that $m(W(F)) > 0$. As $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$, there exists a unique invariant measure μ of $\overline{F}$ in $\mathbb{T}$ and the support of μ satisfies $\operatorname{supp} \mu = \Omega(\overline{F})$ ($\Omega(\overline{F})$ is the set of nonwandering points on $\mathbb{T}$). Since F is not $\mathcal{C}^0$-conjugate to $\mathrm{Id} + \rho(F)$, $\operatorname{supp} \mu \neq \mathbb{T}$. For these two facts see Herman [22, section II.7.2, p. 25]. Since $\Omega(\overline{F})$ is closed, its nonempty complement in $\mathbb{T}$ is open and therefore of positive Lebesgue measure. The lift to $\mathbb{R}$ of this complement can be mapped by F into I_0 (see Lemma 2.17). Thus $m(W(F)) > 0$. ∎

Remark 3.24 Corollary 3.23 is nonempty as show for instance Katok and Hasselblatt [23, prop. 12.2.2., p. 405]: for all $l \in [0,1)$ and all $\beta \in (0,1)$, there exists $F \in \mathcal{D}^{1+\beta}(\mathbb{T})$ such that $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$ and $m(\Omega(\overline{F})) = l$.

Theorem 3.20 gives a new proof of divergence of the energy (for the previous results of that type see Cooper [13], Dittrich, Duclos and Šeba [14], Cooper and Koch [15], Gignoux and Meplan [16]). Assumptions based on the wandering points (instead of the periodic points like in [13, 14, 16]) give simplifications in the proof and more general results. Corollary 3.22 was already obtained by Cooper [13]. Corollary 3.21 is a slight generalization of Corollary 3.22: it allows to treat an infinite number of periodic points[1]. Corollary 3.23 was unexpected since $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$: although there are no periodic points, E is increasing without bound. But classical solutions are excluded: if $a \in \mathcal{A}_2$, then F is $\mathcal{C}^0$-conjugate to $\mathrm{Id} + \rho(F)$ by Denjoy's theorem (see Herman [22, th. VI.5.5, p. 76]). Note that Theorem 3.20 works with a boundary a differentiable a.e.. In particular this includes the piecewise linear model. In Corollaries 3.21 and 3.22, there are no assumptions made on the nature of the periodic points (hyperbolic or not): this is the reason why we cannot have precise estimates on the rate of increase of E (see Subsection 3.2.4 for a particular case). Compared to Theorem 3.14 the improvement is the following: if there exists a wandering set of strictly positive Lebesgue measure, then the model is strongly unstable.

Finally we can extend the condition of Cooper for the growth of E without bound: "there exists a finite number of periodic characteristics" by "there exists a set of wandering characteristics which has an intersection of positive Lebesgue measure with $\{x \in (0, a(0));\ \varphi_0'(x) + \varphi_1(x) \neq 0\} \cup \{x \in (-a(0), 0);\ \varphi_0'(-x) - \varphi_1(-x) \neq 0\}$".

3.2.4 Asymptotics

In this section we will improve results given by Cooper [13], Dittrich, Duclos and Šeba [14], Cooper and Koch [15]; see also Remark 3.26. For the asymptotics of the solution, see Cooper [13, th. 2, p. 79].

We recall that f is related to the initial conditions on I_0 by the formulas (19)-(20). Although we have also worked out the case $\rho(F) := \frac{p}{q} \in \mathbb{Q}^*$, we present here only the case $q = 1$. Assume that there are only two periodic points of F in I_0: $a_1 := -a(0)$ which is attracting and hyperbolic and a_2 which is repelling. Then $a(0)$ is also an attracting hyperbolic periodic point

[1]Consider for instance the function $a \in \mathcal{A}_0$ equal to $\frac{p}{2}$ ($p \in \mathbb{N}^*$) on a half period and nonconstant on the other half period. Then F has full intervals of periodic points.

with $F'(a(0)) = F'(a_1)$. For simplicity take $a(0) := \frac{p}{2}$. Then $I_n = I_0 + np$ and $E(np) = \|f'\|^2_{I_n}$. We define for every $x \in I_0$:

$$l(x) := \prod_{k=0}^{+\infty} \frac{F'(a_1)}{F' \circ F^k(x)} \ .$$

Theorem 3.25 *Assume $a \in \mathcal{A}_1$, $a'' \in L^\infty(\mathbb{R})$, $\rho(F) := p \in \mathbb{N}^*$ and $E(0) > 0$. Assume also that F has only two periodic points in I_0: $a_1 := -a(0)$ and $a_2 \in \dot{I}_0$ such that $F'(a_1) < 1$ and $F'(a_2) \geq 1$. Take $a(0) := \frac{p}{2}$. Then $l \in \mathcal{C}^0(I_0 \setminus \{a_2\})$, $l > 0$ on $I_0 \setminus \{a_2\}$ ($l(a_2) = 0$) and*

$$E(np) = \frac{\left\|\sqrt{l} f'\right\|^2_{I_0}}{F'(a_1)^n}(1 + o(1)) \quad \text{as } n \to +\infty \ . \tag{48}$$

Proof: We use the following notations: $J := [a_1, a_2]$, for all $n \in \mathbb{N}$, $K_n := \left\|\frac{f'}{\sqrt{DF^n}}\right\|^2_J$, $\hat{K}_n := \frac{1}{F'(a_1)^n}$, $r_n := \frac{K_n}{\hat{K}_n}$, $l_n(x) := \prod_{k=0}^{n-1} \frac{F'(a_1)}{F' \circ F^k(x)} > 0$. Thus $r_n = \left\|\sqrt{l_n} f'\right\|^2_J$. We first prove that $l_n(x)$ converges pointwise to $l(x)$ (first step) and that it is uniformly bounded in order to apply the Dominated Convergence Theorem and conclude that $\lim_{n \to +\infty} r_n =: L_1 \in \mathbb{R}_+$ (second step). It follows: $K_n = \hat{K}_n(L_1 + o(1))$ as $n \to +\infty$. Notice that by symmetry the proof needs just to be done on one side, for instance $[a_1, a_2]$, and similar results hold for $[a_2, a(0)]$.

First step: pointwise convergence. Let $v_k(x) := \frac{F'(a_1) - F' \circ F^k(x)}{F' \circ F^k(x)} > -1$. Then for all $x \in [a_1, a_2]$, $l_n(x) = \prod_{k=0}^{n-1}(1 + v_k(x))$. By continuity of F', there exists a set $J_- := [a_1, a_1 + \delta_-]$, $\delta_- > 0$, such that for all $x \in J_-$, $F'(x) \leq \frac{F'(a_1)+1}{2} < 1$. By assumption, for all $x \in [a_1, a_2)$, there exists $n_0(x) \in \mathbb{N}$ such that for all $n \geq n_0(x)$, $F^n(x) \in J_-$. By the Mean Value Theorem, for any x in $[a_1, a_2)$,

$$|v_k(x)| \leq |a_1 - F^k(x)| \cdot \|F''\|_{L^\infty(J)} \cdot \left\|\frac{1}{F'}\right\|_{L^\infty(J)} \quad \text{and}$$

$$\begin{aligned} \forall k \geq n_0(x),\ |a_1 - F^k(x)| &= \left|F^{k-n_0(x)} \circ F^{n_0(x)}(a_1) - F^{k-n_0(x)} \circ F^{n_0(x)}(x)\right| \\ &\leq |a_1 - x| \cdot \|F'\|^{n_0(x)}_{L^\infty(J)} \cdot \|F'\|^{k-n_0(x)}_{L^\infty(J_-)} \ . \end{aligned}$$

It follows that $\sum_{k=0}^{+\infty} |v_k(x)|$ is convergent, therefore $\prod_{k=0}^{+\infty}(1 + v_k(x))$ is absolutely convergent. The pointwise limit of $l_n(x)$ is $l(x)(l > 0$ on $[a_1, a_2))$. Since l_n is clearly uniformly convergent on any compact included in $[a_1, a_2)$, $l \in \mathcal{C}^0([a_1, a_2))$.

Second step: uniform bound. Let $J_+ := [a_2 - \delta_+, a_2]$ with δ_+ small enough such that $\min_{J_+} F' \geq F'(a_1)$. For $x \in J_+ \setminus \{a_2\}$, denote by $n_x \in \mathbb{N}$ the greatest integer such that $n < n_x$ implies that $F^n(x) \in J_+$. We know that for all $k < n_x$, $\frac{F'(a_1)}{F' \circ F^k(x)} \leq 1$: it follows that for all $x \in [a_2 - \delta_+, a_2)$ and for all $n \leq n_x$, $l_n(x) \leq 1$. If $J_+ = J$ (this is the case if $F'(a_1) = \min_J F'$) then $n_x = +\infty$ and immediatly for all $x \in J$ and for all $n \in \mathbb{N}$, $0 \leq l_n(x) \leq 1$. If $J_+ \neq J$, then $n_x < +\infty$ and for all $x \in J_+ \setminus \{a_2\}$, for all $n > n_x$,

$$\begin{aligned} l_n(x) &= l_{n-n_x}(F^{n_x}(x)) l_{n_x}(x) \leq l_{n-n_x}(F^{n_x}(x)) \\ &\leq \sup_{n\in\mathbb{N}} \sup_{y\notin J_+} l_n(y) =: M < +\infty \ , \end{aligned}$$

since the convergence of l_n is uniform on $[a_1, a_2 - \delta_+]$. We have proved that: $l_n(x) \leq \max\{1, M\}$ on J and for all $x \in J \setminus \{a_2\}$, $\lim_{n\to+\infty} l_n(x) = l(x) > 0$. By the Dominated Convergence Theorem we have that

$$\lim_{n\to+\infty} r_n = \left\| \sqrt{l} f' \right\|_J^2 =: L_1 \geq 0 \ .$$

Summing the results from both intervals $[-a(0), a_2]$ and $[a_2, a(0)]$, the relation (48) follows with $\|\sqrt{l} f'\|_{I_0} > 0$. ∎

Remark 3.26 a) This proof can be generalized to an arbitrary finite number of periodic points $\{a_i\}_{1\leq i\leq N}$ by making a partition $\cup_{i=1}^{N-1}[a_i, a_{i+1})$ of I_0 by the periodic points, then considering E as a sum of terms over every interval $[a_i, a_{i+1})$ and finally applying the results proved in Theorem 3.25.

b) Let us compare our result with previous ones (for $p = 1$): in Cooper [13], Dittrich, Duclos and Šeba [14], it is proven that for all $n \in \mathbb{N}^*$, $\dfrac{\ln E(n)}{n} \geq A + \dfrac{c}{n}$ for some constants $0 < A < -\ln F'(a_1)$ and $c \in \mathbb{R}^*$; in Cooper and Koch [15], $\dfrac{\ln E(n)}{n} = -\ln F'(a_1) + o(1)$ as $n \to +\infty$ and now we have: $\dfrac{\ln E(n)}{n} = -\ln F'(a_1) + \dfrac{c}{n} + o\left(\dfrac{1}{n}\right)$ as $n \to +\infty$, with $c := \ln \left\| \sqrt{l} f' \right\|_{I_0}^2$.

c) If $F'(a_2) > 1$ and f' is L^∞ in a neighborhood of a_2, we are even able to prove that the remainder is not only $o\left(\frac{1}{n}\right)$ but $\mathcal{O}\left(\frac{1}{n \ln n}\right)$. We do not give the proof here.

d) The reader can deduce the behavior of $E(t)$ from $E(np)$ by Lemma 2.17.

e) Even for nonsmooth boundary a, there might be an exponential increase of E. The following result can be proven: assume that $a \in \mathcal{A}_0$, $\rho(F) = \frac{p}{q} \in \mathbb{Q}^*$ and there exists a periodic point of F in I_0 called a_1 and satisfying:

$$\limsup_{x\to a_1^+} \frac{F^q(x) - F^q(a_1)}{x - a_1} =: \gamma < 1 \ .$$

Let $a_2 > a_1$ be the nearest periodic point of F and assume that $\|f'\|_{(a_1,a_2)} > 0$. Then for all $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$\forall t \in \mathbb{R}_+, \quad E(t) \geq c_\epsilon \left(\frac{1}{\gamma + \epsilon} \right)^{\frac{t}{p}} .$$

f) By Proposition 3.12, we know that the maximal rate of exponential increase of E satisfies $r \leq -\frac{1}{2a_{\min}} \ln F'_{\min} =: r'$. Assume $p = 1$, then in Theorem 3.25 we obtain $r = -\ln F'(a_1)$. Since $p = q = 1$, then $\frac{1}{2} \in \operatorname{Ran} a$ by Lemma 2.4. Now assume that $F'(a_1) := F'_{\min}$, then the relative error $e := \frac{|r'-r|}{r}$ between the estimate of Proposition 3.12 and the result of Theorem 3.25 is $e = \left| \frac{1}{2a_{\min}} - 1 \right|$, which can be made arbitrarily small if $a_{\min}$ is close to $\frac{1}{2}$.

3.2.5 Instability: strong instability is not the rule

Proposition 3.27 *1) Let $\mathcal{G}^\infty := \mathcal{D}^\infty(\mathbb{T}) \setminus \operatorname{Int} \rho^{-1}(\mathbb{Q})$. Then $\mathcal{G}^\infty$ is a closed set with empty interior and there exists $A \subset \mathcal{G}^\infty \cap \rho^{-1}(\mathbb{R} \setminus \mathbb{Q})$ such that A is a dense G_δ set in $\mathcal{G}^\infty$. Moreover, for all $F \in A \cap \mathcal{F}_\infty$,*

$$\limsup_{t \to +\infty} E(t) = +\infty ,$$

whenever $E(0) > 0$.

2) Let $F_\beta(x) := x + \alpha \sin(2\pi x) + \beta$, $0 < |\alpha| < \frac{1}{2\pi}$, $\beta > |\alpha|$ and α is fixed. Let $K_\alpha := [|\alpha|, |\alpha| + 1] \setminus \operatorname{Int} \{\beta; \rho(F_\beta) \in \mathbb{Q}\}$. Then K_α is a Cantor set and there exists $B \subset K_\alpha \cap \rho^{-1}(\mathbb{R} \setminus \mathbb{Q})$ such that B contains a dense G_δ set in K_α. Moreover, for all $F \in B \cap \mathcal{F}_\infty$, $\limsup_{t \to +\infty} E(t) = +\infty$, whenever $E(0) > 0$.

Proof: A direct consequence of Herman [22, th. XII.1.11, p. 168 and prop. XII.1.13, p. 169] and Corollary 3.15. ∎

Remark 3.28 This is an application of Corollary 3.15 which shows that instability can be even generic. The example given in part 2) is the well known map of Arnol'd [29]. By Proposition 3.5, one knows that the limit inferior of E is finite. So this shows that instability is possible even if there are no periodic points for F and if a is smooth. This answers negatively to a conjecture of Cooper [17]. However according to Cooper and Koch [15], the spectrum of the evolution operator on one period under the conditions of Proposition 3.27 (in that case $\limsup_{t \to +\infty} E(t) = +\infty$) and of Theorem 3.11 (in that case $0 < \inf_{t \in \mathbb{R}} E(t) \leq \sup_{t \in \mathbb{R}} E(t) < +\infty$) is pure point and dense in $\mathbb{T}$; this shows that one cannot predict the behavior of the energy with such a spectral information.

3.2.6 Perturbation of the boundary

In the following proposition we will prove that the instability of our model is in general the rule after perturbation. Let us denote $E_a := E$ the energy of the model with a moving boundary a.

Proposition 3.29 *Let $a \in \mathcal{A}_k$, $k \geq 1$, and $E_a(0) > 0$. The two following assertions are true:*

1) There exists $\tilde{a} \in \mathcal{A}_k$ arbitrarily near a in the $\mathcal{C}^k$-topology such that $E_{\tilde{a}}$ is increasing exponentially fast as t tends to infinity, whenever $E_{\tilde{a}}(0) > 0$.

2) Assume that $F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1}$ is a Morse-Smale diffeomorphism. Then there exists $\epsilon > 0$ such that for all $\tilde{a} \in \mathcal{A}_k$ satisfying $\|a - \tilde{a}\|_k < \epsilon$, $E_{\tilde{a}}$ is still increasing exponentially fast, whenever $E_{\tilde{a}}(0) > 0$.

Proof: Let $a \in \mathcal{A}_k$, $k \geq 1$; then $F := (\mathrm{Id} + a) \circ (\mathrm{Id} - a)^{-1} \in \mathcal{F}_k$. By a theorem of Peixoto (e.g. Nitecki [30, section 1.2, p. 50]), one knows that there exists a dense, open subset of $\mathcal{D}^k(\mathbb{T})$ which consists of diffeomorphisms G with the following properties: the nonwandering set $\Omega(G)$ is finite[2] and all periodic points are hyperbolic[3]. Consequently, there exists $\tilde{F} \in \mathcal{D}^k(\mathbb{T})$ arbitrarily near F in the $\mathcal{C}^k$-topology. Since $\mathcal{F}_k$ is open for $k \geq 1$, we can take $\tilde{F} \in \mathcal{F}_k$. By Lemma 2.3, $\mathcal{A}_k$ and $\mathcal{F}_k$ are homeomorphic and

$$\tilde{a} := \left(\frac{\tilde{F} - \mathrm{Id}}{2}\right) \circ \left(\frac{\tilde{F} + \mathrm{Id}}{2}\right)^{-1} \in \mathcal{A}_k \tag{49}$$

(see formula (11)). Then $E_{\tilde{a}}$ is increasing exponentially fast by Theorem 3.25 or Remark 3.26 e). This proves 1).

If we assume that $F \in \mathcal{F}_k$, $k \geq 1$, is a Morse-Smale diffeomorphism, then if $\tilde{F}$ is a sufficiently small perturbation of F in the $\mathcal{C}^k$-topology, $\tilde{F}$ is still a Morse-Smale diffeomorphism and $\tilde{F} \in \mathcal{F}_k$. Thus, defining $\tilde{a}$ by the formula (49), we can apply Theorem 3.25 or Remark 3.26 e), which say that $E_{\tilde{a}}$ is increasing exponentially if $E_{\tilde{a}}(0) > 0$. ∎

4 Neumann problem

What will be remarkable is the opposite behavior of the Neumann problem w.r.t. to the Dirichlet problem. Technicalities in the proofs are often the same, so we omit them.

[2]Hence $\Omega(G)$ consists only of periodic points.
[3]G is a Morse-Smale diffeomorphism.

4.1 Stability

4.1.1 An upper bound

Note that here a is not necessarily periodic. We recall that

$$f'(x) = \frac{1}{2}(\varphi_0'(x) + \varphi_1(x)) \quad \text{a.e. in } (0, a(0)) , \tag{50}$$

$$f'(x) = \frac{1}{2}(-\varphi_0'(-x) + \varphi_1(-x)) \quad \text{a.e. in } (-a(0), 0) . \tag{51}$$

Proposition 4.1 *If $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0,1)$, $a > 0$, $a_{\max} < +\infty$ and*

$$\varphi_1 \in L^\infty((0, a(0))) , \quad \varphi_0' \in L^\infty((0, a(0))) ,$$

then

$$\forall t \in \mathbb{R}, \quad E(t) \leq 2a_{\max} F'_{\max} \|f'\|^2_{L^\infty(I_0)} .$$

Moreover $E \in \mathrm{Lip}(\mathbb{R})$ and $L(E) \leq 2L(a) \|f'\|^2_{L^\infty(I_0)}$.

Proof: By the conditions on φ_0 and φ_1, $f' \in L^\infty(I_0)$. Then by Lemma 2.17,

$$\begin{aligned} \forall t \in \mathbb{R}, \quad E(t) &\leq F'_{\max} \left\| f' \sqrt{DF^{n(t)}} \right\|^2_{I_0} \\ &\leq F'_{\max} \|f'\|^2_{L^\infty(I_0)} \left(F^{n(t)}(a(0)) - F^{n(t)}(-a(0)) \right) \\ &\leq F'_{\max} \|f'\|^2_{L^\infty(I_0)} 2a_{\max} . \end{aligned}$$

By the relation (23), we have $f' \circ F = f'$ a.e.. So by successive iterations we can show that $f' \in L^\infty(\mathbb{R})$ and $\|f'\|_{L^\infty(\mathbb{R})} = \|f'\|_{L^\infty(I_0)}$. Using the formula $E' = 2a'|f' \circ k|^2$ a.e. (see Lemma 2.15 4)), we get that E is Lipschitz continuous on $\mathbb{R}$ and $L(E) \leq 2L(a)\|f'\|^2_{L^\infty(I_0)}$. ∎

Remark 4.2 The assumption that $\varphi_0', \varphi_1 \in L^\infty((0, a(0)))$ is crucial in the Neumann problem. There are examples for which the energy diverges if it is not the case as is shown in Section 4.2.2.

Note also that if a is stricly increasing on $\mathbb{R}_+$ with $a' \geq \gamma > 0$, then $a_{\max} = +\infty$ and $\lim_{t \to +\infty} E(t) = +\infty$. The proof is similar as the one in Remark 3.2.

4.1.2 Periodicity

With respect to the Dirichlet case (cf. Theorem 3.3) only minor changes are necessary to establish the following theorem:

Theorem 4.3 *1) Let $a \in \mathcal{A}_0$. If $F^q = \mathrm{Id} + p$ for some $q \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$, then*

$$\forall t \in \mathbb{R}, \quad E(t + p) = E(t) .$$

2) Let $a \in \mathrm{Lip}(\mathbb{R})$ such that $L(a) \in [0,1)$, $a > 0$. If there exists $p \in \mathbb{R}_+^*$ such that for all $(\varphi_0, \varphi_1) \in H^1((0,a(0))) \times L^2((0,a(0)))$, the relation

$$E(t+p) = E(t)$$

holds for all $t \in \mathbb{R}$, then a is periodic of period p and if a is not constant in $\mathbb{R}$, there exists $q \in \mathbb{N}$, $q \geq 2$, such that $F^q = \mathrm{Id} + p$.

4.1.3 Asymptotics

In the next theorem we give an asymptotics in a simplified case: instead of a general rational rotation number $\rho(F) = \frac{p}{q}$, we consider $\rho(F) = p \in \mathbb{N}^*$; moreover we restrict the number of periodic points to two. This simplification has the advantage to propose a readable formula for the asymptotics. The general formula can be also worked out. Remember that f' is given on I_0 by the formulas (50)-(51).

Theorem 4.4 *Let $p \in \mathbb{N}^*$. Assume that (i) $a \in \mathcal{A}_1$, (ii) $\frac{p}{2} \in \mathrm{Ran}\ a$, (iii) F has just two fixed points a_1, a_2 in I_0 such that a_1 is attracting and a_2 is repelling; take $a_1 := -a(0)$, (iv) $(\varphi_0, \varphi_1) \in \mathcal{C}^1([0,a(0)]) \times \mathcal{C}^0([0,a(0)])$ and $\varphi_0'(0) = \varphi_0'(a(0)) = 0$. Then E is asymptotically 1-periodic and more precisely*

$$\lim_{t \to +\infty} \Big(E(t) - 2|f'(a_2)|^2 a(t) \Big) = 0 \, .$$

Proof: By assumption (ii), we can choose $a(0) = \frac{p}{2}$ and then for all $n \in \mathbb{Z}$, $I_n = \left[-\frac{p}{2}, \frac{p}{2}\right) + np$. Let f_n' be the restriction of f' to I_n. Then the equation (23) implies that $f_n' = f_0' \circ F^{-n}$. By (iv), $f' \in \mathcal{C}^0(I_0)$, thus $f' \in \mathcal{C}^0(\mathbb{R})$. Obviously: $\|f'\|_{L^\infty(\mathbb{R})} = \|f'\|_{L^\infty(I_0)} < +\infty$. Let $\phi : I_0 \to \mathbb{R}$ be the function equal to $f'(a_1)$ if $x = a_1$ and equal to $f'(a_2)$ otherwise. Assumption (iii) implies that for all $x \in I_0$, $f'(x+np)$ converges pointwise to $\phi(x)$ as $n \to +\infty$. Let $\tau \in I_0$ and define the sequence $t_n := t_n(\tau) := h^{-1}(\tau) + np$, $n \in \mathbb{N}$. Then $h(t_n) = \tau + np \in I_n$ and by obvious changes of variables:

$$\begin{aligned} E(t_n(\tau)) &= \|f'\|^2_{(h(t_n(\tau)),k(t_n(\tau)))} = \left\| f'\sqrt{F'} \right\|^2_{(x_{n-1},\tau+np)} + \|f'\|^2_{(\tau+np,x_n)} \\ &= \left\| f'(\cdot + np)\sqrt{F'} \right\|^2_{(-a(0),\tau)} + \|f'(\cdot + np)\|^2_{(\tau,a(0))} \, . \end{aligned}$$

Since $f' \in L^\infty(\mathbb{R})$ and $\lim_{n \to +\infty} f'(x+np) = \phi(x)$ for all $x \in I_0$, we can apply the Dominated Convergence Theorem:

$$\begin{aligned} \lim_{n \to +\infty} E(t_n(\tau)) &= |f'(a_2)|^2 \left(F(\tau) - F(-a(0))\right) + |f'(a_2)|^2 (a(0) - \tau) \\ &= |f'(a_2)|^2 \left(F(\tau) - \tau\right) =: E_\infty(\tau) \, . \end{aligned}$$

It is easy to see that

$$|E(t_n(\tau)) - E_\infty(\tau)| \leq \|F'\|_{L^\infty(I_0)} \left\| |f'(\cdot + np)|^2 - |f'(a_2)|^2 \right\|_{L^1(I_0)} \, .$$

The right hand side tends to zero when n tends to infinity and is independent of τ, thus $E(t_n)$ converges uniformly w.r.t. τ to $E_\infty(\tau)$. Since $F(\tau) - \tau = 2a(h^{-1}(\tau)) = 2a(t_n(\tau))$, the proof is complete. ∎

4.1.4 Absence of strong instability

Like for the Dirichlet case (cf. Proposition 3.5) we have, by similar arguments:

Proposition 4.5 *Assume that a is of class Q and $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$. Let $V := \widehat{\mathrm{Var}}(\ln F')$. Then there exists a subsequence $\{I_{\pm q_n}\}_{n\in\mathbb{N}}$ of the sequence $\{I_n\}_{n\in\mathbb{Z}}$ defined by (14) such that q_n tends to infinity as n tends to infinity and*

$$\forall n \in \mathbb{N}, \quad \forall t \in h^{-1}(I_{\pm q_n}), \quad e^{-V} E(0) F'_{\min} \le E(t) \le e^{V} E(0) F'_{\max} \,.$$

4.1.5 A sufficient condition of stability

The proofs of the following theorems are similar to Theorems 3.7 and 3.11.

Theorem 4.6 *Assume $a \in \mathcal{A}_0$, $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$ and denote by μ the unique invariant measure of $\overline{F}$. Assume that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that the Radon-Nikodym derivative of μ w.r.t. m satisfies: $0 < \lambda_1 \le \frac{d\mu}{dm} \le \lambda_2 < +\infty$. Then*

$$\forall t \in \mathbb{R}, \quad \frac{\lambda_1}{\lambda_2} E(0) F'_{\min} \le E(t) \le \frac{\lambda_2}{\lambda_1} E(0) F'_{\max} \,. \tag{52}$$

Theorem 4.7 *If $a \in \mathcal{C}^k(\mathbb{T})$, $k \in \mathbb{R}$, $k > 2$, $|a'| < 1$, $a > 0$ and $\rho(F)$ is β-Diophantine with $\beta < k - 2$, then (52) holds.*

Remark 4.8 Like for the Dirichlet problem, if $F \in \mathcal{F}_1$ is Lipschitz conjugate to $\mathrm{Id} + \rho(F)$, then the Neumann model is stable.

4.2 Instability

4.2.1 Universal lower and upper bounds

Note that here a is not necessarily periodic and $n(t)$ is defined in Lemma 2.17. The more general bounds of the Neumann problem are given by the following proposition. The proof is similar to the one of Theorem 3.12.

Proposition 4.9 *If $a \in \mathrm{Lip}(\mathbb{R})$, $L(a) \in [0, 1)$ and $a > 0$, then*

$$\forall t \in \mathbb{R}_+, \quad E(0) \left(F'_{\min}\right)^{n(t)+1} \le E(t) \le E(0) \left(F'_{\max}\right)^{n(t)+1} \,.$$

If moreover $E(0) > 0$, then the two following assertions are true:

1) If $a_{\min} > 0$, then the maximal rate of exponential increase of E defined by $r_+ := \limsup_{t\to+\infty} \frac{\ln E(t)}{t}$ satisfies

$$r_+ \leq \frac{1}{2a_{\min}} \ln F'_{\max} . \tag{53}$$

2) If $\lim_{t\to+\infty} \frac{a(t)}{t} = 0$, then the minimal rate of exponential increase of E defined by $r_- := \liminf_{t\to+\infty} \frac{\ln E(t)}{t}$ satisfies

$$r_- \geq \frac{1}{2a_{\max}} \ln F'_{\min} , \tag{54}$$

with $a_{\max}^{-1} = 0$ if $a_{\max} = +\infty$.

One could think that these estimates are unrealistic by the already existing bound given in Proposition 4.1. The results given in Proposition 4.10 and Theorem 4.13 show that this is not the case.

4.2.2 Singular initial data

Proposition 4.10 *There exist $a \in \mathcal{A}_k$ ($1 \leq k \leq +\infty$ or $k = \omega$) and initial conditions $(\varphi_0, \varphi_1) \in H^1((0, a(0))) \times L^2((0, a(0)))$ with $\varphi_0' \notin L^\infty((0, a(0)))$ (or $\varphi_1 \notin L^\infty((0, a(0)))$) such that:*

$$\lim_{t\to+\infty} E(t) = +\infty$$

and the rate of increase is exponential.

Proof: Let $F \in \mathcal{F}_k$, $k \geq 1$, such that: $F(0) = p$ for some $p \in \mathbb{N}^*$, $F'(0) > 1$ and there exists $x_0 \in (0, a(0)]$ such that for all $x \in [0, x_0]$, $F'(x) = F'(0)$. It is obvious that $\rho(F) = p$ and that we can build such F. To this F there corresponds a unique $a \in \mathcal{A}_k$ as stated in Lemma 2.3. For simplification we introduce $\tilde{F} := F - p$. Then $\tilde{F}(0) = 0$ and $DF^n = D\tilde{F}^n$. For any $n \in \mathbb{N}^*$, let us look for $x \in (0, x_0]$ such that $\tilde{F}^n(x) \leq x_0$. We define now the sequence $y_0 := x$, $y_n := \tilde{F}(y_{n-1})$, $n \geq 1$. Immediately $y_n = F'(0)^n x$. Then for all $n \in \mathbb{N}$, $n \geq 1$,

$$F^n(x) \leq x_0 \Longleftrightarrow x \leq \frac{x_0}{F'(0)^n} =: \epsilon_n \longrightarrow 0 \quad \text{as } n \to +\infty .$$

Assume for instance that $\varphi_0 \in H^1((0, a(0)))$ and $\varphi_1 \in L^2((0, a(0)))$ satisfy $\varphi_0(x) := \frac{8}{3} x^{\frac{3}{4}}$ and $\varphi_1(x) := 0$ for $x \in (0, x_0]$ and are arbitrary otherwise. Then $f'(x) = x^{-\frac{1}{4}}$ on $(0, x_0]$. Thus $f' \in L^2((0, x_0)) \setminus L^\infty((0, x_0))$. Hence, by Lemma 2.17,

$$\begin{aligned}
\forall t \in \mathbb{R}_+^*, \quad E(t) \ &\geq \ F'_{\min} \left\| f' \sqrt{D\tilde{F}^{n(t)}} \right\|^2_{(0,\epsilon_{n(t)})} = F'_{\min} F'(0)^{n(t)} \left\| f' \right\|^2_{(0,\epsilon_{n(t)})} \\
&= \ 2F'_{\min} F'(0)^{n(t)} \frac{\sqrt{x_0}}{\sqrt{F'(0)^{n(t)}}} = 2F'_{\min} \sqrt{x_0} F'(0)^{\frac{n(t)}{2}} .
\end{aligned}$$

This construction is valid for $k \leq +\infty$. We can also easily build examples of $F \in \mathcal{D}^{\omega}(\mathbb{T})$. This lower bound and Proposition 4.9 show that E grows without bound exponentially fast. ∎

Remark 4.11 If in the above proof we assume more generally that $\varphi_0(x)$ behaves on $(0, x_0]$ like x^{α}, $\frac{1}{2} < \alpha < 1$, $E(t)$ grows faster than $cF'(0)^{2(1-\alpha)n(t)}$. The more singular is φ_0 (i.e. α smaller and smaller) the faster the energy increases. In particular, the bound (53) on the maximal rate of exponential increase of E is almost reached for α close to $\frac{1}{2}$.

The physical explanation of this proposition is simple: singularities of the initial data are trapped by an attractor and amplified by the Doppler effect.

4.2.3 Sufficient conditions for the decay of the energy

As an immediate consequence of Theorem 4.4, we have:

Corollary 4.12 *If all the conditions of Theorem 4.4 hold and $f'(a_2) = 0$, then $\lim_{t\to+\infty} E(t) = 0$.*

In the next theorem we give conditions to have an exponential decay of the energy. We recall that for every $x \in I_0$, we defined in Theorem 3.25:

$$l(x) := \prod_{k=0}^{+\infty} \frac{F'(a_1)}{F' \circ F^k(x)} \ .$$

Theorem 4.13 *Assume that*

(i) $a \in \mathcal{A}_1$, $a'' \in L^{\infty}(\mathbb{R})$.

(ii) $\rho(F) = \frac{p}{q} \in \mathbb{Q}^*$ and F has just two periodic points a_1, a_2 on I_0 such that $DF^q(a_1) < 1$ and $DF^q(a_2) \geq 1$. Take $a_1 := -a(0)$.

(iii) $(\varphi_0, \varphi_1) \in \mathcal{C}^1([0, a(0)]) \times \mathcal{C}^0([0, a(0)])$, $\varphi_0'(0) = \varphi_0'(a(0)) = 0$ and $E(0) > 0$.

(iv) There exists an open neighborhood of a_2 denoted by $\mathcal{V}_{\epsilon} := (a_2 - \epsilon, a_2 + \epsilon)$ for some $\epsilon > 0$, such that for all $x \in \mathcal{V}_{\epsilon}$, $f'(x) = 0$.

Then E decreases exponentially to 0 and more precisely:

$$E(mp) = \left\| \frac{f'}{\sqrt{l}} \right\|_{I_0}^2 DF^q(a_1)^m (1 + o(1)) \quad \text{as } m \to +\infty \ ,$$

where l is continuous and strictly positive on $[-a(0), a(0)] \setminus \mathcal{V}_{\epsilon}$.

Proof: Since the proof is very similar to the one of Theorem 3.25, we only sketch it. By (ii), $a(0)$ is an attracting periodic point, $DF^q(-a(0)) = DF^q(a(0))$. Since for all $t \in \mathbb{R}_+$, there exist a unique $m(t) \in \mathbb{N}$ and a unique $r(t) \in \{0, 1, \ldots, q-1\}$ such that $n(t) = m(t)q + r(t)$, it is possible to rewrite the energy as follows:

$$\begin{aligned} E(t) &= \alpha_N(t)\left(\left\|f'\sqrt{DF^{n(t)}}\right\|^2_{(a_1,a_2-\epsilon)} + \left\|f'\sqrt{DF^{n(t)}}\right\|^2_{(a_2+\epsilon,a(0))}\right) \\ &= \beta(t)\left(\left\|f'\sqrt{DF^{qm(t)}}\right\|^2_{(a_1,a_2-\epsilon)} + \left\|f'\sqrt{DF^{qm(t)}}\right\|^2_{(a_2+\epsilon,a(0))}\right) \\ &= \beta(t)E(m(t)p) \end{aligned}$$

for some strictly positive and bounded function β (which is obtained similarly as α_N in Lemma 2.17). By Lemma 2.17, $\lim_{t\to+\infty} \frac{m(t)}{t} = \frac{1}{p}$. Now similar arguments as in the proof of Theorem 3.25 show that l is continuous and strictly positive on $[a_1, a_2 - \epsilon] \cup [a_2 + \epsilon, a(0)]$ and:

$$\begin{aligned} \left\|f'\sqrt{DF^{mq}}\right\|^2_{(a_1,a_2-\epsilon)} &= \left\|\frac{f'}{\sqrt{l}}\right\|^2_{(a_1,a_2-\epsilon)} DF^q(a_1)^m(1+o(1)) \text{ as } m \to +\infty , \\ \left\|f'\sqrt{DF^{mq}}\right\|^2_{(a_2+\epsilon,a(0))} &= \left\|\frac{f'}{\sqrt{l}}\right\|^2_{(a_2+\epsilon,a(0))} DF^q(a_1)^m(1+o(1)) \text{ as } m \to +\infty . \end{aligned}$$

The end of the proof is now obvious. ∎

Remark 4.14 Like for Theorem 3.25 the assumptions of Theorem 4.13 are stronger than needed in order to obtain simpler results.

A Appendix

Lemma A *Let $\Sigma \subset \mathbb{R}^2$ be a domain (i.e. nonempty, open and connected) such that for each $y \in \mathbb{R}$, $J^y := \{x \in \mathbb{R};\ (x,y) \in \Sigma\}$ and for each $x \in \mathbb{R}$, $J_x := \{y \in \mathbb{R};\ (x,y) \in \Sigma\}$ are intervals of $\mathbb{R}$. Let $K_1 := \{x \in \mathbb{R};\ \Sigma \cap (\{x\} \times \mathbb{R}) \neq \emptyset\}$, $K_2 := \{y \in \mathbb{R};\ \Sigma \cap (\mathbb{R} \times \{y\}) \neq \emptyset\}$. Then K_1 and K_2 are nonempty open intervals of $\mathbb{R}$.*

Let $\varphi \in H^1_{\mathrm{loc}}(\Sigma)$ and assume that $\varphi_{xy} = 0$ in $\mathcal{D}'(\Sigma)$. Then there exist $f \in H^1_{\mathrm{loc}}(K_1)$, $g \in H^1_{\mathrm{loc}}(K_2)$ such that:

$$\varphi(x,y) = f(x) + g(y) \quad a.e. \ in\ \Sigma .$$

Proof: Since Σ is nonempty, open and connected, the same holds for K_1 and K_2; so K_1 and K_2 are nonempty open intervals.

Since $\varphi \in H^1_{\mathrm{loc}}(\Sigma)$, $\varphi_x \in L^2_{\mathrm{loc}}(\Sigma) \subset L^1_{\mathrm{loc}}(\Sigma)$. Further $\varphi_{xy} = 0$ in $\mathcal{D}'(\Sigma)$ implies that $\partial_y(\varphi_x) \in L^1(\Sigma)$. Consequently (e.g. Kufner, John and Fučík [24, th. 5.6.3, p. 274]), there exists a function $u \in L^1_{\mathrm{loc}}(\Sigma)$ such that: (i)

$u = \varphi_x$ a.e. in Σ, (ii) there exists $\hat{K}_1 \subset K_1$ with $m(K_1 \setminus \hat{K}_1) = 0$ such that for all $x \in \hat{K}_1$, $J_x \ni y \mapsto u(x,y)$ is absolutely continuous and (iii) $u_y = \varphi_{xy} = 0$ a.e. in Σ.

By Fubini's theorem, (iii) $\Rightarrow$ (iv): there exists $\tilde{K}_1 \subset K_1$ with $m(K_1 \setminus \tilde{K}_1) = 0$ such that for all $x \in \tilde{K}_1$, $u_y(x,y) = 0$ for almost every $y \in J_x$. Let $K_1' := \hat{K}_1 \cap \tilde{K}_1$; $m(K_1 \setminus K_1') = 0$. The statements (ii) and (iv) imply that for all $x \in K_1'$, $J_x \ni y \mapsto u(x,y)$ is constant. Let $\phi : K_1 \to \mathbb{R}$ be the function defined by $\phi(x) := u(x,y)$ if $x \in K_1'$ and $y \in J_x$ (ϕ is arbitrary on $K_1 \setminus K_1'$ which is of zero measure). Let us show that $\phi \in L^2_{\text{loc}}(K_1)$. It is sufficient to prove that for all $x \in K_1$, there exists $r > 0$ such that $\phi \in L^2((x-r, x+r))$. Let $x_0 \in K_1$. Thus $J_{x_0} \neq \emptyset$ and there exists $r > 0$ such that $B^\infty((x_0,y_0),r) \subset \Sigma$ for some $y_0 \in J_x$ with $B^\infty((x_0,y_0),r) := \{(x,y) \in \mathbb{R}^2; \max\{|x-x_0|, |y-y_0|\} < r\}$. Then for all $x \in (x_0 - r, x_0 + r) \cap K_1'$ and for all $y \in (y_0 - r, y_0 + r)$, one has $u(x,y) = \phi(x)$. The statement (i) implies that for almost every $y \in (y_0 - r, y_0 + r)$, one has $u(\cdot, y) \in L^2((x_0 - r, x_0 + r))$. Let $y_1 \in (y_0 - r, y_0 + r)$ satisfying this assertion. Then for all $x \in (x_0 - r, x_0 + r) \cap K_1'$, one has $\phi(x) = u(x, y_1)$; thus $\phi \in L^2((x_0 - r, x_0 + r))$.

Since φ, $\varphi_x \in L^1_{\text{loc}}(\Sigma)$, we use again the theorem in Kufner, John and Fučík [24, th. 5.6.3, p. 274]: there exists a function $v \in L^1_{\text{loc}}(\Sigma)$ such that: (i') $v = \varphi$ a.e. in Σ, (ii') there exists $\hat{K}_2 \subset K_2$ with $m(K_2 \setminus \hat{K}_2) = 0$ such that for all $y \in \hat{K}_2$, $J^y \ni x \mapsto v(x,y)$ is absolutely continuous and (iii') $v_x = \varphi_x = u$ a.e. in Σ.

Again by Fubini's theorem, (iii') $\Rightarrow$ (iv'): there exists $\tilde{K}_2 \subset K_2$ with $m(K_2 \setminus \tilde{K}_2) = 0$ such that for all $x \in \tilde{K}_2$, $v_x(x,y) = u(x,y)$ for almost every $x \in J^y$. By (ii'), with $a_y \in J^y$ arbitrary:

$$\forall y \in \hat{K}_2,\ \forall x \in J^y, \quad v(x,y) = v(a_y, y) + \int_{a_y}^x v_x(z,y)\, dz\,. \tag{55}$$

Using (iv'), (55) implies

$$\begin{aligned}
\forall y \in \hat{K}_2 \cap \tilde{K}_2 =: K_2', \quad \forall x \in J^y, \quad v(x,y) &= v(a_y, y) + \int_{a_y}^x u(z,y)\, dz \\
&= v(a_y, y) + \int_{a_y}^x \phi(z)\, dz\,.
\end{aligned}$$

Let $a_0 \in K_1$ and define:

$$\begin{aligned}
\forall x \in K_1, \quad f(x) &:= \int_{a_0}^x \phi(z)\, dz\,, \\
\forall y \in K_2, \quad g(y) &:= v(a_y, y) + \int_{a_y}^{a_0} \phi(z)\, dz\,.
\end{aligned}$$

Since $\phi \in L^1_{\text{loc}}(K_1)$, f is absolutely continuous on K_1 and $f' = \phi$ a.e. on K_1: it follows that $f \in H^1_{\text{loc}}(K_1)$. We know that $g : K_2 \to \mathbb{R}$ is a function. Thus $\varphi(x,y) = v(x,y) = f(x) + g(y)$ a.e. on Σ. By changing the rôle of f and g, there exist $g_1 \in H^1_{\text{loc}}(K_2)$ and a function $f_1 : K_1 \to \mathbb{R}$ such that

$\varphi(x,y) = f_1(x) + g_1(y)$ a.e. in Σ. Hence $f_1 = f - c$ a.e. and $g_1 = g + c$ a.e. for some $c \in \mathbb{R}$. This yields:

$$\varphi(x,y) = f(x) + g(y) \text{ a.e. in } \Sigma \ ,$$

with $f \in H^1_{\text{loc}}(K_1)$, $g \in H^1_{\text{loc}}(K_2)$. ∎

Lemma B *Let $a \in \text{Lip}(\mathbb{R})$, $L(a) \in [0,1)$ and $a > 0$. If $\varphi \in H^1_{\text{loc}}(\Omega)$ satisfies $\varphi_{tt} - \varphi_{xx} = 0$ in $\mathcal{D}'(\Omega)$, then there exist $f,\ g \in H^1_{\text{loc}}(\mathbb{R})$ such that*

$$\varphi(x,t) = f(t+x) + g(t-x) \quad \textit{a.e. in } \Omega \ .$$

Moreover φ can be continuously extended to $\overline{\Omega}$, the traces of φ on each line $t = constant$ are in $H^1((0, a(constant)))$ and the traces of φ on the boundary $\partial\Omega$ of Ω are in $H^1_{\text{loc}}(\partial\Omega)$.

Proof: The first statement is obvious by a change of variables and by using Lemma A. Then the second statement follows from the fact that f and g are in $H^1_{\text{loc}}(\mathbb{R})$ (and in particular f and g are continuous). ∎

Proof of Theorem 2.13:

First step: by Lemma B, if a weak solution φ exists, then $\varphi(x,t) = f(t+x) + g(t-x)$ a.e. in Ω with $f,\ g \in H^1_{\text{loc}}(\mathbb{R})$.
Second step: the function φ_x has a trace on $x = 0$ by Lemma B and (6) implies that $f' = g'$ a.e.. Then $g = f + c$, where $c \in \mathbb{R}$. Denote again by f the function $f + \frac{c}{2}$. Then $\varphi(x,t) = f(t+x) + f(t-x)$ a.e. in Ω.
Third step: the functions φ and φ_t have a trace on $t = 0$. Then f is given on I_0 by the initial conditions φ_0 and φ_1 and by formulas similar to (19)-(20); thus $f \in H^1(I_0)$.
Fourth step: the function φ_x has a trace on $x = a(t)$, thus (7) implies that

$$f' \circ F = f' \quad \text{a.e. in } \mathbb{R} \ .$$

Fifth step: by successive iterations, for all $n \in \mathbb{Z}$, $f' \circ F^n = f'$ a.e., thus f' is known a.e. on $\mathbb{R}$. Up to a constant (depending on n) f is known on I_n by integration of f'. Using the third step the constants are determined by continuity of f. Obviously $f \in H^1_{\text{loc}}(\mathbb{R})$ and $\varphi \in H^1_{\text{loc}}(\Omega)$.
Sixth step: by the third and fifth steps, f is unique on $\mathbb{R}$. Thus φ is unique. ∎

Acknowledgments.
We wish to thank S. Vaienti for fruitful discussions. This work is partly supported by GACR grant No. 202/96/0218 (J.D.) and by a grant of the French Ministry of Foreign Affairs and the Czech Ministry of Education (N.G.). J.D. thanks the Centre de Physique Théorique of Marseille and the PhyMath and P.D., N.G. thank the Nuclear Physics Institute for hospitality.

References

[1] E. Fermi, "On the origin of the cosmic radiation", *Phys. Rev.* **75** (1949) 8 p. 1169-1174.

[2] S. Ułam, "On some statistical properties of dynamical systems", in *Proceedings of the Fourth Berkeley Symposium on Mathematical and Statistical Problems, vol. 3, p. 315-320*, University of California Press, Berkeley, 1961.

[3] L.D. Pustyl'nikov, "On Ułam's problem", *Theoret. Math. Phys.* **57** (1983) p. 1035-1038.

[4] L.D. Pustyl'nikov, "Poincaré models, rigorous justification of the second element of thermodynamics on the basis of mechanics, and the Fermi acceleration mechanism", *Russian Math. Surveys* **50** (1995) 1 p. 145-189.

[5] L.D. Pustyl'nikov, "A new mechanism for particle acceleration and a relativistic analogue of the Fermi-Ułam model", *Theoret. Math. Phys.* **77** (1988) 1 p. 1110-1115.

[6] L.D. Pustyl'nikov, "A new mechanism of particle acceleration and rotation numbers", *Theoret. Math. Phys.* **82** (1990) 2 p. 180-187.

[7] G. Karner, "On the quantum Fermi accelerator and its relevance to "quantum chaos"", *Lett. Math. Phys.* **17** (1989) 4 p. 329-339.

[8] P. Šeba, "Quantum chaos in the Fermi-accelerator model", *Phys. Rev. A* **41** (1990) 5 p. 2306-2310.

[9] S. T. Dembiński, A. J. Makowski, P. Pepłowski, "Quantum bouncer with chaos", *Phys. Rev. Letters* **70** (1993) 8 p. 1093-1096.

[10] V.V. Dodonov, A.B. Klimov, D.E. Nikonov,"Quantum particle in a box with moving walls", *J. Math. Phys.* **34** (1993) 8 p. 3391-3404.

[11] P. Duclos, P. Šťovíček, "Floquet hamiltonians with pure point spectrum", *Comm. Math. Phys.* **177** (1996) 2 p. 327-347.

[12] N. Balazs, "On the solution of the wave equation with moving boundaries", *J. Math. Anal. Appl.* **3** (1961) p. 472-484.

[13] J. Cooper, "Asymptotic behavior for the vibrating string with a moving boundary", *J. Math. Anal. Appl.* **174** (1993) 1 p. 67-87.

[14] J. Dittrich, P. Duclos, P. Šeba, "Instability in a classical periodically driven string", *Phys. Rev. E* **49** (1994) 4 p. 3535-3538.

[15] J. Cooper, H. Koch, "The spectrum of a hyperbolic evolution operator", *J. Funct. Anal.* **133** (1995) 2 p. 301-328.

[16] C. Gignoux, O. Meplan, "Exponential growth of the energy of a wave in a 1D vibrating cavity. Application to the quantum vacuum" (preprint ISN 95.24, Institut de Sciences Nucléaires, Grenoble, 1995).

[17] J. Cooper, "Long time behavior and energy growth for electromagnetic waves reflected by a moving boundary", *IEEE Transac. Ant. Prop.* **41** (1993) 10 p. 1365-1370.

[18] G.T. Moore, "Quantum field theory of the electromagnetic field in a variable-length one-dimensional cavity", *J. Math. Phys.* **11** (1970) 9 p. 2679-2691.

[19] G. Calucci, "Casimir effect for moving bodies", *J. Phys. A: Math. Gen.* **25** (1992) 13 p. 3873-3882 and corrigenda: *J. Phys. A: Math. Gen.* **26** (1993) 20 p. 5636.

[20] V.V. Dodonov, A.B. Klimov, D.E. Nikonov, "Quantum phenomena in resonators with moving walls", *J. Math. Phys.* **34** (1993) 7 p. 2742-2756.

[21] H. Johnston, S. Sarkar, "A re-examination of the quantum field theory of optical cavities with moving mirrors", *J. Phys. A: Math. Gen.* **29** (1996) 8 p. 1741-1746.

[22] M. Herman, "Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations", *Publ. Math. I.H.E.S.* **49** (1979) p. 5-234 (in french).

[23] A. Katok, B. Hasselblatt, "Introduction to the Modern Theory of Dynamical Systems", Encyclopedia of Mathematics and its Applications, **54**, Cambridge University Press, Cambridge, 1995.

[24] A. Kufner, O. John, S. Fučík, "Function Spaces", Noordhoff International Publishing, Leyden; Academia, Prague, 1977.

[25] O.A. Ladyzhenskaya, "The Boundary Value Problems of Mathematical Physics", Applied Mathematical Sciences, **49**, Springer-Verlag, New York-Berlin, 1985.

[26] J.L. Lions, E. Magenes, "Problèmes aux Limites Non Homogènes et Applications. Vol. I", Travaux et Recherches Mathématiques, **17**, Dunod, Paris, 1968 (in french).

[27] S. Lang, "Introduction to Diophantine Approximations", Second Edition, Springer-Verlag, New York, 1995.

[28] Y. Katznelson, D. Ornstein, "The differentiability of the conjugation of certain diffeomorphisms of the circle", *Erg. Theory Dynam. Systems* **9** (1989) 4 p. 643-680.

[29] V.I. Arnol'd, "Small denominators I. Mappings of the circumference onto itself", *Translations A.M.S. Series 2* **46** (1965) p. 213-284.

[30] Z. Nitecki, "Differentiable Dynamics", The M.I.T. Press, Cambridge-Massachusetts-London, 1971.