A CHARACTERIZATION OF ALL ELLIPTIC SOLUTIONS OF THE AKNS HIERARCHY

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ABSTRACT. An explicit characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy is presented. More precisely, we show that a pair of elliptic functions (p, q) is an algebro-geometric AKNS potential, that is, a solution of some equation of the stationary AKNS hierarchy, if and only if the associated linear differential system $J\Psi' + Q\Psi = E\Psi$, where $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $Q = \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}$, has a fundamental system of solutions which are meromorphic with respect to the independent variable for infinitely many and hence for all values of the spectral parameter $E \in \mathbb{C}$.

Our approach is based on (an extension of) a classical theorem of Picard, which guarantees the existence of solutions which are elliptic of the second kind for n^{th}-order ordinary differential equations with elliptic coefficients associated with a common period lattice. The fundamental link between Picard's theorem and elliptic algebro-geometric solutions of completely integrable hierarchies of nonlinear evolution equations has recently been established in connection with the KdV hierarchy. The current investigation appears to be the first of its kind associated with matrix-valued Lax pairs. As by-products we offer a detailed Floquet analysis of Dirac-type differential expressions with periodic coefficients specifically emphasizing algebro-geometric coefficients and a constructive reduction of singular hyperelliptic curves and their Baker-Akhiezer functions to the nonsingular case.

1. INTRODUCTION

Before describing our approach in some detail, we shall give a brief account of the history of the problem of characterizing elliptic algebro-geometric solutions of completely integrable systems. This theme dates back to a 1940 paper of Ince [50] who studied what is presently called the Lamé–Ince potential

$$q(x) = -n(n+1)\wp(x + \omega_3), \quad n \in \mathbb{N},\ x \in \mathbb{R} \tag{1.1}$$

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in connection with the second-order ordinary differential equation

$$y''(E,x) + q(x)y(E,x) = Ey(E,x), \quad E \in \mathbb{C}. \tag{1.2}$$

Here $\wp(x) = \wp(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods $2\omega_1$ and $2\omega_3$ ($\mathrm{Im}(\omega_3/\omega_1) \neq 0$). In the special case where ω_1 is real and ω_3 is purely imaginary, the potential $q(x)$ in (1.1) is real-valued and Ince's striking result [50], in modern spectral theoretic terminology, yields that the spectrum of the unique self-adjoint operator associated with the differential expression $L_2 = d^2/dx^2 + q(x)$ in $L^2(\mathbb{R})$ exhibits finitely many bands (respectively gaps), that is,

$$\sigma(L_2) = (-\infty, E_{2n}] \cup \bigcup_{m=1}^{n} [E_{2m-1}, E_{2m-2}]\,, \quad E_{2n} < E_{2n-1} < \ldots < E_0. \tag{1.3}$$

What we call the Lamé–Ince potential has, in fact, a long history and many investigations of it precede Ince's work [50]. Without attempting to be complete we refer the interested reader, for instance, to [2], [3], Sect. 59, [6], Ch. IX, [9], Sect. 3.6.4, [18], Sects. 135–138, [19], [20], [22], [39], [48], p. 494–498, [49], p. 118–122, 266–418, 475–478, [51], p. 378–380, [54], [57], p. 265–275, [71], [73], [98], [100], [102], Ch. XXIII as pertinent publications before and after Ince's fundamental paper.

Following the traditional terminology, any real-valued potential q that gives rise to a spectrum of the type (1.3) is called an algebro-geometric KdV potential. The proper extension of this notion to general complex-valued meromorphic potentials q then proceeds via the KdV hierarchy of nonlinear evolution equations obtained from appropriate Lax pairs $(P_{2n+1}(t), L_2(t))$, with $L_2(t) = d^2/dx^2 + q(x,t)$, $P_{2n+1}(t)$ a differential expression of order $2n+1$, whose coefficients are certain differential polynomials in $q(x,t)$ (i.e., polynomials in q and its x-derivatives), and $t \in \mathbb{R}$ an additional deformation parameter. Varying $n \in \mathbb{N} \cup \{0\}$, the collection of all Lax equations

$$\frac{d}{dt} L_2 = [P_{2n+1}, L_2]\,, \text{ that is, } q_t = [P_{2n+1}, L_2] \tag{1.4}$$

then defines the celebrated KdV hierarchy. In particular, $q(x,t)$ is called an algebro-geometric solution of (one of) the n_0^{th} equation in (1.4) if it satisfies for some (and hence for all) fixed $t_0 \in \mathbb{R}$ one of the higher-order stationary KdV equations in (1.4) associated with some $n_1 \geq n_0$. Therefore, without loss of generality, one can focus on characterizing stationary elliptic algebro-geometric solutions of the KdV hierarchy (and similarly in connection with other hierarchies of soliton equations).

The stationary KdV hierarchy, characterized by $q_t = 0$ or $[P_{2n+1}, L_2] = 0$, is intimately connected with the question of commutativity of ordinary differential expressions. In particular, if $[P_{2n+1}, L] = 0$, a celebrated theorem of Burchnall and Chaundy

[16], [17] implies that P_{2n+1} and L_2 satisfy an algebraic relationship of the form

$$P_{2n+1}^2 = \prod_{m=0}^{2n} (L_2 - E_m) \text{ for some } \{E_m\}_{m=0}^{2n} \subset \mathbb{C} \tag{1.5}$$

and hence define a (possibly singular) hyperelliptic curve (branched at infinity)

$$w^2 = \prod_{m=0}^{2n} (E - E_m). \tag{1.6}$$

It is the curve (1.6), which signifies that q in $L_2 = d^2/dx^2 + q(x)$ represents an algebro-geometric KdV potential.

While these considerations pertain to general solutions of the stationary KdV hierarchy, we now concentrate on the additional restriction that q be an elliptic function (i.e., meromorphic and doubly periodic) and hence return to the history of elliptic algebro-geometric potentials q for $L_2 = d^2/dx^2 + q(x)$, or, equivalently, elliptic solutions of the stationary KdV hierarchy. Ince's remarkable algebro-geometric result (1.3) remained the only explicit elliptic algebro-geometric example until the KdV flow $q_t = \frac{1}{4}q_{xxx} + \frac{3}{2}qq_x$ with the initial condition $q(x,0) = -6\wp(x)$ was explicitly integrated by Dubrovin and Novikov [26] in 1975 (see also [29]–[31], [53]), and found to be of the type

$$q(x,t) = -2\sum_{j=1}^{3} \wp(x - x_j(t)) \tag{1.7}$$

for appropriate $\{x_j(t)\}_{1\le j\le 3}$. Given these results it was natural to ask for a systematic account of all elliptic solutions of the KdV hierarchy, a problem posed, for instance, in [70], p. 152.

In 1977, Airault, McKean and Moser, in their seminal paper [1], presented the first systematic study of the isospectral torus $I_{\mathbb{R}}(q_0)$ of real-valued smooth potentials $q_0(x)$ of the type

$$q_0(x) = -2\sum_{j=1}^{M} \wp(x - x_j) \tag{1.8}$$

with an algebro-geometric spectrum of the form (1.3). In particular, the potential (1.8) turned out to be intimately connected with completely integrable many-body systems of the Calogero-Moser-type [19], [67] (see also [20], [22]). This connection with integrable particle systems was subsequently exploited by Krichever [58] in his fundamental construction of elliptic algebro-geometric solutions of the Kadomtsev-Petviashvili equation. The next breakthrough occurred in 1988 when Verdier [99] published new explicit examples of elliptic algebro-geometric potentials. Verdier's examples spurred a flurry of activities and inspired Belokolos and Enol'skii [11], Smirnov

[84], and subsequently Taimanov [90] and Kostov and Enol'skii [56] to find further such examples by combining the reduction process of Abelian integrals to elliptic integrals (see [7], [8], [9], Ch. 7, [10]) with the aforementioned techniques of Krichever [58], [59]. This development finally culminated in a series of recent results of Treibich and Verdier [95], [96], [97], where it was shown that a general complex-valued potential of the form

$$q(x) = -\sum_{j=1}^{4} d_j \,\wp(x - \omega_j) \tag{1.9}$$

($\omega_2 = \omega_1 + \omega_3,\ \omega_4 = 0$) is an algebro-geometric potential if and only if $d_j/2$ are triangular numbers, that is, if and only if

$$d_j = g_j(g_j + 1) \text{ for some } g_j \in \mathbb{Z}, \quad 1 \le j \le 4. \tag{1.10}$$

We shall refer to potentials of the form (1.9), (1.10) as Treibich-Verdier potentials. The methods of Treibich and Verdier are based on hyperelliptic tangent covers of the torus $\mathbb{C}/\Lambda$ (Λ being the period lattice generated by $2\omega_1$ and $2\omega_3$).

The state of the art of elliptic algebro-geometric solutions up to 1993 was recently reviewed in issues 1 and 2 of volume 36 of Acta Applicandae Math., see, for instance, [12], [33], [60], [86], [91], [94] and also in [13], [24], [25], [32], [47], [81], [87], [93]. In addition to these investigations on elliptic solutions of the KdV hierarchy, the study of other soliton hierarchies, such as the modified KdV hierarchy, nonlinear Schrödinger hierarchy, and Boussinesq hierarchy has also begun. We refer, for instance, to [21], [28], [39], [40], [63], [64], [66], [79], [80], [82], [83], [85], [88].

Despite these (basically algebro-geometric) approaches described thus far, an efficient characterization of all elliptic solutions of the KdV hierarchy remained elusive until recently. The final breakthrough in this characterization problem in [43], [44] became possible due to the application of the most powerful analytic tool in this context, a theorem of Picard. This result of Picard (cf. Theorem 6.1) is concerned with the existence of solutions which are elliptic of the second kind of n^{th}-order ordinary differential equations with elliptic coefficients. The main hypothesis in Picard's theorem for a second-order differential equation of the form

$$y''(x) + q(x)y(x) = Ey(x), \quad E \in \mathbb{C}, \tag{1.11}$$

with an elliptic potential q, relevant in connection with the KdV hierarchy (cf. the second-order differential expression L_2 in (1.4)), assumes the existence of a fundamental system of solutions meromorphic in x. Hence we call any elliptic function q which has this property for all values of the spectral parameter a **Picard-KdV potential**. The characterization of all elliptic algebro-geometric solutions of the stationary KdV hierarchy, then reads as follows:

Theorem 1.1. *([43], [44]) q is an elliptic algebro-geometric potential if and only if it is a Picard-KdV potential.*

In particular, Theorem 1.1 sheds new light on Picard's theorem since it identifies the elliptic coefficients q for which there exists a meromorphic fundamental system of solutions of (1.11) precisely as the elliptic algebro-geometric solutions of the stationary KdV hierarchy. Moreover, we stress its straightforward applicability based on an elementary Frobenius-type analysis which decides whether or not (1.11) has a meromorphic fundamental system for each $E \in \mathbb{C}$. Related results and further background information on our approach can be found in [38], [39]–[41], [42], [45].

After this somewhat detailed description of the history of the problem under consideration, we now turn to the content of the present paper. The principal objective in this paper is to prove an analogous characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy and hence to extend the preceding formalism to matrix-valued differential expressions. More precisely, replace the scalar second-order differential equation (1.2) by the 2×2 first-order system

$$J\Psi'(E,x) + Q(x)\Psi(E,x) = E\Psi(E,x), \quad E \in \mathbb{C}, \tag{1.12}$$

where $\Psi(E,x) = (\psi_1(E,x), \psi_2(E,x))^t$ ("t" abbreviating transpose) and

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{1.13}$$

$$Q(x) = \begin{pmatrix} Q_{1,1}(x) & Q_{1,2}(x) \\ Q_{2,1}(x) & Q_{2,2}(x) \end{pmatrix} = \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}. \tag{1.14}$$

Similarly, replace the scalar KdV differential expression L_2 by the 2×2 matrix-valued differential expression $L(t) = Jd/dx + Q(x,t)$, t a deformation parameter. The AKNS hierarchy of nonlinear evolution equations is then constructed via appropriate Lax pairs $(P_{n+1}(t), L(t))$, where $P_{n+1}(t)$ is a 2×2 matrix-valued differential expression of order $n+1$ (cf. Section 2 for an explicit construction of P_{n+1}). In analogy to the KdV hierarchy, varying $n \in \mathbb{N} \cup \{0\}$, the collection of all Lax equations

$$\frac{d}{dt} L = [P_{n+1}, L], \tag{1.15}$$

then defines the AKNS hierarchy of nonlinear evolution equations for $(p(x,t), q(x,t))$. Algebro-geometric AKNS solutions are now introduced as in the KdV context and stationary AKNS solutions, characterized by $p_t = 0, q_t = 0$ or $[P_{n+1}, L] = 0$, again yield an algebraic relationship between P_{n+1} and L of the type

$$P_{n+1}^2 = \prod_{m=0}^{2n+1} (L - E_m) \text{ for some } \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C} \tag{1.16}$$

and hence a (possibly singular) hyperelliptic curve (not branched at infinity)

$$w^2 = \prod_{m=0}^{2n+1} (E - E_m). \tag{1.17}$$

In order to characterize all elliptic solutions of the AKNS hierarchy we follow our strategy in the KdV context and consider 2×2 first-order systems of the form

$$J\Psi'(x) + Q(x)\Psi(x) = E\Psi(x), \quad E \in \mathbb{C}, \tag{1.18}$$

with Q an elliptic 2×2 matrix of the form (1.14). Again we single out those elliptic Q such that (1.18) has a fundamental system of solutions meromorphic in x for all values of the spectral parameter $E \in \mathbb{C}$ and call such Q **Picard-AKNS potentials**. Our principal new result in this paper, a characterization of all elliptic algebro-geometric solutions of the stationary AKNS hierarchy, then simply reads as follows:

Theorem 1.2. *An elliptic potential Q is an algebro-geometric AKNS potential if and only if it is a Picard-AKNS potential (i.e., if and only if for infinitely many and hence for all $E \in \mathbb{C}$,* (1.18) *has a fundamental system of solutions meromorphic with respect to x).*

The proof of Theorem 1.2 in Section 6 (Theorem 6.4) relies on three main ingredients: A purely Floquet theoretic part to be discussed in detail in Sections 4 and 5, the fact that meromorphic algebro-geometric AKNS potentials are Picard potentials using gauge transformations in Section 3, and an elliptic function part described in Section 6. The corresponding Floquet theoretic part is summarized in Theorems 4.7, 4.8, 5.1–5.4. In particular, Theorems 4.7 and 4.8 illustrate the great variety of possible values of algebraic multiplicities of (anti)periodic and Dirichlet eigenvalues in the general case where L is non-self-adjoint. Theorem 5.1 on the other hand reconstructs the (possibly singular) hyperelliptic curve (1.17) associated with the 2×2 periodic matrix Q (not necessarily elliptic), which gives rise to two linearly independent Floquet solutions of $J\Psi' + Q\Psi = E\Psi$ for all but finitely many values of $E \in \mathbb{C}$.

Our use of gauge transformations in Section 3, in principle, suggests a constructive method to relate τ-functions associated with a singular curve $\mathcal{K}_n$ and θ-functions of the associated desingularized curve $\hat{\mathcal{K}}_{\hat{n}}$, which appears to be of independent interest.

The elliptic function portion in Section 6 consists of several items. First of all we describe a matrix generalization of Picard's (scalar) result in Theorem 6.1. In Theorem 6.3 we prove the key result that all $4\omega_j$-periodic eigenvalues associated with Q lie in certain strips

$$S_j = \{E \in \mathbb{C} \,|\, |\operatorname{Im}(|\omega_j|^{-1}\omega_j E)| \leq C_j\}, \quad j = 1, 3 \tag{1.19}$$

for suitable constants $C_j > 0$. Then S_1 and S_3 do not intersect outside a sufficiently large disk centered at the origin. A combination of this fact and Picard's Theorem 6.1 then yields a proof of Theorem 1.2 (see the proof of Theorem 6.4).

We close Section 6 with a series of remarks that put Theorem 1.2 into proper perspective: Among a variety of points, we stress, in particular, its straightforward applicability based on an elementary Frobenius-type analysis, its property of complementing Picard's original result, and its connection with the Weierstrass theory of reduction of Abelian to elliptic integrals. Finally, Section 7 rounds off our presentation with a few explicit examples.

The result embodied by Theorems 1.1 and 1.2 in the context of the KdV and AKNS hierarchies, uncovers a new general principle in connection with elliptic algebro-geometric solutions of completely integrable systems: The existence of such solutions appears to be in a one-to-one correspondence with the existence of a meromorphic (with respect to the independent variable) fundamental system of solutions for the underlying linear Lax differential expression (for all values of the corresponding spectral parameter $E \in \mathbb{C}$).

Even though the current AKNS case is technically much more involved than the KdV case in [44] (and despite the large number of references at the end) we have made every effort to keep this presentation self-contained.

2. The AKNS Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we briefly review the construction of the AKNS hierarchy using a recursive approach. This method was originally introduced by Al'ber [4] in connection with the Korteweg-de Vries hierarchy. The present case of the AKNS hierarchy was first systematically developed in [37].

Suppose that $q = iQ_{1,2}$, $p = -iQ_{2,1} \in C^\infty(\mathbb{R})$ (or meromorphic on $\mathbb{C}$) and consider the Dirac-type matrix-valued differential expression

$$L = J\frac{d}{dx} + Q(x) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}, \tag{2.1}$$

where we abbreviate

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{2.2}$$

$$Q(x) = \begin{pmatrix} Q_{1,1}(x) & Q_{1,2}(x) \\ Q_{2,1}(x) & Q_{2,2}(x) \end{pmatrix} = \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}. \tag{2.3}$$

In order to explicitly construct higher-order matrix-valued differential expressions P_{n+1}, $n \in \mathbb{N}_0$ $(=\mathbb{N}\cup\{0\})$ commuting with L, which will be used to define the stationary AKNS hierarchy, one can proceed as follows (see [37] for more details).

Define functions f_ℓ, g_ℓ, and h_ℓ by the following recurrence relations,

$$f_{-1} = 0, \quad g_0 = 1, \quad h_{-1} = 0,$$

$$f_{\ell+1} = \frac{i}{2} f_{\ell,x} - iqg_{\ell+1}, \quad g_{\ell+1,x} = pf_\ell + qh_\ell, \quad h_{\ell+1} = -\frac{i}{2} h_{\ell,x} + ipg_{\ell+1} \tag{2.4}$$

for $\ell = -1, 0, 1, \dots$. The functions f_ℓ, g_ℓ, and h_ℓ are polynomials in the variables $p, q, p_x, q_x, \dots$ and $c_1, c_2, \dots$ where the c_j denote integration constants. Assigning weight $k+1$ to $p^{(k)}$ and $q^{(k)}$ and weight k to c_k one finds that f_ℓ, $g_{\ell+1}$, and h_ℓ are homogeneous of weight $\ell + 1$.

Explicitly, one computes,

$$
\begin{aligned}
&f_0 = -iq,\\
&f_1 = \frac{1}{2}q_x + c_1(-iq),\\
&f_2 = \frac{i}{4}q_{xx} - \frac{i}{2}pq^2 + c_1(\frac{1}{2}q_x) + c_2(-iq),\\
&g_0 = 1,\\
&g_1 = c_1,\\
&g_2 = \frac{1}{2}pq + c_2,\\
&g_3 = -\frac{i}{4}(p_x q - pq_x) + c_1(\frac{1}{2}pq) + c_3,\\
&h_0 = ip,\\
&h_1 = \frac{1}{2}p_x + c_1(ip),\\
&h_2 = -\frac{i}{4}p_{xx} + \frac{i}{2}p^2 q + c_1(\frac{1}{2}p_x) + c_2(ip),\\
&\text{etc.}
\end{aligned}
\tag{2.5}
$$

Next one defines the matrix-valued differential expression P_{n+1} by

$$P_{n+1} = -\sum_{\ell=0}^{n+1}(g_{n-\ell+1}J + iA_{n-\ell})L^{\ell}, \tag{2.6}$$

where

$$A_\ell = \begin{pmatrix} 0 & -f_\ell \\ h_\ell & 0 \end{pmatrix}, \quad \ell = -1, 0, 1, \dots. \tag{2.7}$$

One verifies that

$$[g_{n-\ell+1}J + iA_{n-\ell}, L] = 2iA_{n-\ell}L - 2iA_{n-\ell+1}, \tag{2.8}$$

where $[\cdot,\cdot]$ denotes the commutator. This implies

$$[P_{n+1}, L] = 2iA_{n+1}. \tag{2.9}$$

The pair (P_{n+1}, L) represents a Lax pair for the AKNS hierarchy. Introducing a deformation parameter t into (p, q), that is, $(p(x), q(x)) \to (p(x,t), q(x,t))$, the AKNS hierarchy (cf., e.g., [68], Chs. 3, 5 and the references therein) is defined as the collection of evolution equations (varying $n \in \mathbb{N}_0$)

$$\frac{d}{dt}L(t) - [P_{n+1}(t), L(t)] = 0 \tag{2.10}$$

or equivalently, by

$$\mathrm{AKNS}_n(p,q) = \begin{pmatrix} p_t(x,t) - 2h_{n+1}(x,t) \\ q_t(x,t) - 2f_{n+1}(x,t) \end{pmatrix} = 0, \tag{2.11}$$

that is, by

$$\mathrm{AKNS}_n(p,q) = \begin{pmatrix} p_t + i(H_{n,x} - 2iEH_n - 2pG_{n+1}) \\ q_t - i(F_{n,x} + 2iEF_n - 2qG_{n+1}) \end{pmatrix} = 0. \tag{2.12}$$

Explicitly, one obtains for the first few equations in (2.11)

$$\begin{aligned} &\mathrm{AKNS}_0(p,q) = \begin{pmatrix} p_t - p_x - 2ic_1 p \\ q_t - q_x + 2ic_1 q \end{pmatrix} = 0 \\ &\mathrm{AKNS}_1(p,q) = \begin{pmatrix} p_t + \frac{i}{2}p_{xx} - ip^2q - c_1p_x - 2ic_2p \\ q_t - \frac{i}{2}q_{xx} + ipq^2 - c_1q_x + 2ic_2q \end{pmatrix} = 0 \\ &\mathrm{AKNS}_2(p,q) = \begin{pmatrix} p_t + \frac{1}{4}p_{xxx} - \frac{3}{2}pp_xq + c_1(\frac{i}{2}p_{xx} - ip^2q) - c_2p_x - 2ic_3p \\ q_t + \frac{1}{4}q_{xxx} - \frac{3}{2}pqq_x + c_1(-\frac{i}{2}q_{xx} + ipq^2) - c_2q_x + 2ic_3q \end{pmatrix} = 0 \\ &\text{etc.} \end{aligned} \tag{2.13}$$

The stationary AKNS hierarchy is then defined by the vanishing of the commutator of P_{n+1} and L, that is, by

$$[P_{n+1}, L] = 0, \quad n \in \mathbb{N}_0, \tag{2.14}$$

or equivalently, by

$$f_{n+1} = h_{n+1} = 0, \quad n \in \mathbb{N}_0. \tag{2.15}$$

Next, we introduce F_n, G_{n+1}, and H_n which are polynomials with respect to $E \in \mathbb{C}$,

$$\begin{aligned} F_n(E,x) &= \sum_{\ell=0}^{n} f_{n-\ell}(x)E^\ell, \\ G_{n+1}(E,x) &= \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x)E^\ell, \\ H_n(E,x) &= \sum_{\ell=0}^{n} h_{n-\ell}(x)E^\ell, \end{aligned} \tag{2.16}$$

and note that (2.15) becomes

$$F_{n,x}(E,x) = -2iEF_n(E,x) + 2q(x)G_{n+1}(E,x), \tag{2.17}$$

$$G_{n+1,x}(E,x) = p(x)F_n(E,x) + q(x)H_n(E,x), \tag{2.18}$$

$$H_{n,x}(E,x) = 2iEH_n(E,x) + 2p(x)G_{n+1}(E,x). \tag{2.19}$$

These equations show that $G_{n+1}^2 - F_nH_n$ is independent of x. Hence

$$R_{2n+2}(E) = G_{n+1}(E,x)^2 - F_n(E,x)H_n(E,x) \tag{2.20}$$

is a monic polynomial in E of degree $2n+2$.

One can use (2.17)–(2.20) to derive differential equations for F_n and H_n separately by eliminating G_{n+1}. One obtains

$$q(2F_nF_{n,xx} - F_{n,x}^2 + 4(E^2 - pq)F_n^2) - q_x(2F_nF_{n,x} + 4iEF_n^2) = -4q^3R_{2n+2}(E), \tag{2.21}$$

$$p(2H_nH_{n,xx} - H_{n,x}^2 + 4(E^2 - pq)H_n^2) - p_x(2H_nH_{n,x} - 4iEH_n^2) = -4p^3R_{2n+2}(E). \tag{2.22}$$

Next, assuming $[P_{n+1}, L] = 0$, one infers

$$P_{n+1}^2 = \sum_{\ell,m=0}^{n+1} (g_{n-\ell+1}J + iA_{n-\ell})(g_{n-m+1}J + iA_{n-m})L^{\ell+m}. \tag{2.23}$$

Hence,

$$P_{n+1}^2 = -G_{n+1}(L,x)^2 + F_n(L,x)H_n(L,x) = -R_{2n+2}(L), \tag{2.24}$$

that is, whenever P_{n+1} and L commute they necessarily satisfy an algebraic relationship. In particular, they define a (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus n of the type

$$\mathcal{K}_n : w^2 = R_{2n+2}(E), \quad R_{2n+2}(E) = \prod_{m=0}^{2n+1} (E - E_m) \text{ for some } \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C}. \tag{2.25}$$

The functions f_ℓ, g_ℓ, and h_ℓ, and hence the matrices A_ℓ and the differential expressions P_ℓ defined above, depend on the choice of the integration constants $c_1, c_2, ..., c_\ell$ (cf. (2.5)). In the following we make this dependence explicit and write $f_\ell(c_1, \dots, c_\ell)$, $g_\ell(c_1, \dots, c_\ell)$, $h_\ell(c_1, \dots, c_\ell)$, $A_\ell(c_1, ..., c_\ell)$, $P_\ell(c_1, ..., c_\ell)$, etc. In particular, we denote homogeneous quantities, where $c_\ell = 0, \ell \in \mathbb{N}$ by $\hat{f}_\ell = f_\ell(0, \dots, 0)$, $\hat{g}_\ell = g_\ell(0, \dots, 0)$, $\hat{h}_\ell = h_\ell(0, \dots, 0)$, $\hat{A}_\ell = A_\ell(0, \dots, 0)$, $\hat{P}_\ell = P_\ell(0, ..., 0)$, etc. In addition, we note that

$$f_\ell(c_1, ..., c_\ell) = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad g_\ell(c_1, ..., c_\ell) = \sum_{k=0}^{\ell} c_{\ell-k}\hat{g}_k, \quad h_\ell(c_1, ..., c_\ell) = \sum_{k=0}^{\ell} c_{\ell-k}\hat{h}_k, \tag{2.26}$$

and

$$A_\ell(c_1, ..., c_\ell) = \sum_{k=0}^{\ell} c_{\ell-k}\hat{A}_k, \tag{2.27}$$

defining $c_0 = 1$. In particular, then

$$P_r(c_1, ..., c_r) = \sum_{\ell=0}^{r} c_{r-\ell}\hat{P}_\ell. \tag{2.28}$$

Next suppose that P_{n+1} is any 2×2 matrix-valued differential expression such that $[P_{n+1}, L]$ represents multiplication by a matrix whose diagonal entries are zero. This implies that the leading coefficient of P_{n+1} is a constant diagonal matrix. Since any constant diagonal matrix can be written as a linear combination of J and I (the identity matrix in $\mathbb{C}^2$), we infer the existence of complex numbers α_{n+1} and β_{n+1} such that

$$S_1 = P_{n+1} - \alpha_{n+1}\hat{P}_{n+1} - \beta_{n+1}L^{n+1} \tag{2.29}$$

is a differential expression of order at most n whenever P_{n+1} is of order $n+1$. Note that

$$[S_1, L] = [P_{n+1}, L] - \alpha_{n+1}[\hat{P}_{n+1}, L] = [P_{n+1}, L] - 2i\alpha_{n+1}\hat{A}_{n+1} \tag{2.30}$$

represents multiplication with zero diagonal elements. An induction argument then shows that there exists S_{n+1} such that

$$S_{n+1} = P_{n+1} - \sum_{\ell=1}^{n+1}(\alpha_\ell\hat{P}_\ell + \beta_\ell L^\ell) \text{ and } [S_{n+1}, L] = [P_{n+1}, L] - 2i\sum_{\ell=1}^{n+1}\alpha_\ell\hat{A}_\ell. \tag{2.31}$$

Since the right-hand side of the last equation is multiplication with a zero diagonal, S_{n+1} is a constant diagonal matrix, that is, there exist complex numbers α_0 and β_0 such that $S_{n+1} = \alpha_0 J + \beta_0 I$. Hence

$$P_{n+1} = \sum_{\ell=0}^{n+1}(\alpha_\ell\hat{P}_\ell + \beta_\ell L^\ell) \text{ and } [P_{n+1}, L] = 2i\sum_{\ell=0}^{n+1}\alpha_\ell\hat{A}_\ell. \tag{2.32}$$

Consequently, if all $\alpha_\ell = 0$, then P_{n+1} is a polynomial of L, and P_{n+1} and L commute irrespective of p and q. If, however, $\alpha_r \neq 0$ and $\alpha_\ell = 0$ for $\ell > r$, then

$$P_{n+1} = \alpha_r P_r(\frac{\alpha_{r-1}}{\alpha_r}, ..., \frac{\alpha_0}{\alpha_r}) + \sum_{\ell=0}^{n+1}\beta_\ell L^\ell. \tag{2.33}$$

In this case P_{n+1} and L commute if and only if

$$\sum_{\ell=0}^{r}\frac{\alpha_\ell}{\alpha_r}\hat{A}_\ell = A_r(\frac{\alpha_{r-1}}{\alpha_r}, ..., \frac{\alpha_0}{\alpha_r}) = 0, \tag{2.34}$$

that is, if and only if (p, q) is a solution of some equation of the stationary AKNS hierarchy. In this case $P_{n+1} - \sum_{\ell=0}^{n+1}\beta_\ell L^\ell$ and L satisfy an algebraic relationship of the type (2.24).

Suppose on the other hand that P_{n+1} is a matrix-valued differential expression such that $(P_{n+1} - K_r(L))^2 = -R_{2n+2}(L)$ for some polynomials K_r and R_{2n+2}. Then L commutes with $(P_{n+1} - K_r(L))^2$ and this enforces that L also commutes with $P_{n+1} - K_r(L)$ and hence with P_{n+1}. Thus we proved the following theorem.

Theorem 2.1. *Let L be defined as in (2.1). If P_{n+1} is a matrix-valued differential expression of order $n+1$ which commutes with L, whose leading coefficient is different from a constant multiple of J^{n+1}, then there exist polynomials K_r and R_{2n+2} of degree $r \leq n+1$ and $2n+2$, respectively, such that $(P_{n+1} - K_r(L))^2 = -R_{2n+2}(L)$.*

Theorem 2.1 represents a matrix-valued generalization of a celebrated result due to Burchnall and Chaundy [16], [17] in the special case of scalar differential expressions.

By the arguments presented thus far in this section it becomes natural to make the following definition. We denote by $M_2(\mathbb{C})$ the set of all 2×2 matrices over $\mathbb{C}$.

Definition 2.2. A function $Q : \mathbb{R} \to M_2(\mathbb{C})$ of the type $Q = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}$ is called an **algebro-geometric AKNS potential** if (p, q) is a stationary solution of some equation of the AKNS hierarchy (2.15).

By a slight abuse of notation we will also call (p, q) an algebro-geometric AKNS potential in this case.

The following theorem gives a sufficient condition for Q to be algebro-geometric.

Theorem 2.3. *Assume that $F_n(E, x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) E^\ell$ with $f_0(x) = -iq(x)$ is a polynomial of degree n in E, whose coefficients are twice continuously differentiable complex-valued functions on (a, b) for some $-\infty \leq a < b \leq \infty$. Moreover, suppose that q has (at most) finitely many zeros on each compact interval on $\mathbb{R}$ with $\pm\infty$ the only possible accumulation points. If*

$$\begin{aligned} \frac{1}{4q(x)^3} \Big\{ & q(x)\big(2F_n(E,x)F_{n,xx}(E,x) - F_{n,x}(E,x)^2 + 4(E^2 - p(x)q(x))F_n(E,x)^2\big) \\ & - q_x(x)\big(2F_n(E,x)F_{n,x}(E,x) + 4iEF_n(E,x)^2\big)\Big\}, \end{aligned} \tag{2.35}$$

is independent of x, then p, q and all coefficients f_ℓ of F_n are in $C^\infty((a, b))$. Next, define

$$g_0(x) = 1,$$

$$g_{\ell+1}(x) = \frac{1}{q(x)}\Big(-\frac{1}{2} f_{\ell,x}(x) - i f_{\ell+1}(x)\Big), \tag{2.36}$$

$$h_\ell(x) = \frac{1}{q(x)}(-g_{\ell+1,x}(x) - ip(x) f_\ell(x)) \tag{2.37}$$

for $\ell = 0, \dots, n$, where $f_{n+1} = 0$. Then the differential expression P_{n+1} defined by (2.6) commutes with L in (2.1). In particular, if $(a, b) = \mathbb{R}$ then (p, q) is an algebro-geometric AKNS potential.

Proof. The expression (2.35) is a monic polynomial of degree $2n+2$ with constant coefficients. We denote it by

$$R_{2n+2}(E) = \sum_{m=0}^{2n+2} \gamma_{2n+2-m} E^m, \quad \gamma_0 = 1. \tag{2.38}$$

We now compare coefficients in (2.35) and (2.38) starting with the largest powers. First of all this yields

$$q_x = i\gamma_1 q - 2f_1, \tag{2.39}$$

which shows that $q \in C^3((a,b))$, and secondly that

$$p = (-4\gamma_2 q^2 - 3q_x^2 + 2qq_{xx} - 8iqf_2 - 4f_1^2 + 8q_x f_1)/(4q^3), \tag{2.40}$$

which shows that $p \in C^1((a,b))$. Comparing the coefficients of $E^{2n-\ell}$ allows one to express $4q^2 f_{\ell,xx}$ as a polynomial in p, q, q_x, the coefficients f_ℓ, their first derivatives, and the second derivatives of $f_0,\dots,f_{\ell-1}$. Therefore, one may show recursively that $f_{\ell,xx} \in C^1((a,b))$ for any $\ell \in \{1,\dots,n\}$. Equations (2.39) and (2.40) then show that $q \in C^4((a,b))$ and $p \in C^2((a,b))$. Thus it follows that the $f_{\ell,xx}$ are in $C^2((a,b))$. An induction argument now completes the proof of the first part of the theorem.

Next, introducing

$$G_{n+1}(E,x) = \sum_{\ell=0}^{n+1} g_{n+1-\ell}(x)E^\ell, \quad H_n(E,x) = \sum_{\ell=0}^{n} h_{n-\ell}(x)E^\ell, \tag{2.41}$$

one finds that F_n, G_{n+1}, and H_n satisfy equations (2.17) and (2.18). Equating (2.35) with R_{2n+2} yields $R_{2n+2} = G_{n+1}^2 - F_n H_n$. Hence $G_{n+1}^2 - F_n H_n$ does not depend on x and therefore differentiating with respect to x results in $2G_{n+1}G_{n+1,x} - F_n H_{n,x} - F_{n,x}H_n = 0$ which shows that equation (2.19) also holds. This in turn proves that f_ℓ, g_ℓ, and h_ℓ satisfy the recurrence relations given above with $f_{n+1} = h_{n+1} = 0$. The commutativity of P_{n+1} and L now follows as before. $\square$

The same proof yields the following result.

Corollary 2.4. *Assume that $F_n(E,x) = \sum_{\ell=0}^n f_{n-\ell}(x)E^\ell$, $f_0(x) = -iq(x)$ is a polynomial of degree n in E whose coefficients are meromorphic in x. If*

$$\begin{aligned}\frac{1}{4q(x)^3}\Big\{&q(x)\big(2F_n(E,x)F_{n,xx}(E,x) - F_{n,x}(E,x)^2 + 4(E^2 - p(x)q(x))F_n(E,x)^2\big)\\ &- q_x(x)\big(2F_n(E,x)F_{n,x}(E,x) + 4iEF_n(E,x)^2\big)\Big\},\end{aligned} \tag{2.42}$$

is independent of x, then (p,q) is a meromorphic algebro-geometric AKNS potential.

Finally, we mention an interesting scale invariance of the AKNS equations (2.11).

Lemma 2.5. *Suppose (p, q) satisfies one of the AKNS equations* (2.11),

$$AKNS_n(p, q) = 0 \tag{2.43}$$

for some $n \in \mathbb{N}_0$. Consider the scale transformation

$$(p(x, t), q(x, t)) \to (\breve{p}(x, t), \breve{q}(x, t)) = (Ap(x, t), A^{-1}q(x, t)), \quad A \in \mathbb{C}\backslash\{0\}. \tag{2.44}$$

Then

$$AKNS_n(\breve{p}, \breve{q}) = 0. \tag{2.45}$$

We omit the straightforward proof which can be found, for instance, in [37].

In the particular case of the nonlinear Schrödinger (NS) hierarchy, where

$$p(x, t) = \pm\overline{q(x, t)}, \tag{2.46}$$

(2.44) further restricts A to be unimodular, that is,

$$|A| = 1. \tag{2.47}$$

Note that the KdV hierarchy as well as the modified Korteweg-de Vries (mKdV) hierarchy are contained in the AKNS hierarchy. In fact, setting all integration constants $c_{2\ell+1}$ equal to zero the n^{th} KdV equation is obtained from the $(2n)^{\text{th}}$ AKNS system by the constraint

$$p(x, t) = 1 \tag{2.48}$$

while the n^{th} mKdV equation is obtained from the $(2n)^{\text{th}}$ AKNS system by the constraint

$$p(x, t) = \pm q(x, t). \tag{2.49}$$

3. Gauge Transformations for the Stationary AKNS Hierarchy

This section is devoted to a study of meromorphic properties of solutions $\Psi_\pm(E, x)$ of $L\Psi = E\Psi$ with respect to $x \in \mathbb{C}$ under the assumption that Q is a meromorphic algebro-geometric AKNS potential associated with a (possibly singular) hyperelliptic curve $\mathcal{K}_n$. Meromorphic properties of $\Psi_\pm(E, x)$ will enter at a crucial stage in the proof of our main characterization result, Theorem 6.4.

In the following we denote the order of a meromorphic function f at the point $x \in \mathbb{C}$ by $\operatorname{ord}_x(f)$.

Proposition 3.1. *If Q is a meromorphic algebro-geometric AKNS potential then*

$$\operatorname{ord}_x(p) + \operatorname{ord}_x(q) \geq -2$$

for every $x \in \mathbb{C}$.

Proof. Assume the contrary (i.e., $\mathrm{ord}_x(p) + \mathrm{ord}_x(q) \leq -1$) and choose E such that $R_{2n+2}(E) = 0$, where R_{2n+2} is the polynomial defining the hyperelliptic curve associated with Q. Define F_n as in (2.16) and denote its order at x by r. Then $\mathrm{ord}_x(-4pqF_n(E, \cdot)^2)$ is strictly smaller than $2r - 2$ while the order of any other term on the left-hand side of (2.21) is at least $2r - 2$. This is impossible since the right and thus the left-hand side of (2.21) vanishes identically. □

Therefore, to discuss algebro-geometric AKNS potentials we only need to consider the case where p and q have Laurent expansions of the form

$$p(x) = \sum_{j=0}^{\infty} p_j (x - x_0)^{j-1+m}, \quad q(x) = \sum_{j=0}^{\infty} q_j (x - x_0)^{j-1-m}, \tag{3.1}$$

with m an integer and at least one of the numbers p_0 and q_0 different from zero. If $B = \mathrm{diag}(-m, 0)$, the change of variables $y = x^B w$ transforms the differential equation $Jy' + Qy = Ey$ into the equation $w' = (R/x + S + \sum_{j=0}^{\infty} A_{j+1} x^j)w$, where

$$R = \begin{pmatrix} m & q_0 \\ p_0 & 0 \end{pmatrix}, \quad S = -iE \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & q_j \\ p_j & 0 \end{pmatrix}. \tag{3.2}$$

We now make the ansatz

$$w(x) = \sum_{j=0}^{\infty} \Omega_j (x - x_0)^{j+T}, \tag{3.3}$$

where T and the Ω_j are suitable constant matrices. This ansatz yields the recurrence relation

$$R\Omega_0 - \Omega_0 T = 0, \tag{3.4}$$

$$R\Omega_{j+1} - \Omega_{j+1}(T + j + 1) = -S\Omega_j - \sum_{l=0}^{j} A_{l+1}\Omega_{j-l} = B_j, \tag{3.5}$$

where the last equality defines B_j. In the following we denote the l^{th} column of Ω_j and B_j by $\omega_j^{(l)}$ and $b_j^{(l)}$, respectively.

Proposition 3.2. *Suppose Q is a meromorphic potential of $L\Psi = E\Psi$ and x_0 is a pole of Q where p and q have Laurent expansions given by* (3.1)*. The equation $L\Psi = E\Psi$ has a fundamental system of solutions which are meromorphic in a vicinity of x_0 if and only if*
(i) the eigenvalues λ and $m - \lambda$ (where, without loss of generality, $\lambda > m - \lambda$) of R are distinct integers and
(ii) $b_{2\lambda-m-1}$ is in the range of $R - \lambda$.

Proof. A fundamental matrix of $w' = (R/x + S + \sum_{j=0}^{\infty} A_{j+1} x^j)w$ may be written as (3.3) where T is in Jordan normal form. If all solutions of $L\Psi = E\Psi$ and hence of $w' = (R/x + S + \sum_{j=0}^{\infty} A_{j+1} x^j)w$ are meromorphic near x_0, then T must be a diagonal matrix with integer eigenvalues. Equation (3.4) then shows that the eigenvalues of T are the eigenvalues of R and that R is diagonalizable. But since at least one of p_0 and q_0 is different from zero, R is not diagonalizable if λ is a double eigenvalue of R, a case which is therefore precluded. This proves (i). Since T is a diagonal matrix, equation (3.5) implies

$$(R+\lambda-m-j-1)\omega_{j+1}^{(2)}=b_j^{(2)} \tag{3.6}$$

for $j=0,1,\ldots$. Statement (ii) is just the special case where $j=2\lambda-m-1$.

Conversely, assume that (i) and (ii) are satisfied. If the recurrence relations (3.4), (3.5) are satisfied, that is, if w is a formal solution of $w'=(R/x+S+\sum_{j=0}^{\infty}A_{j+1}x^j)w$ then it is also an actual solution near x_0 (see, e.g., Coddington and Levinson [23], Sect. 4.3). Since R has distinct eigenvalues it has linearly independent eigenvectors. Using these as the columns of Ω_0 and defining $T=\Omega_0^{-1}R\Omega_0$ yields (3.4). Since T is a diagonal matrix, (3.5) is equivalent to the system

$$b_j^{(1)}=(R-\lambda-j-1)\omega_{j+1}^{(1)}, \tag{3.7}$$

$$b_j^{(2)}=(R+\lambda-m-j-1)\omega_{j+1}^{(2)}. \tag{3.8}$$

Next, we note that $R-\lambda-j-1$ is invertible for all $j\in\mathbb{N}_0$. However, $R+\lambda-m-j-1$ is only invertible if $j\neq 2\lambda-m-1$. Hence a solution of the proposed form exists if and only if $b_{2\lambda-m-1}^{(2)}$ is in the range of $R-\lambda$, which is guaranteed by hypothesis (ii). $\square$

Note that

$$B_0=-EJ\Omega_0-A_1\Omega_0 \tag{3.9}$$

is a first-order polynomial in E. As long as R has distinct eigenvalues and $j\leq 2\lambda-m-1$, we may compute Ω_j recursively from (3.7) and (3.8) and B_j from the equality on the right in (3.5). By induction one can show that Ω_j is a polynomial of degree j and that B_j is a polynomial of degree $j+1$ in E. This leads to the following result.

Theorem 3.3. *Suppose Q is a meromorphic potential of $L\Psi=E\Psi$. The equation $L\Psi=E\Psi$ has a fundamental system of solutions which are meromorphic with respect to the independent variable for all values of the spectral parameter $E\in\mathbb{C}$ whenever this is true for a sufficiently large finite number of distinct values of E.*

Proof. By hypothesis, Q has countably many poles. Let x_0 be any one of them. Near x_0 the functions p and q have the Laurent expansions (3.1). The associated matrix R has eigenvalues λ and $m-\lambda$, which are independent of E. The vector $v=(q_0,-\lambda)^t$ spans $R-\lambda$, and the determinant of the matrix whose columns are v and $b_{2\lambda-m-1}$ is a polynomial in E of degree $2\lambda-m$. Our hypotheses and Proposition 3.2 imply that this determinant has more than $2\lambda-m$ zeros and hence is identically equal to zero. This shows that $b_{2\lambda-m-1}$ is a multiple of v for every value of E. Applying Proposition

3.2 once more then shows that all solutions of $L\Psi = E\Psi$ are meromorphic near x_0 for all $E \in \mathbb{C}$. Since x_0 was arbitrary, this concludes the proof. $\square$

Next, let $\{E_0, ..., E_{2n+1}\}$ be a set of not necessarily distinct complex numbers. We recall (cf. (2.25)),

$$\mathcal{K}_n = \{P = (E, V) \,|\, V^2 = R_{2n+2}(E) = \prod_{m=0}^{2n+1} (E - E_m)\}. \tag{3.10}$$

We introduce the meromorphic function $\phi(\cdot, x)$ on $\mathcal{K}_n$ by

$$\phi(P, x) = \frac{V + G_{n+1}(E, x)}{F_n(E, x)}, \quad P = (E, V) \in \mathcal{K}_n. \tag{3.11}$$

We remark that ϕ can be extended to a meromorphic function on the compactification (projective closure) of the affine curve $\mathcal{K}_n$. This compactification is obtained by joining two points to $\mathcal{K}_n$.

Next we define

$$\psi_1(P, x, x_0) = \exp\left\{\int_{x_0}^{x} dx'[-iE + q(x')\phi(P, x')]\right\}, \tag{3.12}$$

$$\psi_2(P, x, x_0) = \phi(P, x)\psi_1(P, x, x_0), \tag{3.13}$$

where the simple Jordan arc from x_0 to x in (3.12) avoids poles of q and ϕ. One verifies with the help of (2.17)–(2.20), that

$$\phi_x(P, x) = p(x) - q(x)\phi(P, x)^2 + 2iE\phi(P, x). \tag{3.14}$$

From this and (2.6) we find

$$L\Psi(P, x, x_0) = E\Psi(P, x, x_0), \quad P_{n+1}\Psi(P, x, x_0) = iV\Psi(P, x, x_0), \tag{3.15}$$

where

$$\Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}. \tag{3.16}$$

One observes that the two branches $\Psi_\pm(E, x, x_0) = (\psi_{\pm,1}(E, x, x_0), \psi_{\pm,2}(E, x, x_0))^t$ of $\Psi(P, x, x_0)$ represent a fundamental system of solutions of $Ly = Ey$ for all $E \in \mathbb{C} \setminus \{\{E_m\}_{m=0}^{2n+1} \cup \{\mu_j(x_0)\}_{j=1}^{n}\}$, since

$$W(\Psi_-(E, x, x_0), \Psi_+(E, x, x_0)) = \frac{2V_+(E)}{F_n(E, x_0)}. \tag{3.17}$$

Here $W(f, g)$ denotes the determinant of the two columns f and g and $V_+(\cdot)$ (resp. $V_-(\cdot)$) denotes the branch of $V(\cdot)$ on the upper (resp. lower) sheet of $\mathcal{K}_n$ (we follow the notation established in [37]).

In the special case where $\mathcal{K}_n$ is nonsingular, that is, $E_m \neq E_{m'}$ for $m \neq m'$, the explicit representation of $\Psi(P, x, x_0)$ in terms of the Riemann theta function associated with $\mathcal{K}_n$ immediately proves that $\Psi_\pm(E, x, x_0)$ are meromorphic with respect to $x \in \mathbb{C}$

for all $E \in \mathbb{C} \setminus \{\{E_m\}_{m=0}^{2n+1} \cup \{\mu_j(x_0)\}_{j=1}^n\}$. A detailed account of this theta function representation can be found, for instance, in Theorem 3.5 of [37]. In the following we demonstrate how to use gauge transformations to reduce the case of singular curves $\mathcal{K}_n$ to nonsingular ones.

Let (p,q) be meromorphic on $\mathbb{C}$, the precise conditions on (p,q) being immaterial (at least, temporarily) for introducing gauge transformations below. Define L and Q as in (2.1), (2.3) and consider the formal first-order differential system $L\Psi = E\Psi$. Introducing,

$$A(E,x) = \begin{pmatrix} iE & -q(x) \\ -p(x) & -iE \end{pmatrix}, \tag{3.18}$$

$L\Psi = E\Psi$ is equivalent to $\Psi_x(E,x) + A(E,x)\Psi(E,x) = 0$.

Next we consider the gauge transformation,

$$\tilde{\Psi}(E,x) = \Gamma(E,x)\Psi(E,x), \tag{3.19}$$

$$\tilde{A}(E,x) = \begin{pmatrix} iE & -\tilde{q}(x) \\ -\tilde{p}(x) & -iE \end{pmatrix} = \Gamma(E,x)A(E,x)\Gamma(E,x)^{-1} - \Gamma_x(E,x)\Gamma(E,x)^{-1}, \tag{3.20}$$

implying

$$\tilde{\Psi}_x(E,x) + \tilde{A}(E,x)\tilde{\Psi}(E,x) = 0, \text{ that is, } \tilde{L}\tilde{\Psi}(E,x) = E\tilde{\Psi}(E,x), \tag{3.21}$$

with $\tilde{L}$ defined as in (4.1), (4.2) replacing (p,q) by $(\tilde{p},\tilde{q})$. In the following we make the explicit choice (cf., e.g., [55]),

$$\Gamma(E,x) = \begin{pmatrix} E - \tilde{E} - \frac{i}{2}q(x)\phi^{(0)}(\tilde{E},x) & \frac{i}{2}q(x) \\ \frac{i}{2}\phi^{(0)}(\tilde{E},x) & -\frac{i}{2} \end{pmatrix}, \quad E \in \mathbb{C} \setminus \{\tilde{E}\} \tag{3.22}$$

for some fixed $\tilde{E} \in \mathbb{C}$ and

$$\phi^{(0)}(\tilde{E},x) = \psi_2^{(0)}(\tilde{E},x)/\psi_1^{(0)}(\tilde{E},x), \tag{3.23}$$

where $\Psi^{(0)}(\tilde{E},x) = (\psi_1^{(0)}(\tilde{E},x), \psi_2^{(0)}(\tilde{E},x))^t$ is any solution of $L\Psi = \tilde{E}\Psi$. Using (3.14), equation (3.20) becomes

$$\tilde{p}(x) = \phi^{(0)}(\tilde{E},x), \tag{3.24}$$

$$\tilde{q}(x) = -q_x(x) - 2i\tilde{E}q(x) + q(x)^2\phi^{(0)}(\tilde{E},x). \tag{3.25}$$

Moreover, one computes for $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^t$ in terms of $\Psi = (\psi_1, \psi_2)^t$,

$$\tilde{\psi}_1(E,x) = (E - \tilde{E})\psi_1(E,x) + \frac{i}{2}q(x)(\psi_2(E,x) - \phi^{(0)}(\tilde{E},x)\psi_1(E,x)), \tag{3.26}$$

$$\tilde{\psi}_2(E,x) = -\frac{i}{2}(\psi_2(E,x) - \phi^{(0)}(\tilde{E},x)\psi_1(E,x)). \tag{3.27}$$

In addition, we note that

$$\det(\Gamma(E,x)) = -\frac{i}{2}(E-\tilde{E}) \tag{3.28}$$

and therefore,

$$W(\tilde{\Psi}_1(E),\tilde{\Psi}_2(E)) = -\frac{i}{2}(E-\tilde{E})W(\Psi_1(E),\Psi_2(E)), \tag{3.29}$$

where $\Psi_j(E,x)$, $j=1,2$ are two linearly independent solutions of $L\Psi = E\Psi$.

Our first result proves that gauge transformations as defined in this section leave the class of meromorphic algebro-geometric potentials of the AKNS hierarchy invariant.

Theorem 3.4. *Suppose (p,q) is a meromorphic algebro-geometric AKNS potential. Fix $\tilde{E}\in\mathbb{C}$ and define $(\tilde{p},\tilde{q})$ as in (3.24), (3.25), with $\phi^{(0)}(\tilde{E},x)$ defined as in (3.23). Suppose $\phi^{(0)}(\tilde{E},x)$ is meromorphic in x. Then $(\tilde{p},\tilde{q})$ is a meromorphic algebro-geometric AKNS potential.*

Proof. The upper right entry $G_{1,2}(E,x,x')$ of the Green's matrix of L is given by

$$G_{1,2}(E,x,x') = \frac{i\psi_{+,1}(E,x,x_0)\psi_{-,1}(E,x',x_0)}{W(\Psi_-(E,\cdot,x_0),\Psi_+(E,\cdot,x_0))}, \quad x\geq x'. \tag{3.30}$$

Combining (3.11)–(3.13) and (3.14), its diagonal (where $x=x'$) equals

$$G_{1,2}(E,x,x) = \frac{iF_n(E,x)}{2V_+(E)}. \tag{3.31}$$

The corresponding diagonal of the upper right entry $\tilde{G}_{1,2}(E,x,x)$ of the Green's matrix of $\tilde{L}$ is computed to be

$$\tilde{G}_{1,2}(E,x,x) = \frac{i\tilde{\psi}_{+,1}(E,x)\tilde{\psi}_{-,1}(E,x)}{W(\tilde{\Psi}_-(E,\cdot),\tilde{\Psi}_+(E,\cdot))} = \frac{i\tilde{F}_{n+1}(E,x)}{2(E-\tilde{E})V_+(E)}, \tag{3.32}$$

where

$$\begin{aligned}\tilde{F}_{n+1}(E,x) =& 2iF_n(E,x)\{(E-\tilde{E}) + \tfrac{i}{2}q(x)[\phi_+(E,x)-\phi^{(0)}(\tilde{E},x)]\}\times \\ &\times\{(E-\tilde{E}) + \tfrac{i}{2}q(x)[\phi_-(E,x)-\phi^{(0)}(\tilde{E},x)]\},\end{aligned} \tag{3.33}$$

using (3.17), (3.26), (3.29), and

$$\psi_{+,1}(E,x,x_0)\psi_{-,1}(E,x,x_0) = \frac{F_n(E,x)}{F_n(E,x_0)}. \tag{3.34}$$

By (3.11), $\phi_+(E,x)+\phi_-(E,x) = 2G_{n+1}(E,x)/F_n(E,x)$ and $\phi_+(E,x)\phi_-(E,x) = H_n(E,x)/F_n(E,x)$. From this, (2.5), and (2.16), it follows that $\tilde{F}_{n+1}(\cdot,x)$ is a polynomial of degree $n+1$ with leading coefficient

$$iq_x(x) - 2\tilde{E}q(x) - iq(x)^2\phi^{(0)}(\tilde{E},x) = -i\tilde{q}(x). \tag{3.35}$$

Finally, using $\tilde{L}\tilde{\Psi}_{\pm} = E\tilde{\Psi}_{\pm}$, that is, (3.21), one verifies that $\tilde{G}_{1,2}(E,x,x)$ satisfies the differential equation,

$$\tilde{q}\big(2\tilde{G}_{1,2}\tilde{G}_{1,2,xx} - \tilde{G}_{1,2,x}^2 + 4(E^2 - \tilde{p}\tilde{q})\tilde{G}_{1,2}^2\big) - \tilde{q}_x\big(2\tilde{G}_{1,2}\tilde{G}_{1,2,x} + 4iE\tilde{G}_{1,2}^2\big) = \tilde{q}^3. \quad (3.36)$$

Hence $\tilde{F}_{n+1}(E,x)$ satisfies the hypotheses of Corollary 2.4 (with n replaced by $n+1$ and (p,q) replaced by $(\tilde{p},\tilde{q})$) and therefore, $(\tilde{p},\tilde{q})$ is a meromorphic algebro-geometric AKNS potential. ☐

We note here that (3.36) implies also that $\tilde{F}_{n+1}$ satisfies

$$\begin{aligned} &\tilde{q}(2\tilde{F}_{n+1}\tilde{F}_{n+1,xx} - \tilde{F}_{n+1,x}^2 + 4(E^2 - \tilde{p}\tilde{q})\tilde{F}_{n+1}^2) - \tilde{q}_x(2\tilde{F}_{n+1}\tilde{F}_{n+1,x} + 4iE\tilde{F}_{n+1}^2) \\ &= -\,4\tilde{q}^3(E-\tilde{E})^2 R_{2n+2}(E), \end{aligned} \quad (3.37)$$

that is, $(\tilde{p},\tilde{q})$ is associated with the curve

$$\tilde{\mathcal{K}}_{n+1} = \{(E,V)\,|\,V^2 = (E-\tilde{E})^2 R_{2n+2}(E)\}. \quad (3.38)$$

Corollary 3.5. *Suppose (p,q) is a meromorphic algebro-geometric AKNS potential associated with the hyperelliptic curve*

$$\mathcal{K}_n = \{(E,V)\,|\,V^2 = R_{2n+2}(E) = \prod_{m=0}^{2n+1}(E-E_m)\}, \quad (3.39)$$

which has a singular point at $(\tilde{E},0)$, that is, R_{2n+2} has a zero of order $r \geq 2$ at the point $\tilde{E}$. Choose

$$\phi^{(0)}(\tilde{E},x) = \frac{G_{n+1}(\tilde{E},x)}{F_n(\tilde{E},x)} \quad (3.40)$$

(cf. (3.11)) and define $(\tilde{p},\tilde{q})$ as in (3.24), (3.25). Then $\phi^{(0)}(\tilde{E},\cdot)$ is meromorphic and the meromorphic algebro-geometric AKNS potential $(\tilde{p},\tilde{q})$ is associated with the hyperelliptic curve

$$\tilde{\mathcal{K}}_{\tilde{n}} = \{(E,V)\,|\,V^2 = \tilde{R}_{2n-2s+4}(E) = (E-\tilde{E})^{2-2s}R_{2n+2}(E)\} \quad (3.41)$$

for some $2 \leq s \leq (r/2)+1$. In particular, $\tilde{\mathcal{K}}_{\tilde{n}}$ and $\mathcal{K}_n$ have the same structure near any point $E \neq \tilde{E}$.

Proof. Since $V^2 = R_{2n+2}(E)$ we infer that $V_+(E)$ has at least a simple zero at $\tilde{E}$. Hence

$$\phi_{\pm}(E,x) - \phi^{(0)}(\tilde{E},x) = \frac{\pm V_+(E)}{F_n(E,x)} + \frac{G_{n+1}(E,x)F_n(\tilde{E},x) - G_{n+1}(\tilde{E},x)F_n(E,x)}{F_n(E,x)F_n(\tilde{E},x)} \quad (3.42)$$

also have at least a simple zero at $\tilde{E}$. From (3.33) one infers that $\tilde{F}_{n+1}(E,x)$ has a zero of order at least 2 at $\tilde{E}$, that is,

$$\tilde{F}_{n+1}(E,x) = (E-\tilde{E})^s \tilde{F}_{n+1-s}(E,x), \ s \geq 2. \tag{3.43}$$

Define $\tilde{n} = n+1-s$. Then $\tilde{F}_{\tilde{n}}$ still satisfies the hypothesis of Corollary 2.4. Moreover, inserting (3.43) into (3.37) shows that $(E-\tilde{E})^{2s}$ must be a factor of $(E-\tilde{E})^2 R_{2n+2}(E)$. Thus, $2s \leq r+2$ and hence

$$\tilde{q}(2\tilde{F}_{\tilde{n}}\tilde{F}_{\tilde{n},xx} - \tilde{F}_{\tilde{n},x}^2 + 4(E^2 - \tilde{p}\tilde{q})\tilde{F}_{\tilde{n}}^2) - \tilde{q}_x(2\tilde{F}_{\tilde{n}}\tilde{F}_{\tilde{n},x} + 4iE\tilde{F}_{\tilde{n}}^2) = -4\tilde{q}^3 \tilde{R}_{2\tilde{n}+2}(E), \tag{3.44}$$

where

$$\tilde{R}_{2\tilde{n}+2}(E) = (E-\tilde{E})^{2-2s} R_{2n+2}(E) \tag{3.45}$$

is a polynomial in E of degree $0 < 2n-2s+4 < 2n+2$. This proves (3.41). □

In view of our principal result, Theorem 6.4, our choice of $\phi^{(0)}(\tilde{E},x)$ led to a curve $\tilde{\mathcal{K}}_{\tilde{n}}$ which is less singular at $\tilde{E}$ than $\mathcal{K}_n$, without changing the structure of the curve elsewhere. By iterating the procedure from $\mathcal{K}_n$ to $\tilde{\mathcal{K}}_{\tilde{n}}$ one ends up with a curve which is nonsingular at $(\tilde{E},0)$. Repeating this procedure for each singular point of $\mathcal{K}_n$ then results in a nonsingular curve $\hat{\mathcal{K}}_{\hat{n}}$ and a corresponding Baker-Akhiezer function $\hat{\Psi}(P,x,x_0)$ which is meromorphic with respect to $x \in \mathbb{C}$ (this can be seen by using their standard theta function representation, cf., e.g., [37]). Suppose that $\hat{\mathcal{K}}_{\hat{n}}$ was obtained from $\mathcal{K}_n$ by applying the gauge transformation

$$\Gamma(E,x) = \Gamma_N(E,x) ... \Gamma_1(E,x), \tag{3.46}$$

where each of the Γ_j is of the type (3.22). Then the branches of

$$\Psi(P,x) = \Gamma(E,x)^{-1}\hat{\Psi}(P,x,x_0) = \Gamma_1(E,x)^{-1} ... \Gamma_N(E,x)^{-1}\hat{\Psi}(P,x,x_0) \tag{3.47}$$

are linearly independent solutions of $L\Psi = E\Psi$ for all $E \in \mathbb{C}\backslash\{E_0, ..., E_{2n+1}, \mu_1(x_0), ..., \mu_n(x_0)\}$. These branches are meromorphic with respect to x since

$$\Gamma_j(E,x)^{-1} = \frac{2i}{E-\tilde{E}} \begin{pmatrix} -\frac{i}{2} & -\frac{i}{2}q(x) \\ -\frac{i}{2}\phi^{(0)}(\tilde{E},x) & E-\tilde{E}-\frac{i}{2}q(x)\phi^{(0)}(\tilde{E},x) \end{pmatrix} \tag{3.48}$$

maps meromorphic functions to meromorphic functions in view of the fact that q and $\phi^{(0)}(\tilde{E},\cdot) = G_{n+1}(\tilde{E},\cdot)/F_n(\tilde{E},\cdot)$ are meromorphic. Combining these findings and Theorem 3.3 we thus proved the principal result of this section.

Theorem 3.6. *Suppose (p,q) is a meromorphic algebro-geometric AKNS potential. Then the solutions of $L\Psi = E\Psi$ are meromorphic with respect to the independent variable for all values of the spectral parameter $E \in \mathbb{C}$.*

Remark 3.7. In the case of the KdV hierarchy, Ehlers and Knörrer [27] used the Miura transformation and algebro-geometric methods to prove results of the type stated in Corollary 3.5. An alternative approach in the KdV context has recently been found by Ohmiya [72]. The present technique to combine gauge transformations, the polynomial recursion approach to integrable hierarchies based on hyperelliptic curves (such as the KdV, AKNS, and Toda hierarchies), and the fundamental meromorphic function $\phi(\cdot, x)$ on $\mathcal{K}_n$ (cf. (3.11)), yields a relatively straightforward and unified treatment, further details of which will appear elsewhere. To the best of our knowledge this is the first such approach for the AKNS hierarchy.

A systematic study of the construction used in Theorem 3.6 yields explicit connections between the τ-function associated with the possibly singular curve $\mathcal{K}_n$ and the Riemann theta function of the nonsingular curve $\hat{\mathcal{K}}_{\hat{n}}$. This seems to be of independent interest and will be pursued elsewhere.

4. Floquet Theory

Throughout this section we will assume the validity of the following basic hypothesis.

Hypothesis 4.1. *Suppose that $p, q \in L^1_{loc}(\mathbb{R})$ are complex-valued periodic functions of period $\Omega > 0$ and that L is a 2×2 matrix-valued differential expression of the form*

$$L = J\frac{d}{dx} + Q, \tag{4.1}$$

where

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}. \tag{4.2}$$

We note that

$$-J^2 = I \quad \text{and} \quad JQ + QJ = 0, \tag{4.3}$$

where I is the 2×2 identity matrix in $\mathbb{C}^2$.

Given Hypothesis 4.1, we uniquely associate the following densely and maximally defined closed linear operator H in $L^2(\mathbb{R})^2$ with the matrix-valued differential expresssion L,

$$Hy = Ly,\, \mathcal{D}(H) = \{y \in L^2(\mathbb{R})^2 \mid y \in AC_{\text{loc}}(\mathbb{R})^2,\, Ly \in L^2(\mathbb{R})^2\}. \tag{4.4}$$

One easily verifies that L is unitarily equivalent to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \frac{1}{2}\begin{pmatrix} (p+q) & i(p-q) \\ i(p-q) & -(p+q) \end{pmatrix}, \tag{4.5}$$

a form widely used in the literature.

We consider the differential equation $Ly = Ey$ where L satisfies Hypothesis 4.1 and where E is a complex spectral parameter. Define $\phi_0(E, x, x_0, Y_0) = \mathrm{e}^{E(x-x_0)J}Y_0$ for

$Y_0 \in M_2(\mathbb{C})$. The matrix function $\phi(E, \cdot, x_0, Y_0)$ is the unique solution of the integral equation

$$Y(x) = \phi_0(E, x, x_0, Y_0) + \int_{x_0}^{x} \mathrm{e}^{E(x-x')J} JQ(x')Y(x')dx' \tag{4.6}$$

if and only if it satisfies the initial value problem

$$JY' + QY = EY, \quad Y(x_0) = Y_0. \tag{4.7}$$

Since

$$\frac{\partial \phi_0}{\partial x_0}(E, x, x_0, Y_0) = EJ\mathrm{e}^{E(x-x_0)J}Y_0 = E\mathrm{e}^{E(x-x_0)J}JY_0, \tag{4.8}$$

differentiating (4.6) with respect to x_0 yields

$$\begin{aligned} \frac{\partial \phi}{\partial x_0}(E, x, x_0, Y_0) =& \mathrm{e}^{E(x-x_0)J}(EJ - JQ(x_0))Y_0 \\ &+ \int_{x_0}^{x} \mathrm{e}^{E(x-x')J} JQ(x') \frac{\partial \phi}{\partial x_0}(E, x', x_0, Y_0,)dx', \end{aligned} \tag{4.9}$$

that is,

$$\frac{\partial \phi}{\partial x_0}(E, x, x_0, Y_0) = \phi(E, x, x_0, (E + Q(x_0))JY_0), \tag{4.10}$$

taking advantage of the fact that (4.6) has unique solutions.

In contrast to the Sturm-Liouville case, the Volterra integral equation (4.6) is not suitable to determine the asymptotic behavior of solutions as E tends to infinity. The following treatment circumvents this difficulty and closely follows the outline in [62], Section 1.4.

Suppose L satifies Hypothesis 4.1, $p, q \in C^n(\mathbb{R})$, and then define recursively,

$$\begin{aligned} a_1(x) &= iQ(x), \\ b_k(x) &= -i \int_0^x Q(t)a_k(t)dt, \\ a_{k+1}(x) &= -a_{k,x}(x) + iQ(x)b_k(x), \quad k = 1, \dots, n. \end{aligned} \tag{4.11}$$

Next let $A : \mathbb{R}^2 \to \mathbb{M}_2(\mathbb{C})$ be the unique solution of the integral equation

$$A(x, y) = a_{n+1}(x - y) + \int_0^y Q(x - y') \int_{y-y'}^{x-y'} Q(x')A(x', y - y')dx'dy'. \tag{4.12}$$

Introducing

$$\hat{a}_n(E, x) = \int_0^x A(x, y)\mathrm{e}^{-2iEy}dy, \quad \hat{b}_n(E, x) = -i \int_0^x Q(y)\hat{a}_n(E, y)dy \tag{4.13}$$

and

$$u_1(E,x) = I + \sum_{k=1}^{n} b_k(x)(2iE)^{-k} + \hat{b}_n(E,x)(2iE)^{-n}, \tag{4.14}$$

$$u_2(E,x) = \sum_{k=1}^{n} a_k(x)(2iE)^{-k} + \hat{a}_n(E,x)(2iE)^{-n}, \tag{4.15}$$

we infer that

$$Y_1(E,x) = \mathrm{e}^{iEx}\{(I+iJ)u_1(E,x) + (I-iJ)u_2(E,x)\}, \tag{4.16}$$

$$Y_2(E,x) = \mathrm{e}^{-iEx}\{(I-iJ)u_1(-E,x) - (I+iJ)u_2(-E,x)\} \tag{4.17}$$

satisfy the differential equation

$$JY' + QY = EY. \tag{4.18}$$

Since $|A(x,y)|$ is bounded on compact subsets of $\mathbb{R}^2$ one obtains the estimates

$$|\mathrm{e}^{iEx}\hat{a}_n(E,x)|, \quad |\mathrm{e}^{iEx}\hat{b}_n(E,x)| \le CR^2\mathrm{e}^{|x\,\mathrm{Im}(E)|} \tag{4.19}$$

for a suitable constant $C>0$ as long as $|x|$ is bounded by some $R>0$.

The matrix $\hat{Y}(E,x,x_0) = (Y_1(E,x-x_0) + Y_2(E,x-x_0))/2$ is also a solution of $JY'+QY=EY$ and satisfies $\hat{Y}(E,x_0,x_0) = I + Q(x_0)/(2E)$. Therefore, at least for sufficiently large $|E|$, the matrix function

$$\phi(E,\cdot,x_0,I) = \hat{Y}(E,\cdot,x_0)\hat{Y}(E,x_0,x_0)^{-1} \tag{4.20}$$

is the unique solution of the initial value problem $JY'+QY=EY$, $Y(x_0)=I$. Hence, if $p,q\in C^2(\mathbb{R})$, one obtains the asymptotic expansion

$$\begin{aligned}\phi(E,x_0+\Omega,x_0,I) = &\begin{pmatrix} \mathrm{e}^{-iE\Omega} & 0 \\ 0 & \mathrm{e}^{iE\Omega}\end{pmatrix} + \frac{1}{2iE}\begin{pmatrix} \beta\mathrm{e}^{-iE\Omega} & 2q(x_0)\sin(E\Omega) \\ 2p(x_0)\sin(E\Omega) & -\beta\mathrm{e}^{iE\Omega}\end{pmatrix} \\ &+ O(\mathrm{e}^{|\,\mathrm{Im}(E)|\Omega}E^{-2}),\end{aligned} \tag{4.21}$$

where

$$\beta = \int_{x_0}^{x_0+\Omega} p(t)q(t)dt. \tag{4.22}$$

From this result we infer in particular that the entries of $\phi(\cdot,x_0+\Omega,x_0,I)$, which are entire functions, have order one whenever $q(x_0)$ and $p(x_0)$ are nonzero.

Denote by T the operator defined by $Ty = y(\cdot+\Omega)$ on the set of $\mathbb{C}^2$-valued functions on $\mathbb{R}$ and suppose L satisfies Hypothesis 4.1. Then T and L commute and this implies that $T(E)$, the restriction of T to the (two-dimensional) space $V(E)$ of solutions of $Ly=Ey$, maps $V(E)$ into itself. Choosing as a basis of $V(E)$ the columns of $\phi(E,\cdot,x_0,I)$, the operator $T(E)$ is represented by the matrix $\phi(E,x_0+\Omega,x_0,I)$. In particular, $\det(T(E)) = \det(\phi(E, x_0+\Omega, x_0, I)) = 1$. Therefore, the eigenvalues $\rho(E)$ of $T(E)$, the so called Floquet multipliers, are determined as solutions of

$$\rho^2 - \operatorname{tr}(T(E))\rho + 1 = 0. \tag{4.23}$$

These eigenvalues are degenerate if and only if $\rho^2(E) = 1$ which happens if and only if the equation $Ly = Ey$ has a solution of period 2Ω. Hence we now study asymptotic properties of the spectrum of the densely defined closed realization $H_{2\Omega,x_0}$ of L in $L^2([x_0, x_0+2\Omega])^2$ given by

$$\begin{aligned} H_{2\Omega,x_0}y = Ly,\ \mathcal{D}(H_{2\Omega,x_0}) = \{y \in L^2([x_0, x_0+2\Omega])^2 \,|\, y \in AC([x_0, x_0+2\Omega])^2, \\ y(x_0+2\Omega) = y(x_0),\ Ly \in L^2([x_0, x_0+2\Omega])^2\}. \end{aligned} \tag{4.24}$$

Its eigenvalues, which are called the (semi-)periodic eigenvalues of L, and their multiplicities are given, respectively, as the zeros and their multiplicities of the function $\operatorname{tr}(T(E))^2 - 4$. The asymptotic behavior of these eigenvalues is described in the following result.

Theorem 4.2. *Suppose that $p, q \in C^2(\mathbb{R})$. Then the eigenvalues E_j, $j \in \mathbb{Z}$ of $H_{2\Omega,x_0}$ are x_0-independent and satisfy the asymptotic behavior*

$$E_{2j}, E_{2j-1} = \frac{j\pi}{\Omega} + O(\frac{1}{|j|}) \tag{4.25}$$

as $|j|$ tends to infinity, where all eigenvalues are repeated according to their algebraic multiplicities. In particular, all eigenvalues of $H_{2\Omega,x_0}$ are contained in a strip

$$\Sigma = \{E \in \mathbb{C} \,|\, |\operatorname{Im}(E)| \le C\} \tag{4.26}$$

for some constant $C > 0$.

Proof. Denoting $A(E,x) = (E + Q(x))J$ (cf.(3.18)), equation (4.10) implies

$$\partial\phi(E, x, x_0, I)/\partial x = -A(E,x)\phi(E, x, x_0, I), \tag{4.27}$$

$$\partial\phi(E, x, x_0, I)/\partial x_0 = \phi(E, x, x_0, I)A(E,x) \tag{4.28}$$

and hence

$$\partial \operatorname{tr}(T(E))/\partial x_0 = 0. \tag{4.29}$$

Thus the eigenvalues of $H_{2\Omega,x_0}$ are independent of x_0. According to (4.21), $\operatorname{tr}(T(E))$ is asymptotically given by

$$\operatorname{tr}(T(E)) = 2\cos(E\Omega) + \beta\sin(E\Omega)E^{-1} + O(\mathrm{e}^{|\operatorname{Im}(E)|\Omega}E^{-2}). \tag{4.30}$$

Rouché's theorem then implies that two eigenvalues E lie in a circle centered at $j\pi/a$ with radius of order $1/|j|$.

To prove that the eigenvalues may be labeled in the manner indicated, one again uses Rouché's theorem with a circle of sufficiently large radius centered at the origin of the E-plane in order to compare the number of zeros of $\operatorname{tr}(T(E))^2-4$ and $4\cos(E\Omega)^2 - 4$ in the interior of this circle. $\square$

The conditional stability set $\mathcal{S}(L)$ of L in (4.1) is defined to be the set of all spectral parameters E such that $Ly = Ey$ has at least one bounded nonzero solution. This happens if and only if the Floquet multipliers $\rho(E)$ of $Ly = Ey$ have absolute value one. Hence

$$\mathcal{S}(L) = \{E \in \mathbb{C} \,|\, -2 \leq \operatorname{tr}(T(E)) \leq 2\}. \tag{4.31}$$

It is possible to prove that the spectrum of H coincides with the conditional stability set $\mathcal{S}(L)$ of L, but since we do not need this fact we omit a proof. In the following we record a few properties of $\mathcal{S}(L)$ to be used in Sections 5 and 6.

Theorem 4.3. *Assume that $p, q \in C^2(\mathbb{R})$. Then the conditional stability set $\mathcal{S}(L)$ consists of a countable number of regular analytic arcs, the so called spectral bands. At most two spectral bands extend to infinity and at most finitely many spectral bands are closed arcs. The point E is a band edge, that is, an endpoint of a spectral band, if and only if* $\operatorname{tr}(T(E))^2 - 4$ *has a zero of odd order.*

Proof. The fact that $\mathcal{S}(L)$ is a set of regular analytic arcs whose endpoints are odd order zeros of $(\operatorname{tr}(T(E)))^2 - 4$ and hence countable in number, follows in standard manner from the fact that $\operatorname{tr}(T(E))$ is entire with respect to E. (For additional details on this problem, see, for instance, the first part of the proof of Theorem 4.2 in [101].)

From the asymptotic expansion (4.21) one infers that $\operatorname{tr}(T(E))$ is approximately equal to $2\cos(E\Omega)$ for $|E|$ sufficiently large. This implies that the Floquet multipliers are in a neighborhood of $\mathrm{e}^{\pm iE\Omega}$. If $E_0 \in \mathcal{S}(L)$ and $|E_0|$ is sufficiently large, then it is close to a real number. Now let $E = |E_0|\mathrm{e}^{it}$, where $t \in (-\pi/2, 3\pi/2]$. Whenever this circle intersects $\mathcal{S}(L)$ then t is close to 0 or π. When t is close to 0, the Floquet multiplier which is near $\mathrm{e}^{iE\Omega}$ moves radially inside the unit circle while the one close to $\mathrm{e}^{-iE\Omega}$ leaves the unit disk at the same time. Since this can happen at most once, there is at most one intersection of the circle of radius $|E_0|$ with $\mathcal{S}(L)$ in the right half-plane for $|E|$ sufficiently large. Another such intersection may take place in the left half-plane. Hence at most two arcs extend to infinity and there are no closed arcs outside a sufficiently large disk centered at the origin.

Since there are only countably many endpoints of spectral arcs, and since outside a large disk there can be no closed spectral arcs, and at most two arcs extend to infinity, the conditional stability set consists of at most countably many arcs. $\square$

Subsequently we need to refer to components of vectors in $\mathbb{C}^2$. If $y \in \mathbb{C}^2$, we will denote the first and second components of y by y_1 and y_2, respectively, that is, $y = (y_1, y_2)^t$, where the superscript "t" denotes the transpose of a vector in $\mathbb{C}^2$.

The boundary value problem $Ly = zy$, $y_1(x_0) = y_1(x_0 + \Omega) = 0$ in close analogy to the scalar Sturm-Liouville case, will be called the Dirichlet problem for the interval $[x_0, x_0+\Omega]$ and its eigenvalues will therefore be called Dirichlet eigenvalues (associated with the interval $[x_0, x_0 + \Omega]$). In the corresponding operator theoretic formulation one introduces the following closed realization H_{D,x_0} of L in $L^2([x_0, x_0+\Omega])^2$,

$$\begin{aligned} H_{D,x_0}y = Ly,\ \mathcal{D}(H_{D,x_0}) = \{y \in L^2([x_0, x_0+\Omega])^2 \,|\, y \in AC([x_0, x_0+\Omega])^2, \\ y_1(x_0) = y_1(x_0+\Omega) = 0,\ Ly \in L^2([x_0, x_0+\Omega])^2\}. \end{aligned} \tag{4.32}$$

The eigenvalues of H_{D,x_0} and their algebraic multiplicities are given as the zeros and their multiplicities of the function

$$g(E, x_0) = (1,0)\phi(E, x_0+\Omega, x_0, I)(0,1)^t, \tag{4.33}$$

that is, the entry in the upper right corner of $\phi(E, x_0+\Omega, x_0, I)$.

Theorem 4.4. *Suppose $p, q \in C^2(\mathbb{R})$. If $q(x_0) \neq 0$ then there are countably many Dirichlet eigenvalues $\mu_j(x_0)$, $j \in \mathbb{Z}$, associated with the interval $[x_0, x_0+\Omega]$. These eigenvalues have the asymptotic behavior*

$$\mu_j(x_0) = \frac{j\pi}{\Omega} + O(\frac{1}{|j|}) \tag{4.34}$$

as $|j|$ tends to infinity, where all eigenvalues are repeated according to their algebraic multiplicities.

Proof. From the asymptotic expansion (4.21) we obtain that

$$g(E, x_0) = \frac{-iq(x_0)}{E}\sin(E\Omega) + O(\mathrm{e}^{|\operatorname{Im}(E)|\Omega}E^{-2}). \tag{4.35}$$

Rouché's theorem implies that one eigenvalue E lies in a circle centered at $j\pi/\Omega$ with radius of order $1/|j|$ and that the eigenvalues may be labeled in the manner indicated (cf. the proof of Theorem 4.2). $\square$

We now turn to the x-dependence of the function $g(E, x)$.

Theorem 4.5. *Assume that $p, q \in C^1(\mathbb{R})$. Then the function $g(E, \cdot)$ satisfies the differential equation*

$$\begin{aligned} &q(x)(2g(E,x)g_{xx}(E,x) - g_x(E,x)^2 + 4(E^2 - p(x)q(x))g(E,x)^2) \\ &\quad - q_x(x)(2g(E,x)g_x(E,x) + 4iEg(E,x)^2) = -q(x)^3(\operatorname{tr}(T(E))^2 - 4). \end{aligned} \tag{4.36}$$

Proof. Since $g(E,x) = (1,0)\phi(E,x,x+\Omega,I)(0,1)^t$ we obtain from (4.27) and (4.28),

$$g_x(E,x) = (1,0)(\phi(E,x,x+\Omega,I)A(E,x) - A(E,x)\phi(E,x,x+\Omega,I))(0,1)^t, \tag{4.37}$$

$$\begin{aligned} g_{xx}(E,x) = &(1,0)(\phi(E,x,x+\Omega,I)A(E,x)^2 - 2A(E,x)\phi(E,x,x+\Omega,I)A(E,x) \\ &+ A(E,x)^2\phi(E,x,x+\Omega,I) + \phi(E,x,x+\Omega,I)A_x(E,x) \\ &- A_x(E,x)\phi(E,x,x+\Omega,I))(0,1)^t, \end{aligned} \tag{4.38}$$

where we used periodicity of A, that is, $A(E, x+\Omega) = A(E,x)$. This yields the desired result upon observing that $\operatorname{tr}(\phi(z, x+\Omega, x, I)) = \operatorname{tr}(T(E))$ is independent of x. $\square$

Definition 4.6. The algebraic multiplictiy of E as a Dirichlet eigenvalue $\mu(x)$ of $H_{D,x}$ is denoted by $\delta(E,x)$. The quantities

$$\delta_i(E) = \min\{\delta(E,x) \,|\, x \in \mathbb{R}\}, \tag{4.39}$$

and

$$\delta_m(E,x) = \delta(E,x) - \delta_i(E) \tag{4.40}$$

will be called the immovable part and the movable part of the algebraic multiplicity $\delta(E,x)$, respectively. The sum $\sum_{E\in\mathbb{C}} \delta_m(E,x)$ is called the number of movable Dirichlet eigenvalues.

If $q(x) \neq 0$ the function $g(\cdot,x)$ is an entire function with order of growth equal to one. The Hadamard factorization theorem then implies

$$g(E,x) = F_D(E,x)D(E), \tag{4.41}$$

where

$$F_D(E,x) = g_D(x)\mathrm{e}^{h_D(x)E}E^{\delta_m(0,x)} \prod_{\lambda\in\mathbb{C}\setminus\{0\}} (1-(E/\lambda))^{\delta_m(z,x)}\mathrm{e}^{\delta_m(z,x)E}, \tag{4.42}$$

$$D(E) = \mathrm{e}^{d_0E}E^{\delta_i(E)} \prod_{\lambda\in\mathbb{C}\setminus\{0\}} (1-(E/\lambda))^{\delta_i(E)}\mathrm{e}^{\delta_i(E)E}, \tag{4.43}$$

for suitable numbers $g_D(x)$ and $h_D(x)$ and d_0.

Define

$$U(E) = (\mathrm{tr}(T(E))^2 - 4)/D(E)^2. \tag{4.44}$$

Then Theorem 4.2 shows that

$$\begin{aligned}
&-q(x)^3U(E)\\
=&q(x)(2F_D(E,x)F_{D,xx}(E,x) - F_{D,x}(E,x)^2 + 4(E^2 - q(x)p(x))F_D(E,x)^2)\\
&-q_x(x)(2F_D(E,x)F_{D,x}(E,x_0) + 4iEF_D(E,x)^2).
\end{aligned} \tag{4.45}$$

As a function of E the left-hand side of this equation is entire (see Proposition 5.2 in [101] for an argument in a similar case). Introducing $s(E) = \mathrm{ord}_E(\mathrm{tr}(T(E))^2 - 4)$ we obtain the following important result.

Theorem 4.7. *Under the hypotheses of Theorem 4.5, $s(E) - 2\delta_i(E) \geq 0$ for every $E \in \mathbb{C}$.*

We now define the sets $\mathcal{E}_1 = \{E \in \mathbb{C} \,|\, s(E) > 0,\, \delta_i(E) = 0\}$ and $\mathcal{E}_2 = \{E \in \mathbb{C} \,|\, s(E) - 2\delta_i(E) > 0\}$. Of course, $\mathcal{E}_1$ is a subset of $\mathcal{E}_2$ which, in turn, is a subset of the set of zeros of $\mathrm{tr}(T(E))^2 - 4$ and hence isolated and countable.

Theorem 4.8. *Assume Hypothesis 4.1 and that $Ly = Ey$ has degenerate Floquet multipliers ρ (equal to ± 1) but two linearly independent Floquet solutions. Then E is an immovable Dirichlet eigenvalue, that is, $\delta_i(E) > 0$. Moreover, $\mathcal{E}_1$ is contained in the set of all those values of E such that $Ly = Ey$ does not have two linearly independent Floquet solutions.*

Proof. If $Ly = Ey$ has degenerate Floquet multipliers $\rho(E)$ but two linearly independent Floquet solutions then every solution of $Ly = Ey$ is Floquet with multiplier $\rho(E)$. This is true, in particular, for the unique solution y of the initial value problem $Ly = Ey$, $y(x_0) = (0,1)^t$. Hence $y(x_0 + \Omega) = (0, \rho)^t$ and y is a Dirichlet eigenfunction regardless of x_0, that is, $\delta_i(E) > 0$.

If $E \in \mathcal{E}_1$ then $s(E) > 0$ and $Ly = Ey$ has degenerate Floquet multipliers. Since $\delta_i(E) = 0$, there cannot be two linearly independent Floquet solutions. $\square$

Spectral theory for nonself-adjoint periodic Dirac operators has very recently drawn considerable attention in the literature and we refer the reader to [46] and [92].

5. Floquet Theory and Algebro-Geometric Potentials

In this section we will obtain necessary and sufficient conditions in terms of Floquet theory for a function $Q : \mathbb{R} \to M_2(\mathbb{C})$ which is periodic with period $\Omega > 0$ and which has zero diagonal entries to be algebro-geometric (cf. Definition 2.2). Throughout this section we assume the validity of Hypothesis 4.1.

We begin with sufficient conditions on Q and recall the definition of $U(E)$ in (4.44).

Theorem 5.1. *Suppose that $p, q \in C^2(\mathbb{R})$ are periodic with period $\Omega > 0$. If $U(E)$ is a polynomial of degree $2n + 2$ then the following statements hold.*
(i) $\deg(U)$ is even, that is, n is an integer.
(ii) The number of movable Dirichlet eigenvalues (counting algebraic multiplicities) equals n.
(iii) $\mathcal{S}(L)$ consists of finitely many regular analytic arcs.
(iv) $p, q \in C^\infty(\mathbb{R})$.
(v) There exists a 2×2 matrix-valued differential expression P_{n+1} of order $n + 1$ with leading coefficient J^{n+2} which commutes with L and satisfies

$$P_{n+1}^2 = \prod_{E \in F_2} (L - E)^{s(E) - 2\delta_i(E)}. \tag{5.1}$$

Proof. The asymptotic behavior of Dirichlet and periodic eigenvalues (Theorems 4.2 and 4.4) shows that $s(E) \leq 2$ and $\delta(E, x) \leq 1$ when $|E|$ is suitably large. Since $(U(E)$ is a polynomial, $s(E) > 0$ implies that $s(E) = 2\delta_i(E) = 2$. If $\mu(x)$ is a Dirichlet eigenvalue outside a sufficiently large disk, then it must be close to $m\pi/\Omega$ for some integer m and hence close to a point E where $s(E) = 2\delta_i(E) = 2$. Since there is only one Dirichlet eigenvalue in this vicinity we conclude that $\mu(x) = E$ is independent of x. Hence, outside a sufficiently large disk, there is no movable

Dirichlet eigenvalue, that is, $F_D(\cdot, x)$ is a polynomial. Denote its degree, the number of movable Dirichlet eigenvalues, by $\tilde{n}$. By (4.45) $U(E)$ is a polynomial of degree $2\tilde{n}+2$. Hence $\tilde{n} = n$ and this proves parts (i) and (ii) of the theorem.

Since asymptotically $s(E) = 2$, we infer that $s(E) = 1$ or $s(E) \geq 3$ occurs at only finitely many points E. Hence, by Theorem 4.3, there are only finitely many band edges, that is, $\mathcal{S}(L)$ consists of finitely many arcs, which is part (iii) of the theorem.

Let $\gamma(x)$ be the leading coefficient of $F_D(\cdot, x)$. From equation (4.45) we infer that $-\gamma(x)^2/q(x)^2$ is the leading coefficient of $U(E)$ and hence $\gamma(x) = ciq(x)$ for a suitable constant c. Therefore, $F(\cdot, x) = F_D(\cdot, x)/c$ is a polynomial of degree n with leading coefficient $iq(x)$ satisfying the hypotheses of Theorem 2.3. This proves that $p, q \in C^\infty(\mathbb{R})$ and that there exists a 2×2 matrix-valued differential expression P_{n+1} of order $n+1$ and leading coefficient J^{n+2} which commutes with L. The differential expressions P_{n+1} and L satisfy $P_{n+1}^2 = R_{2n+2}(L)$, where

$$R_{2n+2}(E) = U(E)/(4c^2) = \prod_{\lambda \in F_2} (E-\lambda)^{s(\lambda)-2\delta_i(\lambda)}, \tag{5.2}$$

concluding parts (iv) and (v) of the theorem. ☐

Theorem 5.2. *Suppose that $p, q \in C^2(\mathbb{R})$ are periodic of period $\Omega > 0$ and that the differential equation $Ly = Ey$ has two linearly independent Floquet solutions for all but finitely many values of E. Then $U(E)$ is a polynomial.*

Proof. Assume that $U(E)$ in (4.44) is not a polynomial. At any point outside a large disk where $s(E) > 0$ we have two linearly independent Floquet solutions and hence, by Theorem 4.8, $\delta_i(E) \geq 1$. On the other hand, we infer from Theorem 4.2 that $s(E) \leq 2$ and hence $s(E) - 2\delta_i(E) = 0$. Therefore, $s(E) - 2\delta_i(E) > 0$ happens only at finitely many points and this contradiction proves that $U(E)$ is a polynomial. ☐

Theorem 5.3. *Suppose that $p, q \in C^2(\mathbb{R})$ are periodic of period $\Omega > 0$ and that the associated Dirichlet problem has n movable eigenvalues for some $n \in \mathbb{N}$. Then $U(E)$ is a polynomial of degree $2n+2$.*

Proof. If there are n movable Dirichlet eigenvalues, that is, if $\deg(F_D(\cdot, x)) = n$ then (4.45) shows that $U(E) = (\mathrm{tr}(T(E))^2 - 4)/D(E)^2$ is a polynomial of degree $2n+2$. ☐

Next we prove that $U(E)$ being a polynomial, or the number of movable Dirichlet eigenvalues being finite, is also a necessary condition for Q to be algebro-geometric.

Theorem 5.4. *Suppose L satisfies Hypothesis 4.1. Assume there exists a 2×2 matrix-valued differential expression P_{n+1} of order $n+1$ with leading coefficient J^{n+2} which commutes with L but that there is no such differential expression of smaller order commuting with L. Then $U(E)$ is a polynomial of degree $2n+2$.*

Proof. Without loss of generality we may assume that $P_{n+1} = \hat{P}_{c_1,...,c_{n+1}}$ for suitable constants c_j. According to the results in Section 2, the polynomial

$$F_n(E,x) = \sum_{\ell=0}^{n} f_{n-\ell}(c_1, ..., c_{n-\ell})(x)E^\ell \tag{5.3}$$

satisfies the hypotheses of Theorem 2.3. Hence the coefficients f_ℓ and the functions p and q are in $C^\infty(\mathbb{R})$. Also the f_ℓ, and hence P_{n+1}, are periodic with period Ω.

Next, let $\mu(x_0)$ be a movable Dirichlet eigenvalue. Since $\mu(x)$ is a continuous function of $x \in \mathbb{R}$ and since it is not constant, there exists an $x_0 \in \mathbb{R}$ such that $s(\mu(x_0)) = 0$, that is, $\mu(x_0)$ is neither a periodic nor a semi-periodic eigenvalue. Suppose that for this choice of x_0 the eigenvalue $\mu := \mu(x_0)$ has algebraic multiplicity k. Let $V = \ker((H_{D,x_0} - \mu)^k)$ be the algebraic eigenspace of μ. Then V has a basis $\{y_1, ..., y_k\}$ such that $(H_{D,x_0} - \mu)y_j = y_{j-1}$ for $j = 1, ..., k$, agreeing that $y_0 = 0$. Moreover, we introduce $V_m :=$ span $\{y_1, ..., y_m\}$ and $V_0 = \{0\}$. First we show by induction that there exists a number ν such that $(T - \rho)y, (P_{n+1} - \nu)y \in V_{m-1}$, whenever $y \in V_m$.

Let $m = 1$. Then $(H_{D,x_0} - \mu)y = 0$ implies $y = \alpha y_1$ for some constant α and hence y is a Floquet solution with multiplier $\rho = y_{1,2}(x_0 + \Omega)$, that is, $(T - \rho)y = 0$. (We define, in obvious notation, $y_{j,k}$, $k = 1, 2$ to be the k-th component of y_j, $1 \leq j \leq m$.) Since P_{n+1} commutes with both L and T, we find that $P_{n+1}y$ is also a Floquet solution with multiplier ρ. Since $s(\mu) = 0$, the geometric eigenspace of ρ is one-dimensional and hence $P_{n+1}y = \nu y$ for a suitable constant ν.

Now assume that the statement is true for $1 \leq m < k$. Let $y = \sum_{j=1}^{m+1} \alpha_j y_j \in V_{m+1}$. Note that $(T - \rho)y$ satisfies Dirichlet boundary conditions. Hence

$$(H_{D,x_0} - \mu)(T - \rho)y = (T - \rho) \sum_{j=1}^{m+1} \alpha_j (H_{D,x_0} - \mu)y_j = \sum_{j=1}^{m+1} \alpha_j (T - \rho)y_{j-1} \tag{5.4}$$

is an element of V_{m-1}, say equal to $v = \sum_{j=1}^{m-1} \beta_j y_j$. The nonhomogeneous equation $(H_{D,x_0} - \mu)w = v$ has the general solution

$$w = \sum_{j=1}^{m-1} \beta_j y_{j+1} + \alpha y_1, \tag{5.5}$$

where α is an arbitary constant. Since w is in V_m, the particular solution $(T - \rho)y$ of $(H_{D,x_0} - \mu)w = v$ is in V_m too.

Since Q is infinitely often differentiable, so are the functions $y_1, ..., y_k$. Hence L can be applied to $(P_{n+1} - \nu)y$ and one obtains

$$(L - \mu)(P_{n+1} - \nu)y = \sum_{j=1}^{m+1} \alpha_j (P_{n+1} - \nu)y_{j-1} = \sum_{j=1}^{m-1} \gamma_j y_j \tag{5.6}$$

for suitable constants γ_j. Thus there are numbers α_1 and α_2 such that

$$(P_{n+1} - \nu)y = \sum_{j=1}^{m-1} \gamma_j y_{j+1} + \alpha_1 \hat{y} + \alpha_2 y_1, \tag{5.7}$$

where $\hat{y}$ is the solution of $Ly = \mu y$ with $\hat{y}(x_0) = (1,0)^t$. Note that $(P_{n+1}y - \nu y)_1(x_0) = (P_{n+1}y)_1(x_0) = \alpha_1$. Let $w = (T - \rho)y$ and $v = (P_{n+1} - \nu)w$. Then $w \in V_m$ and $v \in V_{m-1}$. Hence,

$$\begin{aligned}(P_{n+1}y - \nu y)_1(x_0 + a) &= (T(P_{n+1} - \nu)y)_1(x_0) = ((P_{n+1} - \nu)Ty)_1(x_0) \\ &= (P_{n+1}(\rho y + w))_1(x_0) \\ &= \rho(P_{n+1}y)_1(x_0) + (P_{n+1}w)_1(x_0) = \rho\alpha_1. \end{aligned}\tag{5.8}$$

On the other hand, $(P_{n+1}y - \nu y)_1(x_0 + a) = \alpha_1/\rho$ since $\hat{y}_1(x_0 + a) = 1/\rho$. Thus $0 = \alpha_1(\rho - 1/\rho)$ which implies $\alpha_1 = 0$ and $(P_{n+1} - \nu)y \in V_m$.

Hence we have shown that T and P_{n+1} map V into itself. In particular, $(P_{n+1}y)_1(x_0) = 0$ for every $y \in V$.

Next observe that the functions $y_1, ..., y_k$ defined above satisfy $(L - \mu)^j y_m = y_{m-j}$, agreeing that $y_m = 0$ whenever $m \leq 0$. Consequently,

$$L^j y_m = \sum_{r=0}^{j} \binom{j}{r} \mu^r (L - \mu)^{j-r} y_m = \sum_{r=0}^{j} \binom{j}{r} \mu^r y_{m+r-j}. \tag{5.9}$$

Moreover,

$$\begin{aligned} P_{n+1}y_m &= -\sum_{j=0}^{n+1} (g_{n+1-j}J + iA_{n-j})\, L^j y_m \\ &= -\sum_{j=0}^{n+1}\sum_{r=0}^{j} \binom{j}{r} \mu^r \begin{pmatrix} ig_{n+1-j}y_{m+r-j,1} - if_{n-j}y_{m+r-j,2} \\ -ig_{n+1-j}y_{m+r-j,2} + ih_{n-j}y_{m+r-j,1} \end{pmatrix}. \end{aligned}\tag{5.10}$$

Since $(P_{n+1}y_j)_1(x_0) = y_{j,1}(x_0) = 0$, evaluating the first component of (5.10) at x_0 yields

$$\begin{aligned} 0 = (P_{n+1}y_m)_1(x_0) &= i\sum_{\ell=0}^{n+1} y_{m-\ell,2}(x_0)\frac{1}{\ell!}\sum_{j=\ell}^{n+1} j...(j - \ell + 1)\mu^{j-\ell} f_{n-j}(x_0) \\ &= i\sum_{\ell=0}^{n} y_{m-\ell,2}(x_0)\frac{1}{\ell!}\frac{\partial^\ell F_n}{\partial E^\ell}(\mu, x_0). \end{aligned}\tag{5.11}$$

Letting m run from 1 through k shows that μ is a zero of $F_n(\cdot, x_0)$ of order at least k. Therefore, there can be at most n movable Dirichlet eigenvalues counting multiplicities. However, if there were less than n movable Dirichlet eigenvalues then, by Theorems 5.1 and 5.3, there would exist a differential expression of order less than

$n+1$ which commutes with L without being a polynomial of L. Hence there are precisely n movable Dirichlet eigenvalues and $\deg(U) = 2n+2$. □

6. A Characterization of Elliptic Algebro-Geometric AKNS Potentials

Picard's theorem yields sufficient conditions for a linear (scalar) n^{th}-order differential equation, whose coefficients are elliptic functions with a common period lattice spanned by $2\omega_1$ and $2\omega_3$, to have a fundamental system of solutions which are elliptic of the second kind. We start by generalizing Picard's theorem to first-order systems. Let T_j, $j = 1,3$, be the operators defined by $T_j y = y(\cdot + 2\omega_j)$. In analogy to the scalar case we call y elliptic of the second kind if it is meromorphic and

$$y(\cdot + 2\omega_j) = \rho_j y(\cdot) \text{ for some } \rho_j \in \mathbb{C} \setminus \{0\},\ j = 1,3. \tag{6.1}$$

Theorem 6.1. *Suppose that the entries of $A : \mathbb{C} \to M_n(\mathbb{C}_\infty)$ are elliptic functions with common fundamental periods $2\omega_1$ and $2\omega_3$. Assume that the first-order differential system $\psi' = A\psi$ has a meromorphic fundamental system of solutions. Then there exists at least one solution ψ_1 which is elliptic of the second kind. If in addition, the restriction of either T_1 or T_3 to the (n-dimensional) space W of solutions of $\psi' = A\psi$ has distinct eigenvalues, then there exists a fundamental system of solutions of $\psi' = A\psi$ which are elliptic of the second kind.*

Proof. T_1 is a linear operator mapping W into itself and thus has an eigenvalue ρ_1 and an associated eigenfunction u_1, that is, $\psi' = A\psi$ has a solution u_1 satisfying $u_1(x + 2\omega_1) = \rho_1 u_1(x)$.

Now consider the functions

$$u_1(x),\ u_2(x) = u_1(x + 2\omega_3),\ \dots,\ u_m(x) = u_1(x + 2(m-1)\omega_3), \tag{6.2}$$

where $m \in \{1, \dots, n\}$ is chosen such that the functions in (6.2) are linearly independent but including $u_1(x + 2m\omega_3)$ would render a linearly dependent set of functions. Then,

$$u_m(x + 2\omega_3) = b_1 u_1(x) + \dots. + b_m u_m(x). \tag{6.3}$$

Next, denote the restriction of T_3 to the span V of $\{u_1, \dots, u_m\}$ by $\tilde{T}_3$. It follows from (6.3) that the range of $\tilde{T}_3$ is again V. Let ρ_3 be an eigenvalue of $\tilde{T}_3$ and v the associated eigenvector, that is, v is a meromorphic solution of the differential equation $\psi' = A\psi$ satisfying $v(x + 2\omega_3) = \rho_3 v(x)$. But v also satisfies $v(x + 2\omega_1) = \rho_1 v(x)$ since every element of V has this property. Hence v is elliptic of the second kind.

The numbers ρ_1 and ρ_3 are the Floquet multipliers corresponding to the periods $2\omega_1$ and $2\omega_3$, respectively. The process described above can be performed for each multiplier corresponding to the period $2\omega_1$. Moreover, the roles of $2\omega_1$ and $2\omega_3$ may of course be interchanged. The last statement of the theorem follows then from the observation that solutions associated with different multipliers are linearly independent. □

What we call Picard's theorem following the usual convention in [3], p. 182–185, [18], p. 338–343, [48], p. 536–539, [57], p. 181–189, appears, however, to have a longer history. In fact, Picard's investigations [76]–[78] in the scalar n^{th}-order case were inspired by earlier work of Hermite in the special case of Lamé's equation [49], p. 118–122, 266–418, 475–478 (see also [9], Sect. 3.6.4 and [102], p. 570–576). Further contributions were made by Mittag-Leffler [65], and Floquet [34]–[36]. Detailed accounts on Picard's differential equation can be found in [48], p. 532–574, [57], p. 198–288. For a recent extension of Theorem 6.1 see [38].

Picard's Theorem 6.1 motivates the following definition.

Definition 6.2. A 2×2 matrix Q whose diagonal entries are zero and whose off-diagonal entries are elliptic functions with a common period lattice is called a **Picard-AKNS potential** if and only if the differential equation $J\psi' + Q\psi = E\psi$ has a meromorphic fundamental system of solutions (with respect to the independent variable) for infinitely many values of the spectral parameter $E \in \mathbb{C}$.

Recall from Theorem 3.3 that $J\psi' + Q\psi = E\psi$ has a meromorphic fundamental system of solutions for all values of E if this is true for a sufficiently large finite number of values of E.

In the following assume, without loss of generality, that $\mathrm{Re}(\omega_1) > 0$, $\mathrm{Re}(\omega_3) \geq 0$, $\mathrm{Im}(\omega_3/\omega_1) > 0$. The fundamental period parallelogram then consists of the points $E = 2\omega_1 s + 2\omega_3 t$, where $0 \leq s, t < 1$.

We introduce $\theta \in (0, \pi)$ by

$$e^{i\theta} = \frac{\omega_3}{\omega_1} \left| \frac{\omega_1}{\omega_3} \right| \tag{6.4}$$

and for $j = 1, 3$,

$$Q_j(\zeta) = t_j Q(t_j \zeta + x_0), \tag{6.5}$$

where $t_j = \omega_j/|\omega_j|$. Subsequently, the point x_0 will be chosen in such a way that no pole of Q_j, $j = 1, 3$ lies on the real axis. (This is equivalent to the requirement that no pole of Q lies on the line through the points x_0 and $x_0 + 2\omega_1$ nor on the line through x_0 and $x_0 + 2\omega_3$. Since Q has only finitely many poles in the fundamental period parallelogram this can always be achieved.) For such a choice of x_0 we infer that the entries of $Q_j(\zeta)$ are real-analytic and periodic of period $\Omega_j = 2|\omega_j|$ whenever ζ is restricted to the real axis. Using the variable transformation $x = t_j \zeta + x_0$, $\psi(x) = \chi(\zeta)$ one concludes that ψ is a solution of

$$J\psi'(x) + Q(x)\psi(x) = E\psi(x) \tag{6.6}$$

if and only if χ is a solution of

$$J\chi'(\zeta) + Q_j(\zeta)\chi(\zeta) = \lambda\chi(\zeta), \tag{6.7}$$

where $\lambda = t_j E$.

Theorem 4.2 is now applicable and yields the following result.

Theorem 6.3. *Let $j = 1$ or 3. Then all $4\omega_j$-periodic (i.e., all $2\omega_j$-periodic and all $2\omega_j$-semi-periodic) eigenvalues associated with Q lie in the strip S_j given by*

$$S_j = \{E \in \mathbb{C} \,|\, |\operatorname{Im}(t_j E)| \leq C_j\} \tag{6.8}$$

for suitable constants $C_j > 0$. The angle between the axes of the strips S_1 and S_3 equals $\theta \in (0, \pi)$.

Theorem 6.3 applies to any elliptic potential Q whether or not it is algebro-geometric. Next we present our principal result, a characterization of all elliptic algebro-geometric potentials of the AKNS hierarchy. Given the preparations in Sections 3–5, the proof of our principal result, Theorem 6.4 below, will be fairly short.

Theorem 6.4. *Q is an elliptic algebro-geometric AKNS potential if and only if it is a Picard-AKNS potential.*

Proof. The fact that any elliptic algebro-geometric AKNS potential is a Picard potential is a special case of Theorem 3.12.

Conversely, assume that Q is a Picard-AKNS potential. Choose $R > 0$ large enough such that the exterior of the closed disk $\overline{D(0,R)}$ of radius R centered at the origin contains no intersection of S_1 and S_3 (defined in (6.8)), that is,

$$(\mathbb{C}\backslash\overline{D(0,R)}) \cap (S_1 \cap S_3) = \emptyset. \tag{6.9}$$

Let $\rho_{j,\pm}(\lambda)$ be the Floquet multipliers of Q_j, that is, the solutions of

$$\rho_j^2 - \operatorname{tr}(T_j)\rho_j + 1 = 0. \tag{6.10}$$

Then (6.9) implies that for $E \in \mathbb{C}\backslash\overline{D(0,R)}$ at most one of the eigenvalues $\rho_1(t_1 E)$ and $\rho_3(t_3 E)$ can be degenerate. In particular, at least one of the operators T_1 and T_3 has distinct eigenvalues. Since by hypothesis Q is Picard, Picard's Theorem 6.1 applies with $A = -J(Q - E)$ and guarantees the existence of two linearly independent solutions $\psi_1(E,x)$ and $\psi_2(E,x)$ of $J\psi' + Q\psi = E\psi$ which are elliptic of the second kind. Then $\chi_{j,k}(\zeta) = \psi_k(t_j\zeta + x_0)$, $k = 1, 2$ are linearly independent Floquet solutions associated with Q_j. Therefore the points λ for which $J\chi' + Q_j\chi = \lambda\chi$ has only one Floquet solution are necessarily contained in $\overline{D(0,R)}$ and hence finite in number. This is true for both $j = 1$ and $j = 3$. Applying Theorem 5.2 then proves that both Q_1 and Q_3 are algebro-geometric. This implies that Q itself is algebro-geometric. $\square$

The following corollary slightly extends the class of AKNS potentials $Q(x)$ considered thus far in order to include some cases which are not elliptic but very closely related to elliptic $Q(x)$. Such cases have recently been considered by Smirnov [89].

Corollary 6.5. *Suppose*

$$Q(x) = \begin{pmatrix} 0 & -iq(x)\mathrm{e}^{-2(ax+b)} \\ ip(x)\mathrm{e}^{2(ax+b)} & 0 \end{pmatrix} \tag{6.11}$$

where $a, b \in \mathbb{C}$ and p, q are elliptic functions with a common period lattice. Then Q is an algebro-geometric AKNS potential if and only if $J\Psi' + Q\Psi = E\Psi$ has a meromorphic fundamental system of solutions (with respect to the independent variable) for all values of the spectral parameter $E \in \mathbb{C}$.

Proof. Suppose that for all values of E the equation $L\Psi = J\Psi' + Q\Psi = E\Psi$ has a meromorphic fundamental system of solutions. Let

$$\mathcal{T} = \begin{pmatrix} \mathrm{e}^{ax+b} & 0 \\ 0 & \mathrm{e}^{-ax+b} \end{pmatrix}. \tag{6.12}$$

Then $\mathcal{T}L\mathcal{T}^{-1} = \tilde{L} + iaI = Jd/dx + \tilde{Q}iaI$, where

$$\tilde{Q} = \begin{pmatrix} 0 & -iq \\ ip & 0 \end{pmatrix}. \tag{6.13}$$

Moreover, $L\Psi = E\Psi$ is equivalent to $\tilde{L}(\mathcal{T}\Psi) = (E - ia)(\mathcal{T}\Psi)$. Hence the equation $\tilde{L}\Psi = (E - ia)\Psi$ has a meromorphic fundamental system of solutions for all E. Consequently, Theorem 6.4 applies and yields that $\tilde{Q}$ is an algebro-geometric AKNS potential. Thus, for some n there exists a differential expression $\tilde{P}$ of order $n+1$ with leading coefficient $-J^n$ such that $[\tilde{P}, \tilde{L}] = 0$. Define $P = \mathcal{T}^{-1}\tilde{P}\mathcal{T}$. The expression P is a differential expression of order $n+1$ with leading coefficient $-J^n$ and satisfies $[P, L] = \mathcal{T}^{-1}[\tilde{P}, \tilde{L} + iaI]\mathcal{T} = 0$, that is, Q is an algebro-geometric AKNS potential. The converse follows by reversing the above proof. □

We add a series of remarks further illustrating the significance of Theorem 6.4.

Remark 6.6. While an explicit proof of the algebro-geometric property of (p, q) is in general highly nontrivial (see, e.g., the references cited in connection with special cases such as the Lamé-Ince and Treibich-Verdier potentials in the introduction), the fact of whether or not $J\Psi'(x) + Q(x)\Psi(x) = E\Psi(x)$ has a fundamental system of solutions meromorphic in x for all but finitely many values of the spectral parameter $E \in \mathbb{C}$ can be decided by means of an elementary Frobenius-type analysis (see, e.g., [39] and [40]). To date, Theorem 6.4 appears to be the only effective tool to identify general elliptic algebro-geometric solutions of the AKNS hierarchy.

Remark 6.7. Theorem 6.4 complements Picard's Theorem 6.1 in the special case where $A(x) = -J(Q(x) - E)$ in the sense that it determines the elliptic matrix functions Q which satisfy the hypothesis of the theorem precisely as (elliptic) algebro-geometric solutions of the stationary AKNS hierarchy.

Remark 6.8. Theorem 6.4 is also relevant in the context of the Weierstrass theory of reduction of Abelian to elliptic integrals, a subject that attracted considerable interest, see, for instance, [7], [8], [9], Ch. 7, [10], [11], [21], [29]–[31], [53], [56], [58],

[63], [83], [84], [90]. In particular, the theta functions corresponding to the hyperelliptic curves derived from the Burchnall-Chaundy polynomials (2.25), associated with Picard potentials, reduce to one-dimensional theta functions.

7. Examples

With the exception of the studies by Christiansen, Eilbeck, Enol'skii, and Kostov in [21] and Smirnov in [89], not too many examples of elliptic solutions (p, q) of the AKNS hierarchy associated with higher (arithmetic) genus curves of the type (2.25) have been worked out in detail. The genus $n = 1$ case is considered, for example, in [52], [75]. Moreover, examples for low genus n for special cases such as the nonlinear Schrödinger and mKdV equation (see (2.46) and (2.49)) are considered, for instance, in [5], [8], [61], [64], [74], [88]. Subsequently we will illustrate how the Frobenius method, whose essence is captured by Proposition 3.2, can be used to establish existence of meromorphic solutions and hence, by Theorem 6.4, proves their algebro-geometric property. The notation established in the beginning of Section 3 will be used freely in the following.

Example 7.1. Let

$$p(x) = q(x) = n(\zeta(x) - \zeta(x - \omega_2) - \eta_2), \tag{7.1}$$

where $n \in \mathbb{N}$. The potential (p, q) has two poles in the fundamental period parallelogram. Consider first the pole $x = 0$. In this case we have

$$R = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}, \tag{7.2}$$

whose eigenvalues are $\pm n$, that is, $\lambda = n$. Moreover, since $p = q$ is odd, we have $p_{2j-1} = q_{2j-1} = 0$. One proves by induction that $b_{2j}^{(2)}$ is a multiple of $(1, 1)^t$ and that $b_{2j-1}^{(2)}$ is a multiple of $(1, -1)^t$. Hence $b_{2n-1}^{(2)}$ is a multiple of $(1, -1)^t$, that is, it is in the range of $R - n$. Hence every solution of $L\Psi = E\Psi$ is meromorphic at zero regardless of E.

Next consider the pole $x = \omega_2$ and shift coordinates by introducing $\xi = x - \omega_2$. Then we have $p(x) = q(x) = n(\zeta(\xi + \omega_2) - \zeta(\xi) - \eta_2) = -p(\xi)$ and hence

$$R = \begin{pmatrix} 0 & -n \\ -n & 0 \end{pmatrix}. \tag{7.3}$$

One can use again a proof by induction to show that $b_{2n-1}^{(2)}$ is in the range of $R - n$, which is spanned by $(1, 1)^t$.

Hence we have shown that the matrix

$$Q(x) = \begin{pmatrix} 0 & -in(\zeta(x) - \zeta(x - \omega_2) - \eta_2) \\ in(\zeta(x) - \zeta(x - \omega_2) - \eta_2) & 0 \end{pmatrix} \tag{7.4}$$

is a Picard-AKNS and therefore an algebro-geometric AKNS potential.

Example 7.2. Here we let $p = 1$ and $q = n(n+1)\wp(x)$, where $n \in \mathbb{N}$. Then we have just one pole in the fundamental period parallelogram. In this case we obtain

$$R = \begin{pmatrix} 1 & n(n+1) \\ 1 & 0 \end{pmatrix} \tag{7.5}$$

and $\lambda = n+1$. Since q is even we infer that $q_{2j-1} = 0$. A proof by induction then shows that $b_{2j}^{(2)}$ is a multiple of $(n-2j, 1)^t$ and that $b_{2j-1}^{(2)}$ is a multiple of $(1,0)^t$. In particular, $b_{2n}^{(2)}$ is a multiple of $(-n, 1)^t$, which spans the range of $R - \lambda$. This shows that

$$Q(x) = \begin{pmatrix} 0 & -in(n+1)\wp(x) \\ i & 0 \end{pmatrix} \tag{7.6}$$

is a Picard-AKNS and hence an algebro-geometric AKNS potential.

Incidentally, if $p = 1$, then $J\Psi' + Q\Psi = E\Psi$ is equivalent to the scalar equation $\psi_2'' - q\psi_2 = -E^2\psi_2$ where $\Psi = (\psi_1, \psi_2)^t$ and $\psi_1 = \psi_2' - iE\psi_2$. Therefore, if $-q$ is an elliptic algebro-geometric potential of the KdV hierarchy then by Theorem 5.7 of [44] ψ_2 is meromorphic for all values of E. Hence Ψ is meromorphic for all values of E and therefore Q is a Picard-AKNS and hence an algebro-geometric AKNS potential. Conversely if Q is an algebro-geometric AKNS potential with $p = 1$ then $-q$ is an algebro-geometric potential of the KdV hierarchy (cf. (2.48)). In particular, $q(x) = n(n+1)\wp(x)$ is the celebrated class of Lamé potentials associated with the KdV hierarchy (cf., e.g., [39] and the references therein).

Example 7.3. Suppose $e_2 = 0$ and hence $g_2 = 4e_1^2$ and $g_3 = 0$. Let $u(x) = -\wp'(x)/(2e_1)$. Then, near $x = 0$,

$$u(x) = \frac{1}{e_1 x^3} - \frac{e_1}{5}x + O(x^3), \tag{7.7}$$

and near $x = \pm\omega_2$,

$$u(x) = e_1(x \mp \omega_2) - \frac{3e_1^3}{5}(x \mp \omega_2)^5 + O((x \mp \omega_2)^7). \tag{7.8}$$

Now let $p(x) = 3u(x)$ and $q(x) = u(x - \omega_2)$. Then p has a third-order pole at 0 and a simple zero at ω_2 while q has a simple zero at zero and a third-order pole at ω_2. Let us first consider the point $x = 0$. We have

$$R = \begin{pmatrix} -2 & e_1 \\ 3/e_1 & 0 \end{pmatrix}, \tag{7.9}$$

and hence $\lambda = 1$. Moreover, $p_2 = q_2 = 0$, $p_4 = -3e_1/5$, and $q_4 = -3e_1^3/5$. Since $\lambda = 1$ we have to show that $b_3^{(2)}$ is a multiple of $(q_0, -1)^t$. We get, using $p_2 = q_2 = 0$,

$$b_3^{(2)} = (q_0 E^4/6 + q_4, -E^4/6 - q_0 p_4)^t, \tag{7.10}$$

which is a multiple of $(q_0, -1)^t$ if and only if $q_4 = p_4 q_0^2$, a relationship which is indeed satisfied in our example.

Next consider the point $x = \omega_2$. Changing variables to $\xi = x - \omega_2$ and using the periodic properties of u we find that $p(x) = 3q(\xi)$ and $q(x) = p(\xi)/3$. Thus q has a pole at $\xi = 0$ and one obtains $m = 2$, $p_0 = 3e_1$, $q_0 = 1/e_1$, $p_2 = q_2 = 0$, $p_4 = -9e_1^3/5$, and $q_4 = -e_1/5$. Since $\lambda = 3$, we have to compute again $b_3^{(2)}$ and find, using $p_2 = q_2 = 0$,

$$b_3^{(2)} = (-q_0 E^4/6 + 3q_4, -E^4/2 - q_0 p_4)^t, \tag{7.11}$$

which is a multiple of $(q_0, -3)^t$ if and only if $9q_4 = p_4 q_0^2$, precisely what we need.

Hence, if $e_2 = 0$ and $u(x) = -\wp'(x)/(2e_1)$, then

$$Q(x) = \begin{pmatrix} 0 & -iu(x-\omega_2) \\ 3iu(x) & 0 \end{pmatrix} \tag{7.12}$$

is a Picard-AKNS and therefore an algebro-geometric AKNS potential.

Example 7.4. Again let $e_2 = 0$. Define $p(x) = \frac{2}{3}(\wp''(x) - e_1^2)$ and $q(x) = -\wp(x - \omega_2)/e_1^2$. First consider $x = 0$. We have $m = -3$, $p_0 = 4$, $q_0 = 1$, $p_2 = q_2 = 0$, $p_4 = -2e_1^2/5$, and $q_4 = -e_1^2/5$. This yields $\lambda = 1$ and we need to show that $b_4^{(2)}$ is a multiple of $(1, -1)^t$. We find, using $p_2 = q_2 = 0$ and $q_0 = \lambda = 1$,

$$b_4^{(2)} = i(-E^5/24 - p_4/4 - q_4, E^5/24 + 5p_4/4 - q_4)^t. \tag{7.13}$$

This is a multiple of $(1, -1)^t$ if $2q_4 = p_4$, which is indeed satisfied.

Next consider $x = \omega_2$. Now q has a second-order pole, that is, we have $m = 1$. Moreover,

$$q(x) = \frac{-1}{e_1^2}\left(\frac{1}{(x-\omega_2)^2} + \frac{e_1^2}{5}(x-\omega_2)^2 + O((x-\omega_2)^2)\right) \tag{7.14}$$

and

$$p(x) = -2e_1^2 + 96e_1^4(x-\omega_2)^4 + O((x-\omega_2)^6). \tag{7.15}$$

We now need $b_2^{(2)}$ to be a multiple of $(q_0, -2)^t$, which is satisfied for $q_2 = p_2 = 0$.

Hence, if $e_2 = 0$, then

$$Q(x) = \begin{pmatrix} 0 & i\wp(x-\omega_2)/e_1^2 \\ 2i(\wp''(x) - e_1^2)/3 & 0 \end{pmatrix} \tag{7.16}$$

is a Picard-AKNS and thus an algebro-geometric AKNS potential.

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