Nonlinear S-transform

Sergio Albeverio
Fakultät für Mathematik, Ruhr–Universität,
D 44780 Bochum (Germany);
BiBoS Research Centre, Bielefeld;
SFB 237 (Essen–Bochum–Düsseldorf);
CERFIM, Locarno
e-mail Sergio.Albeverio@rz.ruhr-uni-bochum.de

Yuri Kondratiev
BiBoS Research Centre,
D 33615 Bielefeld (Germany);
Institute of Mathematics, Kiev (Ukraine)
e-mail kondratiev@physu.physik.uni-bielefeld.de

Yuri Kozitsky
Institute of Mathematics, Maria Curie-Sklodowska University
PL 20-031 Lublin (Poland);
Institute for Condensed Matter Physics, Lviv (Ukraine)
e-mail jkozi@golem.umcs.lublin.pl

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Abstract

A nonlinear generalization of the S-transform known in Gaussian analysis is introduced. It is defined as a nonlinear operator on topological spaces of entire functions of infinitely many variables with growth of at most second order and finite type restricted to a certain value. This nonlinear S-transform (NLST) has fixed points a certain family of which is found and analyzed. In particular, their stability is described in terms of spectral properties of the NLST Fréchet derivative. This is utilized to study the convergence of sequences of entire functions generated by the NLST, which is specified here to describe the temperature Gibbs states of hierarchically interacting quantum anharmonic oscillators. Two families of models of such oscillators – $\mathcal{M}_+$ and $\mathcal{M}_-$, are considered. In the case of $\mathcal{M}_-$, it is proven that for all values of the model parameters, the convergence occurs only to the stable fixed point which corresponds to the absence of any critical point anomalies for such models at all temperatures. In the case of $\mathcal{M}_+$, however, it is proven that varying the parameters of the model within the frames of the family one may obtain the convergence to both types of fixed points – stable and unstable. The convergence to the unstable fixed point corresponds to the appearance of a strong dependence between the oscillators which is peculiar to the critical point of the model.

Contents

1 Introduction

Methods of infinite dimensional analysis created during the last decades have found important applications in several areas of modern mathematics. Among them is a rigorous construction of models of quantum statistical physics [1]. Conversely, the problems posed by quantum statistical physics stimulate further development of infinite dimensional analysis methods. In particular, the task of the great importance here is to generalize to the infinite dimensional case appropriate tools of finite dimensional nonlinear analysis, especially those used to study such essentially nonlinear phenomena as phase transitions. This paper is intended as a step in such a direction.

The role played by the S-transform in White Noise Analysis and more generally in Gaussian Analysis is well known (see e.g. [3] and [8]). In some sense, it can be considered as an infinite dimensional version of the Laplace transform adapted to the use in Gaussian spaces. In Section 2, we introduce a natural nonlinear generalization of the S-transform in a separable Hilbert space with a Gaussian measure. Such an operation is defined on special topological spaces of entire functions of infinitely many variables with growth of at most second order and finite type restricted to a certain value. Similar types of entire functions are known to be useful in different domains of infinite dimensional analysis, see e.g. [3].

The nonlinear S-transform (NLST) introduced here has fixed points a family of which is found and analyzed in Section 2. We describe their stability studying the spectra of the NLST Fréchet derivative at these points. Having this information, one may consider the convergence of sequences of entire functions generated by the NLST similarly to the case of dynamical systems with discrete time. To discuss in more details the asymptotic behaviour of such sequences we specify in Section 3 this transform to describe the temperature Gibbs states of interacting quantum anharmonic oscillators. The interaction is put to be hierarchical – just as in Dyson's hierarchical models (see e.g. [5], [14], [4]). This allows to extend to the infinite dimensional case the tools recently used in the case of classical hierarchical models [9].

The properties of the considered models depend on the values of certain parameters. Having in mind the stability of the fixed points we distinguish two families of such models – $\mathcal{M}_+$ and $\mathcal{M}_-$, to which they are referred to by the values of these parameters. In the case of $\mathcal{M}_+$, we have proven that varying the parameters within the frames of the family one may obtain the convergence to both types of the fixed points – unstable and stable (Theorem 3.1). The convergence to the unstable fixed point corresponds to the appearance (when the inverse temperature reaches a certain value) of a strong dependence between the oscillators which is peculiar to the critical point of the model. Meanwhile, the convergence to the stable fixed point corresponds to the absence of any critical point anomalies. In the case of $\mathcal{M}_-$, we prove that for all values of the parameters, the convergence only to the stable fixed point is possible (Theorem 3.2). The proof of these theorems relies on a number of lemmas formulated in Section 3. The families $\mathcal{M}_+$ and $\mathcal{M}_-$ are described in Section 4, where we show in particular that they are not empty. Section 5 contains the proofs of these lemmas. It should be noted that a more detailed description of the hierarchical models considered here can be found in our paper [2]. We point out, however, that the models studied in this work are essentially different from those investigated in the paper [11], where the case of bounded spins was considered.

2 General Setup

Consider a real separable Hilbert space $\mathcal{H}$ with an inner product $(.\ ,\ .)$ and a norm $|.|$. Let $\mathcal{H}_c$ denote its complexification. The bilinear extension to $\mathcal{H}_c$ of the inner product will also be written as $(.\ ,\ .)$; the linear extension to $\mathcal{H}_c$ of a linear operator A, defined on $\mathcal{H}$, will be denoted by A. Let a linear strictly positive trace class operator S be defined on $\mathcal{H}$. Thus there exists a complete orthonormal set of eigenvectors $\{\omega_j, j \in \mathbb{N}\}$ and a set of corresponding eigenvalues $\{s_j, j \in \mathbb{N}\}$ such that

$$S\omega_j = s_j\omega_j, \quad j \in \mathbb{N}; \quad \sum_{j=1}^{\infty} s_j < \infty; \; s_j > 0. \tag{2.1}$$

For this pair $\{S, \mathcal{H}\}$, one can construct (see e.g. [3] , p.107) the Gaussian measure γ_S, with zero mean value and S as a covariance operator, defined on the Borel σ-algebra $\mathcal{B}(\mathcal{H})$. Then for $\varphi \in \mathcal{H}$,

$$\int_{\mathcal{H}} \exp(i(\varphi, \omega)) d\gamma_S(\omega) = \exp(-\frac{1}{2}(S\varphi, \varphi)). \tag{2.2}$$

For an entire functions $F : \mathcal{H}_{\mathbf{c}} \rightarrow \mathbf{C}$ and some positive α, we denote

$$\begin{aligned} \|F\|_\alpha &= \sup\{|F(\zeta)| \exp(-\frac{1}{2}\alpha|\zeta|^2) \mid \zeta \in \mathcal{H}_{\mathbf{c}}\}, \qquad (2.3) \\ |\zeta|^2 &= |\xi|^2 + |\eta|^2, \quad \xi, \eta \in \mathcal{H}. \end{aligned}$$

Then

$$\mathcal{F} := \{F |\ \exists \alpha > 0 \ : \|F\|_\alpha < \infty\},$$

and for $\alpha \geq 0$

$$\mathcal{F}_\alpha := \{F \in \mathcal{F} \mid \|F\|_{\alpha'} < \infty, \ \forall \alpha' > \alpha\}. \tag{2.4}$$

The latter set endowed with the topology generated by the family of norms $\{\|.\|_{\alpha'} \mid \alpha' > \alpha\}$, and the linear operations defined pointwise, becomes a complete locally convex space.

For certain positive σ and θ, and for appropriate $F \in \mathcal{F}$, we introduce the mapping $F \mapsto \mathbf{G}_S^{\sigma,\theta}(F)$ defined by

$$\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) = \int_{\mathcal{H}} F^2(\sigma\zeta + \theta\omega) d\gamma_S(\omega), \quad \zeta \in \mathcal{H}_c, \tag{2.5}$$

which can be expressed by means of the generalized S-transform $\mathbf{S}_{\sigma,\theta}$, used in White Noise Analysis (see e.g. [8]), as

$$\mathbf{G}_S^{\sigma,\theta}(F) = \mathbf{S}_{\sigma,\theta}(F^2).$$

Note that the parameter σ defines a rescaling of the argument of F, whereas θ is responsible for the coupling between taking a power and the integration: $\mathbf{S}_{\sigma,\theta}$ turns into identity for $\theta = 0$. To define $\mathbf{G}_S^{\sigma,\theta}$ as an operator on appropriate spaces $\mathcal{F}_\alpha$, we will need a technical background, which is explained just below. By means of Theorem 1.6 [3] (p.107), one can prove the following statement – a version of the Fernique theorem.

Proposition 2.1 *Let S and γ_S be as above and let A be a symmetric bounded linear operator on $\mathcal{H}$ such that the operator SA obeys the condition*

$$\mathrm{trace}(SA) < 1. \tag{2.6}$$

Then the expression

$$\exp(\frac{1}{2}(A\omega,\omega))d\gamma_S(\omega) = K_B d\gamma_B(\omega) \tag{2.7}$$

defines on $\mathcal{B}(\mathcal{H})$ the Gaussian measure γ_B with the covariance

$$B = (1 - SA)^{-1}S, \tag{2.8}$$

and

$$K_B = \int_{\mathcal{H}} \exp(\frac{1}{2}(A\omega,\omega))d\gamma_S(\omega) = [\det(1 - SA)]^{-\frac{1}{2}} \leq [1 - \mathrm{trace}(SA)]^{-\frac{1}{2}}.$$

For a symmetric bounded linear operator A and some $G \in \mathcal{F}$, the function

$$F(\zeta) = \exp(\frac{1}{2}(A\zeta,\zeta))G(\zeta) \tag{2.9}$$

also belongs to $\mathcal{F}$, and we may act by $\mathbf{G}_S^{\sigma,\theta}$ on this product. The result is as follows.

Lemma 2.1 *Let S, $\mathbf{G}_S^{\sigma,\theta}$, and A be as above, let also*

$$\mathrm{trace}(SA) < \frac{1}{2\theta^2}. \tag{2.10}$$

Then for F given by (2.9):

$$\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) = K_C \exp\Big(\sigma^2(A\zeta,\zeta) + 2\sigma^2\theta^2(A\zeta, CA\zeta)\Big) \mathbf{G}_C^{\sigma,\theta}(G)((1+2\theta^2 CA)\zeta), \tag{2.11}$$

where

$$C = (1-2\theta^2 SA)^{-1}S; \tag{2.12}$$

$$K_C = [\det(1-2\theta^2 SA)]^{-\frac{1}{2}}. \tag{2.13}$$

Proof. Inserting F as given by (2.9) into (2.5), we obtain

$$\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) = \exp\Big(\sigma^2(A\zeta,\zeta)\Big) \int_{\mathcal{H}} G^2(\sigma\zeta+\theta\omega)\exp\Big(2\sigma\theta(A\zeta,\omega)+\theta^2(A\omega,\omega)\Big)\, d\gamma_S(\omega).$$

Now we apply Proposition 2.1:

$$\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) = \exp\Big(\sigma^2(A\zeta,\zeta)\Big) K_C \int_{\mathcal{H}} G^2(\sigma\zeta+\theta\omega)\exp\left(2\sigma\theta(A\zeta,\omega)\right) d\gamma_C(\omega), \tag{2.14}$$

with C given by (2.12) as it is prescribed by (2.8). The next step is to shift the measure γ_C (see e.g. [3], p.151) in order to include the exp(.) under the integral in (2.14) into the shifted measure. Let $y \in \mathcal{H}$ be chosen such that $C^{-1}y \in \mathcal{H}$. Then the shifted measure $\gamma_{C,y}$ is determined by its Radon-Nikodym derivative as follows:

$$\frac{d\gamma_{C,y}}{d\gamma_C}(\omega) = \exp(-(C^{-1}y,\omega) - \frac{1}{2}(C^{-1}y,y)),$$

and

$$\int \Psi(\omega)d\gamma_{C,y}(\omega) = \int \Psi(\omega-y)d\gamma_C(\omega). \tag{2.15}$$

This yields:

$$\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) = \exp(\sigma^2(A\zeta,\zeta))K_C \int_{\mathcal{H}} G^2(\sigma\zeta+\theta\omega)\exp\{2\sigma\theta(A\zeta,\omega) + (C^{-1}y,\omega) + \frac{1}{2}(C^{-1}y,y)\}d\gamma_{C,y}(\omega).$$

Putting here

$$y = -2\sigma\theta CA\zeta, \tag{2.16}$$

one arrives at

$$\begin{aligned}\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) &= \exp(\sigma^2(A\zeta,\zeta) + 2\sigma^2\theta^2(A\zeta, CA\zeta)) \\ & K_C \int_{\mathcal{H}} G^2(\sigma\zeta + \theta\omega) d\gamma_{C,y}(\omega).\end{aligned} \tag{2.17}$$

Then by means of (2.15), one obtains from (2.17)

$$\begin{aligned}\mathbf{G}_S^{\sigma,\theta}(F)(\zeta) &= \exp(\sigma^2(A\zeta,\zeta) + 2\sigma^2\theta^2(A\zeta, CA\zeta)) \\ & K_C \int_{\mathcal{H}} G^2(\sigma\zeta + \theta(\omega + 2\sigma\theta CA\zeta)) d\gamma_C(\omega),\end{aligned}$$

which gives (2.11) when (2.5) is taken into account. ■

Now we may define $\mathbf{G}_S^{\sigma,\theta}$ as a continuous operator mapping appropriate spaces of entire functions into themselves. Denote the operator norm of S by $\|S\|$ and suppose $\sigma < 1/\sqrt{2}$. Let also

$$\alpha_{\max} := \min\{\frac{1-2\sigma^2}{2\theta^2\|S\|} \; ; \; \frac{1}{2\theta^2 \mathrm{trace} S}\}. \tag{2.18}$$

Lemma 2.2 *For $\alpha < \alpha_{\max}$, the mapping $\mathbf{G}_S^{\sigma,\theta}$ defined by (2.5) is continuous and Fréchet differentiable on $\mathcal{F}_\alpha$ where its Fréchet derivative is a bounded linear operator.*

Proof. For $\alpha' \in (\alpha, \alpha_{\max})$, and two given functions $F_1, F_2 \in \mathcal{F}_\alpha$, let us estimate

$$\Phi(\zeta) := \int_{\mathcal{H}} F_1(\sigma\zeta + \theta\omega) F_2(\sigma\zeta + \theta\omega) d\gamma_S(\omega).$$

The definitions (2.3), (2.4) yield

$$|F_i(\zeta)| \leq \|F_i\|_{\alpha'} \exp(\frac{1}{2}\alpha'|\zeta|^2), \quad \zeta = \xi + i\eta \in \mathcal{H}_c, \; i \; = \; 1, \; 2.$$

Therefore,

$$\begin{aligned}|\Phi(\zeta)| &\leq \int_{\mathcal{H}} |F_1(\sigma\zeta + \theta\omega)||F_2(\sigma\zeta + \theta\omega)| d\gamma_S(\omega) \\ &\leq \|F_1\|_{\alpha'}\|F_2\|_{\alpha'} \int_{\mathcal{H}} \exp\left\{\alpha'(\sigma^2|\xi|^2 + 2\sigma\theta(\xi,\omega) + \sigma^2|\eta|^2 + \theta^2|\omega|^2)\right\} d\gamma_S(\omega).\end{aligned} \tag{2.19}$$

Computing the integral over $\mathcal{H}$, one obtains

$$\begin{aligned} |\Phi(\zeta)| &\leq \|F_1\|_{a'}\|F_2\|_{\alpha'}K\exp(\alpha'\sigma^2|\eta|^2+\alpha'\sigma^2((1-2\alpha'\theta^2 S)^{-1}\xi,\xi)) \\ &\leq K\|F_1\|_{\alpha'}\|F_2\|_{\alpha'}\exp\left\{\frac{\alpha'\sigma^2}{1-2\alpha'\theta^2\|S\|}|\zeta|^2\right\}; \end{aligned} \tag{2.20}$$

where

$$K=(\det(1-2\alpha'\theta^2 S))^{-\frac{1}{2}}\leq[1-2\alpha'\theta^2\mathrm{trace}S]^{-\frac{1}{2}}.$$

The latter yields in turn for $\|\Phi\|_{\alpha'}$ with $\alpha'<\alpha_{\max}$,

$$\|\int_{\mathcal{H}}F_1(\sigma\zeta+\theta\omega)F_2(\sigma\zeta+\theta\omega)d\gamma_S(\omega)\|_{\alpha'}\leq K\|F_1\|_{\alpha'}\|F_2\|_{\alpha'}. \tag{2.21}$$

Putting here $F_1=F_2=F$, one gets

$$\|\mathbf{G}_S^{\sigma,\theta}(F)\|_{\alpha'}\leq K\|F\|_{\alpha'}^2. \tag{2.22}$$

For $F\in\mathcal{F}_\alpha$, $G\in\mathcal{F}_\alpha$, we have

$$\begin{aligned} &\mathbf{G}_S^{\sigma,\theta}(F+G)(\zeta)-\mathbf{G}_S^{\sigma,\theta}(F)(\zeta)= \\ &2\int_{\mathcal{H}}F(\sigma\zeta+\theta\omega)G(\sigma\zeta+\theta\omega)d\gamma_S(\omega)+\mathbf{G}_S^{\sigma,\theta}(G)(\zeta). \end{aligned} \tag{2.23}$$

The latter term can be estimated by means of (2.22). Therefore, the Fréchet derivative of the operator $\mathbf{G}_S^{\sigma,\theta}$ at the "point" $F\in\mathcal{F}_\alpha$ is the following bounded linear operator on $\mathcal{F}_\alpha$:

$$\{(\mathbf{G}_S^{\sigma,\theta}(F))'G\}(\zeta)=2\int_{\mathcal{H}}F(\sigma\zeta+\theta\omega)G(\sigma\zeta+\theta\omega)d\gamma_S(\omega). \tag{2.24}$$

Its boundedness can be shown by putting $F_1=F$, $F_2=G$ in (2.21). ∎

Remark 2.1 *We can extend the mapping $\mathbf{G}_S^{\sigma,\theta}$ preserving all its properties stated above on $\mathcal{F}_{\alpha_{\max}}$, provided $\alpha_{\max}$ is strictly less than $(2\theta^2\mathrm{trace}S)^{-1}$, or on the other subsets of $\mathcal{F}$, e.g. consisting of functions of the type of (2.9) with $G\in\mathcal{F}_0$ and such A, that the constant (2.13) exists.*

Definition 2.1 *The subset of $\mathcal{F}$ where the mapping $\mathbf{G}_S^{\sigma,\theta}$ can be defined possessing the properties stated by Lemma 2.2 is denoted by $\mathcal{F}_S$*

The general problem we are concerned here is the existence of fixed points of this operator and their stability.

Definition 2.2 *The function $F \in \mathcal{F}$ is said to be a fixed point of $\mathbf{G}_S^{\sigma,\theta}$ if*

$$\mathbf{G}_S^{\sigma,\theta}(F) = F \tag{2.25}$$

Among all possible fixed points, one can distinguish a family which can be described in an evident way. This family consists of Gaussian fixed points and is set as follows. Let $\mathcal{Q}_{\text{fin}} = \{Q \subset I\!N \mid |Q| < \infty\}$, where $|Q|$ means the cardinality of Q. For a certain nonempty $Q \in \mathcal{Q}_{\text{fin}}$, let $\mathcal{H}_Q$ be a linear span of the family $\{\omega_j, j \in Q\}$, where $\{\omega_j\}$ are given by (2.1). Let also W_Q be the orthogonal projector on $\mathcal{H}_Q$ and S_Q be a restriction of the operator S onto $\mathcal{H}_Q$. For a nonempty $Q \in \mathcal{Q}_{\text{fin}}$, we put

$$A_Q = \frac{1 - 2\sigma^2}{2\theta^2} S_Q^{-1} W_Q, \tag{2.26}$$

and $A_Q = 0$ for an empty Q. Note that S_Q^{-1} exists for each nonempty $Q \in \mathcal{Q}_{\text{fin}}$, a fact which follows from the strict positivity of S.

Proposition 2.2 *Each element of the family*

$$\mathcal{A} = \{F_Q(\zeta) = (2\sigma^2)^{\frac{|Q|}{2}} \exp(\frac{1}{2}(A_Q\zeta, \zeta)) \mid Q \in \mathcal{Q}_{\text{fin}}\} \tag{2.27}$$

is a fixed point of $\mathbf{G}_S^{\sigma,\theta}$.

Proof. For Q being empty, we have $F_\emptyset \equiv 1$, which obviously is a fixed point of the NLST given by (2.5). The case of a nonempty $Q \in \mathcal{Q}_{\text{fin}}$ can be handle out by means of (2.11) with $G \equiv 1$ by imposing appropriate conditions on A. ∎

Remark 2.2 *There exists also the fixed point $F \equiv 0$, which will be discussed below.*

Remark 2.3 *When solving the equation (2.25), we have the possibility to put $A = S^{-1}$ without any projecting on finite-dimensional subspaces $\mathcal{H}_Q$. But in this case, A becomes unbounded, a fact which causes the function $\exp(\frac{1}{2}(A\zeta, \zeta))$ to be outside of $\mathcal{F}$. Nevertheless, such a function can be considered as a fixed point in some weak sense. These generalized fixed points will be considered in a separate work.*

Now we study the stability of the fixed points that belong to $\mathcal{A}$. This can be done by means of the Fréchet derivative of $\mathbf{G}_S^{\sigma,\theta}$ at the point F_Q which can be obtained by inserting F_Q into the right hand side of (2.24). Then the stability conditions can be described in terms of the spectral properties of the Fréchet derivative, which can be shown to be a compact operator. Thus to describe its spectrum, we consider the eigenvalue problem

$$(\mathbf{G}_S^{\sigma,\theta}(F_Q))'G = \lambda G. \tag{2.28}$$

Let $G(\zeta) = F_Q(\zeta)P(\zeta)$, where P is a polynomial. Inserting such G into (2.24) one gets from (2.28)

$$2(2\sigma^2)^{|Q|}\int_{\mathcal{H}} \exp\left[(A_Q(\sigma\zeta+\theta\omega),\sigma\zeta+\theta\omega)\right]P(\sigma\zeta+\theta\omega)d\gamma_S(\omega)$$
$$= \lambda(2\sigma)^{\frac{|Q|}{2}}\exp(\frac{1}{2}(A_Q\zeta,\zeta))P(\zeta).$$

By means of Lemma 2.1, the latter can be transformed into the equation:

$$2\int_{\mathcal{H}} P(\sigma T_Q\zeta+\theta\omega)d\gamma_{C_Q}(\omega) = \lambda P(\zeta), \tag{2.29}$$

where

$$\sigma T_Q = \sigma(1-W_Q)+\frac{1}{2\sigma}W_Q, \tag{2.30}$$

$$C_Q = (1-W_Q)S+\frac{1}{2\sigma^2}W_Q S. \tag{2.31}$$

Denote $\deg P = k$, then each a polynomial P can be written in the form

$$P = P_k+\delta P; \quad \deg P_k = k,\ P_k(\alpha\zeta) = \alpha^k P_k(\zeta),\ \deg\delta P < k. \tag{2.32}$$

Inserting such P into the equation (2.29), we find

$$2P_k(\sigma T_Q\zeta)+P'(\zeta) = \lambda P_k(\zeta)+\lambda\delta P(\zeta), \tag{2.33}$$

with some polynomial P' such that $\deg P' < k$. Let $\mathcal{H}_{Qc} = \{\zeta = \xi+i\eta \mid \xi \in \mathcal{H}_Q \text{ or } \eta \in \mathcal{H}_Q\}$ and $\bar{\mathcal{H}}_{Qc} = \{\zeta = \xi+i\eta \mid \xi \in \mathcal{H}\setminus\mathcal{H}_Q \text{ or } \eta \in \mathcal{H}\setminus\mathcal{H}_Q\}$. We set

$$\begin{aligned}\mathcal{P}_Q &= \{P \mid P_k(\zeta) = 0,\ \forall\zeta \in \mathcal{H}_{Qc}\} \\ \tilde{\mathcal{P}}_Q &= \{P \mid P_k(\zeta) = 0,\ \forall\zeta \in \bar{\mathcal{H}}_{Qc}\}.\end{aligned} \tag{2.34}$$

Then one has

$$\begin{aligned}P_k(\sigma T_Q\zeta) &= P_k(\sigma\zeta), \quad P \in \mathcal{P}_Q; \\ P_k(\sigma T_Q\zeta) &= P_k(\frac{1}{2\sigma}\zeta),\ P \in \tilde{\mathcal{P}}_Q.\end{aligned} \tag{2.35}$$

Inserting such P into (2.33) and taking into account the homogeneity of P_k (2.32) as well as (2.35), we arrive at

$$\lambda_k = 2\sigma^k; \quad P \in \mathcal{P}_Q, \ k = \deg P, \ k \in \mathbb{Z}_+, \tag{2.36}$$

$$\tilde{\lambda}_k = \frac{2}{(2\sigma)^k}; \quad P \in \tilde{\mathcal{P}}_Q, \ k = \deg P, \ k \in \mathbb{N}. \tag{2.37}$$

Definition 2.3 *The fixed point F_Q is said to be stable if all eigenvalues of $(\mathbf{G}_S^{\sigma,\theta}(F_Q))'$ are at most one. It is said to be l-unstable ($l \in \mathbb{N}$) if there exist l eigenvalues which are greater than one.*

For every unstable fixed point, there exist the "directions" in which an arbitrary small perturbation δF results that the sequence $\{F_n, \ n \in \mathbb{Z}_+\}$, defined recursively

$$F_n = \mathbf{G}_S^{\sigma,\theta}(F_{n-1}) \tag{2.38}$$

with $F_0 = F_Q + \delta F$, will never approach the fixed point F_Q . Such directions are defined by the eigenpolynomials which correspond to λ greater than one. In our case, $\lambda_0 = 2 > 1$. This means that the sequence $\{F_n, \ n \in \mathbb{Z}_+\}$ with $F_0(\zeta) = K_0 \exp(\frac{1}{2}(A_Q\zeta, \zeta)), K_0 \neq K_Q := (2\sigma^2)^{|Q|/2}$ does not converge to the fixed point F_Q. For $|K_0| > K_Q$, this sequence diverges because $|K_n| \to +\infty$. For $|K_0| < K_Q$, it converges to the fixed point $F \equiv 0$ mentioned in Remark 2.2. The latter attracts all sequences with "small" values of $|F_n(0)|$. To exclude such a trivial instability, we redefine the operator (2.5) in such a way that it would map the manifold $\{F \in \mathcal{F}_S \mid F(0) = 1\}$ into itself. Put instead of (2.5)

$$\hat{\mathbf{G}}_S^{\sigma,\theta}(F)(\zeta) = \frac{\int_{\mathcal{H}} F^2(\sigma\zeta + \theta\omega) d\gamma_S(\omega)}{\int_{\mathcal{H}} F^2(\theta\omega) d\gamma_S(\omega)}. \tag{2.39}$$

Accordingly, the family of fixed points $\mathcal{A}$ (2.27) transforms into

$$\hat{\mathcal{A}} = \{F_Q(\zeta) = \exp(\frac{1}{2}(A_Q\zeta, \zeta)) \mid Q \in \mathcal{Q}_{\rm fin}\}. \tag{2.40}$$

Let ${\mathcal{P}_Q}^{(e)}$ and $\tilde{\mathcal{P}}_Q^{(e)}$ be the subsets of corresponding families introduced in (2.34) consisting of even polynomials. Denote also

$$\mathcal{F}^{(e)} = \{F \in \mathcal{F}_S \mid F(\zeta) = F(-\zeta), \ F(0) = 1\}. \tag{2.41}$$

Then $\hat{\mathcal{A}} \subset \mathcal{F}^{(e)}$, and $\hat{\mathbf{G}}_S^{\sigma,\theta}$ maps $\mathcal{F}^{(e)}$ into itself. The stability of each $F_Q \in \hat{\mathcal{A}}$ with respect to the perturbations within $\mathcal{F}^{(e)}$ is defined by the eigenvalues

$$\lambda_k = 2\sigma^{2k}, \ P \in {\mathcal{P}_Q}^{(e)}, \ 2k = \deg P, \ k \in \mathbb{N}; \tag{2.42}$$

$$\tilde{\lambda}_k = \frac{2}{(2\sigma)^{2k}}, \ P \in \tilde{\mathcal{P}}_Q^{(e)}, \ 2k = \deg P, \ k \in I\!N. \tag{2.43}$$

Note that $\sigma < 1/\sqrt{2}$, and hence all $\lambda_k < 1$, $k \in I\!N$. At the same time, $\tilde{\lambda}_1 > 1$, which means that F_Q is unstable whenever $\tilde{\mathcal{P}}_Q^{(e)}$ is nonempty. The only case where $\tilde{\mathcal{P}}_Q^{(e)}$ is empty is $A_Q = 0$ which occurs for Q being empty. Thus the only one stable fixed point in $\hat{\mathcal{A}}$ is $F_\emptyset \equiv 1$. For the values of σ defined by the condition $\tilde{\lambda}_k = 1$, bifurcations of the fixed points take place. Then new fixed points appear when $\sigma = \sigma_k = 2^{-1+1/2k}$, $k \geq 2$. In what follows, the sequences $\{F_n \in \mathcal{F}^{(e)}, \ n \in Z\!\!\!Z_+\}$ defined recursively by

$$F_n = \hat{\mathbf{G}}_S^{\sigma,\theta}(F_{n-1}), \ n \in I\!N, \tag{2.44}$$

with $\sigma \in (0, 1/\sqrt{2})$, may converge in a certain sense to the stable fixed point $F_\emptyset \equiv 1$, or to some unstable one $F_Q \in \hat{\mathcal{A}}$. The example of such a sequence can be constructed by choosing $F_0(z) = \exp(\frac{1}{2}(B_0 z, z))$ with a suitable operator B_0. In this case, all F_n have the form $F_n(\zeta) = \exp(\frac{1}{2}(B_n\zeta, \zeta))$, and the sequence $\{B_n, n \in Z\!\!\!Z_+\}$ may be described explicitly. We do not consider this simple example and will be concerned on the case of non-Gaussian F_0. When speaking about Gaussian or non-Gaussian sequences $\{F_n\}$, we mean their connection with corresponding sequences of measures. In fact, the case of peculiar interest is the choice of the function F_0 being the Laplace transform of some non-Gaussian measure. Then all elements of the sequence $\{F_n, n \in Z\!\!\!Z_+\}$ may possess this property. Consider this case in details.

For $\{S, \mathcal{H}\}$ introduced above, we define $\mathcal{H}_+ = \{\omega \in \mathcal{H} \mid S^{-1}\omega \in \mathcal{H}\}$. Then for every $\xi \in \mathcal{H}$, there exists the pairing $< \xi, \omega >:= (\xi, S^{-1}\omega)$. On the other hand, we may turn $\mathcal{H}_+$ into a separable Hilbert space by setting its inner product and norm as follows:

$$(\omega, \tilde{\omega})_+ = (S^{-1}\omega, S^{-1}\tilde{\omega}); \ \ |.|_+^2 = (.,.)_+. \tag{2.45}$$

Thus $\mathcal{H}$ and $\mathcal{H}_+$ form the topologically dual pair with respect to $< .,. >$. To point out this duality, we will write hereafter $\mathcal{H} = \mathcal{H}_-$. For every $\omega \in \mathcal{H}_+$, set $|\omega|_0^2 =< \omega, \omega >$ and let $\mathcal{H}_0$ be the completion of $\mathcal{H}_+$ in the topology defined by the norm $|.|_0$. Then $\mathcal{H}_+$ is densely embedded into $\mathcal{H}_0$ and the embedding operator $\sqrt{S}$ belongs to the Hilbert-Schmidt class. The same is true for the pair $\mathcal{H}_0$ and $\mathcal{H}_-$, thus we have constructed the rigging

$$\mathcal{H}_- \supset \mathcal{H}_0 \supset \mathcal{H}_+. \tag{2.46}$$

Let $\mathcal{H}_{0c}$ be the complexification of $\mathcal{H}_0$ and $< . , . >$ be the bilinear extension to $\mathcal{H}_{0c}$ of the product $< ., . >$. For any measure ν defined on Borel subsets of $\mathcal{H}_0$, we introduce the Laplace transformation as follows

$$F_\nu(\varphi) = \int_{\mathcal{H}_0} \exp(< \varphi, \omega >) d\nu(\omega), \quad \varphi \in \mathcal{H}_{0c}. \tag{2.47}$$

Define $\mathcal{F}^{(e)}_{\mathrm{Lap}}$ as a set of functions from $\mathcal{F}^{(e)}$ each of which being restricted on $\mathcal{H}_{0c}$ from $\mathcal{H}_c = \mathcal{H}_{-c}$ possesses the representation (2.47) with some Borel measure ν.

In the subsequent sections, we will construct the sequences of non-Gaussian functions $\{F_n \mid F_0 \in \mathcal{F}^{(e)}_{\mathrm{Lap}}\}$ converging to the fixed points from $\hat{\mathcal{A}}$. Each such a sequence defines the sequence of non-Gaussian measures converging to a certain Gaussian measure. It is naturally to consider the sequences of measures for which the assertion about such a convergence is analogous to a central limit theorem. We have chosen the ones describing temperature Gibbs states of certain quantum statistical models for a special choice of S, $\mathcal{H}$, and F_0.

3 Quantum Hierarchical Model

We will construct the non-Gaussian sequences $\{F_n \mid F_0 \in \mathcal{F}^{(e)}_{\mathrm{Lap}}\}$ which describe temperature Gibbs states of models of hierarchically interacting quantum anharmonic oscillators. Each a sequence of this type is characterized by a number of parameters. A part of them describe the individual properties of an oscillator, they define the starting element F_0 of the sequence $\{F_n\}$. The other parameters describe the interaction between the oscillators and define in turn the properties of the operator $\hat{\mathbf{G}}^{\sigma,\theta}_S$ given by (2.39) in such a way that the case of noninteracting oscillators corresponds to $\theta = 0$. Among the parameters, there is one which plays a special role here – the inverse temperature. It defines the properties of both F_0 and $\hat{\mathbf{G}}^{\sigma,\theta}_S$. The interaction between the oscillators grows proportionally to this nonnegative parameter. When the inverse temperature exceeds its critical value $\beta_* > 0$, the oscillations become strongly dependent which is known as a critical point behaviour of the model. At this temperature, the sequence $\{F_n\}$ converges to some unstable fixed point. For $\beta < \beta_*$, this sequence converges to the stable fixed point which corresponds to the case of weakly dependent oscillators. The assertions about these properties of $\{F_n\}$ can be considered as central limit theorems for strongly or weakly dependent random processes. These are among the results of this section.

We specify the pair $\{S, \mathcal{H}\}$ or the rigging (2.46) putting $\mathcal{H}_0 = L^2(I_\beta)$ and

$$\mathcal{H}_- = W_2^{-1}(I_\beta) \supset L^2(I_\beta) \supset W_2^1(I_\beta) = \mathcal{H}_+, \tag{3.1}$$

where I_β is the circle of length $\beta > 0$ (which can be interpreted as the interval $[0, \beta]$ with identified ends), $L^2(I_\beta)$ is the standard Hilbert space of square-integrable (with respect to the Lebesgue measure) real valued functions on I_β, $W_2^{-1}(I_\beta)$ and $W_2^1(I_\beta)$ are corresponding Sobolev spaces. Then the operator S is:

$$S = (-\Delta_\beta + 1)^{-1}, \tag{3.2}$$

where Δ_β stands for the Laplace-Beltrami operator on the circle I_β. The starting element F_0 is to be chosen as the Laplace transform (2.47) of the following measure ν_0. For $m > 0$, we put $S_m = (-m\Delta_\beta + 1)^{-1}$ and let γ_{S_m} be the Gaussian measure on $\mathcal{B}(\mathcal{H}_-)$ which obeys (2.2) with this S_m. Let also Ω_β be the set of all continuous real valued functions on I_β, and

$$\tilde{V}(q) = -\frac{1}{2}q^2 + V(q), \quad V(q) = \frac{1}{2}aq^2 + \frac{1}{4}bq^4, \tag{3.3}$$

where $a \in I\!R$, $b > 0$. Then one can prove the following assertion (see e.g. [1] or [7])

Proposition 3.1 (i) *the set $\Omega_\beta \subset \mathcal{H}_0 = L^2(I_\beta)$ has the property:*

$$\gamma_{S_m}(\Omega_\beta) = 1;$$

(ii) *there exists the measure ν_0 on $\mathcal{B}(\mathcal{H}_0)$ such that $\nu_0(\Omega_\beta) = 1$, which is absolutely continuous with respect to γ_{S_m} possessing the Radon-Nikodym derivative*

$$\frac{d\nu_0}{d\gamma_{S_m}}(\omega) = K \exp\left\{ -\int_{I_\beta} \tilde{V}(\omega(\tau))d\tau \right\},$$

with an appropriate normalizing constant K.

Having such a measure, we can prove also the statement

Proposition 3.2 *Let the sequence $\{F_n, n \in \mathbb{Z}_+\}$ be defined by (2.44) with the starting element F_0 being the Laplace transform of the measure ν_0 described above. Then all elements of this sequence belong to the family $\mathcal{F}_0^{(e)}$ defined by (2.41). Each F_n is the Laplace transform of a Borel measure ν_n*

defined on $\mathcal{B}(\mathcal{H}_0)$ such that $\nu_n(\Omega_\beta) = 1$, and the sequence of measures $\{\nu_n\}$ obeys the recursion relation

$$d\nu_n(\omega) = \frac{\exp[(\theta^2/2\sigma^2)|\omega|_0^2] d(\nu_{n-1} * \nu_{n-1})(\omega/\sigma)}{\int_{\mathcal{H}_0} \exp(\frac{1}{2}\theta^2|\omega|_0^2) d(\nu_{n-1} * \nu_{n-1})(\omega)}; \quad n \in \mathbb{N}. \tag{3.4}$$

As it has been already remarked above and is seen from (2.39), (3.4), the parameter θ describes the dependence (interaction) between the oscillations and its value $\theta = 0$ corresponds to the absence of this dependence. One may fix $\theta > 0$ by putting

$$\theta = \sqrt{\frac{1}{2} - \sigma^2}. \tag{3.5}$$

The other positive values of θ can be considered by means of a rescaling in (2.39) or in (3.4). The choice of θ (3.5) is suitable in view of the form of A_Q (2.26). For $\sigma \in (1/\sqrt{2}\ , 1/2)$, we also put:

$$\sigma = 2^{-(1+\delta)/2}, \quad \delta \in (0, 1). \tag{3.6}$$

A more detailed description of the hierarchical model of quantum anharmonic oscillators is given in the next section and in [2]. For S, θ, σ given by (3.2), (3.5), and (3.6) respectively, we put $\mathbf{G}_\beta = \hat{\mathbf{G}}_S^{\sigma,\theta}$.

Definition 3.1 *The hierarchical model of quantum anharmonic oscillators is defined as the sequence of functions $\{F_n, n \in \mathbb{Z}_+\}$ satisfying the following relation:*

$$\begin{aligned} F_n(\zeta) &= \mathbf{G}_\beta(F_{n-1})(\zeta) \qquad\qquad (3.7)\\ &= \frac{\int_{\mathcal{H}_-} F_{n-1}^2(2^{-(1+\delta)/2}\zeta + 2^{-(1+\delta)/2}\sqrt{2^\delta - 1}\omega) d\gamma_S(\omega)}{\int_{\mathcal{H}_-} F^2(2^{-(1+\delta)/2}\sqrt{2^\delta - 1}\omega) d\gamma_S(\omega)}, \ \zeta \in \mathcal{H}_{-c}, \\ F_0(\varphi) &= \int_{\mathcal{H}_0} \exp(< \varphi, \omega >) d\nu_0(\omega), \ \varphi \in \mathcal{H}_{0c}, \end{aligned}$$

or, equivalently, as a sequence of measures $\{\nu_n, n \in \mathbb{Z}_+\}$ described by Proposition 3.2 with θ and σ given by (3.5), (3.6), and with the starting element ν_0 described by Proposition 3.1. This model is labeled by the tuple $\{m, a, b, \delta |\ m > 0, a \in \mathbb{R}, b > 0, \delta > 0\}$.

Consider the sequence $\{F_n\}$ from the point of view of analytic properties of its elements. As an entire function, each F_n can be written in the form of the following expansion

$$F_n = \sum_{l=0}^{\infty} \frac{1}{(2l)!} F_{2l}^{(n)} \tag{3.8}$$

which is absolutely convergent on the whole $\mathcal{H}_{-c}$. Since $F_n(0) = 1$, the function

$$U^{(n)} = \log F_n, \tag{3.9}$$

is holomorphic in some neighbourhood of zero in $\mathcal{H}_{-c}$, and hence the expansion

$$U^{(n)} = \sum_{l=1}^{\infty} \frac{1}{(2l)!} U_{2l}^{(n)} \tag{3.10}$$

converges absolutely in this neighbourhood. Both $F_{2l}^{(n)}$ and $U_{2l}^{(n)}$ are homogeneous $2l$-linear continuous functions on the corresponding tensor products of the spaces $\mathcal{H}_{-c}$. The embedding operator $\mathcal{H}_{0c} \to \mathcal{H}_{-c}$ belongs to the Hilbert-Schmidt class, thus the kernel theorem [3] implies that for $\varphi \in \mathcal{H}_{0c}$, these functions possess the integral representations:

$$F_{2l}^{(n)} = \int_{I_\beta^{2l}} F_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \varphi(\tau_1) \ldots \varphi(\tau_{2l}) d\tau_1 d\tau_2 \ldots d\tau_{2l}, \tag{3.11}$$

$$U_{2l}^{(n)} = \int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \varphi(\tau_1) \ldots \varphi(\tau_{2l}) d\tau_1 d\tau_2 \ldots d\tau_{2l}, \tag{3.12}$$

which kernels belong to $L^2(I_\beta^{2l})$. In addition, one can observe that these kernels are invariant with respect to the shifts on I_β, i.e. for every $\vartheta \in I_\beta$,

$$U_{2l}^{(n)}(\tau_1 + \vartheta, \tau_2 + \vartheta, \ldots, \tau_{2l} + \vartheta) = U_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}), \tag{3.13}$$

where the addition is modulo β. Such a symmetry yields that the following integrals do not depend on $\tau \in I_\beta$:

$$\hat{U}_n(0) := \int_{I_\beta} U_2^{(n)}(\tau, \tau') d\tau'; \tag{3.14}$$

$$X_n := \int_{I_\beta} U_4^{(n)}(\tau, \tau', \tau'', \tau'') d\tau' d\tau''. \tag{3.15}$$

Every model introduced by Definition 3.1 is labeled by the tuple of parameters $\{m, a, b, \delta\}$. In order to establish the conditions under the which the sequences of functions describing this model converge to corresponding fixed points, we distinguish two families of such models by imposing restrictions on these parameters.

For $\delta \in (0, 1/2)$, we set $\bar{\varepsilon} = \min\{\frac{1}{6}, \frac{1-2\delta}{4}\}$, and

$$\bar{u} = \frac{2^\delta - 2^{-\bar{\varepsilon}}}{2^\delta - 1}. \tag{3.16}$$

We also put

$$\kappa(u) = \frac{2^{-\delta}}{1-(1-2^{-\delta})u}, \quad u \in (0, (1-2^{-\delta})^{-1}). \tag{3.17}$$

Then $\bar{\kappa} := \kappa(\bar{u})$ obeys the estimate

$$\bar{\kappa} \leq \min\{2^{1/6}, \quad 2^{(1-2\delta)/4}\}. \tag{3.18}$$

For given $u \in (0, \bar{u})$, let

$$\begin{aligned} w &= w(u) = 2\sqrt{2}u(u-1), \\ \bar{w} &= \min\{\frac{2^{1-\delta}}{3+2^{\delta}}, \quad w(\bar{u})\}. \end{aligned} \tag{3.19}$$

The quantities β_0^-, β_0^+ are defined as the smallest values of β obeying the equations

$$\begin{aligned} \hat{U}_0(0) &= 1, \quad \text{for} \quad \beta = \beta_0^-; \\ \hat{U}_0(0) &= \bar{u}, \quad \text{for} \quad \beta = \beta_0^+; \end{aligned} \tag{3.20}$$

where $\hat{U}_0$ is given by (3.14).

Definition 3.2 *The hierarchical model of quantum anharmonic oscillators is said to belong to the family* $\mathcal{M}_+$ *if*
(i) $\delta \in (0, 1/2)$;
(ii) β_0^-, β_0^+ *defined by (3.20) exist;*
(iii) *for all* β *belonging to the closed interval with the ends* β_0^+, β_0^-,

$$|X_0| < \bar{w}. \tag{3.21}$$

Definition 3.3 *The hierarchical model of quantum anharmonic oscillators is said to belong to the family* $\mathcal{M}_-$ *if*
(i) $\delta > 0$;
(ii) $\hat{U}_0(0) < 1$ *for all* $\beta > 0$.

In both definitions, the first condition is imposed directly on the model parameter δ. The other ones are rather implicit. We clarify them below specifying the restrictions imposed directly on the parameters m, a, and b. Now let us describe more precisely the fixed points of $\mathbf{G}_\beta$ to which the sequences of functions are shown to converge. For our choice of S, θ, and σ

(3.2), (3.5) , (3.6), the operator A_Q is simply $S_Q^{-1}W_Q$ (see (2.26)). As it is shown below, there is only one choice of nonempty Q, that defines a fixed point to which the considered convergence for the models from the family $\mathcal{M}_+$ is possible. In this case Q consists of only one element: $Q = \{1\}$. For our choice of S, $\omega_1 \equiv 1/\sqrt{\beta}$ that corresponds to the eigenvalue $s_1 = 1$ (see (3.2)). Then $|\omega_1|_- = |\omega_1|_0 = 1$, $\|S\|^2 = s_1 = 1$, and $\alpha_{\max} = 1$ (see Remark 2.1). For such Q

$$\begin{aligned}\exp(\frac{1}{2}(A_Q\varphi,\varphi)_-) &= \exp(\frac{1}{2}(\varphi,\omega_1)_0^2) \\ &= \exp\left\{\frac{1}{2\beta}(\int_{I_\beta}\varphi(\tau)d\tau)^2\right\} \in \hat{\mathcal{A}}.\end{aligned} \tag{3.22}$$

Theorem 3.1 *For every model from the family $\mathcal{M}_+$, there exists β_* such that for $\beta = \beta_*$, the corresponding sequence $\{F_n, n \in \mathbb{Z}_+\}$ converges γ_S-almost everywhere to the fixed point*

$$F_*(\varphi) = \exp\left\{\frac{1}{2\beta_*}(\int_{I_{\beta_*}}\varphi(\tau)d\tau)^2\right\}. \tag{3.23}$$

For $\beta < \beta_*$, this sequence converges in the same sense to the fixed point $F_\emptyset \equiv 1$.

Theorem 3.2 *For every model from the family $\mathcal{M}_-$, the sequence $\{F_n, n \in \mathbb{Z}_+\}$ converges γ_S-almost everywhere to the fixed point $F_\emptyset \equiv 1$ for all positive β.*

The proof of these theorems relies on a sequence of lemmas, which are given below and proved in a separate section.

To exploit the symmetry property (3.13), we introduce

$$\mathcal{K} = \{k \mid k = \frac{2\pi}{\beta}\kappa, \ \kappa \in \mathbb{Z}\}, \tag{3.24}$$

and

$$\hat{U}_n(k) = \int_{I_\beta} U_2^{(n)}(0,\tau)\cos(k\tau)d\tau, \quad k \in \mathcal{K} \tag{3.25}$$

Then the value of the latter for $k = 0$ coincides with that defined by (3.14). We also set

$$\begin{aligned}Y_n &= \int_{I_\beta^3} U_4^{(n)}(\tau_1,\tau_2,\tau_3,\tau_4)d\tau_2 d\tau_3 d\tau_4 \\ &= \frac{1}{\beta}\int_{I_\beta^4} U_4^{(n)}(\tau_1,\tau_2,\tau_3,\tau_4)d\tau_1 d\tau_2 d\tau_3 d\tau_4.\end{aligned} \tag{3.26}$$

Lemma 3.1 *For every model from the family $\mathcal{M}_+$, there exists β_* such that for $\beta = \beta_*$, the sequences $\{X_n, n \in \mathbb{Z}_+\}$, $\{Y_n, n \in \mathbb{Z}_+\}$ converge to zero, and the sequence $\{\hat{U}_n(0), n \in \mathbb{Z}_+\}$ converges to one. For $\beta < \beta_*$, all these sequences converge to zero.*

Lemma 3.2 *For every model from the family $\mathcal{M}_-$, the sequences $\{Y_n\}$, $\{\hat{U}_n(0)\}$ converge to zero for all positive β.*

Lemma 3.3 *For all $n \in \mathbb{Z}_+$ and $l \in \mathbb{N}$, the following estimates hold true (Lebesgue)-almost everywhere,*

$$\begin{aligned} & \int_{I_\beta^2} U_4^{(n)}(\tau, \tau, \tau_1, \tau_2) d\tau_1 d\tau_2 \\ \leq \; & \int_{I_\beta^2} U_4^{(n)}(\tau, \tau', \tau_1, \tau_2) d\tau_1 d\tau_2; \end{aligned} \tag{3.27}$$

$$(-1)^{l-1} U_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \geq 0; \tag{3.28}$$

$$F_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l}) \geq 0. \tag{3.29}$$

Lemma 3.4 *For every $\beta \leq \beta_*$ and arbitrary nonzero $k \in \mathcal{K}$, the sequence $\{\hat{U}_n(k), n \in \mathbb{Z}_+\}$ converges to zero.*

Lemma 3.5 *For every $n \in \mathbb{N}$, the following representation holds true*

$$F_n(z\xi_1) = \prod_{j=1}^{\infty} (1 + c_j z^2), \tag{3.30}$$

where

$$c_j > 0, \;\; \sum_j c_j < \infty, \;\; c_1 > c_j, \;\; \forall j > 1;$$

$z \in \mathbb{C}$, and $\xi_1 = \xi_1(\tau) \equiv 1, \;\; \forall \tau \in I_\beta$.

Lemma 3.6 *Let $Y_n \to 0$ when $n \to \infty$, then*

$$\int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1, \ldots, \tau_{2l}) d\tau_1 \ldots d\tau_{2l} \to 0, \tag{3.31}$$

for all $l \geq 3$. Let $\hat{U}_n(0) \to 0$, then $Y_n \to 0$, i.e. (3.31) holds for all $l \geq 2$.

Proof. From the representation (3.30), one has ($l \geq 1$):

$$\int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1, \ldots, \tau_{2l}) d\tau_1 \ldots d\tau_{2l} = 2(2l-1)!(-1)^{l-1} \sum_{j=1}^{\infty} c_j^l. \tag{3.32}$$

Taking into account (3.14), (3.26) and the fact that $c_1 > c_j, \quad \forall j > 1$, one obtains from (3.32):

$$\begin{aligned} |Y_n| &< 6c_1 \hat{U}_n(0), \\ \left| \int_{I_\beta^{2l}} U_{2l}^{(n)}(\tau_1, \ldots, \tau_{2l}) d\tau_1 \ldots d\tau_{2l} \right| &< \frac{(2l-1)!}{6} c_1^{l-2} \beta |Y_n|. \end{aligned}$$

■

For a continuous complex valued function $\varphi : I_\beta \to \mathbb{C}$, i.e. for $\varphi \in C(I_\beta)$, we set

$$\|\varphi\|_{C(I_\beta)} = \sup\{|\varphi(\tau)| \mid \tau \in I_\beta\}.$$

Lemma 3.7 *For every $\varphi \in C(I_\beta)$ and $n \in \mathbb{Z}_+$, $F_n(z\varphi)$ is an entire function of $z \in \mathbb{C}$ which obeys the estimate*

$$|F_n(z\varphi)| \leq \exp(\frac{1}{2}\hat{U}_n(0)|z|^2 \|\varphi\|^2_{C(I_\beta)}) \tag{3.33}$$

Proof. To estimate $F_n(\varphi)$ for $\varphi \in C(I_\beta)$, we exploit its representation (3.8). Having in mind that $C(I_\beta) \subset L^2(I_\beta)$, we may use the representation (3.11), the positivity property (3.29), and obtain

$$\begin{aligned} |F_n(\varphi)| &\leq \sum_{l=0}^{\infty} \frac{1}{(2l)!} \|\varphi\|^{2l}_{C(I_\beta)} \int_{I_\beta^{2l}} |F_{2l}^{(n)}(\tau_1, \tau_2, \ldots, \tau_{2l})| d\tau_1 d\tau_2 \ldots d\tau_{2l} \\ &= F_n(\|\varphi\|_{C(I_\beta)} \xi_1) \\ &= \prod_{j=1}^{\infty} (1 + c_j \|\varphi\|^2_{C(I_\beta)}). \end{aligned} \tag{3.34}$$

Here we have used Lemma 3.5 as well. Thus for every fixed $\varphi \in C(I_\beta)$ and arbitrary $z \in \mathbb{C}$, we have

$$\begin{aligned} |F_n(z\varphi)| &\leq \prod_{j=1}^{\infty} (1 + c_j |z|^2 \|\varphi\|^2_{C(I_\beta)}) \\ &\leq \prod_{j=1}^{\infty} \exp(c_j |z|^2 \|\varphi\|^2_{C(I_\beta)}) = \exp\left(\frac{1}{2}\hat{U}_n(0)|z|^2 \|\varphi\|^2_{C(I_\beta)}\right) \end{aligned} \tag{3.35}$$

For every $z \in \mathbb{C}$, $F_n(z\varphi)$ is an entire function of $z\varphi \in \mathcal{H}_{-c}$, thus it is an entire function of z for every fixed $\varphi \in C(I_\beta) \subset \mathcal{H}_{-c}$. ■

Proof of Theorems 3.1 and 3.2. First we note that the set $C(I_\beta)$ is a complexification of Ω_β, and for the choice of S (3.2), one has from Proposition 3.1

$$\gamma_S(\Omega_\beta) = 1. \tag{3.36}$$

Then we choose $\varphi \in C(I_\beta) \subset \mathcal{H}_{0c}$ such that the series in (3.10) absolutely converges. We put $\beta = \beta_*$ and estimate the difference $\log F_n(\varphi) - \log F_*(\varphi)$ for chosen φ. For this purpose, we use the representation implied by the kernel theorem.

$$\begin{aligned} & |\log F_n(\varphi) - \log F_*(\varphi)| = \left| U^{(n)}(\varphi) - \frac{1}{2\beta_*}\Big(\int_{I_{\beta_*}} \varphi(\tau)d\tau\Big)^2 \right| \\ \leq \;& \frac{1}{2}\|\varphi\|^2_{C(I_{\beta_*})} \int_{I^2_{\beta_*}} \left| U_2^{(n)}(\tau_1,\tau_2) - \frac{1}{\beta_*}\right| d\tau_1 d\tau_2 \\ + \;& \sum_{l=2}^{\infty} \frac{1}{(2l)!}\|\varphi\|^{2l}_{C(I_{\beta_*})} \int_{I^{2l}_{\beta_*}} |U_{2l}^{(n)}(\tau_1,\tau_2,\ldots,\tau_{2l})| d\tau_1 d\tau_2 \ldots \tau_{2l}. \end{aligned} \tag{3.37}$$

Then taking into account (3.14), Lemmas 3.1, 3.6, and the continuity of the function $\log x$, we obtain the convergence of the sequence $\{F_n(\varphi)\}$ for this φ. Now let φ be chosen arbitrarily in $C(I_{\beta_*})$. Then there exists $z \in \mathbb{C}$ such that for chosen φ, $z\varphi$ belongs to the subset of $\mathcal{H}_{-c}$ where the series in (3.10) absolutely converges, and hence the estimate (3.37) holds. We denote the set of all such z by Z_φ (it obviously has a limiting point) and use Lemma 3.7. Therefore, we have a sequence $\{F_n(z\varphi)\}$ of entire functions of $z \in \mathbb{C}$ with growth of at most second order which converges to $F_*(z\varphi)$ on the set Z_φ. This allows to use the Vitaly theorem which implies that this sequence also converges for $z = 1$. The latter means that the sequence $\{F_n(\varphi)\}$ converges to the fixed point F_* for every $\varphi \in C(I_{\beta_*})$. Therefore, in view of (3.36), the stated converges holds for $\beta = \beta_*$. In the case of $\beta < \beta_*$, the sequence $\{\hat{U}_n(0)\}$ converges to zero and we should prove the convergence to the fixed point $F_\emptyset$. Minor changes of the scheme just applied yield the proof of the stated convergence also for such β as well as the proof of Theorem 3.2q. ■

4 Description of the Families $\mathcal{M}_+$ and $\mathcal{M}_-$

We start with a more detailed explanation what does the notion of a hierarchical model of quantum oscillators mean here. As it can be understood from

Definition 3.1, the model is set by m, a, and b. These are the parameters of the one-particle Hamiltonian which is the operator

$$H = -\frac{1}{2m}(\frac{d}{dx})^2 + V(x) \;\; = \;\; -\frac{1}{2m}(\frac{d}{dx})^2 + \frac{1}{2}ax^2 + \frac{1}{4}bx^4, \quad x \in I\!R \qquad (4.1)$$

that acts in the Hilbert space $\mathcal{H}^{\rm osc} = L^2(I\!R, dx)$ and describes the dynamics of a quantum particle. Here m stands for the particle's mass and the function V describes the potential energy of its one-dimensional oscillations around the equilibrium position $x = 0$. This particle is said to be an oscillator, and for $b = 0$, the oscillator is known being harmonic. In this case, the Hamiltonian spectrum can be computed explicitly. Thus the parameter b, which is assumed here being positive, defines the rate of the oscillator's anharmonicity. Let q denote the position operator which describes the oscillator's displacements from the equilibrium and acts in $\mathcal{H}^{\rm osc}$ as follows: $(q\psi)(x) = x\psi(x)$. For all positive β, the operators $q^l \exp(-\beta H)$, $l \in Z\!\!\!Z_+$ are of trace class, hence we may set

$$Z = {\rm trace}(e^{-\beta H}), \quad < q^2 >= \frac{1}{Z}{\rm trace}(q^2 e^{-\beta H}). \qquad (4.2)$$

Let a countable variety (a chain) of copies of the oscillator described above be considered. The oscillators in the chain are assumed to interact in such a way that the potential energy of the interaction between any two of them depends only on their displacements and their position in the chain. This dependence can be characterized as hierarchical. To describe it, we introduce

$$\Lambda_{n,j} = \{s \in I\!N |\; 2^n(j-1) + 1 \leq s \leq 2^n j\}, \; j \in I\!N, \; n \in Z\!\!\!Z_+. \qquad (4.3)$$

Then

$$\Lambda_{n,j} = \bigcup_{s \in \Lambda_{l,j}} \Lambda_{n-l,s}, \; l = 1, 2, \ldots, n. \qquad (4.4)$$

The interaction between the oscillators located at $j \in I\!N$ and $j' \in I\!N$ depends on the number of the hierarchy level on which both these points are located in the same subset $\Lambda_{n,s}$, that is on $n(j,j') = \min\{n \mid j \in \Lambda_{n,s} \text{ and } j' \in \Lambda_{n,s}\}$. It appears as a part of the local Hamiltonian $H_{\Lambda_{n,j}}$ which describes the dynamics of the particles located in $\Lambda_{n,j}$. The family of all such operators $\{H_{\Lambda_{n,j}}, n \in Z\!\!\!Z_+, j \in I\!N\}$ is defined recursively by means of (4.4):

$$H_{\Lambda_{n,j}} = -\frac{1}{2}(2^\delta - 1)2^{-n(1+\delta)}(\sum_{s \in \Lambda_{n,j}} q_s)^2 + \sum_{s \in \Lambda_{1,j}} H_{\Lambda_{n-1,s}}, \qquad (4.5)$$

where δ was introduced in (3.6), and the starting element $H_{\Lambda_{0,j}}$ is given by (4.1). Each $H_{\Lambda_{n,j}}$ acts in the space $\mathcal{H}^{\rm osc}_{\Lambda_{n,j}}$ which is the tensor product of copies of $\mathcal{H}^{\rm osc}$ indexed by $s \in \Lambda_{n,j}$. The Gibbs state of the particles located in $\Lambda_{n,j}$ may be described by means of temperature Green functions which are defined as follows. For $0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_l$, $l \in I\!N$, we set

$$\Gamma^{\beta,\Lambda_{n,j}}_{A_1,\ldots,A_l}(\tau_1,\ldots,\tau_l) = \frac{1}{Z_{\Lambda_{n,j}}} \quad (4.6)$$
$$\mathrm{trace}\left\{A_1 \exp(-(\tau_2-\tau_1)H_{\Lambda_{n,j}})\ldots A_l \exp(-(\beta-\tau_l+\tau_1)H_{\Lambda_{n,j}})\right\},$$

where

$$Z_{\Lambda_{n,j}} = \mathrm{trace}\left\{\exp(-\beta H_{\Lambda_{n,j}})\right\},$$

all $A_i,\ i = 1,\ldots,l$ are densely defined in $\mathcal{H}^{\rm osc}_{\Lambda_{n,j}}$, and such that the trace in (4.6) exists. Put

$$A_i = A_{\Lambda_{n,j}} = 2^{-\frac{1}{2}n(1+\delta)} \sum_{s\in\Lambda_{n,j}} q_s,\ \forall i. \quad (4.7)$$

Note that $\Lambda_{n,j}$ consists of 2^n points, which means that the normalization of such sums is abnormal (comparing with that used in central limit theorems for weakly dependent random elements) due to the presence of the factor $2^{-n\delta/2}$ with $\delta > 0$. Therefore, if the dependence between the displacements q_s is weak, then one may expect that the sequence $\{A_{\Lambda_{n,j}}, n \in I\!N\}$ is degenerated at zero. The latter means that the sequences of Green functions defined by (4.6) with such A_i converge to zero for almost all values of their variables and for all $l \in I\!N$. To simplify the notations, we denote

$$\Gamma^{(n)}_{2l}(\tau_1,\ldots,\tau_{2l}) = \Gamma^{\beta,\Lambda_{n,j}}_{A_1,\ldots,A_{2l}}(\tau_1,\ldots,\tau_{2l}), \quad (4.8)$$

where A_i are given by (4.7). The following assertion establishes the relationship between the operator and functional integral approaches in quantum statistical mechanics. It directly follows from [1] and was used e.g. in [2].

Proposition 4.1 *Let the sequence of functions $\{F_n, n \in Z\!\!\!Z_+\}$ be set by Definition 3.1. Then for all $\varphi \in L^2(I_\beta)$*

$$F_n(\varphi) = \sum_{l=1}^{\infty} \frac{1}{2l!} \int_{I_\beta^{2l}} \Gamma^{(n)}_{2l}(\tau_1,\ldots,\tau_{2l})\varphi(\tau_1)\ldots\varphi(\tau_{2l})d\tau_1\ldots d\tau_{2l}. \quad (4.9)$$

In other words, the functions introduced in (3.11) and those defined by (4.6) – (4.8) are the same.

Now let us describe the possibilities for the models we consider to belong to the families introduced above. This will be done by a number of lemmas which are mainly proved in our paper [2]. The proof is based on the lattice approximation method (we do not give here the technical details which can be found e.g. in [13]).

Lemma 4.1 *Let $\hat{U}_n(0)$ be defined by (3.14). Then $\hat{U}_0(0)$ as a function of β possesses the properties:*
(i) *it is continuously differentiable on* $(0, +\infty)$*;*
(ii) $\lim_{\beta\to 0} \hat{U}_0(0) = 0$*;*
(iii) *for every positive* β*,*

$$\hat{U}_0(0) \geq \phi(\beta) := \beta < q^2 > f\left(\frac{\beta}{4m < q^2 >}\right), \tag{4.10}$$

where the function f is defined [6] implicitly by the relation

$$f(y \tanh y) = \frac{1}{y} \tanh y, \quad y \in (0, +\infty).$$

Lemma 4.2 *Let $a < 0$, and ϕ be defined above, then*

$$\lim_{\beta\to\infty} \phi(\beta) \geq U_{\min} := \frac{4ma^2}{9b^2}.$$

Corollary 4.1 *Let the parameters m, a, b meet the estimate*

$$U_{\min} > \bar{u}; \quad a < 0, \tag{4.11}$$

then the condition (iii) *of Definition 3.2 is satisfied. Moreover, $\beta_0^- < \beta_0^+$.*

Lemma 4.3 *Let*

$$b < \frac{\sqrt{2}}{3}\bar{u}(\bar{u} - 1), \tag{4.12}$$

then the condition (3.21) is satisfied.

For $b \geq 0$, the operator H (4.1) is such that H^{-1} is compact. Then there exists a complete set of its eigenfunctions $\{\psi_s\}$, hence we may set $H\psi_s = \epsilon_s\psi_s$, $q_{ss'} = (\psi_s, q\psi_{s'})_{\mathcal{H}^{\rm osc}}$, and

$$\Delta_H = \min\{\epsilon_s - \epsilon_{s-1}, \ s \in I\!N\}. \tag{4.13}$$

Lemma 4.4 *Let*

$$\frac{1}{m\Delta_H^2} < 1, \tag{4.14}$$

then the condition (iii) *of Definition 3.3 is satisfied.*

Set

$$< q^2 >_{\text{class}} = \frac{\int_{I\!R} q^2 \exp(-V(q))dq}{\int_{I\!R} \exp(-V(q)dq}, \tag{4.15}$$

where V is given by (4.1) with $b > 0$ and $a \in I\!R$.

Lemma 4.5 *Let*

$$\beta > \frac{\bar{u}}{< q^2 >_{\text{class}}}, \tag{4.16}$$

then there exists $m(\beta)$ *such that for* $m \geq m(\beta)$*, the condition* $\hat{U}_0(0) = \bar{u}$ *is satisfied.*

Lemma 4.6 *For given* $b > 0$ *and* $a \in I\!R$*, there exists* $m_-(a, b)$ *such that for* $m < m_-(a, b)$*, the condition* (iii) *of Definition 3.3 is satisfied.*

Let us discuss the consequences of these lemmas having in mind the description of the families $\mathcal{M}_+$, $\mathcal{M}_-$. Thus in order to belong to $\mathcal{M}_+$, the model must be "weakly" anharmonic, that is b must be small enough (see (4.12)). For δ close to $\frac{1}{2}$, $\bar{u}$ is close to one, then the anharmonicity parameter b ought to be close to zero. Consider the case of negative a, and b obeying the restriction (4.12). Then the function V has two minima (wells) located at the points $\pm\hat{q}$, where $\hat{q} = \sqrt{-a/b}$. Their depth is $\hat{v} = -V(\hat{q}) = a^2/4b$. Combining (4.11) and (4.12), we obtain the following sufficient condition to belong to the family $\mathcal{M}_+$:

$$m\hat{v} > \frac{3\sqrt{2}}{16}\bar{u}^2(\bar{u} - 1). \tag{4.17}$$

Therefore, even for small values of the oscillator's mass, one can place the corresponding model into $\mathcal{M}_+$ by setting the wells of V to be deep enough. For the case of arbitrary negative a, we can place the model into the family $\mathcal{M}_+$ by letting the oscillator's mass to be large enough as it is prescribed by Lemma 4.5. On the other hand, such a model can be placed into the family $\mathcal{M}_-$ by letting this mass to be small enough (Lemma 4.6).

5 Proof of Lemmas

First we describe some properties of the sequences of functions considered.

Lemma 5.1 *For arbitrary $n \in I\!N$,*

$$\hat{U}_n(0) \leq \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0); \tag{5.1}$$

$$\begin{aligned} 0 &\leq \hat{U}_n(k) \leq \kappa(\hat{U}_{n-1}(k))\hat{U}_{n-1}(k) \\ &+ (1-2^{-\delta})(\kappa(\hat{U}_{n-1}(k))^3 2^{2\delta-1}|X_{n-1}|, \quad k \neq 0; \end{aligned} \tag{5.2}$$

$$\hat{U}_n(0) \geq \kappa(\hat{U}_{n-1}(0))\hat{U}_{n-1}(0) + \frac{1}{2}(1-2^{-\delta})(\kappa(\hat{U}_{n-1}(0)))^3 2^{2\delta-1}X_{n-1}; \tag{5.3}$$

$$2^{2\delta-1}(\kappa(\hat{U}_{n-1}(0)))^4 X_{n-1} \leq X_n < 0; \tag{5.4}$$

$$0 \leq \hat{U}_n(k) \leq \hat{U}_n(0), \tag{5.5}$$

where $\kappa(u)$ and X_n are defined by (3.17) and (3.15) respectively.

The proof of this Lemma was done in our paper [2] by means of methods derived in [9] within the lattice approximation approach (see e.g. [13]).

Proposition 5.1 *For every $n \in Z\!\!\!Z_+$ and $k \in \mathcal{K}$, $\hat{U}_n(k)$ is a continuously differentiable function of β.*

The proof of this Proposition can be done recursively on the base of Lemma 4.1. The following inductive lemma is the main tool used below.

Lemma 5.2 *Let the model belong to the family $\mathcal{M}_+$, and $\mathcal{I}_n$, $n \in Z\!\!\!Z_+$, be the triple of statements (i_n^1, i_n^2, i_n^3)*

$$\begin{aligned} i_n^1 &= \{\exists \beta_n^+ \in [\beta_0^-, \beta_0^+] \ : \ \hat{U}_n(0) = \bar{u} \text{ for } \beta = \beta_n^+; \ \hat{U}_n(0) < \bar{u} \ \forall \beta < \beta_n^+\}; \\ i_n^2 &= \{\exists \beta_n^- \in [\beta_0^-, \beta_0^+] \ : \ \hat{U}_n(0) = 1 \text{ for } \beta = \beta_n^-; \ \hat{U}_n(0) < 1 \ \ \forall \beta < \beta_n^-\}; \\ i_n^3 &= \{\forall \beta \in (0, \beta_n^+) \ : \ |X_n| < \bar{w}\}. \end{aligned}$$

Then
(i) $\mathcal{I}_0$ is true;
(ii) $\mathcal{I}_{n-1}$ implies $\mathcal{I}_n$.

Proof. $\mathcal{I}_0$ is true in view of Definition 3.2 and Lemma 4.1 statements (i) and (ii). Let us prove the implication. We set $\kappa(\hat{U}_n(0)) = \kappa_n$, then for $\beta = \beta_n^+$, $\kappa_n = \bar{\kappa}$ and $\kappa_n < \bar{\kappa}$ for $\beta < \beta_n^+$. Remind that $\bar{\kappa} = \kappa(\bar{u})$ is defined by (3.17) and obeys the estimate (3.18). Let $\beta = \beta_{n-1}^+$, then (3.19), (5.3), and the statement i_{n-1}^3 yield

$$\begin{aligned} \hat{U}_n(0) &\geq \bar{\kappa}\bar{u} + \frac{1}{2}(1 - 2^{-\delta})\bar{\kappa}^3 2^{2\delta-1} X_{n-1} \\ &> \bar{\kappa}\bar{u}[1 - \sqrt{2}(1 - 2^{-\delta})\bar{\kappa}^2 2^{2\delta-1}\bar{w}] > \bar{u}. \end{aligned} \tag{5.6}$$

For $\beta = \beta_{n-1}^-$, the estimate (5.1) gives

$$\hat{U}_{n-1}(0) \leq 1. \tag{5.7}$$

Taking into account the continuity of $\hat{U}_n(0)$ as a function of β (arising from Proposition 5.1), the estimates (5.6) and (5.7), one obtains that there exists at least one value $\tilde{\beta}_n^+ \in (\beta_{n-1}^-, \beta_{n-1}^+)$ such that $\hat{U}_n(0) = \bar{u}$. The smallest such $\tilde{\beta}_n^+$ is denoted by β_n^+. That is, $\beta_n^+ = \inf \tilde{\beta}_n^+$. The mentioned continuity of $\hat{U}_n(0)$ yields $\hat{U}_n(0) < \bar{u}$ for $\beta < \beta_n^+$. This implies that the statement i_n^1 is true. Similarly, we can prove the existence of $\beta_n^- \in [\beta_{n-1}^-, \beta_{n-1}^+)$ by putting $\beta_n^- = \inf\{\beta \in [\beta_{n-1}^- < \beta_{n-1}^+) \mid \hat{U}_n(0) = 1\}$. Thus the statement i_n^2 is also true. For $\beta < \beta_n^+$, we have $\kappa_n < \bar{\kappa}$ which yields

$$|X_n| < 2^{2\delta-1}\bar{\kappa}^4 |X_{n-1}| \leq |X_{n-1}| < \bar{w}, \tag{5.8}$$

that is i_n^3 is true. The proof is concluded with the remark that

$$[\beta_n^-, \beta_n^+] \subset [\beta_{n-1}^-, \beta_{n-1}^+] \subset [\beta_0^-, \beta_0^+]. \tag{5.9}$$

■

Lemma 5.3 *Let Δ_H be defined by (4.13), then for all β and $k \in \mathcal{K}$:*

$$\hat{U}_0(k) \leq \frac{1}{m(\Delta_H^2 + k^2)}. \tag{5.10}$$

Proof. For the operator H (4.1), we have Proposition 4.1. Then for $U_2^{(0)}(0,\tau)$, one obtains from (4.9)

$$U_2^{(0)}(0,\tau) = \frac{1}{Z}\sum_{s,s'} (q_{ss'})^2 \exp(-\tau(\epsilon_s - \epsilon_{s'}) - \beta\epsilon_{s'}).$$

Inserting this expression into the definition of $\hat{U}_0(k)$ (3.14), one gets

$$\hat{U}_0(k) = \frac{1}{Z}\sum_{s,s'}(q_{ss'})^2 \frac{(\epsilon_s - \epsilon_{s'})(e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s})}{(\epsilon_s - \epsilon_{s'})^2 + k^2}. \tag{5.11}$$

Taking into account that $q_{ss} = 0$ (due to Z_2 - symmetry of H), one can omit the case $s = s'$ when summing in (5.11). Then estimating $(\epsilon_s - \epsilon_{s'})^2$ with $s \neq s'$ by Δ_H^2 (4.13), one arrives at

$$\begin{aligned}\hat{U}_0(k) &\leq \frac{1}{\Delta_H^2 + k^2}\frac{1}{Z}\sum_{s,s'}(q_{ss'})^2(\epsilon_s - \epsilon_{s'})(e^{-\beta\epsilon_{s'}} - e^{-\beta\epsilon_s}) \\ &= \frac{1}{\Delta_H^2 + k^2}\frac{1}{Z}\mathrm{trace}([q,[H,q]]e^{-\beta H}) \\ &= \frac{1}{m(\Delta_H^2 + k^2)}. \end{aligned} \tag{5.12}$$

Here $[.,.]$ stands for the commutator. ■

Proof of Lemma 4.4. It follows directly from the estimate (5.10) with $k = 0$.

Proof of Lemma 3.1. Let some $u \in (1, \bar{u})$ be chosen and let D_n denote the closure of the set $\{\beta \in [\beta_n^-, \beta_n^+] \mid 1 \leq \hat{U}_n(0) \leq u\}$. It is nonempty: $\beta_n^- \in D_n$. For chosen u, we define $\varepsilon > 0$ by putting $\kappa(u) = 2^\varepsilon$. Then this ε obeys the restriction

$$\varepsilon < \bar{\varepsilon} := \min\{\frac{1}{6}\ ,\ \frac{1-2\delta}{4}\}, \tag{5.13}$$

which follows from (3.16), (3.18), and from the monotonicity of $\kappa(u)$ given by (3.17). Then the set

$$D_* = \bigcap_{n \in \mathbb{Z}_+} D_n, \tag{5.14}$$

is nonempty and closed. Thus for $\beta \in D_*$, we have for all $n \in \mathbb{Z}_+$;

$$1 \leq \hat{U}_n(0) \leq u\ ; \tag{5.15}$$

$$\kappa(\hat{U}_n(0)) \leq 2^\varepsilon. \tag{5.16}$$

Applying the estimate (5.4), one gets

$$|X_n| \leq 2^{(2\delta - 1 + 4\varepsilon)n}\bar{w}, \tag{5.17}$$

with $2\delta - 1 + 4\varepsilon$ being negative (see (5.13)). This yields

$$X_n \to 0, \quad \forall \beta \in D_*. \tag{5.18}$$

Now we prove that $\hat{U}_n(0) \to 1$ for $n \to \infty$ and $\beta \in D_*$. Rewrite the estimate (5.3) as follows:

$$\hat{U}_n(0) - 1 \geq \chi_{n-1}(\hat{U}_{n-1}(0) - 1) + (2^\delta - 1)\kappa_{n-1}T_{n-1}, \tag{5.19}$$

where

$$\chi_{n-1} = 1 + \frac{1 - 2^{-\delta}}{1 - (1 - 2^{-\delta})\hat{U}_{n-1}(0)};$$

and

$$T_{n-1} = (\hat{U}_{n-1}(0) - 1)^2 + \frac{1}{2}2^{\delta-1}\kappa_{n-1}^2 X_{n-1}. \tag{5.20}$$

Then $\chi_{n-1} > 2^\delta$ for $\beta \in D_*$, which is caused by the fact $\hat{U}_n(0) \geq 1$. For such β, let us prove that $T_n \leq 0$, $\forall n \in \mathbb{Z}_+$. Assume that for some $p \in \mathbb{N}$, we had $T_{p-1} > 0$, then (5.19) would yield

$$\hat{U}_p(0) - 1 > \chi_{p-1}(\hat{U}_{p-1}(0) - 1) \geq 2^\delta(\hat{U}_{p-1}(0) - 1).$$

In this case, one obtains from (5.20)

$$\begin{aligned}
T_p &= (\hat{U}_p(0) - 1)^2 + \frac{1}{2}2^{\delta-1}\kappa_p^2 X_p \\
&> 2^{2\delta}(\hat{U}_{p-1}(0) - 1)^2 + \frac{1}{2}2^{\delta-1}2^{2\varepsilon}2^{2\delta-1+4\varepsilon}X_{p-1} \\
&= 2^{2\delta}\left[(\hat{U}_{p-1}(0) - 1)^2 + \frac{1}{2}2^{\delta-1}2^{6\varepsilon-1}X_{p-1}\right] \\
&> 2^{2\delta}\left[(\hat{U}_{p-1}(0) - 1)^2 + \frac{1}{2}2^{\delta-1}X_{p-1}\right] \\
&\geq 2^{2\delta}\left[(\hat{U}_{p-1}(0) - 1)^2 + \frac{1}{2}2^{\delta-1}\kappa_{p-1}^2 X_{p-1}\right] \\
&= 2^{2\delta}T_{p-1}.
\end{aligned}$$

To derive these estimates, we have used: (5.15), (5.4), (5.13). Therefore, assuming the positivity of T_{p-1}, we have obtained the same property of T_n for all $n \geq p$, as well as $T_n \geq 2^{2n\delta}T_{p-1}$ that being used in (5.19) would yield $\hat{U}_{n+1}(0) - 1 > 2^{2n\delta}T_{p-1}(2^\delta - 1)\kappa_p$. The latter runs counter to the boundedness of the sequence $\{\hat{U}_n(0)\}$ (5.15). Thus $T_n \leq 0$ for all $n \in I\!N$. Taking this into account as well as the estimate (5.17), we obtain from (5.20):

$$\hat{U}_n(0) - 1 = O(2^{n(2\delta-1+4\varepsilon)/2}). \tag{5.21}$$

Set $\beta_* = \inf D_*$, then for $\beta = \beta_* : \hat{U}_n(0) \to 1$, $X_n \to 0$, which proves the first part of the lemma.

For $\beta < \beta_*$, there exists $p \in I\!N$ such that (5.15) does not occur. There are two possibilities of its failure: (i) $\hat{U}_p(0) < 1$; (ii) $\hat{U}_p(0) > u$. For chosen β, (5.9) implies $\beta < \beta_n^+ \ \ \forall n \in I\!N$. Thus the second possibility is contradictory to the statement i_p^1 of Lemma 5.2. Hence it remains only $\hat{U}_p(0) < 1$ which gives $\kappa_p < 1$, and then the estimate (5.1) yields

$$\hat{U}_{p+1}(0) < \hat{U}_p(0) < 1.$$

Repeating these arguments we conclude that the bounded sequence $\{\hat{U}_n(0),\ n \geq p\}$ is monotone and hence convergent to some limit $\hat{U} < 1$. Note that $\kappa_n < 1$, thus (5.1) may be used to get

$$\hat{U}_{n+1}(0) \leq \kappa(\hat{U}_n(0))\hat{U}_n(0) < \hat{U}_n(0).$$

The left and the right hand parts of this double inequality converge to the same limit $\hat{U}$, then the middle part ought to converge to this limit too. Therefore,

$$\hat{U} = \kappa(\hat{U})\hat{U},$$

that has only one solution: $\hat{U} = 0$, which is the limit of the sequence $\{\hat{U}_n(0)\}$ for $\beta < \beta_*$. As for the sequence $\{X_n\}$, its convergence to zero for $\beta < \beta_*$ has been established above by the estimate (5.17). For Y_n given by (3.26), Lemma 3.3 implies $|Y_n| \leq |X_n|$ which yields $Y_n \to 0$ for $\beta \leq \beta_*$. ■

Proof of Lemma 3.2. For every model from the family $\mathcal{M}_-$, we have $\hat{U}_n(0) < 1$ for all positive β. Therefore, the sequence $\{\hat{U}_n(0), n \in Z\!\!Z_+\}$ is monotone and hence convergent to zero similarly to the case of $\beta < \beta_*$ just considered. The latter yields the same convergence of the sequence $\{Y_n\}$, as it follows from Lemma 3.6. ■

Few words ought to be said about the proof of Lemma 3.3. All inequalities (3.27) – (3.29) are proven within the lattice approximation method mentioned earlier by proving similar estimates for classical systems of unbounded (φ^4 – like) spins. The estimate corresponding to (3.27) is proven in [2] . The sign rule (3.28) is proven by means of the Ising approximation of φ^4-spins (see e.g. [13]) on the base of Shlosman's results for the Ising model

[12]. The estimate (3.29) corresponds to the Griffiths' inequalities valid for φ^4-spin systems.

Proof of Lemma 3.4. For $\beta < \beta_*$, we have proven $\hat{U}_n(0) \to 0$ when $n \to \infty$. Thus the estimate (5.5) implies the asserted convergence to hold for such β. The proof for the case $\beta = \beta_*$ will be done as follows. First we prove that for $\beta = \beta_*$, there exists a finite subset of $\mathcal{K}$, $\mathcal{K}^{\rm one}$ such that for $k \in \mathcal{K}^{\rm one}$, $\hat{U}_n(k) \to 1$ when $n \to \infty$, and $\hat{U}_n(k) \to 0$ for $k \in \mathcal{K}^{\rm zero} := \mathcal{K} \backslash \mathcal{K}^{\rm one}$. Then we show that $\mathcal{K}^{\rm one}$ consists of only one point: $\mathcal{K}^{\rm one} = \{0\}$.

Let $\alpha = -(2\delta - 1 + 4\varepsilon) > 0$, and

$$v_n = 1 - 2^{-n\alpha} 2^{\delta} \bar{w}, \quad n \in \mathbb{Z}_+. \tag{5.22}$$

Then all v_n are positive (see (3.19) and $v_n \to 1$ when $n \to \infty$. We set

$$\begin{aligned} \mathcal{K}_n &= \{k \in \mathcal{K} |\ \hat{U}_n(k) \le v_n\}; \\ \bar{\mathcal{K}}_n = \mathcal{K} \setminus \mathcal{K}_n &= \{k \in \mathcal{K} |\ \hat{U}_n(k) > v_n\}. \end{aligned} \tag{5.23}$$

Let us prove that $\mathcal{K}_{n-1} \subset \mathcal{K}_n$ for all $n \in \mathbb{N}$. Applying the estimate (5.2), we obtain

$$\begin{aligned} &\hat{U}_n(k) - \hat{U}_{n-1}(k) \le -\frac{1 - 2^{-\delta}}{1 - (1 - 2^{-\delta})\hat{U}_{n-1}(k)} \\ &\left\{\hat{U}_{n-1}(k)(1 - \hat{U}_{n-1}(k)) - \kappa(\hat{U}_{n-1}(k))^2 2^{\delta - 1} 2^{-\alpha(n-1)} \bar{w}\right\}. \end{aligned} \tag{5.24}$$

Let $k \in \mathcal{K}_{n-1}$, then $\kappa(\hat{U}_{n-1}(k)) < 1$ and

$$\begin{aligned} & \hat{U}_{n-1}(k)(1 - \hat{U}_{n-1}(k)) - \kappa(\hat{U}_{n-1}(k))^2 2^{\delta-1} 2^{-\alpha(n-1)} \bar{w} \\ > \ & \hat{U}_{n-1}(k)(1 - \hat{U}_{n-1}(k)) - 2^{-\alpha(n-1)} 2^{\delta-1} \bar{w} \\ := \ & p_{n-1}(\hat{U}_{n-1}(k)). \end{aligned} \tag{5.25}$$

The polynomial $p_{n-1}(v)$ has two roots

$$v_{\pm} = \frac{1}{2}\left\{1 \pm \sqrt{1 - 2^{-\alpha(n-1)} 2^{\delta+1} \bar{w}}\right\}. \tag{5.26}$$

Due to (3.19), both roots are real, and $v_+ > v_{n-1}$ (see (5.22). Let $\hat{U}_{n-1}(k) \le v_-$, then for all $n \in \mathbb{N}$, $\hat{U}_{n-1}(k) < 2^{\delta}\bar{w} < 1$, which yields $\kappa(\hat{U}_{n-1}(k)) < 1$. Applying these estimates in (5.2), we get $\forall n \in \mathbb{N}$

$$\begin{aligned} \hat{U}_n(k) &< 2^{\delta}\bar{w} + 2^{2\delta-1}(1 - 2^{-\delta})\bar{w} \\ &= 2^{\delta-1}(1 + 2^{\delta})\bar{w} \le 1 - 2^{\delta}\bar{w} < v_n, \end{aligned}$$

where we have used $\bar{w} \leq 2^{1-\delta}(3+2^{\delta})^{-1}$, which follows from (3.19). Thus in this case $k \in \mathcal{K}_n$. Now let us consider the case $v_- < \hat{U}_{n-1}(k) < v_+$. Then $p_{n-1}(\hat{U}_{n-1}(k))$ in (5.25) is positive, yielding (by (5.24)) $\hat{U}_n(k) < \hat{U}_{n-1}(k) \leq v_{n-1} < v_n$. Hence $k \in \mathcal{K}_n$ also in this case. Having in mind that $v_+ > v_{n-1}$, we conclude that all possible values of $\hat{U}_{n-1}(k)$, for $k \in \mathcal{K}_{n-1}$, have been taken into account. Then $k \in \mathcal{K}_{n-1}$ implies $k \in \mathcal{K}_n$, i.e. $\mathcal{K}_{n-1} \subset \mathcal{K}_n$, for all $n \in I\!N$.

Set

$$\mathcal{K}^{\mathrm{zero}} = \bigcup_{n \in \mathbb{Z}_+} \mathcal{K}_n \; ; \qquad \mathcal{K}^{\mathrm{one}} = \mathcal{K} \setminus \mathcal{K}^{(0)} = \bigcap_{n \in \mathbb{Z}+} \bar{\mathcal{K}}_n. \tag{5.27}$$

Then:

$$\begin{aligned} &\text{(i)} \;\; \mathcal{K}^{\mathrm{one}} \;\; \text{is finite;} \\ &\text{(ii)} \;\; \hat{U}_n(k) \to 1 \;, \qquad \forall k \in \mathcal{K}^{\mathrm{one}}; \\ &\text{(iii)} \;\; \hat{U}_n(k) \to 0 \;, \qquad \forall k \in \mathcal{K}^{\mathrm{zero}}. \end{aligned}$$

To prove (i), we note that $\bar{w} < 2^{-1-\delta}$, hence $v_0 \geq 1/2$. Then taking into account (5.10), we get

$$\begin{aligned} \mathcal{K}_0 \;\; &\supset \;\; \{k \in \mathcal{K} |\; k^2 \geq \frac{2}{m} - \Delta_H^2\}, \\ \bar{\mathcal{K}}_0 \;\; &\subset \;\; \{k \in \mathcal{K} |\; k^2 < \frac{2}{m} - \Delta_H^2\}. \end{aligned} \tag{5.28}$$

The latter yields that $\bar{\mathcal{K}}_0$ is finite, but $\mathcal{K}_0 \subset \mathcal{K}_1 \subset \ldots \subset \mathcal{K}_n \subset \ldots \subset \mathcal{K}^{\mathrm{zero}}$ implies $\mathcal{K}^{\mathrm{one}} \subset \bar{\mathcal{K}}_0$. This means $\mathcal{K}^{\mathrm{one}}$ is finite as well. To prove (ii), we remark that accordingly to the definition of $\bar{\mathcal{K}}_n$ and due to (5.5), we have for $k \in \bar{\mathcal{K}}_n$:

$$v_n < \hat{U}_n(k) \leq \hat{U}_n(0).$$

The latter implies that the value $k = 0$ belongs to $\mathcal{K}^{\mathrm{one}}$. For $\beta = \beta_*$, we had $\hat{U}_n(0) \to 1$. But the sequence $\{v_n\}$ also converges to unity. This proves (ii). To prove (iii), we let $k \in \mathcal{K}^{\mathrm{zero}}$. Therefore, there exists $n_0 \in I\!N$ such that $k \in \mathcal{K}_{n-1}$ for all $n > n_0$. Then either $\hat{U}_{n-1}(k) \leq v_- < 2^{-\alpha(n-1)}2^{\delta}\bar{w}$, or $v_- < \hat{U}_{n-1}(k) < v_+$. The latter yields $\hat{U}_n(k) < \hat{U}_{n-1}(k)$. The validity of the former assumption leads directly to the convergence we are proving. The validity of the latter one means that the sequence $\{\hat{U}_n(k), \; n > n_0\}$ is monotone and bounded from below by zero. This yields in turn that such a sequence converges to zero similarly to the case of $\beta < \beta_*$ where the convergence of the sequence $\{\hat{U}_n(0)\}$ to zero has been proved.

The proof of the lemma is concluded by showing that $\mathcal{K}^{(1)} = \{0\}$. Up to now, we have proved that the sequences $\{\hat{U}_n(k) | k \in \mathcal{K}\}$ converge to the characteristic function of the set $\mathcal{K}^{\rm one}$. Thus the sequence of functions $\{U_2^{(n)}(\tau, \tau') | \tau, \tau' \in I_{\beta_*}\}$ converges in $L^2(I_{\beta_*}^2)$ to

$$U_2(\tau, \tau') = \frac{1}{\beta_*} \left\{ 1 + \sum_{k \in \mathcal{K}^{\rm one} \setminus \{0\}} \cos k(\tau - \tau') \right\}, \tag{5.29}$$

with finite $\mathcal{K}^{\rm one}$. Therefore, there exists a subsequence $\{U_2^{(n_j)}(\tau, \tau')\}$ converging to this $U_2(\tau, \tau')$ almost everywhere on $I_{\beta_*}^2$. Thus the sign rule (3.28) yields in this case that $U_2(\tau, \tau')$ ought to be nonnegative also almost everywhere on $I_{\beta_*}^2$. It is not difficult to show that the only case where this nonnegativity is preserved is just $\mathcal{K}^{\rm one} = \{0\}$ (see (5.29). ■

Proof of Lemma 3.5 . Within the lattice approximation approach on the base of the Lieb - Sokal theorem [10], we can show that for all $n \in \mathbb{Z}_+$, the function $f_n(z) := F_n(z\xi_1)$ is different from zero on the following set $\{z \in \mathbb{C} \,|\, {\rm Re} z \neq 0\}$. This function is entire, even, and of the order of growth which is less then two. Then it should possess purely imaginary zeros, hence its Hadamard's representation ought to be exactly the same as (3.32) ■

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