Embedded singular continuous spectrum for one-dimensional Schrödinger operators

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Abstract

We investigate one-dimensional Schrödinger operators with sparse potentials (i.e. the potential consists of a sequence of bumps with rapidly growing barrier separations). These examples illuminate various phenomena related to embedded singular continuous spectrum.

1 Introduction

In this paper, we will study one-dimensional Schrödinger equations on the half-line

$$-y''(x) + V(x)y(x) = Ey(x), \quad x \in [0, \infty). \tag{1}$$

We are interested in the spectral properties of the corresponding operators $H_\alpha = -\frac{d^2}{dx^2} + V(x)$ on $L_2([0,\infty))$ with boundary conditions $y(0)\cos\alpha + y'(0)\sin\alpha = 0$, $\alpha \in [0,\pi)$ (see e.g. [25] for the basic theory). (1) describes the motion of a quantum mechanical particle, and important physical properties of this system depend directly on the spectral characteristics of the operators H_α (for more background information, consult e.g. [14]).

Here, we will analyze two classes of potentials V related to the question of the occurrence of singular continuous spectrum which is embedded in the absolutely continuous spectrum. We will construct potentials so that $\sigma_{ac}(H_\alpha) = [0,\infty)$ and, for a set of boundary conditions α of positive measure, we have that $\sigma_{sc}(H_\alpha) \cap (0,\infty) \neq \emptyset$ (see Theorem 3.3 below). There is also a "complementary" construction: Using similar ideas, we will obtain potentials with $\sigma_{sc}(H_\alpha) = [0,\infty)$, $\sigma_{ac}(H_\alpha) \cap (0,\infty) \neq \emptyset$ for all α (= Theorem 3.5). I do not know of any previous examples for these types of spectral behavior.

It is more difficult to obtain embedded singular *continuous* spectrum than, say, embedded point spectrum because singular continuous spectrum is related to the *subordinacy* of the generalized eigenfunction (the notion of subordinacy was introduced and analyzed in [4]; for subsequent developments, see [6, 15]). Here, the term "generalized eigenfunction" simply refers to a solution of (1) which satisfies the boundary condition at $x = 0$. More specifically, one encounters the following problem: The singular part of the spectral measure is supported on the set where the generalized eigenfunction is subordinate [4], but even if this set is large, it does not automatically follow that there actually is singular continuous spectrum. Of course, as for the point spectrum, the situation is completely different: If the generalized eigenfunction is square integrable, then, trivially, the corresponding energy is an eigenvalue. In fact, potentials with embedded point spectrum which is dense in $[0,\infty)$ have been known for a relatively long time [12, 21].

We will illuminate these remarks with our second class of potentials. These examples have purely absolutely continuous spectrum on $(0,\infty)$, but, as we will prove, the set of energies for which the generalized eigenfunction is subordinate has (local) Hausdorff dimension 1 (= Theorem 4.2b)! (Recall that, by general principles, this set always has Lebesgue measure zero.) Again, this is, to the best of my knowledge, the first explicit example for these phenomena.

All potentials in this paper will be sparse potentials (i.e. mainly $V = 0$). This is no coincidence: Sparse potentials lead to non-trivial asymptotics of the solutions of (1), and there are powerful methods which allow a detailed analysis of these asymptotics. Here, we will rely mainly on further extensions of the techniques recently developed in [8, 16]. However, sparse potentials were already studied in the celebrated work [13]; further papers using sparse potentials in one way or another are [5, 7, 17, 20, 22]. Although the methods of [8] and [16] are very similar in spirit, the actual implementation of the basic strategy differs in some respects (actually, the results of [8] are stronger). It turns out that both viewpoints are needed here. The examples with embedded singular continuous spectrum make heavy use of the ideas of [8], whereas the second class of examples will be analyzed with an extension of the techniques developed in [16].

This paper is organized as follows: In the next Section, we fix the notations and explain the basic strategy for an effective analysis of Schrödinger equations with sparse potentials (as developed in [8, 16]). Sections 3 and 4 investigate in detail the examples described above.

While I was proofreading this paper, I obtained an interesting preprint by Molchanov [11] that also discusses sparse potentials. Molchanov's work has some overlap with [8, 16] and also with Section 3 of this paper.

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2 Preliminaries

The potentials we will study will have the form

$$V(x) = \sum_{n=1}^{\infty} g_n V_n(x - a_n) \tag{2}$$

with $g_n > 0, V_n \in L_1([-B_n, B_n])$; the intervals $[a_n - B_n, a_n + B_n]$ are assumed to be disjoint. Let $L_n = a_n - B_n - a_{n-1} - B_{n-1}$ (with $a_0 = B_0 := 0$) and

$$I_n = \int_{-B_n}^{B_n} |V_n(x)|\, dx. \tag{3}$$

The a_n's can of course be recovered from the L_n, B_n, so it suffices to specify these latter parameters.

Fix $\alpha \in [0, \pi)$, and, for $k > 0$, let $y(x,k)$ be the solution of (1) with $E = k^2$ and $y(0,k) = -\sin\alpha, y'(0,k) = \cos\alpha$. Note that y satisfies the boundary condition α at $x = 0$. The (modified) Prüfer variables $R(x,k), \varphi(x,k)$ are defined by

$$\begin{pmatrix} y \\ y' \end{pmatrix} = R \begin{pmatrix} \sin\varphi \\ k\cos\varphi \end{pmatrix}.$$

Here, we demand that $R > 0$ and φ be continuous in x. Clearly, R, φ also depend on α, but this dependence will not be made explicit in the notation. We write $R_n(k) = R(a_n - B_n, k), \varphi_n(k) = \varphi(a_n - B_n, k)$ (so R_n, φ_n are the Prüfer variables immediately before the nth barrier).

R, φ obey the equations

$$(\ln R)' = \frac{V}{2k} \sin 2\varphi, \tag{4}$$

$$\varphi' = k - \frac{V}{k} \sin^2 \varphi. \tag{5}$$

In particular, we have $R(x) \equiv R_n$ on $x \in [a_{n-1} + B_{n-1}, a_n - B_n]$ and

$$\varphi_n(k) = \varphi(a_{n-1} + B_{n-1}, k) + kL_n. \tag{6}$$

Fix a compact interval $J = [k_1, k_2] \subset (0, \infty)$. Then, for $k \in J$, we can integrate (4), (5) over the interval $[a_n - B_n, a_n + B_n]$ and use an elementary Taylor expansion in the parameter $g_n I_n$. This routine calculation yields (compare [8])

$$\begin{aligned} \ln \frac{R_{n+1}(k)}{R_n(k)} &= \frac{g_n}{2k} \int_{-B_n}^{B_n} dx\, V_n(x) \sin 2\theta_n(x,k) \\ &-\frac{g_n^2}{k^2} \int_{-B_n}^{B_n} dx\, V_n(x) \cos 2\theta_n(x,k) \int_{-B_n}^{x} dt\, V_n(t) \sin^2 \theta_n(t,k) \\ &+O(g_n^3 I_n^3). \end{aligned} \tag{7}$$

where $\theta_n(x,k) := k(x + B_n) + \varphi_n(k)$. The remainder $O(g_n^3 I_n^3)$ is bounded by $C(J) g_n^3 I_n^3$ where C is independent of $k \in J$ and $n \in \mathbb{N}$. Finally, let

$$C_n = \max_{k \in J} \left| \frac{d\varphi(a_{n-1} + B_{n-1}, k)}{dk} \right|.$$

The crucial observation is that, because of (6), for appropriate probability measures on J, the Prüfer angles $\varphi_n(k)$ (evaluated modulo π) are approximately independent random variables, provided that $L_n \gg C_n$. This property can be exploited by computing moments [8] or by investigating the joint distribution of the φ_n's [16]. With either method, one can analyze (7) rather accurately.

In order to get quantitative conditions on the L_n's, we need a priori control on the C_n's.

Lemma 2.1 ([8]) *Assume $g_n I_n \to 0$ and $(L_n + B_n)/L_{n+1} \to 0$. Then there is a constant $C = C(J)$ such that $C_n \le C L_{n-1}$ and*

$$\max_{k \in J} \left| \frac{d^2\varphi(a_{n-1} + B_{n-1}, k)}{dk^2} \right| \le C \left(1 + \sum_{m=1}^{n-1} g_m I_m L_m^2 \right).$$

Proof. This follows from a Gronwallization of the differential equations satisfied by $\partial\varphi/\partial k, \partial^2\varphi/\partial k^2$; see [8, Proposition 5.1] for details. □

3 Embedded singular continuous spectrum

Suppose $g_n I_n \in l_3$. Then (7) can be written as

$$\begin{aligned} \ln R_{N+1}(k) &= \frac{1}{2k} \mathrm{Im} \sum_{n=1}^{N} X_n(k) - \frac{1}{2k^2} \mathrm{Re} \sum_{n=1}^{N} Y_n(k) \\ &+\frac{1}{8k^2} \mathrm{Re} \sum_{n=1}^{N} Z_n(k) + \frac{1}{8k^2} \sum_{n=1}^{N} g_n^2 |\widehat{V}_n(2k)|^2 + \rho_N(k) \end{aligned} \tag{8}$$

where $|\rho_N(k)| \le C$ for all $N \in \mathbb{N}, k \in J$. Here, we have set

$$\widehat{V}_n(2k) = \int_{-B_n}^{B_n} V_n(x) e^{2ikx}\, dx \tag{9}$$

and

$$\begin{aligned}
X_n(k) &= g_n \widehat{V}_n(2k) e^{2i(\varphi_n(k)+kB_n)}, \\
Y_n(k) &= g_n^2 e^{2i(\varphi_n(k)+kB_n)} \int_{-B_n}^{B_n} dx\, V_n(x) e^{2ikx} \int_{-B_n}^{x} dt\, V_n(t), \\
Z_n(k) &= g_n^2 (\widehat{V}_n(2k))^2 e^{4i(\varphi_n(k)+kB_n)}.
\end{aligned}$$

Since X_n, Y_n, Z_n contain the highly oscillatory factors $\exp(2i\varphi_n)$, we expect that the leading term of (8) will be $\sum g_n^2 |\widehat{V}_n(2k)|^2$. Thus, by choosing the shape of the bumps V_n carefully, we can get non-trivial k-dependence of the asymptotics of $R_N(k)$ $(N \to \infty)$. This idea was already used in [8] to construct potentials with $\sigma_{ac} = [E_1, E_2] \subset (0, \infty)$, $\sigma_{sc} = [0, \infty) \setminus (E_1, E_2)$ (see [8, Theorem 6.3]). As we will see, it is more difficult to obtain *embedded* singular continuous spectrum.

The barriers V_n will have the form

$$V_n(x) = \chi_{(-B_n, B_n)}(x) W(x) \tag{10}$$

where W is the Fourier transform of the characteristic function of a Cantor type set F. So, construct F as follows: Let $\delta_n > 0$ be sufficiently small prescribed numbers. Fix $F_0 = [a, b] \subset (0, \infty)$ and let $F_1 = F_0 \setminus (c_1^{(0)} - \delta_0, c_1^{(0)} + \delta_0)$ where $c_1^{(0)} = (a+b)/2$ is the center of F_0. In general, if F_n is a disjoint union of 2^n closed intervals with centers $c_m^{(n)}$ $(m = 1, \ldots, 2^n)$, set $F_{n+1} = F_n \setminus \bigcup_{m=1}^{2^n} (c_m^{(n)} - \delta_n, c_m^{(n)} + \delta_n)$. The set $F = \bigcap F_n$ is closed, nowhere dense and has Lebesgue measure $|F| = b - a - \sum_{n=0}^{\infty} 2^{n+1} \delta_n$. We assume that $|F| > 0$, and we define

$$W(x) = \int_F \cos 2kx\, dk. \tag{11}$$

Lemma 3.1 *Let $F, W(x)$ be as above. Suppose $\sup_{n \in \mathbb{N}} \delta_n 2^{\gamma n} < \infty$ for some $\gamma > 1$. Then $W(x) = O((1+|x|)^{-1+1/\gamma})$.*

Proof. Let $f_n(x) = \int_{F_n} \cos 2kx\, dk$. Obviously, $|f_n(x) - f(x)| \le |F_n \setminus F| \to 0$. Furthermore, by construction of the F_n,

$$\begin{aligned}
f_{n+1}(x) - f_n(x) &= -\int_{F_n \setminus F_{n+1}} \cos 2kx\, dk = -\sum_{m=1}^{2^n} \int_{c_m^{(n)} - \delta_n}^{c_m^{(n)} + \delta_n} \cos 2kx\, dk \\
&= -\frac{1}{2x} \sum_{m=1}^{2^n} \left(\sin 2x(c_m^{(n)} + \delta_n) - \sin 2x(c_m^{(n)} - \delta_n) \right) \\
&= -\frac{1}{x} \sin 2\delta_n x \sum_{m=1}^{2^n} \cos 2c_m^{(n)} x.
\end{aligned}$$

In particular, the infinite sum $\sum(f_{n+1}(x)-f_n(x))$ is absolutely convergent, and thus

$$\begin{aligned} |f(x)| &= \left|f_0(x)+\sum_{n=0}^{\infty}(f_{n+1}(x)-f_n(x))\right| \\ &\leq \frac{1}{|x|}\left(1+\sum_{n=0}^{\infty}2^n|\sin 2\delta_n x|\right). \end{aligned} \tag{12}$$

For $|x|>2^{-\gamma}$, define $N(x)\in\mathbb{N}_0$ by demanding $2^{-\gamma}<2^{-\gamma N(x)}|x|\leq 1$. Now the assertion is obtained by considering separately the sums $\sum_{n<N(x)}$ and $\sum_{n\geq N(x)}$ in (12). In the first sum, we use $|\sin 2\delta_n x|\leq 1$; in the second sum, we estimate $|\sin 2\delta_n x|\leq 2\delta_n|x|$, and we use that, by hypothesis, $\delta_n\leq C2^{-\gamma n}$. The details of these straightforward computations are left to the reader. □

In order to control the first three sums of (8), we will need the following result obtained with the methods discussed in [24]:

Proposition 3.2 *Let X_n $(n\in\mathbb{N})$ be (complex-valued) random variables. Suppose that there are numbers $\rho_n\geq 0$ such that $\sum\rho_n\ln^2 n<\infty$ and*

$$E|X_n|^2+\sum_{m=1}^{n-1}|EX_m\overline{X}_n|\leq\rho_n.$$

Then $\lim_{N\to\infty}\sum_{n=1}^{N}X_n$ exists almost surely.

Sketch of the proof. This result follows essentially from [24, Theorem 2.4.2]. Of course, we need a version for complex-valued X_n's here, but this extension presents no difficulties. We then obtain the Proposition by taking

$$g(j,N)=2\sum_{n=j+1}^{j+N}\rho_n,\quad h(j,N)=2\sum_{n=j+1}^{j+N}\rho_n\ln^2 n$$

in [24, Theorem 2.4.2] and noting that

$$E\left|\sum_{n=j+1}^{j+N}X_n\right|^2\leq\sum_{n=j+1}^{j+N}E|X_n|^2+2\sum_{n=j+1}^{j+N}\sum_{m=1}^{n-1}|EX_m\overline{X}_n|.$$

□

We are now ready to formulate and prove our first main result:

Theorem 3.3 *Let F be a Cantor type set satisfying the assumptions of Lemma 3.1 with $\gamma>6$. Let $g_n=n^{-1/2}, B_n=n^{\beta}$ with $(2-4/\gamma)^{-1}<\beta<\gamma/8$, and assume that $n^{\beta/(2\gamma)}L_{n-1}/L_n\to 0$. Then the half-line Schrödinger operators H_α with potential V given by (2), (10), (11) satisfy $\sigma_{ac}(H_\alpha)=\sigma_{ess}(H_\alpha)=[0,\infty)$, $\sigma_p(H_\alpha)\cap(0,\infty)=\emptyset$ and, for a set of boundary conditions α of positive measure, $\sigma_{sc}(H_\alpha)\cap(0,\infty)\neq\emptyset$.*

Remarks. 1. The following proof also works under more general assumptions on γ, g_n, B_n, L_n. However, these general conditions are very clumsy, and it seems pointless to make them explicit.

2. The proof will show that, more precisely, $\sigma_{sc}(H_\alpha) \cap \mathbb{R} \setminus F^2 = \emptyset$ for almost all α, and $\sigma_{sc}(H_\alpha) \cap F^2 \neq \emptyset$ for a set of α's of positive measure. Here, $F^2 = \{k^2 : k \in F\}$. Note that the method of [8] (which would establish that the spectrum is purely absolutely continuous on $(0, \infty) \setminus F^2$ for *every* boundary condition) does not work here because of the slow decay of $W(x)$.

Proof. The assertion on σ_{ess} follows from the fact that $V(x) \to 0$. Furthermore, a standard Gronwall estimate shows that because of the rapid growth of the barrier separations L_n, (1) has no L_2-solutions if $E > 0$.

Moving on to the non-trivial parts of the proof, we fix $f \in C_0^\infty(0, \infty)$ with $f \geq 0$, supp $f \cap F = \emptyset$ and $\int f(k)\, dk = 1$. In the first part of the proof, we will show with the aid of Proposition 3.2 that almost surely with respect to the probability measure $dP(k) = f(k)\, dk$, the right-hand side of (8) remains bounded as N goes to infinity. Using (again) a Gronwall estimate, one can easily extend this statement to $x \in \bigcup_n [a_n - B_n, a_n + B_n]$, i.e., $R(x, k)$ is bounded on all of $x \in [0, \infty)$. Running through this argument for two different initial angles $\varphi(0, k)$ then shows that *all* solutions are bounded for almost every $k \in$ supp f. Since f is arbitrary and F is nowhere dense, this will establish the claim on σ_{ac} (by [23, Theorem 5]).

Let

$$\epsilon_n = \left(\int_{|x| > B_n} |W(x)|^2\, dx \right)^{1/2}.$$

We will need

Lemma 3.4 *a) Let $f(k)$ be a bounded function. Then there is a constant $C = C(f)$, such that*

$$\int \left| \widehat{V}_n(2k) - \pi \chi_{F \cup -F}(k)/2 \right|^2 f(k)\, dk \leq C \epsilon_n^2, .$$

b) $\lim_{n \to \infty} \widehat{V}_n(2k) = \pi \chi_{F \cup -F}(k)/2$ *for almost every k.*

Proof. a) A calculation using (9), (10), (11), and the Plancherel formula [19, Theorem 9.13(b)] yields

$$\int \left| \widehat{V}_n(2k) - \pi \chi_{F \cup -F}(k)/2 \right|^2 dk = \pi \epsilon_n^2.$$

This obviously implies the assertion.

b) Since $\gamma > 6 > 2$, Lemma 3.1 shows that $W \in L_p$ for some $p < 2$. Thus the assertion follows from Zygmund's Theorem on the pointwise convergence of Fourier transforms [26] and the fact that $B_n \to \infty$. □

We are now ready to compute the moments of X_n, Y_n, Z_n. Note that Lemma 3.1 implies that (I_n was defined in eq. (3))

$$I_n \leq C n^{\beta/\gamma}, \qquad \epsilon_n \leq C n^{\beta/\gamma - \beta/2}. \tag{13}$$

In particular, we have that $\sum (g_n I_n)^3 < \infty$; thus we may indeed use (8).

Let us first verify that X_n satisfies the assumptions of Proposition 3.2. Since we are interested in asymptotic properties (namely, the boundedness of $\sum X_n$ almost everywhere), we may restrict n to large values $n \geq n_0$. In the sequel, we will do so (if necessary) without explicit mention.

An integration by parts shows that (for $m < n$)

$$\begin{aligned}
|EX_m \overline{X}_n| &= g_m g_n \left| \int \widehat{V}_m(2k)\overline{\widehat{V}_n(2k)} e^{2i(\varphi_m(k)-\varphi_n(k)+k(B_m-B_n))} f(k)\, dk \right| \\
&= \frac{g_m g_n}{2} \left| \int \frac{\widehat{V}_m \overline{\widehat{V}_n} f}{\varphi_m' - \varphi_n' + B_m - B_n} \frac{d}{dk} e^{2i(\varphi_m - \varphi_n + k(B_m - B_n))}\, dk \right| \\
&\leq \frac{g_m g_n}{2} \int \left| \frac{d}{dk} \left(\frac{\widehat{V}_m(2k)\overline{\widehat{V}_n(2k)} f(k)}{\varphi_m'(k) - \varphi_n'(k) + B_m - B_n} \right) \right| dk. \qquad (14)
\end{aligned}$$

Evaluating the derivative with the product rule, we get four different terms. The most dangerous term involves the derivative of the denominator. Lemma 2.1 shows that we can estimate this contribution by

$$C g_m g_n \sum_{s=1}^{n-1} g_s I_s \frac{L_s^2}{L_n^2} \int |\widehat{V}_m(2k)\widehat{V}_n(2k)| f(k)\, dk. \tag{15}$$

Here and in the sequel, C denotes a constant which depends only on f. The actual value of C may change from one formula to the next.

We claim that

$$\sum_{s=1}^{n-1} g_s I_s L_s^2 \leq C n^{-1/2+\beta/\gamma} L_{n-1}^2.$$

This estimate will be used extensively throughout the rest of this Section. It says that the sum is dominated by its last term, and this is true because of the rapid growth of L_s. Here is the formal argument: Since $L_{n-1}/L_n \to 0$, we can find $C, \alpha > 0$ so that $L_s^2/L_{n-1}^2 \leq C e^{-\alpha(n-s)}$ for all $s \leq n-1$. Hence

$$\begin{aligned}
\sum_{s=1}^{n-1} g_s I_s L_s^2 &\leq C L_{n-1}^2 e^{-\alpha n} \sum_{s=1}^{n-1} s^{-1/2+\beta/\gamma} e^{\alpha s} \\
&\leq C L_{n-1}^2 e^{-\alpha n} \int_1^n s^{-1/2+\beta/\gamma} e^{\alpha s}\, ds \leq C n^{-1/2+\beta/\gamma} L_{n-1}^2.
\end{aligned}$$

(Recall that C does not necessarily have the same value in every formula.) Here, we have estimated the integral by an integration by parts.

Since supp $f \cap F = \emptyset$, we get from Lemma 3.4a)

$$\begin{aligned}
\int |\widehat{V}_m \widehat{V}_n| f &\leq \left(\int |\widehat{V}_m|^2 f\right)^{1/2} \left(\int |\widehat{V}_n|^2 f\right)^{1/2} \\
&= \left(\int \left|\widehat{V}_m - \pi\chi_{F\cup -F}/2\right|^2 f \int \left|\widehat{V}_n - \pi\chi_{F\cup -F}/2\right|^2 f\right)^{1/2} \\
&\leq C\epsilon_m\epsilon_n.
\end{aligned}$$

Thus we can bound (15) by

$$Cm^{-1/2+\beta/\gamma-\beta/2} n^{-1+2\beta/\gamma-\beta/2} \frac{L_{n-1}^2}{L_n^2}.$$

The other contributions from (14) (where d/dk acts on the numerator) are much easier to deal with. We use the obvious estimates $|\widehat{V}_i| \leq I_i$, $|\widehat{V}_i'| \leq 2B_i I_i$. It follows that these terms can be bounded by Cn^c/L_n for some c. In conclusion, we see that $EX_m\overline{X}_n$ satisfies an estimate of the form

$$|EX_m\overline{X}_n| \leq C \sum_{m=1}^{n-1} \left(\frac{n^c}{L_n} + m^{-1/2+\beta/\gamma-\beta/2} n^{-1+2\beta/\gamma-\beta/2} \frac{L_{n-1}^2}{L_n^2} \right).$$

We assume that the exponent of m is larger than -1; the proof in the other case is completely analogous. Then summing over m yields

$$\sum_{m=1}^{n} |EX_m\overline{X}_n| \leq C \left(\frac{n^{c+1}}{L_n} + n^{-1/2+3\beta/\gamma-\beta} \frac{L_{n-1}^2}{L_n^2} \right). \tag{16}$$

Note that, as promised, the first term on the right-hand side is completely harmless because of the rapid growth of the L_n (faster than exponential).

In order to control $E|X_n|^2$, we use again the fact that supp $f \cap F = \emptyset$ and apply Lemma 3.4a):

$$\begin{aligned}
E|X_n|^2 &= g_n^2 \int |\widehat{V}_n(2k)|^2 f(k)\, dk \\
&= g_n^2 \int \left|\widehat{V}_n(2k) - \pi\chi_{F\cup -F}(k)/2\right|^2 f(k)\, dk \\
&\leq Cg_n^2\epsilon_n^2 \leq Cn^{-1+2\beta/\gamma-\beta}.
\end{aligned}$$

Now it is straightforward to verify that this estimate and (16) indeed ensure that Proposition 3.2, with

$$\rho_n = C\left(n^{-1+2\beta/\gamma-\beta} + \frac{n^{c+1}}{L_n} + n^{-1/2+3\beta/\gamma-\beta} \frac{L_{n-1}^2}{L_n^2} \right),$$

applies to $\sum X_n$.

In the remainder of this Section, it will be convenient to simplify the notation by using the following convention: We will not write out terms which *obviously* decay so rapidly that they do not pose any difficulties. An example would be the contribution n^{c+1}/L_n in (16); note, however, that this term could even be the dominant one in (16), namely, if the L_n grow unreasonably fast. In any event, we will alert the reader by writing $\lesssim$ instead of $\leq$ whenever this convention has been applied.

The above strategy can also be used to control $\sum Y_n, \sum Z_n$. For instance, the leading term in the estimate on $|EY_m\overline{Y}_n|$ $(m < n)$ is

$$C(g_m I_m g_n I_n)^2 \sum_{s=1}^{n-1} g_s I_s \frac{L_s^2}{L_n^2} \leq C m^{-1+2\beta/\gamma} n^{-3/2+3\beta/\gamma} \frac{L_{n-1}^2}{L_n^2},$$

and summing this over m $(m < n)$ yields the bound $Cn^{-3/2+5\beta/\gamma}L_{n-1}^2/L_n^2$. We omit the details of these computations, since the argument is sufficiently close to the discussion above. It turns out that Proposition 3.2 also proves that $\sum Y_n, \sum Z_n$ are bounded (in fact, convergent) for almost all $k \in \operatorname{supp} f$.

It remains to show that $\limsup_{N\to\infty} \sum_{n=1}^N g_n^2 |\widehat{V}_n(2k)|^2 < \infty$ for almost all k with respect to $dP = f\, dk$. To this end, let

$$S(k) := \lim_{N\to\infty} \sum_{n=1}^N g_n^2 |\widehat{V}_n(2k)|^2 \in [0, \infty].$$

Now notice that by monotone convergence and, as above, the fact that $\operatorname{supp} f \cap F = \emptyset$ we have that $\int S(k) f(k)\, dk \leq C \sum g_n^2 \epsilon_n^2 < \infty$, thus indeed $S(k) < \infty$ for almost every $k \in \operatorname{supp} f$. As explained above, this completes the proof of $\sigma_{ac} = [0, \infty)$.

In order to prove the existence of singular continuous spectrum, we use the strategy of the proof of [8, Theorem 1.6(2)]. So pick $f \in C_0^\infty(0, \infty)$ with $f \geq 0$, $\int f(k)\, dk = 1$ and $F \subset \operatorname{supp} f$. It suffices to show that (with $dP(k) = f(k)\, dk$)

$$E \left| \sum_{n=1}^N X_n \right|^2 = o(\ln^2 N), \tag{17}$$

and similarly for Y_n, Z_n. For if this holds, then, by the Chebyshev inequality,

$$P\left(\left| \sum_{n=1}^N X_n \right| \geq \delta \ln N \right) \leq \frac{E|\sum_{n=1}^N X_n|^2}{\delta^2 \ln^2 N} \to 0 \quad (N \to \infty)$$

for any $\delta > 0$. In particular, we can find a subsequence $N_i \to \infty$ so that $\sum_{i=1}^\infty E|\sum_{n=1}^{N_i} X_n|^2 \ln^{-2} N_i < \infty$, and now the Borel-Cantelli Lemma says that

$$P\left(\left| \sum_{n=1}^{N_i} X_n \right| \geq \delta \ln N_i \text{ for infinitely many } i \right) = 0.$$

Of course, analogous statements hold if X_n is replaced with Y_n or Z_n.

On the other hand, Lemma 3.4b) implies that $\lim_{n\to\infty} \widehat{V}_n(2k) = \pi/2$ for almost every $k \in F$. If δ from above is small enough, we thus see from (8) that

$$\lim_{i\to\infty} R_{N_i}(k) \geq \lim_{i\to\infty} \left(c_1 \sum_{n=1}^{N_i} g_n^2 - c_2 \delta \ln N_i \right) = \infty$$

for almost every $k \in F$. Hence the absolutely continuous part of the spectral measure ρ_α gives zero weight to $F^2 = \{k^2 : k \in F\}$ [10, Theorem 1.2]. As noted at the beginning of the proof, this also holds for the point part of ρ_α. Moreover, α was arbitrary, so the spectral averaging formula (see e.g. [3]) becomes

$$0 < |F^2| = \int_0^\pi \rho_\alpha(F^2)\, d\alpha = \int_0^\pi \rho_\alpha^{(sc)}(F^2)\, d\alpha.$$

This forces $\rho_\alpha^{(sc)}(F^2) > 0$ for a set of α's of positive measure, as desired.

It remains to prove (17). To this end, let $S_N = \sum_{n=1}^N X_n$. Then

$$E|S_N|^2 \leq E|S_{N-1}|^2 + E|X_N|^2 + 2|ES_{N-1}\overline{X}_N|. \tag{18}$$

Clearly, $E|X_N|^2 \leq Cg_N^2 = CN^{-1}$. In order to estimate $ES_{N-1}\overline{X}_N$, we use the integration by parts argument from the first part of this proof. This gives

$$|ES_{N-1}\overline{X}_N| \leq Cg_N \int \left| \frac{d}{dk} \left(\frac{f(k)\overline{\widehat{V}_N(2k)} S_{N-1}(k)}{\varphi_N'(k) + B_N} \right) \right| dk.$$

This time, there are two potentially dangerous terms: the first one coming from the derivative of the denominator and the second one involving dS_{N-1}/dk. The first contribution is treated as above to obtain the bound

$$Cg_N \sum_{s=1}^{N-1} g_s I_s \frac{L_s^2}{L_N^2} \int |f\widehat{V}_N S_{N-1}|\, dk \leq CN^{-1+\beta/\gamma} \frac{L_{N-1}^2}{L_N^2} \left(E|S_{N-1}|^2 \right)^{1/2}.$$

The last estimate follows by the Cauchy-Schwarz inequality, Lemma 3.4a), and the usual bound on $\sum_{s=1}^{N-1} g_s I_s L_s^2$.

As for the second term, we note that the leading term of

$$\frac{dS_{N-1}}{dk} = \sum_{n=1}^{N-1} g_n \frac{d}{dk} \widehat{V}_n(2k) e^{2i(\varphi_n(k)+kB_n)}$$

comes from differentiating $e^{2i\varphi_n}$. Now the usual techniques show that the corresponding contribution to $ES_{N-1}\overline{X}_N$ can be bounded by $Cg_N^2 L_{N-1}/L_N$.

As before, we need not worry about the remaining terms which can be bounded by an expression of the form CN^c/L_N, and the rapid growth of L_N

guarantees that these terms are unimportant. So, if we put everything together, (18) becomes

$$E|S_N|^2 \lesssim E|S_{N-1}|^2 + CN^{-1} + CN^{-1+\beta/\gamma}\frac{L_{N-1}^2}{L_N^2}\left(E|S_{N-1}|^2\right)^{1/2}.$$

By an inductive argument, one can now prove that

$$\left(E|S_N|^2\right)^{1/2} \lesssim C\sum_{n=1}^{N} n^{-1+\beta/\gamma}\frac{L_{n-1}^2}{L_n^2} + C\left(\sum_{n=1}^{N} n^{-1}\right)^{1/2}.$$

In fact, the statement needed here is exactly [8, Lemma 6.2]; it would be pointless to repeat that proof here. In any event, using the assumptions of Theorem 3.3, we see that we have the required bound $E|S_N|^2 = o(\ln^2 N)$.

The proof of (17) for Y_n and Z_n is similar. Again, we sketch the argument for Y_n and leave the proof for Z_n to the reader.

An elementary estimate yields $E|Y_N|^2 \le CN^{-2+4\beta/\gamma}$. As usual, $ES_{N-1}\overline{Y}_N$ (here, $S_n = \sum_{t=1}^n Y_t$, of course) is treated with an integration by parts, and the most serious attention has to be paid to the contribution involving $(d/dk)(1/(\varphi_N' + B_N))$. This term can be bounded by

$$C(g_N I_N)^2 \sum_{s=1}^{N-1} g_s I_s \frac{L_s^2}{L_N^2}\int f|S_{N-1}|\,dk \le CN^{-3/2+3\beta/\gamma}\frac{L_{N-1}^2}{L_N^2}\left(E|S_{N-1}|^2\right)^{1/2}.$$

If we collect all terms and use the inductive argument from above, we finally get the following estimate

$$\left(E|S_N|^2\right)^{1/2} \lesssim C\sum_{n=1}^{N} n^{-3/2+3\beta/\gamma}\frac{L_{n-1}^2}{L_n^2} + C\left(\sum_{n=1}^{N} n^{-2+4\beta/\gamma}\right)^{1/2}.$$

Again, a routine verification shows that this latter expression is of order $o(\ln N)$ (in fact, it is even bounded), as desired. $\square$

One can interchange the roles of F and $(0,\infty)\setminus F$ to obtain potentials with embedded *absolutely* continuous spectrum. More precisely, proceed as follows: Pick an even function $g \in \mathcal{S}(\mathbb{R})$ (the Schwartz class) with $g > 0$ and $g(k) = 1$ if $k \in F$. Define

$$W(x) = \int_0^\infty (g(k) - \chi_F(k))\cos 2kx\,dk. \tag{19}$$

Theorem 3.5 *Assume that F, γ, g_n, B_n, L_n satisfy the assumptions of Theorem 3.3. Then, for all α, the half-line Schrödinger operator H_α with potential V given by (2), (10), (19) satisfies $\sigma_{sc}(H_\alpha) = \sigma_{ess}(H_\alpha) = [0,\infty)$, $\sigma_p(H_\alpha) \cap (0,\infty) = \emptyset$ and $\sigma_{ac}(H_\alpha) = F^2$.*

Large parts of this *proof* are similar to the corresponding arguments of the proof of Theorem 3.3. Therefore, these parts of the proof will only be sketched.

As before, it is easy to verify the assertions on σ_{ess}, σ_p. We also have an analogue of Lemma 3.4 where the function $\pi\chi_{F\cup -F}(k)/2$ now is replaced with $\widehat{W}(k) := \pi(g(k) - \chi_{F\cup -F}(k))/2$. Note that $\widehat{W}(k) = 0$ if $k \in F$. Moreover, if $I \subset (0,\infty)$ is a compact set with $I \cap F = \emptyset$, then $\inf_{k\in I} |\widehat{W}(k)| > 0$. Finally, we still have the estimates (13), because $g \in \mathcal{S}(\mathbb{R})$ implies that the part $\int g(k) \cos 2kx\, dk$ of W decays faster than any power of x.

To prove the assertion on σ_{sc}, fix $f \in C_0^\infty(0,\infty)$ with $f \geq 0$, supp $f \cap F = \emptyset$, and $\int f\, dk = 1$. Now we can repeat the arguments from the last part of the proof of Theorem 3.3. In this way, we see that $\rho_{ac}^{(\alpha)}((0,\infty) \setminus F^2) = 0$ for all α, hence, since F is nowhere dense and $\sigma_{ess} = [0,\infty), \sigma_p \cap (0,\infty) = \emptyset$, we must have $\sigma_{sc}(H_\alpha) = [0,\infty)$, as claimed.

Now let $dP(k) = |F|^{-1}\chi_F(k)\, dk$. We want to estimate $EX_m\overline{X}_n$ $(m \leq n)$, where the expectation is computed with this probability measure. To this end, we first observe that there exist functions $f_N(k) \in C_0^\infty(0,\infty)$ $(N \in \mathbb{N})$ with $0 \leq f_N \leq 1$ and

$$\int |\chi_F(k) - f_N(k)|\, dk \leq C2^{(1-\gamma)N}, \quad \int |f_N'(k)|\, dk = 2^{N+1} \tag{20}$$

(γ is from Lemma 3.1). To see this, simply approximate χ_{F_N} by an appropriate smooth function. Here, F_N is the set obtained in the Nth step of the construction of F (see the discussion preceding Lemma 3.1).

Using these approximations of χ_F, we get (for $m < n$)

$$\begin{aligned} |EX_m\overline{X}_n| &= g_m g_n |F|^{-1} \left| \int \chi_F(k)\widehat{V}_m(2k)\overline{\widehat{V}_n(2k)} e^{2i(\varphi_m(k)-\varphi_n(k)+k(B_m-B_n))}\, dk \right| \\ &\leq g_m g_n |F|^{-1} \left(I_m I_n \int |\chi_F(k) - f_N(k)|\, dk \quad + \right. \\ &\qquad \left. \left| \int f_N(k)\widehat{V}_m(2k)\overline{\widehat{V}_n(2k)} e^{2i(\varphi_m(k)-\varphi_n(k)+k(B_m-B_n))}\, dk \right| \right). \end{aligned}$$

Clearly, the first term can be bounded by $Cg_m g_n I_m I_n 2^{(1-\gamma)N}$. In the second term, we use again an integration by parts. This time, the terms obtained by differentiating f_N and, as usual, $1/(\varphi_m' - \varphi_n' + B_m - B_n)$ deserve the most serious attention. We use the by now familiar procedures and also (20) for the term involving f_N'. We get bounds of the form $Cg_m g_n 2^N I_m I_n / L_n$ and

$$\begin{aligned} & C g_m g_n n^{-1/2+\beta/\gamma} \frac{L_{n-1}^2}{L_n^2} \int f_N |\widehat{V}_m \widehat{V}_n|\, dk \\ & \leq C m^{-1/2} n^{-1+\beta/\gamma} \frac{L_{n-1}^2}{L_n^2} \left(I_m I_n \int |f_N - \chi_F|\, dk + \int \chi_F |\widehat{V}_m \widehat{V}_n|\, dk \right) \\ & \leq C m^{-1/2} n^{-1+\beta/\gamma} \frac{L_{n-1}^2}{L_n^2} \left((mn)^{\beta/\gamma} 2^{(1-\gamma)N} + \epsilon_m \epsilon_n \right), \end{aligned}$$

respectively. Now we can take, say, $N = n$ and sum the resulting bounds over m (again, we will treat explicitly only the case where the exponent of m is larger than -1). We thus obtain

$$\sum_{m=1}^{n-1} |EX_m \overline{X}_n| \lesssim C n^{-1/2+3\beta/\gamma-\beta} \frac{L_{n-1}^2}{L_n^2}.$$

Moreover, the analogue of Lemma 3.4a) implies that $E|X_n|^2 = g_n^2 |F|^{-1} \int_F |\widehat{V}_n(2k)|^2\, dk \leq C g_n^2 \epsilon_n^2$. These estimates together with the assumptions of the Theorem show that Proposition 3.2 is applicable. By similar arguments, the same holds for Y_n and Z_n.

Finally, by the analogue of Lemma 3.4a) again,

$$\sum_{n=1}^{\infty} g_n^2 \int_F |\widehat{V}_n(2k)|^2\, dk \leq C \sum_{n=1}^{\infty} g_n^2 \epsilon_n^2 < \infty,$$

so the monotone convergence theorem implies that $\sum g_n^2 |\widehat{V}_n(2k)|^2 < \infty$ for almost every $k \in F$. We have thus shown that for almost all $k \in F$, (1) has only bounded solutions. Therefore every subset $F' \subset F$ with $|F'| > 0$ also satisfies $\rho_{ac}^{(\alpha)}(F'^2) > 0$ [23, Theorem 5]. It is easy to see from the properties of F that this forces $\sigma_{ac} \supset F^2$. On the other hand, we know already that $\sigma_{ac} \subset F^2$, hence $\sigma_{ac} = F^2$, as claimed. □

4 Subordinate solutions

To begin with, recall the results from [4]. Write $\|y\|_x := (\int_0^x |y(t)|^2\, dt)^{1/2}$. A solution $y \not\equiv 0$ of (1) is called subordinate if $\lim_{x\to\infty} \|y\|_x / \|w\|_x = 0$ for every linearly independent solution w of the same equation. The generalized eigenfunction $v_\alpha(x, E)$ is, by definition, the solution of (1) with the initial values $v_\alpha(0, E) = -\sin\alpha, v'_\alpha(0, E) = \cos\alpha$. Note that v_α satisfies the boundary condition described by α.

The following basic result relates the notion of subordinacy to the boundary behavior of the m-function (for more information on the m-function, see e.g. [1]).

Theorem 4.1 ([4]) *$v_\alpha(\cdot, E)$ is subordinate if and only if* $\lim_{\epsilon\to 0+} |m_\alpha(E + i\epsilon)| = \infty$.

Motivated by Theorem 4.1, we consider the sets

$$S_\alpha = \{E \in \mathbb{R} : v_\alpha(\cdot, E) \text{ is subordinate}\}.$$

Then, by Theorem 4.1 and basic facts on the m-function, S_α supports the singular part of the spectral measure ρ_α (see [4]). Note that each S_α has

Lebesgue measure zero because the m-function has a finite limit almost everywhere. Therefore, the result below is of interest. It shows that S_α can be large without there being any singular continuous spectrum (namely, if $g_n \in l_2 \setminus l_1$ in Theorem 4.2).

These phenomena can *not* occur for point or absolutely continuous spectra. That is to say, the existence (or non-existence) of point and absolutely continuous spectrum, respectively, can always be read off from the size of the corresponding minimal supports. This is due to the fact that point and absolutely continuous measures are equivalent to counting and Lebesgue measure, respectively, restricted to these supports. Singular continuous measures do not, in general, have such a universal property. See also [2, 9] for a discussion of related issues in the context of Schrödinger operators.

Theorem 4.2 *Assume that $B_n = B$, $V_n(x) = W(x)$ (where $W \in L_1([-B, B])$, $W \not\equiv 0$), $g_n \to 0$, and $L_{n-1}/L_n \to 0$. Consider the Schrödinger operators H_α with potential V given by (2). Then for all boundary conditions α, we have:*
a) [8] If $g_n \in l_2$, then the spectrum of H_α is purely absolutely continuous on $(0, \infty)$.
b) If $g_n \notin l_1$, then for any open set $U \subset (0, \infty)$, the set $S_\alpha \cap U$ has Hausdorff dimension 1.

Remarks. 1. We could also treat n-dependent barriers V_n with our methods, but in this case one needs additional assumptions on the Fourier transforms $\widehat{V}_n$.

2. If $g_n \notin l_2$, the spectrum is purely singular continuous on $(0, \infty)$ by [8, Theorem 1.6(2)]. Note that part b) of the Theorem continues to hold. However, in this case this statement follows quite easily from the results of [6, 15], and the following rather involved proof would clearly be inappropriate here.

Proof. a) This is [8, Theorem 1.6(1)].

b) Here is the strategy of the proof. Fix α, and assume, without loss of generality, that U is contained in a compact subset of $(0, \infty)$. Also, as above, it will be convenient to work with wavenumbers $k = \sqrt{E}$ instead of energies E. Since I_n is constant, (7) yields

$$\ln R_{N+1}(k) = \frac{1}{2k} \mathrm{Im}\; \widehat{W}(2k) \sum_{n=1}^{N} g_n e^{2i\varphi_n(k)} + \rho_N(k) \tag{21}$$

where $|\rho_N(k)| \leq C(U) \sum_{n=1}^{N} g_n^2$ for all $k \in U$ and

$$\widehat{W}(2k) = \int_{-B}^{B} W(x) e^{2ik(x+B)}\, dx.$$

We will construct a probability measure P on U so that the sum on the right-hand side of (21) goes to $-\infty$ almost surely with respect to P. Moreover, P will give zero weight to sets of Hausdorff dimension less than 1. In this way, we will obtain "small" solutions on a set of Hausdorff dimension 1.

To carry out this program, pick numbers $N_n \in \mathbb{N}$, such that $N_n \to \infty$ and $N_n N_{n-1} L_{n-1}/L_n \to 0$ as $n \to \infty$. Since $\widehat{W}$ is analytic, its zeros are isolated. Thus it clearly suffices to prove the claim for the case when U is an open interval with $\inf_{k\in U} |\widehat{W}(2k)| > 0$ (and, as above, $\inf U > 0$, $\sup U < \infty$). For $n \geq n_0$, we let

$$l_n := \frac{\pi}{N_n L_n}\left(1 + \frac{\theta_n}{N_n}\right), \tag{22}$$

with as yet unspecified $n_0 \in \mathbb{N}$ and $\theta_n \in [0,1]$. We also set $l_{n_0-1} = |U|$. Now a straightforward computation shows that it is possible to choose first n_0 sufficiently large and then inductively the θ_n ($n \geq n_0$), such that

$$r_n := \frac{l_{n-1}}{l_n N_n} \in \mathbb{N} \quad \forall n \geq n_0.$$

For instance, we have that for $n > n_0$ and for fixed θ_{n-1} the quantity

$$\frac{l_{n-1}}{l_n N_n} = \frac{L_n}{N_{n-1}L_{n-1}} \frac{1 + \theta_{n-1}/N_{n-1}}{1 + \theta_n/N_n}$$

runs over an interval of size $\geq L_n/(L_{n-1}N_{n-1}(N_n+1))$ as θ_n runs over $[0,1]$. Since this latter expression tends to infinity, the corresponding interval must contain an integer, provided n is large enough. A similar argument works in the case $n = n_0$.

In order to simplify the notation, we will assume that $n_0 = 1$. The reader can verify easily that our arguments are valid in the general case as well.

By definition of r_1 and l_0, the original interval $U = I^{(0)}$ can be divided into $r_1 N_1$ subintervals $I_i^{(1)}$ of equal length l_1. Each of these subintervals $I_i^{(1)}$, in turn, can be divided into $r_2 N_2$ sub-subintervals $I_j^{(2)}$ of equal length l_2 etc. So we obtain a sequence of partitions $U = \bigcup_j I_j^{(n)}$ which become increasingly finer. Every interval $I_j^{(n)}$ belonging to the nth partition has length l_n and contains exactly $r_{n+1}N_{n+1}$ elements of the $(n+1)$st partition.

Now we define correspondingly discrete approximations ψ_n of the Prüfer angles φ_n. The variable ψ_n will take the values $t\pi/N_n$ ($t = 0, 1, \ldots, N_n - 1$), and ψ_n will be constant on every $I_j^{(n)}$. Let $n \in \mathbb{N}$, fix one of the intervals $I_i^{(n-1)}$, and consider the corresponding subintervals $I_j^{(n)} \subset I_i^{(n-1)}$. By construction, there are $r_n N_n$ such intervals $I_j^{(n)}$. To fix the notation, let us assume that we obtain these intervals if j runs from 1 to $r_n N_n$. We further assume that this labeling is the natural one in the sense that if $j < j'$, then $I_j^{(n)}$ lies left of $I_{j'}^{(n)}$. Now consider the sth group of N_n such intervals (where $s \in \{1, \ldots, r_n\}$), i.e. consider

$$\{I_j^{(n)} : j = (s-1)N_n + 1, \ldots, sN_n\}. \tag{23}$$

Determine $t_0 \in \{0, 1, \ldots, N_n - 1\}$ so that $|t_0\pi/N_n - \varphi_n(k_0)|$ is minimal. In this definition, k_0 is the left endpoint of the first interval (i.e. $j = (s-1)N_n + 1$ in

(23)), and the difference $t_0\pi/N_n - \varphi_n(k_0)$ has to be evaluated modulo π. Set $\psi_n(k) = (t_0 - 1 + m)\pi/N_n$ if k lies in the mth interval (i.e. $j = (s-1)N_n + m$ in (23)). Here, m takes the values $m = 1, \ldots, N_n$. Carry out this procedure for all $I_i^{(n-1)}$ to define $\psi_n(k)$ on all of $k \in U$.

In fact, strictly speaking, this does *not* define ψ_n on all of U, since we have not distinguished between the open and closed intervals $I_j^{(n)}$. So the reader might worry (rightly, at this point) that $\psi_n(k)$ is not well-defined at the endpoints of the $I_j^{(n)}$. However, as we will see shortly, the measure P we are going to construct will be continuous, and thus the above procedure does define $\psi_n(k)$ almost everywhere. This is all we will need. For definiteness, let us agree that in the sequel, it is understood that $I_j^{(n)}$ denotes the open interval.

Note that, by Lemma 2.1,

$$\varphi_n(k+l) - \varphi_n(k) = lL_n\left(1 + O(L_{n-1}/L_n)\right).$$

Using this, (22), and $N_n L_{n-1}/L_n \to 0$, one can verify easily that

$$|\psi_n(k) - \varphi_n(k)| \leq \frac{C}{N_n} \tag{24}$$

where C depends only on U.

Now let $p_n(t)$ be probabilities, i.e. we require that $p_n(t) \geq 0$ and $\sum_{t=0}^{N_n-1} p_n(t) = 1$ for all n. Notice that, by construction, $\psi_m(k)$ is constant on every interval $k \in I_j^{(n)}$ with $n \geq m$ because every such interval is contained in some $I_i^{(m)}$. Thus it makes sense to look for a measure P with the property that for all $I_j^{(n)}$,

$$P(I_j^{(n)}) = \prod_{m=1}^{n} \frac{p_m(t_m)}{r_m} \tag{25}$$

where the t_m describe the values of the ψ_m on $I_j^{(n)}$. More precisely, $t_m \in \{0, 1, \ldots, N_m - 1\}$ is defined by requiring that the value of ψ_m on $I_j^{(n)}$ be given by $t_m\pi/N_m$.

Lemma 4.3 *a) There is a unique probability (Borel) measure P on $I^{(0)} = U$, such that $P(I_j^{(n)})$ is given by (25) for all n, j.*
b) The ψ_n become independent random variables with distributions p_n, i.e.

$$P\left(\psi_1 = \frac{t_1\pi}{N_1}, \ldots, \psi_n = \frac{t_n\pi}{N_n}\right) = p_1(t_1)\cdots p_n(t_n).$$

c) Suppose $p_n(t) \leq C/N_n$ for all n, t. Then $P(A) = 0$ for every set $A \subset U$ with Hausdorff dimension $\dim A < 1$.

Remark. We gave the Lemma in a slightly more general form than actually needed here (for instance, we do not need uniqueness) because it is easy to obtain these stronger statements.

Proof. a) To begin with, note that any P satisfying (25) must be continuous: On the one hand, for any fixed n, the measures of the (open!) intervals $I_j^{(n)}$ already add up to 1, hence the endpoints of the $I_j^{(n)}$ are a set of measure zero. On the other hand, since $r_n \to \infty$, there cannot be any atoms different from these endpoints, either.

Now let P_n be the right continuous (say), monotonically increasing step function with the following properties: $P_n(\inf U) = 0$, and P_n is constant on every $I_j^{(n)}$ and has a jump of size $P(I_j^{(n)})$ at the right endpoint of $I_j^{(n)}$. It is easy to verify that the P_n increase to a limit function P which generates a probability measure consistent with (25). This proves existence.

For the uniqueness part, we need only show that (25) determines $P(a,b)$ for arbitrary $a < b$ (since we know already that P is continuous). This, however, is obvious, because any open interval (a,b) can be approximated by an increasing sequence of finite unions of the original intervals $I_j^{(n)}$.

b) This follows from (25). Note that by construction the conditions $\psi_i = t_i\pi/N_i$ $(i = 1,\dots,n)$ hold on precisely $r_1 \cdots r_n$ intervals $I_j^{(n)}$.

c) By general facts on Hausdorff measures [18, Section 3.3], the assertion will follow if we can show that for any fixed $\gamma < 1$, we have that for P-almost every k,

$$\lim_{\delta\to 0+} \sup_J \frac{P(J)}{|J|^\gamma} = 0. \tag{26}$$

Here, the sup is taken over all intervals J of length $|J| \le \delta$ which contain k. In fact, we will show that in the case at hand, (26) holds for *all* $k \in U$, and the limit is uniform in k.

So let $J \subset U$ be an interval of length $|J| \le \delta$. Let n be the largest integer for which J is contained in the union of two intervals $I_j^{(n)} \cup I_{j+1}^{(n)}$. Then J contains at least one of the intervals $I_i^{(n+1)}$ (for if this is not true, we get a contradiction to the definition of n). In particular, $2 \le 2|J|/l_{n+1}$, and thus J intersects at most $2 + (|J|/l_{n+1}) \le 3|J|/l_{n+1}$ intervals $I_i^{(n+1)}$. Clearly, each of these intervals satisfies

$$P(I_i^{(n+1)}) \le \frac{C}{r_{n+1}N_{n+1}} \max_j P(I_j^{(n)}),$$

and therefore,

$$P(J) \le \frac{3C|J|}{r_{n+1}N_{n+1}l_{n+1}} \max_j P(I_j^{(n)}) = \frac{3C|J|}{l_n} \max_j P(I_j^{(n)}).$$

Using $|J| \le 2l_n$, the definitions of r_n, l_n, and the obvious estimate $\max P(I_j^{(n)}) \le$

$C^n \prod_{m=1}^n (r_m N_m)^{-1}$, we deduce that for fixed $\gamma < 1$,

$$\frac{P(J)}{|J|^\gamma} \leq \frac{12\pi C^{n+1}}{l_0} (N_n L_n)^{\gamma - 1}.$$

Since L_n grows faster than exponentially (recall $L_n / L_{n-1} \to \infty$) and n goes to infinity as δ tends to zero, we see that we indeed have (26). □

As our next step in the proof of Theorem 4.2, we note

Lemma 4.4 *Let P be as in Lemma 4.3. Then for P-almost every k we have that*

$$\ln R_{N+1}(k) = \frac{1}{2k} \mathrm{Im}\ \widehat{W}(2k) \sum_{n=1}^N g_n E(e^{2i\psi_n}) + o\left(\sum_{n=1}^N g_n\right).$$

Remarks. 1. There is an analogue of this Lemma for n-dependent barriers V_n. The proof is more difficult in this case.

2. The estimate on the error term is not uniform in k.

Proof. If we replace the angles φ_n in (21) with their discrete counterparts ψ_n, then, by (24), we obtain an error of order

$$O\left(\sum_{n=1}^N g_n / N_n\right) = o\left(\sum_{n=1}^N g_n\right)$$

(recall $N_n \to \infty$). It remains to show that, almost surely, $\sum_{n=1}^N X_n = o(\sum_{n=1}^N g_n)$ where $X_n := g_n(e^{2i\psi_n} - E(e^{2i\psi_n}))$. By Lemma 4.3b), the X_n are independent random variables. Moreover, $EX_n = 0$ and $E|X_n|^2 \leq g_n^2$, hence, by [24, Theorem 3.3.1] with $p = 2$, it suffices to show that $\sum g_n^2 (\sum_{i=1}^n g_i)^{-2} < \infty$. This, however, is easy to check:

$$\begin{aligned}
\sum_{n=2}^\infty \frac{g_n^2}{\left(\sum_{i=1}^n g_i\right)^2} &\leq C \sum_{n=2}^\infty \frac{g_n}{\left(\sum_{i=1}^n g_i\right)^2} \\
&= C \sum_{n=2}^\infty \left(1 - \frac{g_n}{\sum_{i=1}^n g_i}\right) \left(\frac{1}{\sum_{i=1}^{n-1} g_i} - \frac{1}{\sum_{i=1}^n g_i}\right) \\
&\leq C \sum_{n=2}^\infty \left(\frac{1}{\sum_{i=1}^{n-1} g_i} - \frac{1}{\sum_{i=1}^n g_i}\right) < \infty.
\end{aligned}$$

□

Now we are at last in a position to prove the claim of the Theorem. First of all, recall the remarks made at the beginning of this proof. So, let $U \subset (0, \infty)$ be an open interval with $\inf U > 0$, $\sup U < \infty$, and $\inf_{k \in U} |\widehat{W}(2k)| > 0$. We further demand that the phase of $\widehat{W}$ does not vary much if k runs over U. More precisely, write $\widehat{W}(2k) = |\widehat{W}(2k)| e^{i\beta(2k)}$; we suppose that

$$\sup_{k, k' \in U} |\beta(2k) - \beta(2k')| \leq \pi/8. \tag{27}$$

This, of course, can be ensured by taking U sufficiently small.

Next, we want to use Lemma 4.3 to construct appropriate probability measures on U. Thus we need to specify the $p_n(t)$ from (25). To this end, first choose $t_0(n) \in \{0, 1, \ldots, N_n - 1\}$ such that

$$\left| \frac{2t_0(n)\pi}{N_n} + \beta(2k) + \pi \right| \leq \frac{\pi}{4} \tag{28}$$

for all $k \in U$ (the left-hand side of (28) is to be evaluated modulo 2π). This is possible by (27), at least if we assume, without loss of generality, that $N_n \geq 8$. We choose the N_n as even numbers and set

$$p_n(t_0(n)) = \ldots = p_n(t_0(n) - 1 + N_n/2) = 2/N_n,$$

and $p_n(t) = 0$ otherwise. (In this definition, the argument of p_n has to be evaluated modulo N_n.) Now, Lemma 4.3 yields a probability measure P, such that the ψ_n are independent with distributions determined by the $p_n(t)$, according to Lemma 4.3b).

We compute

$$\begin{aligned} E(e^{2i\psi_n}) &= \frac{2}{N_n} \sum_{t=t_0(n)}^{t_0(n)-1+N_n/2} e^{2\pi i t/N_n} = -\frac{4}{N_n} \frac{e^{2\pi i t_0(n)/N_n}}{e^{2\pi i/N_n} - 1} \\ &= \frac{2i}{\pi} e^{2\pi i t_0(n)/N_n} + O(N_n^{-1}). \end{aligned}$$

Note that by (28), we have Im $ie^{2\pi i t_0(n)/N_n} e^{i\beta(2k)} \leq -1/\sqrt{2}$ for all $n \in \mathbb{N}, k \in U$. Thus by combining Lemma 4.3c) and Lemma 4.4 we see that on a k-set of Hausdorff dimension 1, the Prüfer radius R satisfies

$$\ln R_{N+1}(k) \leq -c \sum_{n=1}^{N} g_n \tag{29}$$

for all large enough $N \geq N_0 = N_0(k)$. The constant $c > 0$ depends only on U. If we consider the corresponding set of energies $E = k^2$, we still have a set of Hausdorff dimension 1 because the map $k \to E = k^2$ is bi-Lipschitz on every compact subset of $(0, \infty)$. Recall that R is given by $R(x,k)^2 = v_\alpha(x,k)^2 + v'_\alpha(x,k)^2/k^2$ (cf. Section 2) where v_α is the generalized eigenfunction introduced at the beginning of this Section. Now let y be any solution of the Schrödinger equation (1) which is linearly independent of $v_\alpha(\cdot, k)$, and let $\tilde{R}^2 = y^2 + y'^2/k^2$ and $\tilde{\varphi}$ be the corresponding Prüfer variables. Constancy of the Wronskian $W(v_\alpha, y) = v_\alpha y' - v'_\alpha y$ yields $R\tilde{R}\sin(\varphi - \tilde{\varphi}) = w \neq 0$, hence $\tilde{R} \geq |w|/R$. It is easy to see that for the potentials under consideration, one has inequalities of the type $\|y\|_x^2 \geq c\int_0^x \tilde{R}^2$ (with $c > 0$, of course) for all sufficiently large x. Hence (29) together with the estimate $\tilde{R} \geq |w|/R$ guarantee that the solution v_α is subordinate. $\square$

The methods of this proof clearly extend to more general Hausdorff measures. More specifically, let $h(t)$ be an increasing, right continuous function on $[0,\infty)$ with $h(0)=0$ and $h(t)>0$ for $t>0$. Then one can define a (generalized) Hausdorff measure μ_h (see [18, Section 2.1]; the details are not of interest here). The usual γ-dimensional Hausdorff measures are obtained with the choice $h(t)=t^\gamma$. Now only a minor modification in the proof of Lemma 4.3c) is needed to prove the following result: Let h be as above with $\lim_{t\to 0+} h(t)/t=\infty$. If the L_n grow sufficiently rapidly (this condition can be put in a more quantitative version, of course), then $\mu_h(S_\alpha \cap U)=\infty$ for all α and all open sets $U\subset(0,\infty)$. In fact, we even have that $S_\alpha \cap U$ is not σ-finite with respect to μ_h.

I do not think that this extension of Theorem 4.2 gives much additional insight, but it does provide explicit examples where the set S_α is arbitrarily large in the measure theoretic sense (given the restriction that $|S_\alpha|=0$).

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