Existence and Stability of Propagating Fronts for an Autocatalytic Reaction-Diffusion System

S. Focant

Physique Théorique
Université Catholique de Louvain
2, ch. du cyclotron
B-1348 Louvain-la-Neuve, Belgique

Th. Gallay

Analyse Numérique et EDP
Université de Paris-Sud
Bâtiment 425
F-91405 Orsay, France

Abstract. We study a one-dimensional reaction-diffusion system which describes an isothermal autocatalytic chemical reaction involving both a quadratic ($A + B \to 2B$) and a cubic ($A + 2B \to 3B$) autocatalysis. The parameters of this system are the ratio $D = D_B/D_A$ of the diffusion constants of the reactant A and the autocatalyst B, and the relative activity k of the cubic reaction. First, for all values of $D > 0$ and $k \geq 0$, we prove the existence of a family of propagating fronts (or travelling waves) describing the advance of the reaction. In particular, in the quadratic case $k = 0$, we recover the results of Billingham and Needham [BN]. Then, if D is close to 1 and k is sufficiently small, we prove using energy functionals that these propagating fronts are stable against small perturbations in exponentially weighted Sobolev spaces. This extends to our system part of the stability results which are known for the scalar Fisher equation.

1. Introduction

We consider the reaction-diffusion system

$$\begin{aligned} \partial_t u(x,t) &= \partial_x^2 u(x,t) - u(x,t)v(x,t) - ku(x,t)v(x,t)^2 \ , \\ \partial_t v(x,t) &= D\partial_x^2 v(x,t) + u(x,t)v(x,t) + ku(x,t)v(x,t)^2 \ , \end{aligned} \tag{1.1}$$

where u, v are nonnegative functions of $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, and $D > 0$, $k \geq 0$ are constant parameters. This system describes (in dimensionless variables) an isothermal autocatalytic chemical reaction of mixed order, involving both a quadratic ($A{+}B \to 2B$) and a cubic ($A + 2B \to 3B$) autocatalysis, see [HPSS], [BN]. Here, u and v are the concentrations of the reactant A and the autocatalyst B, and $D = D_B/D_A$ is the ratio of the diffusion constants. The parameter k measures the contribution of the cubic autocatalysis to the whole reaction. Of particular interest are the purely quadratic case $k = 0$, and the purely cubic case "$k = +\infty$" which corresponds, after a rescaling, to Eq.(1.1) with $k = 1$ and without quadratic terms.

The dynamics of the system (1.1) on a bounded domain $\Omega \subset \mathbb{R}$ with homogeneous boundary conditions is well understood [Ma], [HY]. For any initial data $u_0, v_0 \in \mathrm{L}^\infty(\Omega)$, the solution $(u(t), v(t)) \equiv (u(\cdot,t), v(\cdot,t)) \in \mathrm{L}^\infty(\Omega)^2$ is uniformly bounded for all times and converges as $t \to +\infty$ to a uniform steady state (u^*, v^*) satisfying $u^*v^* = 0$. On the other hand, if $u_0, v_0 \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$, the solution $(u(t), v(t))$ of the system (1.1) on the whole real line stays bounded for all times [BKX] and converges uniformly to zero as $t \to +\infty$. In the purely cubic case, a very detailed description of this convergence can be found in [BX], [BKX]. Finally, if $u_0, v_0 \in \mathrm{L}^\infty(\mathbb{R})$ only, then the solution $(u(t), v(t))$ exists for all times, and stays uniformly bounded if $D \leq 1$ [MP]; if $D > 1$, uniform boundedness is an open problem, but an upper bound is known which diverges extremely slowly as $t \to +\infty$ [CX]. Of course, very little is known about the behavior of the solutions in this general situation.

In this paper, we investigate the existence and stability of propagating fronts (or travelling waves) for the system (1.1). These are uniformly translating solutions connecting the stable steady state $(u, v) = (0, 1)$ at $x = -\infty$ to the unstable state $(u, v) = (1, 0)$ at $x = +\infty$. Thus we look for solutions of (1.1) of the form $u(x,t) = \alpha(x - ct)$, $v(x,t) = \beta(x - ct)$, where $c > 0$ is the velocity of the front. The nonnegative functions α, β satisfy the system

$$\begin{aligned} \alpha''(x) + c\alpha'(x) - \alpha\beta - k\alpha\beta^2 &= 0 \ , \\ D\beta''(x) + c\beta'(x) + \alpha\beta + k\alpha\beta^2 &= 0 \ , \end{aligned} \tag{1.2}$$

together with the boundary conditions

$$(\alpha(-\infty), \beta(-\infty)) \;=\; (0,1)\ , \quad (\alpha(+\infty), \beta(+\infty)) \;=\; (1,0)\ . \tag{1.3}$$

Remark. It is easy to verify that the nonnegative time-independent solutions of (1.1) are exactly the uniform steady states $(u,v) = (a,b)$ with $a \geq 0$, $b \geq 0$ satisfying $ab = 0$. Moreover, a necessary condition for the existence of a trajectory of (1.2) connecting (a_-, b_-) at $x = -\infty$ to (a_+, b_+) at $x = +\infty$ is that $a_- + b_- = a_+ + b_+$. Indeed, adding the two equations in (1.2) and integrating with respect to x, we obtain the conservation law

$$\alpha'(x) + c\alpha(x) + D\beta'(x) + c\beta(x) \;=\; \text{const.}\ , \tag{1.4}$$

and the assertion follows by taking the limits $x \to \pm\infty$. Therefore, any heteroclinic orbit of (1.2) must connect $(0,a)$ to $(a,0)$ for some $a > 0$. Now, the system (1.1) is invariant under the scaling transformation

$$u(x,t) \to \lambda^2 u(\lambda x, \lambda^2 t)\ , \quad v(x,t) \to \lambda^2 v(\lambda x, \lambda^2 t)\ , \quad k \to k/\lambda^2\ , \tag{1.5}$$

for all $\lambda > 0$, so there is no loss of generality in assuming that $a = 1$. This explains the choice of the boundary conditions (1.3).

Existence of solutions to (1.2), (1.3) has been studied by Billingham and Needham [BN] for all $D > 0$ in the cases $k = 0$ and $k = +\infty$. First, they show that any solution satisfies the following properties for all $x \in \mathbb{R}$:

$$\begin{array}{ll} 0 < \alpha(x) < 1\ , & \alpha'(x) > 0\ , \\ 0 < \beta(x) < 1\ , & \beta'(x) < 0\ , \end{array} \quad \text{and} \quad \begin{cases} \alpha(x) + \beta(x) \leq 1 & \text{if } D \leq 1, \\ \alpha(x) + \beta(x) \geq 1 & \text{if } D \geq 1. \end{cases} \tag{1.6}$$

Then, in the purely quadratic case, they prove that a travelling wave exists if and only if $c \geq 2\sqrt{D}$, and is unique up to a translation in the variable x. Finally, in the purely cubic case, they argue that a propagating front exists if and only if $c \geq v_2^*(D)$, where $v_2^*(D)$ is some increasing function of D satisfying $v_2^*(1) = 1/\sqrt{2}$, $v_2^*(D) = \mathcal{O}(D)$ as $D \to 0$, and $v_2^*(D) = \mathcal{O}(\sqrt{D})$ as $D \to +\infty$. In this latter case, their argument relies in part on numerical calculations.

On the other hand, if $D = 1$, the existence of travelling waves has been proved for all $k \geq 0$. Indeed, it follows from (1.6) that $\alpha(x) + \beta(x) = 1$ in this case, so that β satisfies the single equation

$$\beta''(x) + c\beta'(x) + \beta(x)(1 - \beta(x))(1 + k\beta(x)) \;=\; 0\ , \tag{1.7}$$

together with the boundary conditions $\beta(-\infty) = 1$, $\beta(+\infty) = 0$. This problem can be studied by usual phase space techniques, and is known to have a nonnegative solution if and only if $c \geq c^*(k)$, where

$$c^*(k) \;=\; \begin{cases} 2 & \text{if } k \leq 2 \;, \\ \sqrt{k/2} + \sqrt{2/k} & \text{if } k > 2 \;, \end{cases} \tag{1.8}$$

see [BBDKL], [vS]. If $k \leq 2$, the front with minimal speed is often called "pulled" or "linear", because its velocity c^* and its decay rate at infinity can be determined from the linearized equation ahead of the front. In the converse case, it is called "pushed" or "nonlinear".

In the general case $D > 0$, $k \geq 0$, the situation seems more complicated, and our results are still incomplete. If $D > 1$, we can still prove the existence of a minimal propagation speed for the travelling waves:

Theorem 1.1. *Let $D \geq 1$, $k \geq 0$. Then there exists $c^* = c^*(D,k) > 0$ such that Eqs.(1.2), (1.3) have a nonnegative solution if and only if $c \geq c^*$. This solution is unique up to a translation and satisfies (1.6). Moreover, $c^*(D,k)$ is a non-decreasing function of k and satisfies*

$$\left.\begin{array}{lcc} k \leq 2 & : & 2\sqrt{D} \\ k > 2 & : & \sqrt{D}\left(\sqrt{k/2} + \sqrt{2/k}\right) \end{array}\right\} \;\leq\; c^*(D,k) \;\leq\; \left\{\begin{array}{lcl} 2\sqrt{D} & : & k \leq \frac{3D-1}{3D-2} \\ \bar{c}(D,k) & : & k > \frac{3D-1}{3D-2} \end{array}\right. \tag{1.9}$$

where $\bar{c}(D,k)$ is given by Eq.(2.13) below. In particular, $\bar{c}(D,k) \leq \sqrt{D}(\sqrt{k} + \sqrt{1/k})$ for all $D \geq 1$ and all $k > (3D-1)/(3D-2)$.

An immediate consequence of Theorem 1.1 is:

Corollary 1.2. *Let $D \geq 1$. There exists $k^* = k^*(D)$ such that the minimal speed $c^*(D,k)$ defined in Theorem 1.1 satisfies $c^* = 2\sqrt{D}$ if $k < k^*(D)$ and $c^* > 2\sqrt{D}$ if $k > k^*(D)$. Moreover, one has*

$$\frac{3D-1}{3D-2} \;\leq\; k^*(D) \;\leq\; 2 \;,$$

for all $D \geq 1$.

This result says that the front with minimal speed is "linear" if $k < k^*$ and "non-linear" if $k > k^*$. A numerical determination of the curve $k^*(D)$ is shown in Fig. 1 below.

On the other hand, if $D < 1$, we do not know whether the set of values of c for which a propagating front exists is always an interval of the form $[c^*, \infty)$. However, for all $D < 1$ and all $k \geq 0$, we do know that a nonnegative solution to Eqs.(1.2), (1.3) exists if c is sufficiently large and does not exist if $c < 2\sqrt{D}$:

Theorem 1.3. *Let $0 < D < 1$, $k \geq 0$. Then Eqs.(1.2), (1.3) have a nonnegative solution if*

$$c \geq \begin{cases} 2\sqrt{D} & \text{if } k \leq 2 \;, \\ \sqrt{D}(\sqrt{k/2} + \sqrt{2/k}) & \text{if } k > 2 \;. \end{cases}$$

This solution is unique up to a translation and satisfies (1.6). Conversely, there exists no nonnegative solution of Eqs.(1.2), (1.3) if $c < 2\sqrt{D}$.

Remark. In particular, Theorem 1.1 and Theorem 1.3 show that, if $D \geq 1$ and $k \leq (3D-1)/(3D-2)$, or if $D < 1$ and $k \leq 2$, a nonnegative propagating front exists if and only if $c \geq 2\sqrt{D}$. This extends the result obtained by Billingham and Needham [BN] for $k = 0$.

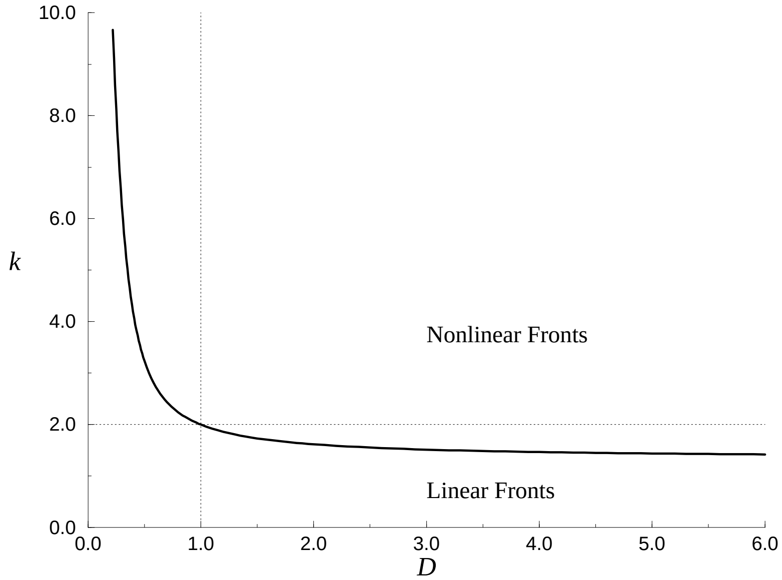

Fig. 1: The curve $k^*(D)$ separating the parameter regions where $c^*(D, k) = 2\sqrt{D}$ (linear fronts) and $c^*(D, k) > 2\sqrt{D}$ (nonlinear fronts). The existence of this curve is asserted by Corollary 1.2 for $D \geq 1$ only, but numerically this separatrix can be observed for all $D > 0$. Note that $k^*(D) > 2$ if $0 < D < 1$, in agreement with Theorem 1.3.

It follows from Theorem 1.1 and Theorem 1.3 that, for all $D > 0$ and all $k \geq 0$, the system (1.1) has a one-parameter family of uniformly translating front solutions, indexed by the velocity c. A natural question to address is whether these propagating fronts are stable against sufficiently small perturbations in appropriate function spaces. Again, the case $D = 1$ is easier and can be treated separately. Indeed, if $D = 1$, the system (1.1) can be written as

$$\begin{aligned} \partial_t w(x,t) &= \partial_x^2 w(x,t) , \\ \partial_t v(x,t) &= \partial_x^2 v(x,t) + v(x,t)(w(x,t) - v(x,t))(1 + kv(x,t)) , \end{aligned} \tag{1.10}$$

where $w(x,t) = u(x,t) + v(x,t)$. In these new variables, the propagating fronts are given by $w(x,t) = 1$, $v(x,t) = \beta(x - ct)$, where β is the solution of (1.7). Therefore, if we consider initial data of the form $w_0 = 1 + f$, $v_0 = \beta + g$, where f, g are sufficiently localized perturbations, then the solution of (1.10) will satisfy $w(x,t) = 1 + \mathcal{O}(t^{-1/2})$ as $t \to +\infty$, so that the behavior of $v(x,t)$ for large times will be governed by the nonlinear diffusion equation

$$\partial_t v(x,t) = \partial_x^2 v(x,t) + v(x,t)(1 - v(x,t))(1 + kv(x,t)) , \tag{1.11}$$

up to a remainder which can be controlled rigorously [Fo1]. Now, the stability of the travelling wave solutions of Eq.(1.11) has been intensively studied by many authors [AW], [Sa], [Ki], [EW], [BK], [Ga2]. In particular, for all $k \ge 0$ and all $c \ge c^*(k)$, each individual front is known to be asymptotically stable against perturbations which decay to zero sufficiently fast — at least as fast as the front itself — as $x \to +\infty$. The decay rate in time of the perturbations is polynomial or exponential depending on the choice of the function space. In addition, if $k > 2$ and $c = c^*$ (the "pushed" case), the family of all translates of the front is orbitally stable against perturbations which decay even *slower* than the front itself as $x \to +\infty$.

In the general case $D \neq 1$, the reduction to a single equation is no longer possible, and much less is known about the stability of propagating fronts. We shall restrict ourselves in the sequel to the situation where D is close to 1 and k is close to 0, see the remarks after Theorem 1.4 below for a discussion of these limitations. In this parameter region, we know from Theorem 1.1, Theorem 1.3 that a propagating front (α, β) exists if and only if $c \ge 2\sqrt{D}$. Since $\alpha(x) \to 0$ and $\beta(x) \to 1$ as $x \to -\infty$, we shall assume (without loss of generality) that $\beta(x) - \alpha(x) \ge 3/4$ for all $x \le 0$. Setting

$$u(x,t) = \alpha(x - ct) + f(x - ct, t) , \quad v(x,t) = \beta(x - ct) + g(x - ct, t) ,$$

and inserting into (1.1), we obtain the evolution equations for the perturbation (f,g) in the moving frame

$$\begin{aligned} \partial_t f &= \partial_x^2 f + c\partial_x f - \beta(1+k\beta)f - \alpha(1+2k\beta)g - N(f,g) \;, \\ \partial_t g &= D\partial_x^2 g + c\partial_x g + \alpha(1+2k\beta)g + \beta(1+k\beta)f + N(f,g) \;, \end{aligned} \tag{1.12}$$

where $N(f,g) = fg + k(2\beta fg + \alpha g^2 + fg^2)$.

As in the scalar case, it is necessary to use weighted spaces which force the perturbations (f,g) to decay to zero sufficiently fast as $x \to +\infty$ [Sa]. For any $s > 0$, we consider the Hilbert spaces X_s, Y_s of real functions on $\mathbb{R}$ defined by the norms

$$\|h\|_{X_s}^2 = \int_{\mathbb{R}} |h(x)|^2 (1 + e^{2sx})\, dx \;, \quad \|h\|_{Y_s}^2 = \|h\|_{X_s}^2 + \|h'\|_{X_s}^2 \;, \tag{1.13}$$

where $'$ denotes the (space) derivative. We also note $X_s^2 = X_s \times X_s$, $Y_s^2 = Y_s \times Y_s$. Then a direct calculation shows that the origin $(f,g) = (0,0)$ in (1.12) is linearly stable in X_s^2 only if $Ds^2 - cs + 1 \le 0$, see [Ga1], [GR] for a similar discussion. Therefore, we must choose $s = 1/\sqrt{D}$ if $c = 2\sqrt{D}$. If $c > 2\sqrt{D}$, s can be chosen in a whole interval, but the biggest perturbation space corresponds to the choice

$$s = \frac{1}{2D}(c - \sqrt{c^2 - 4D}) \;, \tag{1.14}$$

which we shall always assume in the sequel. Note that this value corresponds to the exponential decay rate of both $\alpha(x)$, $\beta(x)$ as $x \to +\infty$.

With these definitions, we can state our main result:

Theorem 1.4. *There exist $d_0 > 0$ and $k_0 > 0$ such that, for all $D \in [1-d_0, 1+d_0]$, all $k \in [0,k_0]$ and all $c \ge 2\sqrt{D}$, there exist $\epsilon_0 > 0$ and $K_0 \ge 1$ such that the following holds: for all $(f_0,g_0) \in Y_s^2$ satisfying $\|(f_0,g_0)\|_{Y_s^2} \le \epsilon_0$, where s is given by (1.14), Eq.(1.12) has a unique global solution $(f,g) \in \mathcal{C}^0([0,+\infty), Y_s^2) \cap \mathcal{C}^1((0,+\infty), Y_s^2)$ satisfying $(f(0),g(0)) = (f_0,g_0)$. Moreover, one has*

$$\|(f(t),g(t))\|_{Y_s^2} \le K_0 \|(f_0,g_0)\|_{Y_s^2} \;, \tag{1.15}$$

for all $t \ge 0$, and

$$\lim_{t\to+\infty} \|(\partial_x f(t), \partial_x g(t))\|_{X_s^2} = 0 \;. \tag{1.16}$$

Remarks.

a) In particular, Eqs.(1.15), (1.16) imply that the perturbation $(f(t), g(t))$ of the front converges to zero uniformly in the following sense

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \bigl(|f(x,t)| + |g(x,t)|\bigr)\,(1+e^{sx}) \;=\; 0\;.$$

b) In addition to (1.16), the proof will show that

$$\int_0^\infty \|(\partial_x f(t), \partial_x g(t))\|_{X_s^2}^2\, dt \;<\; \infty\;,$$

but Theorem 1.4 does not give any pointwise decay rate in time for the perturbations. However, our proof could be extended to provide some (non-optimal) decay rate at the expense of using higher order energy functionals, see for example [FS]. On the other hand, a detailed study of the linearized equation (1.12) (with $N(f,g) = 0$) including a careful determination of the optimal decay rate in this case can be found in [Fo2].

c) A important open problem is whether the result of Theorem 1.4 can be extended to other values of the parameters D, k, using possibly different perturbation spaces. The difficulty we encounter when D is far from 1 is related to the "Turing phenomenon" in the theory of pattern formation [Tu], which shows how a stable equilibrium point of a reaction system can be destabilized by diffusion if the components in the system have very different diffusion rates. The question is therefore whether this mechanism actually leads to instabilities in our system if either $D \ll 1$ or $D \gg 1$. On the other hand, the difficulty we have when $k \gg 1$ is more technical in nature: the spectral analysis of the linearized operator in (1.12) becomes difficult when k is large, because we have to preclude the existence of unstable eigenvalues. This problem already exists when $D = 1$, but in this case the reduction to a single equation allows one to use some results from the theory of Schrödinger operators which do not extend to systems. Another interesting question is therefore whether Theorem 1.4 holds true for all $k \ge 0$, at least if D is close to 1, and whether the orbital stability result for the family of translates of the "pushed front", which is known for $D = 1$, $k > 2$, $c = c^*(k)$, has any analogue in the general case $D \ne 1$.

The rest of this paper is organized as follows. In section 2, we prove the existence of propagating fronts (Theorem 1.1 and Theorem 1.3) using phase space techniques. In section 3, we construct energy functionals which allow us to show that these front solutions are stable against sufficiently small perturbations in Y_s^2 (Theorem 1.4).

2. Existence of Propagating Fronts

In this section, we study the existence of nonnegative solutions to the system (1.2) satisfying the boundary conditions (1.3). Our method, which follows closely Billingham and Needham [BN], relies on the construction of invariant regions in a three-dimensional phase space. When $D < 1$, $k \geq 0$, we show that a propagating front exists if c is large enough, and does not exist if c is too small, thus proving Theorem 1.3. A similar result holds when $D > 1$, but in addition we can show that the existence of a propagating front for $c > 0$, $k \geq 0$ implies the existence for all $c' \geq c$, $k' \leq k$, thus proving Theorem 1.1.

We first note that any solution of (1.2), (1.3) corresponds to a heteroclinic orbit of a *three-dimensional* dynamical system, see [BN]. Indeed, setting $\beta' = w$ and using the conservation law (1.4) to eliminate α', we see that α, β, w satisfy the equations

$$\alpha' = c(1-\alpha-\beta) - Dw \ , \quad \beta' = w \ , \quad w' = -\frac{1}{D}(cw + \alpha\beta(1+k\beta)) \ , \tag{2.1}$$

together with the boundary conditions $(\alpha, \beta, w)(-\infty) = P_1 = (0,1,0)$, $(\alpha, \beta, w)(+\infty) = P_2 = (1,0,0)$.

Straightforward calculations show that the fixed point P_1 is a *saddle* for the dynamical system (2.1), with one positive and two negative eigenvalues given by

$$\lambda_\pm = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 1 + k} \ , \quad \lambda_0 = -\frac{c}{D} \ . \tag{2.2}$$

The eigenvector corresponding to $\lambda_+ > 0$ is

$$v_+ = \left(\frac{c + D\lambda_+}{c + \lambda_+} \ , \ -1 \ , \ -\lambda_+ \right) \ . \tag{2.3}$$

On the other hand, the fixed point P_2 is a *sink*, with three negative eigenvalues given by

$$\mu_\pm = \frac{1}{2D}(-c \pm \sqrt{c^2 - 4D}) \ , \quad \mu_0 = -c \ . \tag{2.4}$$

Note that $\mu_\pm > \mu_0$ if $D > 1/2$ and $1 + (D-1)c^2 > 0$. The eigenvectors corresponding to $\mu_\pm$, μ_0 are respectively

$$w_\pm = \left(-\frac{c + D\mu_\pm}{c + \mu_\pm} \ , \ 1 \ , \ \mu_\pm \right) \ , \quad w_0 = (1,0,0) \ . \tag{2.5}$$

Let $\mathcal{V}$ be the one-dimensional global unstable manifold of P_1, and let $\mathcal{V}_+ \subset \mathcal{V}$ be the invariant manifold which coincides with $\mathcal{V} \cap \mathcal{Q}_+$ in a small neighborhood of P_1, where

$\mathcal{Q}_+ = \{(\alpha,\beta,w)\,|\,\alpha > 0\,,\ \beta > 0\}$. In view of the preceding remarks, proving the existence of a nonnegative propagating front amounts to showing that $\mathcal{V}_+ \subset \bar{\mathcal{Q}}_+ \cap \Omega(P_2)$, where $\Omega(P_2)$ is the basin of attraction of P_2. Following [BN], we shall prove this, for some values of the parameters D, c, k, by constructing an invariant region $\dot{\mathcal{R}} \subset \bar{\mathcal{Q}}_+ \cap \Omega(P_2)$ such that $\mathcal{V}_+ \subset \dot{\mathcal{R}}$. Since everything is known when $D = 1$, we shall only consider the cases $0 < D < 1$ and $D > 1$.

2.1. The Case $D < 1$

Let $D \in (0,1)$. Given $A > 0$, we define the (closed) region $\mathcal{R}_1 \subset \bar{\mathcal{Q}}_+$ by

$$\mathcal{R}_1 \;=\; \big\{(\alpha,\beta,w)\,|\,0 \le \beta \le 1\;,\ 0 \le \alpha \le 1-\beta\;,\ -A\beta(1-\beta) \le w \le 0\big\}\;. \tag{2.6}$$

We also note $\dot{\mathcal{R}}_1 = \mathcal{R}_1 \setminus \{P_1\}$.

Lemma 2.1. *Assume that the parameters $D < 1$, $c > 0$, $k \ge 0$, $A > 0$ satisfy*

$$DA^2 - cA + 1 \;\le\; 0\;, \quad k - 2DA^2 \;\le\; 0\;. \tag{2.7}$$

Then $\mathcal{R}_1$ is invariant under the flow of (2.1), and $\mathcal{V}_+ \subset \dot{\mathcal{R}}_1 \subset \Omega(P_2)$.

Proof. To prove the invariance of $\mathcal{R}_1$, we show that the vector field (2.1) at any point $P \in \partial\mathcal{R}_1$ is directed into $\mathcal{R}_1$ or is parallel to the surface of $\mathcal{R}_1$. This is easy to verify for the faces $\alpha = 0$, $\alpha = 1-\beta$, and $w = 0$. On the last face $w = -A\beta(1-\beta)$, we have

$$\begin{aligned} w' + A\beta'(1-2\beta) \;&=\; w\left(\frac{-c}{D} + A(1-2\beta)\right) - \frac{1}{D}\alpha\beta(1+k\beta) \\ &=\; A\beta(1-\beta)\left(\frac{c}{D} - A(1-2\beta)\right) - \frac{1}{D}\alpha\beta(1+k\beta) \\ &\ge\; \frac{1}{D}\beta(1-\beta)\left(-(DA^2 - cA + 1) + \beta(2DA^2 - k)\right)\;, \end{aligned} \tag{2.8}$$

since $\alpha \le 1-\beta$. Obviously, the right-hand side of (2.8) is nonnegative if (2.7) holds, hence $\mathcal{R}_1$ is invariant under the flow of (2.1).

Due to this invariance, to prove that $\mathcal{V}_+ \subset \mathcal{R}_1$ it suffices to verify the inclusion in a small neighborhood $\mathcal{N}$ of P_1. Since $\mathcal{V}_+ \cap \mathcal{N} \subset \{P_1 + \epsilon v_+ + \mathcal{O}(\epsilon^2)\,|\,0 < \epsilon < \epsilon_0\}$ for some $\epsilon_0 > 0$, it follows from (2.3), (2.6) that $\mathcal{V}_+ \cap \mathcal{N} \subset \mathcal{R}_1$ if

$$\frac{c + D\lambda_+}{c + \lambda_+} \;<\; 1\;, \quad \text{and} \quad \lambda_+ \;<\; A\;. \tag{2.9}$$

The first inequality is always satisfied since $D < 1$, $\lambda_+ > 0$, and the second one follows from (2.2), (2.7). Indeed, adding the two inequalities (2.7), we obtain $DA^2 + cA - (1 + k) \geq 0$, while $\lambda_+^2 + c\lambda_+ - (1+k) = 0$ by (2.2). Since $D < 1$, this implies $A > \lambda_+$. Therefore, we have shown that $\mathcal{V}_+ \subset \mathcal{R}_1$, hence $\mathcal{V}_+ \subset \dot{\mathcal{R}}_1$ since $P_1 \notin \mathcal{V}_+$.

Finally, let $P \in \dot{\mathcal{R}}_1$, and let $\gamma(x) = (\alpha, \beta, w)(x)$ be the solution of (2.1) satisfying $\gamma(0) = P$. Since $\gamma(x) \in \mathcal{R}_1$ for all $x \geq 0$, we have $\alpha'(x) \geq 0$, $\beta'(x) \leq 0$ for all $x \geq 0$, hence $\alpha(x) \to \tilde{\alpha} \in [0,1]$, $\beta(x) \to \tilde{\beta} \in [0,1)$ as $x \to +\infty$ (note that $\tilde{\beta} \leq \beta(0) < 1$.) From (2.1), we deduce that $w(x) \to \tilde{w} = -c^{-1}\tilde{\alpha}\tilde{\beta}(1 + k\tilde{\beta})$ as $x \to +\infty$. Now, the fixed point $(\tilde{\alpha}, \tilde{\beta}, \tilde{w})$ has to satisfy $\tilde{w} = 0$, $\tilde{\alpha}\tilde{\beta} = 0$ and $\tilde{\alpha} + \tilde{\beta} = 1$, hence $\tilde{\alpha} = 1$, $\tilde{\beta} = 0$. This proves that $\gamma(x) \to P_2$ as $x \to +\infty$, hence $\dot{\mathcal{R}}_1 \subset \Omega(P_2)$. $\square$

Proof of Theorem 1.3. From Lemma 2.1, we know that a nonnegative propagating front exists if the conditions (2.7) can be satisfied for some $A > 0$. This front is unique up to a translation in x since the unstable manifold $\mathcal{V}_+$ is one-dimensional, and the properties (1.6) follow from (2.6) or can be verified directly as in [BN]. Now, if $k \leq 2$, we choose $A = 1/\sqrt{D}$, and (2.7) holds for all $c \geq 2\sqrt{D}$. If $k > 2$, we choose $A = \sqrt{k/(2D)}$, and (2.7) holds for all $c \geq \sqrt{D}(\sqrt{k/2} + \sqrt{2/k})$. This proves the first part of the result.

Conversely, if $c < 2\sqrt{D}$, the eigenvalues $\mu_\pm$ in (2.4) become complex, and it is easy to show that no trajectory of (2.1) can stay in $\bar{\mathcal{Q}}_+$ and converge to P_2 as $x \to +\infty$, except on the invariant line $\beta = w = 0$ which does not intersect $\mathcal{V}_+$. Therefore, no nonnegative front solution can exist if $c < 2\sqrt{D}$. This concludes the proof of Theorem 1.3. $\square$

Remark. Following the same lines, one verifies that the region

$$\tilde{\mathcal{R}}_1 = \big\{(\alpha, \beta, w) \,|\, 0 \leq \beta \leq 1 \;,\; (1-\beta)(1 - E\beta) \leq \alpha \leq 1 - \beta \;,\; -A\beta(1-\beta) \leq w \leq 0\big\}$$

satisfies the conclusion of Lemma 2.1 if $E > 0$ and if the additional condition $A(1 - D) - (c - A)E \leq 0$ is fulfilled (see also Lemma 2.2 below.) In particular, if $D > 1/2$, $c \geq 2\sqrt{D}$ and $k \leq 2$, we can choose $A = 1/\sqrt{D}$ and $E = (1-D)/(2D-1)$. This shows that the propagating front satisfies the bound

$$(1-\beta)\left(1 - \frac{1-D}{2D-1}\beta\right) \leq \alpha \leq 1 - \beta \;. \tag{2.10}$$

2.2. The Case $D > 1$: Bounds on the Critical Speed

Let $D > 1$, $A > 0$, $E > 0$. We define the region $\mathcal{R}_2 \subset \bar{\mathcal{Q}}_+$ by

$$\mathcal{R}_2 \;=\; \bigl\{(\alpha,\beta,w)\,|\,0 \le \beta \le 1\;,\; 1-\beta \le \alpha \le (1-\beta)(1+E\beta)\;,\; -A\beta(1-\beta) \le w \le 0\bigr\}\;.$$

We also note $\dot{\mathcal{R}}_2 = \mathcal{R}_2 \setminus \{P_1\}$.

Lemma 2.2. *Assume that the parameters $D > 1$, $c > 0$, $k \ge 0$, $A > 0$, $E > 0$ satisfy*

$$DA^2 - cA + 1 \;\le\; 0\;, \quad E(1+k) + k - 2DA^2 \;\le\; 0\;, \quad A(D-1) - E(c-A) \;\le\; 0\;. \tag{2.11}$$

Then $\mathcal{R}_2$ is invariant under the flow of (2.1), and $\mathcal{V}_+ \subset \dot{\mathcal{R}}_2 \subset \Omega(P_2)$.

Proof. We proceed as in the proof of Lemma 2.1. First, it is easy to verify that the vector field (2.1) is directed into $\mathcal{R}_2$ on the faces $\alpha = 1-\beta$ and $w = 0$. When $w = -A\beta(1-\beta)$, we have as in (2.8)

$$\begin{aligned}
w' + A\beta'(1-2\beta) \;&=\; A\beta(1-\beta)\Bigl(\frac{c}{D} - A(1-2\beta)\Bigr) - \frac{1}{D}\alpha\beta(1+k\beta)\\
&\ge\; \frac{1}{D}\beta(1-\beta)\left(cA - DA^2(1-2\beta) - (1+E\beta)(1+k\beta)\right)\\
&\ge\; \frac{1}{D}\beta(1-\beta)\left(-(DA^2-cA+1) + \beta(2DA^2 - k - E(1+k))\right) \;\ge\; 0\;,
\end{aligned}$$

since $\alpha \le (1-\beta)(1+E\beta)$ and $\beta^2 \le \beta$. Finally, when $\alpha = (1-\beta)(1+E\beta)$, we find

$$\begin{aligned}
\alpha' + \beta'(1-E+2E\beta) \;&=\; -cE\beta(1-\beta) + w(1-D-E+2E\beta)\\
&\le\; \beta(1-\beta)(-cE + A(D-1+E)) \;\le\; 0\;,
\end{aligned}$$

since $-A\beta(1-\beta) \le w \le 0$. This proves that $\mathcal{R}_2$ is invariant under the flow of (2.1).

To show that $\mathcal{V}_+ \subset \dot{\mathcal{R}}_2$, it is sufficient to verify that

$$1 \;<\; \frac{c+D\lambda_+}{c+\lambda_+} \;<\; 1+E\;, \quad \text{and} \quad \lambda_+ < A\;, \qquad (2.12),$$

see (2.9). The first inequality is obvious since $D > 1$. To prove that $\lambda_+ < A$, we multiply the last inequality in (2.11) by A and we add it to the sum of the other two; the result is

$$A^2(1-E) + cA(1+E) - (1+k)(1+E) \;\ge\; 0\;,$$

hence $A^2 + cA - (1+k) > 0$, which implies $A > \lambda_+$. The second inequality in (2.12) follows, since by (2.11) $cE \ge A(D-1+E) > \lambda_+(D-1-E)$. This proves that $\mathcal{V}_+ \subset \dot{\mathcal{R}}_2$.

Finally, if $\gamma(x) = (\alpha, \beta, w)(x)$ is any trajectory of (2.1) in $\dot{\mathcal{R}}_2$, then $\beta'(x) \le 0$ and $\alpha'(x) + D\beta'(x) \le 0$, hence $\beta(x)$ and $\alpha(x)$ converge as $x \to +\infty$. Thus, proceeding as in the case $D < 1$, one shows that $\gamma(x) \to P_2$ as $x \to +\infty$. This proves that $\dot{\mathcal{R}}_2 \subset \Omega(P_2)$. $\square$

Lemma 2.2 ensures the existence of a nonnegative propagating front if the conditions (2.11) can be satisfied for some $A > 0$, $E > 0$. These conditions are equivalent to

$$c \ge DA + 1/A \quad \text{and} \quad c \ge A\left(1 + \frac{D-1}{E}\right) ,$$

for some $E \le (1+k)^{-1}(2DA^2 - k)$. Therefore, given $D > 1$, $k \ge 0$, they can be fulfilled if and only if $c \ge \bar{c}(D,k)$, where

$$\bar{c}(D,k) = \min_{A^2 > k/(2D)} \max\left\{ DA + 1/A \ , \ A\left(1 + \frac{(D-1)(1+k)}{2DA^2 - k}\right)\right\} . \tag{2.13}$$

In particular, setting $A^2 = 1/D$, we see that $\bar{c}(D,k) = 2\sqrt{D}$ if $k \le (3D-1)/(3D-2)$. Similarly, setting $A^2 = k/D$, we obtain $\bar{c}(D,k) \le \sqrt{D}(\sqrt{k} + 1/\sqrt{k})$ for all $D > 1$, $k \ge 1$. Finally, straightforward calculations show that

$$\lim_{D \to 1+} \bar{c}(D,k) = c^*(k) \ , \quad \lim_{D \to +\infty} \frac{\bar{c}(D,k)}{\sqrt{D}} = c^*(2k) ,$$

where $c^*(k)$ is given by (1.8), and that $\lim_{k \to +\infty} \bar{c}(D,k)/\sqrt{k}$ exists for all $D > 1$. This proves the upper bounds in (1.9) for the critical speed $c^*(D,k)$.

Remark. If $k \le (3D-1)/(3D-2)$, the conditions (2.11) are fulfilled for all $c \ge 2\sqrt{D}$ if $A = 1/\sqrt{D}$ and $E = (D-1)/(2D-1)$; this shows that the propagating front satisfies the bound

$$(1-\beta) \le \alpha \le (1-\beta)\left(1 + \frac{D-1}{2D-1}\beta\right) . \tag{2.14}$$

To prove the lower bounds in (1.9), we use a similar argument. First, we verify as in the case $D < 1$ that there exists no nonnegative propagating front if $c < 2\sqrt{D}$. Thus, we assume that $c \ge 2\sqrt{D}$, and we define the region $\mathcal{R}_3 \subset \bar{\mathcal{Q}}_+$ by

$$\mathcal{R}_3 = \left\{(\alpha, \beta, w) \,|\, 0 \le \beta \le 1 \ , \ 1 - \beta \le \alpha \ , \ w \le -B\beta(1-\beta)\right\} , \tag{2.15}$$

for some $B > 0$. Then, if

$$DB^2 \geq 1 \ , \quad DB^2 - cB + 1 > 0 \ , \quad k - 2DB^2 \geq 0 \ , \tag{2.16}$$

the vector field (2.1) on $\partial\mathcal{R}_3$ is always directed into $\mathcal{R}_3$, except on the face $\beta = 0$. Indeed, this is easy to verify for the faces $\beta = 1$ and $\alpha = 1 - \beta$. If $w = -B\beta(1-\beta)$, we have as in (2.8)

$$\begin{aligned} w' + B\beta'(1-2\beta) \ &= \ B\beta(1-\beta)\Big(\frac{c}{D} - B(1-2\beta)\Big) - \frac{1}{D}\alpha\beta(1+k\beta) \\ &\leq \ \frac{1}{D}\beta(1-\beta)\big(-(DB^2 - cB + 1) + \beta(2DB^2 - k)\big) \ \leq \ 0 \ , \end{aligned}$$

since $\alpha \geq 1 - \beta$. In addition, the conditions (2.16) ensure that the unstable manifold $\mathcal{V}_+$ is contained in $\mathcal{R}_3$ in a neighborhood of P_1, namely

$$1 \ < \ \frac{c + D\lambda_+}{c + \lambda_+} \ , \quad \text{and} \quad B \ < \ \lambda_+ \ ,$$

see (2.9). The first inequality is obvious since $D > 1$, and the second one follows by adding the last two inequalities in (2.16) : the result is $DB^2 + cB - (1+k) \leq 0$, hence $B^2 + cB - (1+k) < 0$, which implies $B < \lambda_+$.

According to these results, any trajectory on the unstable manifold $\mathcal{V}_+$ either remains in $\mathcal{R}_3$ for all $x \in \mathbb{R}$ or leaves $\mathcal{R}_3$ (and the positive sector $\bar{\mathcal{Q}}_+$) by crossing the plane $\beta = 0$. Now, the conditions (2.16) also imply that no trajectory of (2.1) can stay in $\mathcal{R}_3$ and converge to P_2 as $x \to +\infty$, except on the invariant line $\beta = w = 0$ (which does not intersect $\mathcal{V}_+$.) Indeed, since the eigenvalues (2.4) satisfy $\mu_0 < \mu_\pm < 0$, any trajectory in $\mathcal{R}_3 \setminus \{\beta = w = 0\}$ converging to P_2 becomes tangent to one of the eigenvectors $w_\pm$ as $x \to +\infty$. In view of (2.5), (2.15), this is possible only if $B \leq |\mu_-|$, in contradiction with the assumptions $DB^2 \geq 1$, $DB^2 - cB + 1 > 0$ which imply $B > |\mu_-|$. Therefore, if (2.16) holds, the invariant manifold $\mathcal{V}_+$ necessarily crosses the plane $\beta = 0$ and no nonnegative propagating front can exist. In particular, if $k > 2$, we set $B = \sqrt{k/(2D)}$, and we conclude from (2.16) that no nonnegative propagating front exists if $c < \sqrt{D}(\sqrt{k/2} + \sqrt{2/k})$. This proves the lower bounds in (1.9) for the critical speed $c^*(D,k)$.

2.3. The case $D > 1$: Existence of the Critical Speed

Let $D > 1$. In this section, we show that the existence of a nonnegative propagating front for some value of the parameters c, k implies the same property for all $c' \geq c$, $k' \leq k$. Thus we fix $c > 0$, $k \geq 0$, and we assume that α, β is a nonnegative solution of (1.2), (1.3). As in [BN], it is easy to verify that $\alpha'(x) > 0$, $\beta'(x) < 0$ and $\alpha(x)+\beta(x) > 1$ for all $x \in \mathbb{R}$. Setting $w(x) = \beta'(x)$ as usual, we consider the bounded region $\mathcal{R}_4 \subset \bar{\mathcal{Q}}_+$ delimited by the following four surfaces:

$$S_1 = \{(\alpha,\beta,w)\,|\,w=0\}\ , \qquad S_2 = \{(\alpha,\beta,w)\,|\,\alpha = 1-\beta\}\ ,$$

$$S_3 = \{(\lambda\alpha(x),\beta(x),w(x))\,|\,x\in\mathbb{R}\,,\ 0\leq\lambda\leq 1\}\ ,$$

$$S_4 = \{(\alpha(x),\beta(x),\mu w(x))\,|\,x\in\mathbb{R}\,,\ 0\leq\mu\leq 1\}\ .$$

We also note $\dot{\mathcal{R}}_4 = \mathcal{R}_4 \setminus \{P_1\}$.

Lemma 2.3. *For all $c' \geq c$, $0 \leq k' \leq k$, the region $\mathcal{R}_4$ above is invariant under the flow of (2.1)', and $\mathcal{V}'_+ \subset \dot{\mathcal{R}}_4 \subset \Omega'(P_2)$.*

Remark. Here and in the sequel, (2.1)' denotes the vector field (2.1) with c, k replaced by c', k', and similarly for $\mathcal{V}'_+$ and $\Omega'(P_2)$.

Proof. We proceed as in the proof of Lemma 2.1 or Lemma 2.2. First, it is straightforward to verify that the vector field (2.1)' is directed into $\mathcal{R}_4$ on the surfaces S_1 and S_2. If $P \in S_3 \cap \partial\mathcal{R}_4$, then $P = (\lambda\alpha(x),\beta(x),w(x))$ for some $x \in \mathbb{R}$, $\lambda \in [0,1]$. Using the equations (2.1) satisfied by α,β,w, it is easy to show that the vector

$$N_3(x,\lambda) \;=\; \big(0\ ,\ D^{-1}\alpha(cw+\alpha\beta(1+k\beta))\ ,\ \alpha w\big)\ ,$$

is normal to S_3 at P and directed outside $\mathcal{R}_4$. On the other hand, for any c', k' the vector field (2.1)' at P is given by

$$V'_3(x,\lambda) \;=\; \big(c'(1-\lambda\alpha-\beta) - Dw\ ,\ w\ ,\ -D^{-1}(c'w+\lambda\alpha\beta(1+k'\beta))\big)\ .$$

Taking the scalar product, we thus obtain

$$N_3(x,\lambda)\cdot V'_3(x,\lambda) \;=\; \frac{\alpha w}{D}\big((c-c')w + (1-\lambda)\alpha\beta + (k-\lambda k')\alpha\beta^2\big) \;\leq\; 0\ ,$$

since $\lambda \leq 1$, $c' \geq c$ and $k' \leq k$, hence the vector field (2.1)' is directed into $\mathcal{R}_4$ on $S_3 \cap \partial\mathcal{R}_4$. Similarly, if $P = (\alpha(x),\beta(x),\mu w(x)) \in S_4$ for some $x \in \mathbb{R}$, $\mu \in [0,1]$, an exterior normal vector at P is given by $N_4(x,\mu) = (w^2\,,\, Dw^2 - cw(1-\alpha-\beta)\,,\, 0)$, and the vector field (2.1)' at P reads

$$V_4'(x,\mu) \;=\; \big(c'(1-\alpha-\beta) - D\mu w\;,\; \mu w\;,\; -D^{-1}(c'\mu w + \alpha\beta(1+k'\beta))\big)\;.$$

Therefore, we have $N_4(x,\mu)\cdot V_4'(x,\mu) = w^2(c' - c\mu)(1-\alpha-\beta) \le 0$, since $\mu \le 1$ and $c' \ge c$. This shows that $\mathcal{R}_4$ is invariant under the flow of (2.1)' for all $c' \ge c$, $k' \le k$.

To prove that $\mathcal{V}_+' \subset \dot{\mathcal{R}}_4$, we may clearly assume that either $c' > c$ or $k' < k$, since $\mathcal{V}_+ \subset \dot{\mathcal{R}}_4$ by construction. Then, as in (2.9), it is sufficient to verify that

$$1 \;<\; \frac{c' + D\lambda_+'}{c' + \lambda_+'} \;<\; \frac{c + D\lambda_+}{c + \lambda_+}\;, \quad \text{and } \lambda_+' \;<\; \lambda_+\;,$$

where $\lambda_+' = \lambda_+(c',k')$ is given by (2.2). The last inequality is satisfied because $\lambda_+(c,k)$ is strictly decreasing in c and increasing in k, and the other relations follow since $D > 1$. Therefore, $\mathcal{V}_+' \subset \dot{\mathcal{R}}_4$ for all $c' \ge c$, $k' \le k$.

Finally, if $\bar\gamma(x) = (\bar\alpha,\bar\beta,\bar w)(x)$ is any trajectory of (2.1)' in $\dot{\mathcal{R}}_4$, then $\bar\beta'(x) \le 0$ and $\bar\alpha'(x) + D\bar\beta'(x) \le 0$, hence $\bar\beta(x)$ and $\bar\alpha(x)$ converge as $x \to +\infty$. Proceeding as in the previous cases, one shows that $\bar\gamma(x) \to P_2$ as $x \to +\infty$. This proves that $\dot{\mathcal{R}}_4 \subset \Omega'(P_2)$. $\square$

Using this result, we are now able to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For any $D > 1$, $k \ge 0$, let $I(D,k)$ be the set of values of $c \ge 2\sqrt{D}$ for which there exists a nonnegative solution of (1.2), (1.3). It is not difficult to verify that this set is closed, hence by Lemma 2.3 $I(D,k) = [c^*, +\infty)$ for some $c^*(D,k) \ge 2\sqrt{D}$. It follows also from Lemma 2.3 that $I(D,k) \subset I(D,k')$ if $k \ge k'$, hence the minimal speed $c^*(D,k)$ is a nondecreasing function of $k \ge 0$. Finally, the upper and lower bounds (1.9) for $c^*(D,k)$ have been established in Section 2.2. This concludes the proof of Theorem 1.1. $\square$

3. Stability of the Propagating Fronts

Throughout this section, we assume that $D > 0$, $k \in [0,1]$, $c \ge 2\sqrt{D}$, and we denote by α, β the solution of (1.2), (1.3) whose existence is ensured by Theorem 1.1 or Theorem 1.3. To prove Theorem 1.4, we shall control the behavior of the solutions of (1.12) in the function space Y_s^2 defined by the norm (1.13), (1.14). We begin with a standard local existence result:

Lemma 3.1. *Let $D > 0$, $k \in [0,1]$, $c \ge 2\sqrt{D}$, and let $(f_0, g_0) \in Y_s^2$. Then there exists a time $t_1 > 0$ such that (1.12) has a unique solution $(f,g) \in \mathcal{C}^0([0,t_1], Y_s^2) \cap \mathcal{C}^1((0,t_1], Y_s^2)$ satisfying $(f(0), g(0)) = (f_0, g_0)$.*

Proof. It is straightforward to verify that the linear operator

$$\mathcal{A} = \begin{pmatrix} \partial_x^2 + c\partial_x - \beta(1+k\beta) & -\alpha(1+2k\beta) \\ \beta(1+k\beta) & D\partial_x^2 + c\partial_x + \alpha(1+2k\beta) \end{pmatrix}$$

is the generator of an analytic semigroup in Y_s^2. Moreover, the nonlinearity $N : Y_s^2 \to Y_s$ in (1.12) is locally Lipschitz, uniformly on any bounded subset of Y_s^2. Therefore, by Theorem 6.3.1 in [Pa], there exists a time $t_1 > 0$ such that Eq.(1.12) has a unique classical solution $(f,g) \in \mathcal{C}^0([0,t_1], Y_s^2) \cap \mathcal{C}^1((0,t_1], Y_s^2)$ with initial data (f_0, g_0). $\square$

Remark. In addition, the proof shows that, for any bounded subset $B \subset Y_s^2$, the existence time $t_1 > 0$ is bounded away from zero uniformly for all $(f_0, g_0) \in B$. It follows that the solution (f,g) either exists for all $t \in \mathbb{R}_+$ or leaves any bounded subset of Y_s^2 in finite time.

In the sequel, we fix $d_0 > 0$, $k_0 > 0$ sufficiently small, and we assume that $D \in [1-d_0, 1+d_0]$, $k \in [0, k_0]$, $c \ge 2\sqrt{D}$. For $\epsilon > 0$ sufficiently small, we make the following assumption:

Hypothesis $\mathcal{H}_\epsilon$: There exists a classical solution (f,g) of (1.12) defined on some time interval $[0,T]$ and satisfying

$$\|(f(t), g(t))\|_{Y_s^2} \le \epsilon ,$$

for all $t \in [0,T]$.

Under this assumption, we shall study the time evolution of some energy functionals which control the size of the solution (f,g) in Y_s^2. In particular, if ϵ, d_0, k_0 are sufficiently small, we shall show that the norm $\|(f,g)\|_{Y_s^2}$ remains bounded on any time interval

$[0, T]$ by a quantity depending only on the initial data. This result will be the main step in the proof of Theorem 1.4. As in [KR], [GR], it is convenient here to split the problem in two parts: First, we shall construct *weighted* functionals, with weight e^{sx}, which control the perturbations (f, g) ahead of the propagating front. Then, we shall introduce *unweighted* functionals to describe the behavior of (f, g) behind the front.

3.1. Weighted Functionals

Let $\rho(x) = e^{-sx}$, where s is given by (1.14). If (f, g) is any solution of (1.12) in Y_s^2, we define the weighted functions F, G, H by

$$F(x,t) \;=\; \rho(x)^{-1} f(x,t) \;, \quad G(x,t) \;=\; \rho(x)^{-1} g(x,t) \;, \quad H \;=\; (1+ds^2)F + G \;, \tag{3.1}$$

where $d = D - 1$. Then $F, G, H \in \mathrm{H}^1(\mathrm{I\!R})$, and a direct calculation shows that G, H satisfy the system

$$\begin{aligned} \partial_t G \;&= (1+d)\partial_x^2 G + \mu \partial_x G - (\hat{\gamma} + \hat{\beta})G + \hat{\beta} H + \rho \tilde{N}(G,H) \;, \\ \partial_t H \;&= \partial_x^2 H + (\mu + 2ds)\partial_x H - (1+ds^2)H + d(\partial_x^2 G - 2s\partial_x G) \\ &\qquad + ds^2 \left((\hat{\gamma} + \hat{\beta})G - \hat{\beta} H - \rho \tilde{N}(G,H) \right) \;, \end{aligned} \tag{3.2}$$

where

$$\mu \;=\; \sqrt{c^2 - 4D} \;, \quad \hat{\beta} \;=\; \frac{\beta(1+k\beta)}{(1+ds^2)} \;, \quad \hat{\gamma} \;=\; 1 - \alpha(1+2k\beta) \;. \tag{3.3}$$

The nonlinearity $\tilde{N}$ in (3.2) is defined by

$$\tilde{N}(G,H) \;=\; \rho^{-2} N\left(\rho \frac{H-G}{1+ds^2}, \rho G \right) \;. \tag{3.4}$$

Remarks.

1. Both functions $\hat{\beta}, \hat{\gamma}$ in (3.3) are close to β if $|d|$ and k are sufficiently small. Indeed, since $s^2 \le 1/D = 1/(1+d)$ by (1.14), we have $1 + ds^2 = 1 + \mathcal{O}(|d|)$, hence

$$\hat{\beta}/\beta \;=\; 1 + \mathcal{O}(|d| + k) \;. \tag{3.5}$$

On the other hand, using the bounds (2.10), (2.14), it is straightforward to verify that

$$(1 - |d| - 2k)\beta \;\le\; \hat{\gamma} \;\le\; \beta \left(1 + \frac{|d|}{1+2d} \right) \;, \tag{3.6}$$

if $d > -1/2$, hence $\hat{\gamma}/\beta = 1 + \mathcal{O}(|d| + k)$. In the sequel, we shall always assume that $|d|$, k are sufficiently small so that $\hat{\beta} > 0$, $\hat{\gamma} > 0$ for all $x \in \mathbb{R}$.

2. In the limit $x \to +\infty$, the equations (3.2) reduce to

$$\begin{aligned} \partial_t G &= (1+d)\partial_x^2 G + \mu \partial_x G \;, \\ \partial_t H &= \partial_x^2 H + (\mu + 2ds)\partial_x H - (1 + ds^2)H + d(\partial_x^2 G - 2s\partial_x G) \;. \end{aligned}$$

For this limiting system, the "energy" $\int (G^2 + H^2)dx$ is non-increasing in time if $d = D-1$ is sufficiently small. Indeed, the diagonal term $-(1 + ds^2)H$ has a good sign, and the only cross term $d(\partial_x^2 G - 2s\partial_x G)$ is a derivative and is multiplied by the small parameter d. This almost diagonal form in the limit $x \to +\infty$ explains our choice of the variables G, H instead of F, G.

3. In the sequel, we shall use the estimate

$$\frac{4}{3}\beta(x) \;\ge\; \begin{cases} \rho(x) & \text{if } x \ge 0 \;, \\ 1 & \text{if } x \le 0 \;, \end{cases} \tag{3.7}$$

which holds for all $c \ge 2\sqrt{D}$ if $|d|$ and k are sufficiently small. To prove (3.7), we first note that $\beta'(x) + s\beta(x) > 0$ for all $x \in \mathbb{R}$. Indeed, using (1.2), (1.14), we see that the function $z = \beta' + s\beta$ satisfies the equation

$$Dz' + (c - Ds)z + (\alpha + k\alpha\beta - 1)\beta \;=\; 0 \;.$$

On the other hand, the bounds (2.10), (2.14) imply that $\alpha + k\alpha\beta - 1 \le \beta(-1 + k + |d|)$, hence $Dz' + (c - Ds)z > 0$ for all $x \in \mathbb{R}$ if $k + |d| < 1$. Since $z(-\infty) = s > 0$, this differential inequality implies that $z(x) = \beta'(x) + s\beta(x) > 0$ for all $x \in \mathbb{R}$, hence $\beta(x)/\rho(x)$ is an increasing function. Now, recall that in the introduction we used the translation invariance of the problem to impose that $\beta(x) - \alpha(x) \ge 3/4$ for all $x \le 0$. In particular, we have $\beta(0) \ge 3/4$, hence $\beta(x)/\rho(x) \ge \beta(0) \ge 3/4$ if $x \ge 0$. Since β is a decreasing function, we also have $\beta(x) \ge \beta(0) \ge 3/4$ for all $x \le 0$. This proves (3.7).

To control the evolution of G, H in $\mathrm{H}^1(\mathbb{R})$, we introduce the energy functionals

$$\begin{aligned} E_0(t) &= \frac{1}{2}\int_{\mathbb{R}} (G^2 + H^2)\, dx \;, \\ E_1(t) &= \frac{1}{2}\int_{\mathbb{R}} \Big(DG'^2 + (\hat{\gamma} + \hat{\beta})G^2 + H'^2 + (1 + ds^2)H^2\Big)\, dx \;, \end{aligned}$$

where $G' = \partial_x G$, $H' = \partial_x H$.

Lemma 3.2. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ above is satisfied, then there exists $K_1 > 0$ such that, for all $t \in (0,T]$,*

$$\dot{E}_0(t) \le -K_1 E_1(t) \; . \tag{3.8}$$

Remarks.

1. The constant K_1 in (3.8) is independent of $d \in [-d_0, d_0]$, $k \in [0, k_0]$, $c \ge 2\sqrt{D}$ and $T > 0$.

2. Here and in the sequel, we denote by $\dot{}$ the time derivative to distinguish it from the space derivative $'$. Unless stated otherwise, all the integrals are taken over the whole real line $\mathbb{R}$.

Proof. Since $(f,g) \in \mathcal{C}^0([0,T], Y_s^2) \cap \mathcal{C}^1((0,T], Y_s^2)$, we have $E_0 \in \mathcal{C}^0([0,T]) \cap \mathcal{C}^1((0,T])$. Using the first equation in (3.2) and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int G^2\,dx &= \int \Big(-(1+d)G'^2 - (\hat{\gamma}+\hat{\beta})G^2 + \hat{\beta}GH + \rho\tilde{N}(G,H)G\Big)\,dx \\ &\le \int \Big(-(1+d)G'^2 - (\hat{\gamma}+\hat{\beta}/2)G^2 + \frac{1}{2}\hat{\beta}H^2 + \rho\tilde{N}(G,H)G\Big)\,dx \; . \end{aligned} \tag{3.9}$$

Similarly, using the second equation in (3.2) and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} H^2\,dx &= \int \big(-H'^2 - (1+ds^2)H^2 - dG'H' - 2dsG'H\big)\,dx \\ &\quad + ds^2\int \Big((\hat{\gamma}+\hat{\beta})GH - \hat{\beta}H^2 - \rho\tilde{N}(G,H)H\Big)\,dx \; . \end{aligned}$$

Since $-dG'H' \le |d|(G'^2 + H'^2)/2$, $-2dsG'H \le |d|s(G'^2 + H^2)$, $ds^2GH \le |d|s^2(G^2 + H^2)/2$, and since $s^2 \le (1+d)^{-1}$, there exists $C_0 > 0$ such that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} H^2\,dx &\le -\int (H'^2 + H^2)\,dx - ds^2\int \rho\tilde{N}(G,H)H\,dx \\ &\quad + C_0|d|\int \Big(H'^2 + H^2 + G'^2 + (\hat{\gamma}+\hat{\beta})(G^2+H^2)\Big)\,dx \; . \end{aligned} \tag{3.10}$$

To bound the nonlinear terms in (3.9), (3.10), we first note that, due to the hypothesis $\mathcal{H}_\epsilon$, there exists $C_1 > 0$ such that

$$\rho(x)|G(x,t)| \le C_1\epsilon\beta(x) \; , \tag{3.11}$$

for all $x \in \mathrm{I\!R}$, $t \in [0,T]$. Indeed, if $x \le 0$, using (3.1), (3.7) and the embedding of $\mathrm{H}^1(\mathrm{I\!R})$ into $\mathrm{L}^\infty(\mathrm{I\!R})$, we find

$$\rho(x)^2|G(x)|^2 \;=\; |g(x)|^2 \;\le\; \|g\|_\infty^2 \;\le\; \|g\|_2\|g'\|_2 \;\le\; \frac{1}{2}\|g\|_{Y_s}^2 \;\le\; \frac{8}{9}\beta(x)^2\|g\|_{Y_s}^2 \;.$$

If $x \ge 0$, we have by (3.7)

$$\rho(x)^2|G(x)|^2 \;\le\; \frac{16}{9}\beta(x)^2|G(x)|^2 \;\le\; \frac{16}{9}\beta(x)^2\|G\|_\infty^2 \;\le\; \frac{16}{9}\beta(x)^2\|G\|_2\|G'\|_2 \;.$$

Since $|G'(x)| \le (s|g(x)| + |g'(x)|)e^{sx}$, we have $\|G\|_2\|G'\|_2 \le C\|g\|_{Y_s}^2$ for some $C > 0$. Therefore, there exists $C_1 > 0$ such that $\rho(x)|G(x)| \le C_1\beta(x)\|g\|_{Y_s}$ for all $x \in \mathrm{I\!R}$, and (3.11) follows from the hypothesis $\mathcal{H}_\epsilon$.

Now, using (3.4), (1.12), we have

$$\int \rho|\tilde{N}(G,H)G|\,dx \;\le\; \int \rho|G|\left(\frac{|H-G||G|}{1+ds^2}(1+2k\beta+k\rho|G|) + k\alpha G^2\right)\,dx \;.$$

Since $|H-G||G| \le (3G^2+H^2)/2$ and $(1+ds^2)^{-1} = 1 + \mathcal{O}(|d|)$, it follows from (3.11) that there exists $C_2 > 0$ such that

$$\begin{aligned}\int \rho|\tilde{N}(G,H)G|\,dx \;&\le\; C_1\epsilon\int\beta\left(\frac{3G^2+H^2}{2(1+ds^2)}(1+2k+kC_1\epsilon) + kG^2\right)\,dx \\ &\le\; C_2\epsilon\int\beta(G^2+H^2)\,dx \;.\end{aligned} \tag{3.12}$$

In a similar way, we obtain

$$\int \rho|\tilde{N}(G,H)H|\,dx \;\le\; C_2\epsilon\int\beta(G^2+H^2)\,dx \;. \tag{3.13}$$

Therefore, combining (3.9), (3.10), (3.12), (3.13), and using (3.5), (3.6), we see that there exists $C_3 > 0$ such that

$$\begin{aligned}\dot{E}_0 \;\le\; &-\int\left(G'^2 + (\hat{\gamma}+\hat{\beta}/2)G^2 + H'^2 + \frac{1}{2}H^2\right)\,dx \\ &+ C_3(|d|+k+\epsilon)\int\left(G'^2 + (\hat{\gamma}+\hat{\beta})G^2 + H'^2 + H^2\right)\,dx \;.\end{aligned}$$

In particular, if $|d|$, k and ϵ are sufficiently small, there exists $K_1 > 0$ (independent of $c \ge 2\sqrt{D}$ and $T > 0$) such that

$$\dot{E}_0 \;\le\; -\frac{K_1}{2}\int\left(DG'^2 + (\hat{\gamma}+\hat{\beta})G^2 + H'^2 + (1+ds^2)H^2\right)\,dx \;=\; -K_1E_1 \;.$$

This concludes the proof of Lemma 3.2. □

Lemma 3.3. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ is satisfied, then there exists $K_2(c) > 0$ such that, for all $t \in (0,T]$,*

$$\dot{E}_1(t) \;\leq\; K_2(c)E_1(t)\ . \tag{3.14}$$

Remark. The constant $K_2(c)$ is independent of $T > 0$, and behaves like c^2 for large values of c, uniformly in $d \in [-d_0, d_0]$ and $k \in [0, k_0]$.

Proof. We start from the identity

$$I_1 \;\equiv\; \frac{1}{2}\frac{d}{dt}\int\Big((1+d)G'^2 + (\hat{\gamma}+\hat{\beta})G^2\Big)\;dx \;=\; \int\Big(-(1+d)G'' + (\hat{\gamma}+\hat{\beta})G\Big)\,\dot{G}\,dx\ .$$

Using (3.2), we replace $-(1+d)G'' + (\hat{\gamma}+\hat{\beta})G$ with $-\dot{G} + \mu G' + \hat{\beta}H + \rho\tilde{N}(G,H)$ in the right-hand side. We obtain

$$I_1 \;=\; \int\Big(-\dot{G}^2 + \mu G'\dot{G} + \hat{\beta}H\dot{G} + \rho\tilde{N}(G,H)\dot{G}\Big)\;dx\ .$$

Since $\mu G'\dot{G} \leq \mu^2 G'^2 + \dot{G}^2/4$ and $\hat{\beta}H\dot{G} \leq \hat{\beta}(H^2+\dot{G}^2)/2$, we have

$$I_1 \;\leq\; \int\Big(-\frac{3}{4}\dot{G}^2 + \mu^2 G'^2 + \frac{1}{2}\hat{\beta}(H^2+\dot{G}^2) + \rho\tilde{N}(G,H)\dot{G}\Big)\;dx\ . \tag{3.15}$$

Similarly, we have the identity

$$I_2 \;\equiv\; \frac{1}{2}\frac{d}{dt}\int\big(H'^2 + (1+ds^2)H^2\big)\;dx \;=\; \int\big(-H'' + (1+ds^2)H\big)\,\dot{H}\,dx\ .$$

Replacing $-H'' + (1+ds^2)H$ in the right-hand side with its expression obtained from the second equation in (3.2), we find

$$\begin{aligned} I_2 \;=\; &\int\Big(-\dot{H}^2 + (\mu+2ds)H'\dot{H} + dG''\dot{H} - 2dsG'\dot{H}\Big)\;dx \\ &+ ds^2\int\Big((\hat{\gamma}+\hat{\beta})G - \hat{\beta}H - \rho\tilde{N}(G,H)\Big)\,\dot{H}\,dx\ . \end{aligned}$$

In view of (3.2), we also have

$$G'' \;=\; \frac{1}{(1+d)}\Big(\dot{G} - \mu G' + (\hat{\gamma}+\hat{\beta})G - \hat{\beta}H - \rho\tilde{N}(G,H)\Big)\ . \tag{3.16}$$

Replacing (3.16) into the expression of I_2, we find

$$\begin{aligned} I_2 \;=& \int \left(-\dot{H}^2 + (\mu + 2ds)H'\dot{H} - 2dsG'\dot{H} + ds^2(\hat{\gamma}+\hat{\beta})G\dot{H} - ds^2\hat{\beta}H\dot{H} \right) \, dx \\ &+ \frac{d}{1+d} \int \left(\dot{G} - \mu G' + (\hat{\gamma}+\hat{\beta})G - \hat{\beta}H \right) \dot{H} \, dx \\ &- \left(\frac{d}{1+d} + ds^2 \right) \int \rho \tilde{N}(G,H)\dot{H} \, dx \;. \end{aligned} \tag{3.17}$$

Since $(\mu + 2ds)H'\dot{H} \le \dot{H}^2/2 + (\mu^2 + 4d^2s^2)H'^2$ and $|\mu G'\dot{H}| \le \mu^2 G'^2/2 + \dot{H}^2/2$, it is easy to verify that there exists $C_4 > 0$ such that

$$\begin{aligned} I_2 \;\le& \int \left(-\frac{1}{2}\dot{H}^2 + \mu^2 H'^2 \right) \, dx + C_4|d| \int \rho |\tilde{N}(G,H)\dot{H}| \, dx \\ &+ C_4|d| \int \left(\dot{H}^2 + H'^2 + H^2 + \dot{G}^2 + (1+\mu^2)G'^2 + (\hat{\gamma}+\hat{\beta})G^2 \right) \, dx \;. \end{aligned} \tag{3.18}$$

To bound the nonlinear terms in (3.15), (3.18), we proceed as in the proof of Lemma 3.2. We obtain

$$\begin{aligned} \int \rho |\tilde{N}(G,H)\dot{G}| \, dx \;&\le\; C_2\epsilon \int \beta (G^2 + \dot{G}^2 + H^2) \, dx \;, \\ \int \rho |\tilde{N}(G,H)\dot{H}| \, dx \;&\le\; C_2\epsilon \int \beta (G^2 + \dot{H}^2 + H^2) \, dx \;. \end{aligned} \tag{3.19}$$

Therefore, combining (3.15), (3.18), (3.19) and using (3.5), (3.6), we see that there exists $C_5 > 0$ such that

$$\begin{aligned} \dot{E}_1 \;\le& \int \left(-\frac{1}{4}\dot{G}^2 - \frac{1}{2}\dot{H}^2 + \mu^2(G'^2 + H'^2) + \frac{1}{2}H^2 \right) \, dx \\ &+ C_5(|d| + k + \epsilon) \int \left(\dot{H}^2 + H'^2 + H^2 + \dot{G}^2 + (1+\mu^2)G'^2 + (\hat{\gamma}+\hat{\beta})G^2 \right) \, dx \;. \end{aligned}$$

In particular, if $|d|$, k and ϵ are sufficiently small, there exists $C_6 > 0$ such that $\dot{E}_1 \le C_6(1+\mu^2)E_1$, hence (3.8) holds with $K_2(c) = C_6(1+\mu^2)$. This concludes the proof of Lemma 3.3. $\square$

3.2. Unweighted Functionals

To control the perturbation (f,g) behind the front, we define $h = f+g$, and we consider the equations satisfied by f,h. From (1.12), we obtain

$$\begin{aligned}\partial_t f &= \partial_x^2 f + c\partial_x f - \delta f - \alpha(1+2k\beta)h - N(f,h-f)\ ,\\ \partial_t h &= D\partial_x^2 h + c\partial_x h - d\partial_x^2 f\ ,\end{aligned} \tag{3.20}$$

where $\delta = \beta - \alpha + k\beta(\beta - 2\alpha)$. As in the weighted case, the variables f,h have been chosen so that the system (3.20) becomes almost diagonal in the limit $x \to -\infty$. To control the evolution of f,h in $\mathrm{H}^1(\mathbb{R})$, we define the functionals

$$\begin{aligned}E_2(t) &= \frac{1}{2}\int_{\mathbb{R}} (f^2 + h^2 + 2\alpha^2 h^2)\,dx\ ,\\ E_3(t) &= \frac{1}{2}\int_{\mathbb{R}} (f'^2 + h'^2) + \frac{K}{2}\int_{\mathbb{R}} (f^2 + 2\alpha^2 h^2)\,dx\ ,\end{aligned}$$

where $K > 6$ is an arbitrary constant.

Before computing the time derivative of E_2, E_3, we note that the additional term $\int \alpha^2 h^2 dx$ in E_2 satisfies

$$\frac{d}{dt}\int \alpha^2 h^2\,dx = -2D\int \alpha^2 h'^2\,dx + 2\int \sigma h^2\,dx - 2d\int \alpha^2 f'' h\,dx\ , \tag{3.21}$$

where $\sigma(x) = (D(\alpha^2)'' - c(\alpha^2)')/2$. In the sequel, we shall use the following two properties of σ:

i) Since $\alpha(x) \sim e^{\lambda_+ x}$ as $x \to -\infty$, where λ_+ is given by (2.2), we have

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sigma(x)}{\alpha^2(x)} &= 2D\lambda_+^2 - c\lambda_+ = (2D+1)\lambda_+^2 - 1 - k\\ &\le (2D+1)(\sqrt{D+1+k} - \sqrt{D})^2 - 1 - k\\ &= -2\sqrt{2}(\sqrt{2}-1)^2 + \mathcal{O}(|d| + k)\ ,\end{aligned}$$

where we used the fact that $\lambda_+ \le \sqrt{D+1+k} - \sqrt{D}$ for all $c \ge 2\sqrt{D}$. Therefore, if $|d|$ and k are sufficiently small, there exists $x_0 \in \mathbb{R}$ such that $\sigma(x) \le -\alpha^2(x)/3$ for all $x \le x_0$. Using the translation invariance of the problem, we may assume (without loss of generality) that $x_0 \ge 0$, hence

$$\sigma(x) \le -\frac{\alpha^2(x)}{3}\ , \quad \text{for all } x \le 0\ . \tag{3.22}$$

This condition is obviously compatible with the previous requirement that $\beta(x)-\alpha(x) \geq 3/4$ when $x \leq 0$.

ii) For all $x \in \mathbb{R}$, we have

$$\sigma(x) \;\leq\; 2D(1+k)\alpha^2(x) \;. \tag{3.23}$$

Indeed, let $z(x) = \alpha'(x) - \sqrt{1+k}\,\alpha(x)$. Then $z(x) \sim e^{\lambda_+ x}(\lambda_+ - \sqrt{1+k})$ as $x \to -\infty$. Since $\lambda_+ < \sqrt{1+k}$ by (2.2), it follows that $z(x) < 0$ if x is sufficiently negative. On the other hand, in view of (1.2), the function z satisfies the differential inequality

$$\begin{aligned} z' + (c+\sqrt{1+k})z \;&=\; \alpha'' + c\alpha' - (1+k)\alpha - c\sqrt{1+k}\,\alpha \\ &=\; -\alpha(1-\beta)(1+k+k\beta) - c\sqrt{1+k}\,\alpha \;<\; 0 \;, \end{aligned}$$

which implies that $z(x)$ stays negative for all $x \in \mathbb{R}$, hence $0 < \alpha'(x) < \sqrt{1+k}\,\alpha(x)$ for all $x \in \mathbb{R}$. Using (1.2) again, we conclude

$$\sigma \;=\; D\alpha'^2 + D\alpha\alpha'' - c\alpha\alpha' \;=\; D\alpha'^2 + D\alpha^2\beta(1+k\beta) - c(1+D)\alpha\alpha' \;\leq\; 2D(1+k)\alpha^2 \;,$$

which proves (3.23).

Now, we shall control the time evolution of E_2 and E_3.

Lemma 3.4. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ is satisfied, then there exist $K_3 > 0$ and $K_4 > 0$ such that, for all $t \in (0,T]$,*

$$\dot{E}_2(t) \;\leq\; -K_3 E_3(t) + K_4 E_1(t) \;. \tag{3.24}$$

Remark. The constants K_3 and K_4 are independent of $d \in [-d_0, d_0]$, $k \in [0, k_0]$, $c \geq 2\sqrt{D}$ and $T > 0$.

Proof. Using (3.20) and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\,dx \;&=\; \int \big(-f'^2 - \delta f^2 - \alpha(1+2k\beta)fh - N(f,h-f)f\big)\;dx \\ &\leq\; \int \Big(-f'^2 - \delta f^2 + (\frac{1}{2} + k\beta)(f^2+\alpha^2h^2) - N(f,h-f)f\Big)\;dx \;. \end{aligned} \tag{3.25}$$

Since $-\delta = \alpha - \beta + k\beta(2\alpha-\beta) \leq \alpha - \beta + k\alpha^2 \leq \alpha - \beta + k$, we have $-\delta \leq 1+k$ for all $x \in \mathbb{R}$ and, due to our choice of the origin, $-\delta \leq -3/4 + k$ for all $x \leq 0$. Therefore,

$$\begin{aligned} \int \Big(\frac{1}{2} + k\beta - \delta\Big) f^2\,dx \;&\leq\; \int_{-\infty}^0 \Big(-\frac{1}{4} + 2k\Big) f^2\,dx + \int_0^\infty \Big(\frac{3}{2} + 2k\Big) f^2\,dx \\ &=\; \int_{\mathbb{R}} \Big(-\frac{1}{4} + 2k\Big) f^2\,dx + \frac{7}{4}\int_0^\infty f^2\,dx \;. \end{aligned} \tag{3.26}$$

To bound the nonlinear term in (3.25), we observe that

$$\begin{aligned}|N(f,h-f)f| &= \left|f^2(h-f)(1+2k\beta)+k\alpha f(h-f)^2+kf^2(h-f)^2\right| \\ &\le \|h-f\|_\infty \left(f^2(1+2k+k\|h-f\|_\infty)+k\alpha|f(h-f)|\right) .\end{aligned}$$

Since $\alpha|f(h-f)| \le \alpha f^2 + (f^2+\alpha^2h^2)/2$ and since $\|h-f\|_\infty = \|g\|_\infty \le \|g\|_{Y_s} \le \epsilon$ by the hypothesis $\mathcal{H}_\epsilon$, there exists $C_1 > 0$ such that

$$\int |N(f,h-f)f|\,dx \;\le\; C_1\epsilon \int (f^2+\alpha^2h^2)\,dx\ . \tag{3.27}$$

Combining (3.25), (3.26), (3.27), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int f^2\,dx \;\le& \int\left(-f'^2-\frac{1}{4}f^2+\frac{1}{2}\alpha^2h^2\right)\,dx + (C_1\epsilon+2k)\int (f^2+\alpha^2h^2)\,dx \\ &+\frac{7}{4}\int_0^\infty f^2\,dx\ .\end{aligned} \tag{3.28}$$

Next, using (3.20) and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\int h^2\,dx \;=\; \int(-Dh'^2+df'h')\,dx \;\le\; \int\left((-D+\frac{|d|}{2})h'^2+\frac{|d|}{2}f'^2\right)\,dx\ . \tag{3.29}$$

Finally, integrating by parts in (3.21), we obtain

$$\begin{aligned}\frac{d}{dt}\int \alpha^2h^2\,dx \;=& \int\left(-2D\alpha^2h'^2+2\sigma h^2+2d\alpha^2f'h'+2d(\alpha^2)'f'h\right)\,dx \\ \le& \int\left((-2D+|d|)\alpha^2h'^2+(2\sigma+|d|(\alpha^2)')h^2+|d|(\alpha^2+(\alpha^2)')f'^2\right)\,dx\ .\end{aligned} \tag{3.30}$$

In view of (3.22), (3.23), we have

$$\begin{aligned}2\int_{\mathbb{R}}\sigma h^2\,dx \;\le& -\frac{2}{3}\int_{-\infty}^0 \alpha^2h^2\,dx + 4D(1+k)\int_0^\infty \alpha^2h^2\,dx \\ \le& -\frac{2}{3}\int_{\mathbb{R}}\alpha^2h^2\,dx + \left(\frac{2}{3}+4D(1+k)\right)\int_0^\infty \alpha^2h^2\,dx\ .\end{aligned} \tag{3.31}$$

Replacing (3.31) into (3.30) and recalling that $(\alpha^2)' = 2\alpha\alpha' \le 2\sqrt{1+k}\,\alpha^2$, we thus find

$$\begin{aligned}\frac{d}{dt}\int \alpha^2h^2\,dx \;\le& -\int\left(2\alpha^2h'^2+\frac{2}{3}\alpha^2h^2\right)\,dx + 5\int_0^\infty \alpha^2h^2\,dx \\ &+ C_2(|d|+k)\int \alpha^2(h'^2+h^2+f'^2)\,dx\ ,\end{aligned} \tag{3.32}$$

for some $C_2 > 0$.

Therefore, combining (3.28), (3.29), (3.32), we see that there exists $C_3 > 0$ such that

$$\begin{aligned}\dot{E}_2 \;\le\; &-\int \left(f'^2 + \frac{1}{4}f^2 + \frac{1}{6}\alpha^2 h^2 + (1+2\alpha^2)h'^2\right)\,dx \\ &+\; C_3(|d|+k+\epsilon)\int (f'^2 + f^2 + h'^2 + \alpha^2 h^2)\,dx + 5\int_0^\infty (f^2+\alpha^2 h^2)\,dx\;.\end{aligned} \tag{3.33}$$

In particular, if $|d|$, k and ϵ are sufficiently small, there exists $K_3 > 0$ (depending on K) such that

$$\dot{E}_2 \;\le\; -K_3 E_3 + 5\int_0^\infty (f^2+h^2)\,dx\;.$$

It remains to show that $\int_0^\infty (f^2+h^2)\,dx \le CE_1$ for some $C > 0$. Using (3.1), (3.7), we have for all $x \ge 0$

$$f^2 + h^2 \;\le\; 3(f^2+g^2) \;=\; 3\rho^2(F^2+G^2) \;\le\; 3\rho(F^2+G^2) \;\le\; 4\beta(F^2+G^2)\;.$$

Thus, if $|d|$ and k are sufficiently small, it follows from (3.1), (3.5) that there exist $C_4 > 0$, $C_5 > 0$ such that $f^2+h^2 \le C_4\beta(G^2+H^2) \le C_5(\hat{\beta}G^2 + (1+ds^2)H^2)$, for all $x \ge 0$, hence $\int_0^\infty (f^2+h^2)\,dx \le 2C_5E_1$. This concludes the proof of Lemma 3.4. □

Lemma 3.5. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ is satisfied, then for all $t \in (0,T]$,*

$$\dot{E}_3(t) \;\le\; KK_4E_1(t)\;. \tag{3.34}$$

Proof. Using (3.20) and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int f'^2\,dx \;=\; \int \left(-f''^2 + \delta f f'' + \alpha(1+2k\beta)hf'' + N(f,h-f)f''\right)\,dx\;.$$

Since $|\delta| \le 1+k$, we have $\delta f f'' \le f''^2/4 + (1+k)^2 f^2$ and $\alpha(1+2k\beta)hf'' \le f''^2/4 + \alpha^2(1+2k)^2h^2$. Moreover, as in (3.27), we have

$$\int |N(f,h-f)f''|\,dx \;\le\; C_1\epsilon\int (f''^2+f^2+\alpha^2h^2)\,dx\;.$$

Therefore, there exists $C_6 > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\int f'^2\,dx \;\le\; \int\left(-\frac{1}{2}f''^2 + f^2 + \alpha^2h^2\right)\,dx + C_6(\epsilon+k)\int (f''^2+f^2+\alpha^2h^2)\,dx\;. \tag{3.35}$$

Similarly, using (3.20) and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\int h'^2\,dx \;=\; \int(-Dh''^2 + df''h'')\,dx \;\le\; -D\int h''^2\,dx + \frac{|d|}{2}\int(f''^2 + h''^2)\,dx\ .$$

Combining these results, we see that, if $|d|$, k and ϵ are sufficiently small, then

$$\frac{1}{2}\frac{d}{dt}\int(f'^2 + h'^2)\,dx \;\le\; (1 + C_6(\epsilon + k))\int(f^2 + \alpha^2 h^2)\,dx\ . \tag{3.36}$$

On the other hand, using (3.28) and (3.32), we obtain as in (3.33)

$$\begin{aligned}\frac{K}{2}\frac{d}{dt}\int(f^2 + 2\alpha^2 h^2)\,dx \;&\le\; -K\int\left(f'^2 + \frac{1}{4}f^2 + \frac{1}{6}\alpha^2 h^2 + 2\alpha^2 h'^2\right)\,dx\\ &+\; C_7(|d| + k + \epsilon)\int(f'^2 + f^2 + \alpha^2 h'^2 + \alpha^2 h^2)\,dx + 5K\int_0^\infty(f^2 + \alpha^2 h^2)\,dx\ ,\end{aligned}$$

for some $C_7 > 0$. Since $K > 6$ by assumption, it follows that, if $|d|$, k and ϵ are sufficiently small, then

$$\dot{E}_3 \;=\; \frac{1}{2}\frac{d}{dt}\int(f'^2 + h'^2)\,dx + \frac{K}{2}\frac{d}{dt}\int(f^2 + \alpha^2 h^2)\,dx \;\le\; 5K\int_0^\infty(f^2 + \alpha^2 h^2)\,dx\ .$$

Proceeding as in the proof of Lemma 3.5, we thus obtain $\dot{E}_3 \le KK_4E_1$. This concludes the proof of Lemma 3.5. □

3.3. Proof of Theorem 1.4.

We first state two Corollaries which are direct consequences of the preceding Lemmas.

Corollary 3.6. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ is satisfied, then there exists $K_0(c) > 0$ independent of T such that, for all $t \in [0, T]$,*

$$\|(f(t), g(t))\|_{Y_s^2} \le K_0(c)\|(f_0, g_0)\|_{Y_s^2}\ . \tag{3.37}$$

Proof. According to the four preceding Lemmas, we can choose $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ sufficiently small so that, if the hypothesis $\mathcal{H}_\epsilon$ is satisfied, then the differential inequalities (3.8), (3.14), (3.24) and (3.34) hold for $t \in [0, T]$. In particular, the function $E_4 \equiv K_1(E_1 + E_2 + E_3) + (K_2 + K_4 + KK_4)E_0$ is non-increasing in time for $t \in [0, T]$.

On the other hand, it is straightforward to verify that there exist $K_5 > 0$ and $K_6 > 0$ (independent of c, T) such that

$$K_5 \|(f,g)\|_{Y_s^2}^2 \;\le\; E_4 \;\le\; K_6 \|(f,g)\|_{Y_s^2}^2 \;,$$

hence (3.37) holds with $K_0 = (K_6/K_5)^{1/2}$. Since $K_2 = \mathcal{O}(c^2)$ as $c \to +\infty$, the same is true for K_6, hence $K_0 = \mathcal{O}(c)$ as $c \to +\infty$. □

Corollary 3.7. *There exist $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ such that, if the hypothesis $\mathcal{H}_\epsilon$ holds for all $T > 0$, then the solution (f,g) of (1.12) satisfies*

$$\lim_{t\to+\infty} \|(\partial_x f(t), \partial_x g(t))\|_{X_s^2} \;=\; 0 \;. \tag{3.38}$$

Proof. We choose $d_0 > 0$, $k_0 > 0$ and $\epsilon > 0$ as in Corollary 3.6. Using (3.8) and (3.14), we first note that the positive functions $E_0(t)$ and $K_1E_1(t)+K_2E_0(t)$ are non-increasing in time, hence converge as $t \to +\infty$. In particular, $E_1(t)$ has a nonnegative limit as $t \to +\infty$. Now, by (3.8), we have

$$\int_0^{+\infty} E_1(t)\, dt \;=\; \frac{1}{K_1}(E_0(0) - E_0(+\infty)) \;<\; +\infty \;,$$

hence $E_1(t)$ converges to 0 as $t \to +\infty$. Similarly, combining (3.8), (3.24) and (3.34), we see that the positive functions $K_1E_2(t) + K_4E_0(t)$ and $K_1E_3(t) + KK_4E_0(t)$ are non-increasing in time. Since $E_0(t)$ converges, the same is true for $E_2(t)$ and $E_3(t)$. By (3.8) and (3.24), we have

$$\int_0^{+\infty} E_3(t)\, dt \;\le\; \frac{1}{K_1K_3}\,(K_4(E_0(0) - E_0(+\infty)) + K_1(E_2(0) - E_2(+\infty))) \;<\; +\infty \;,$$

hence $E_3(t)$ converges to 0 as $t \to \infty$. Since $\|(\partial_x f, \partial_x g)\|_{X_s^2}^2 \le K_7(E_1 + E_3)$ for some $K_7 > 0$, (3.38) follows. □

We are now able to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $d_0 > 0$, $k_0 > 0$, $\epsilon > 0$ be as in the proof of Corollary 3.6, and let $D \in [1-d_0, 1+d_0]$, $k \in [0,k_0]$, $c \ge 2\sqrt{D}$. We set $\epsilon_0 = \epsilon/(2K_0)$, where K_0 is given by Corollary 3.6. If $(f_0, g_0) \in Y_s^2$ satisfies $\|(f_0,g_0)\|_{Y_s^2} \le \epsilon_0$, then by Lemma 3.1 there exists a time $t_1 > 0$ such that (1.12) has a unique classical solution

$(f,g) \in \mathcal{C}^0([0,t_1],Y_s^2) \cap \mathcal{C}^1((0,t_1],Y_s^2)$ satisfying $(f(0),g(0)) = (f_0,g_0)$. By Corollary 3.6, it follows that

$$\|(f(t),g(t))\|_{Y_s^2} \le K_0 \|(f_0,g_0)\|_{Y_s^2} \le \epsilon/2 \,, \tag{3.39}$$

for all $t \in [0,t_1]$. Indeed, assume that there exists $T \in (0,t_1]$ such that $\|(f(t),g(t))\|_{Y_s^2} < \epsilon$ for all $t \in [0,T)$ and $\|(f(T),g(T))\|_{Y_s^2} = \epsilon$. Then the hypothesis $\mathcal{H}_\epsilon$ is satisfied on $[0,T]$, and Corollary 3.6 implies that (3.39) holds for $t \in [0,T]$, which is a contradiction. Therefore, $\|(f(t),g(t))\|_{Y_s^2} < \epsilon$ for all $t \in [0,t_1]$, and (3.39) follows from Corollary 3.6. This shows that the solution (f,g) of (1.12) in Y_s^2 satisfies (3.39) whenever it exists. Using the Remark after Lemma 3.1, we conclude that $(f(t),g(t))$ exists for all $t \ge 0$ and satisfies (3.39). This proves (1.15), and (1.16) follows from Corollary 3.7. This concludes the proof of Theorem 1.4. $\square$

References

[AW] D.G. Aronson and H.F. Weinberger: Multidimensional Nonlinear Diffusion Arising in Population Genetics, Adv. in Math. **30** (1978), 33–76.

[BBDKL] E. Ben-Jacob, H. Brand, G. Dee, L. Kramer and J.S. Langer: Pattern propagation in nonlinear dissipative systems, Physica **14D** (1985), 348–364.

[BX] L. Berlyand and J. Xin: Large time asymptotics of solutions to a model combustion system with critical nonlinearity, Nonlinearity **8** (1995), 161–178.

[BN] J. Billingham and D.J. Needham: The development of travelling waves in quadratic and cubic autocatalysis with unequal diffusion rates. I. Permanent form travelling waves, Phil. Trans. R. Soc. Lond. A **334** (1991), 1–24.

[BK] J. Bricmont and A. Kupiainen: Stability of Moving Fronts in the Ginzburg-Landau Equation, Comm. Math. Phys. **159** (1994), 287–318.

[BKX] J. Bricmont, A. Kupiainen and J. Xin: Global large time self-similarity of a thermal-diffusive combustion system with critical nonlinearity, J. Diff Eqns. **130** (1996), 9–35.

[CX] P. Collet and J. Xin: Global existence and large time asymptotic bounds of L^∞ solutions of thermal diffusive combustion systems on $\mathbb{R}^n$, Ann. Scuola Norm. Sup. Pisa Cl. Sci., to appear.

[EW] J.-P. Eckmann and C.E. Wayne: The Nonlinear Stability of Front Solutions for Parabolic Partial Differential Equations, Comm. Math. Phys. **161** (1994), 323–334.

[FS] E.B. Fabes and D.W. Stroock: A New Proof of Moser's Parabolic Harnack Inequality Using the Old Ideas of Nash, Arch. Rat. Mech. Anal. **96** (1986), 327–338.

[Fo1] S. Focant: Equations de réaction-diffusion et étude de la stabilité de fronts, senior's thesis, Université Catholique de Louvain, 1993.

[Fo2] S. Focant, in preparation.

[Ga1] Th. Gallay: Periodic patterns and travelling fronts for the Ginzburg-Landau equation, in IUTAM/ISIMM-Proceedings Nonlinear Waves in Hannover 1994, World Scientific Press, 1995.

[Ga2] Th. Gallay: Local Stability of Critical Fronts in Nonlinear Parabolic Partial Differential Equations, Nonlinearity **7** (1994), 741–764.

[GR] Th. Gallay and G. Raugel: Stability of travelling waves for a damped hyperbolic equation, to appear in ZAMP (1997).

[HY] A. Haraux and A. Youkana: On a result of K. Masuda concerning reaction-diffusion equations, Tôhoku Math. J. **40** (1988), 159–163.

[HPSS] D. Horváth, V. Petrov, S.K. Scott and K. Showalter: Instabilites in propagating reaction-diffusion fronts, J. Chem. Phys. **98** (8) (1993), 6332–6343.

[Ki] K. Kirchgässner: On the Nonlinear Dynamics of Travelling Fronts, J. Diff. Eqns. **96** (1992), 256–278.

[KR] K. Kirchgässner and G. Raugel: Stability of Fronts for a KPP system : The Non-Critical Case, in *Dynamics of nonlinear waves in dissipative systems: reduction, bifurcation and stability*, G. Dangelmayr, B. Fiedler, K. Kirchgässner, A. Mielke, Pitman Research Notes in Mathematics Series **352**, Longman (1996).

[MP] R.H Martin and M. Pierre: Nonlinear reaction-diffusion systems, in *Nonlinear equations in the applied sciences*, W.F Ames and C. Rogers (Eds), Academic Press, Boston (1992).

[Ma] K. Masuda: On the global existence and asymptotic behavior of solutions of reaction-diffusion equations, Hokkaido Math. J. **12** (1983), 360 –370.

[Pa] A. Pazy: *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag (1983).

[Sa] D.H. Sattinger: On the Stability of Waves of Nonlinear Parabolic Systems, Adv. Math. **22** (1976), 312–355.

[Tu] A. Turing: The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. Lond. **B 237** (1952), 37–72.

[vS] W. van Saarloos: Front propagation into unstable states. II. Linear versus nonlinear marginal stability and rate of convergence Phys. Rev. A **39** (12) (1989), 6367-6390.