An Explicit Solution of the BCS Model and Demonstration of Symmetry Breaking

by

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Abstract: The BCS model, whose partition function is given by $Tr e^{-\beta H(\lambda)}$ where $H(\lambda) = \frac{1}{L^d}\sum_{k,\sigma} e(k) a^+_{k\sigma} a_{k\sigma} + \frac{\lambda}{L^{3d}}\sum_{k,p} a^+_{k\uparrow} a^+_{-k\downarrow} a_{-p\downarrow} a_{p\uparrow}$ is solved explicitely as a quartic model without using mean field theory. The Greens functions and their generating functional can be computed explicitely in arbitrary dimension d and for arbitrary energy momentum relation $e(k)$. Renormalization effects as well as symmetry breaking can be seen explicitely. The more general interaction $\frac{1}{L^{3d}}\sum_{\sigma\tau}\sum_{k,p,q}\sum_{l=0}^{N} \lambda_l\, \bar y_l(k') y_l(p')\, [\delta_{k,p} + \delta_{k,-p} + \delta_{q,0}]\, a^+_{k\sigma} a^+_{q-k\tau} a_{q-p\tau} a_{p\sigma}$ which contains a forward, an exchange and a BCS term can also be treated. For pure BCS, one obtains a $6(N+1)$ dimensional integral representation for the two point functions and for the generating functional. The existence and symmetry of a gap is determined by the global minimum of an effective potential of $6(N+1)$ variables. We show, in 3 space dimensions, that the usual mean field approach may be misleading if the electron electron interaction contains higher angular momentum terms. In particular, if the interaction is given by a single attractive even l term $\frac{\lambda_l}{L^{3d}}\sum_{m=-l}^{l} Y_{lm}(k')\bar Y_{lm}(p')\, a^+_{k\uparrow} a^+_{-k\downarrow} a_{-p\downarrow} a_{p\uparrow}$, then, if $e(k)$ has $SO(3)$ symmetry, the expectations $\langle a^+_{k\sigma} a_{k\sigma}\rangle$ also have to have $SO(3)$ symmetry. The property which makes the model explicitely solvable is the fact that the interaction is given by a finite sum of products of quadratic factors $\sum_{\sigma\tau}\sum_{l=0}^{N}\big(\sum_k \bar y_l(k')\, a^+_{k\sigma} a^+_{-k\tau}\big)\big(\sum_p y_l(p')\, a_{-p\tau} a_{p\sigma}\big)$.

I. Introduction

In this paper, we present an explicit solution of the BCS model. The quartic Hamiltonian is not approximated by a mean field Hamiltonian. The partition function of the BCS model in d dimensions at temperature $T = \frac{1}{\beta} > 0$ is given by

$$Z = Z(\beta, L, \lambda) = Tr\, e^{-\beta H(\lambda)} \tag{I.1}$$

where

$$H(\lambda) = H_0 + I = \tfrac{1}{L^d} \sum_{\mathbf{k},\sigma} e_{\mathbf{k}}\, a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \tfrac{\lambda}{L^d} \tfrac{1}{L^{2d}} \sum_{\mathbf{k},\mathbf{p}} a^+_{\mathbf{k}\uparrow} a^+_{-\mathbf{k}\downarrow} a_{\mathbf{p}\uparrow} a_{-\mathbf{p}\downarrow} \tag{I.2}$$

The spatial momenta range over some subset of $\left(\frac{2\pi}{L}\mathbf{Z}\right)^d$, say, $\mathbf{k} \in M = \{\mathbf{k} \in (\frac{2\pi}{L}\mathbf{Z})^d \,|\, |e_{\mathbf{k}}| \leq \omega_D\}$ where L^d is the spatial volume of the system and $e_{\mathbf{k}}$ denotes the energy momentum relation. We assume $e_{\mathbf{k}} = e_{-\mathbf{k}}$. A positive λ corresponds to an attractive interaction.

We also consider the more general interaction

$$I\left(\{\lambda_\ell\}\right) = \tfrac{1}{L^{2d}} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{\mathbf{k},\mathbf{p}} \tfrac{1}{L^d} U(\mathbf{k}'-\mathbf{p}')\, a^+_{\mathbf{k}\sigma} a^+_{-\mathbf{k}\tau} a_{\mathbf{p}\sigma} a_{-\mathbf{p}\tau} \tag{I.3}$$

where, for $e_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu$, $\mathbf{k}' = \sqrt{2m\mu}\frac{\mathbf{k}}{|\mathbf{k}|} =: k_F \frac{\mathbf{k}}{|\mathbf{k}|}$ and

$$U(\mathbf{k}'-\mathbf{p}') = \sum_{\ell=0}^{n} \lambda_\ell P_\ell\left(\tfrac{\mathbf{k}'\mathbf{p}'}{k_F^2}\right) = \sum_{\ell=0}^{n} \sum_{m=-\ell}^{\ell} \lambda_\ell Y_{\ell m}\left(\tfrac{\mathbf{k}'}{k_F}\right) \bar{Y}_{\ell m}\left(\tfrac{\mathbf{p}'}{k_F}\right), \quad d=3 \tag{I.4}$$

$$U(\mathbf{k}'-\mathbf{p}') = \sum_{\ell=0}^{n} \lambda_\ell \cos\left[\ell(\varphi_{\mathbf{k}} - \varphi_{\mathbf{p}})\right] = \frac{1}{2} \sum_{\ell=-n}^{n} \lambda_{|\ell|} e^{i\ell\varphi_{\mathbf{k}}} e^{-i\ell\varphi_{\mathbf{p}}} + \frac{\lambda_0}{2}, \quad d=2 \tag{I.5}$$

which reduces to (I.2) for $n=0$. To simplify notation, suppose that

$$U(\mathbf{k}'-\mathbf{p}') = \sum_{l=0}^{N} \lambda_l\, y_l(\mathbf{k}')\, \bar{y}_l(\mathbf{p}') \tag{I.6}$$

where $l = (\ell, m)$ or $l = \ell$ depending on the dimension of space.

In section IV.2, by specializing Theorem III.1, in which a more general model with forward, exchange and BCS interaction is solved, we prove the following $6(N+1)$ dimensional integral representation for the $\langle a^+ a\rangle$ expectation values:

$$\begin{aligned}\langle a^+_{p_0,\mathbf{p},\uparrow} a_{p_0',\mathbf{p}',\uparrow}\rangle_{\beta,L} &= \frac{Tr\, e^{-\beta[H_0 + I(\{\lambda_l\})]} a^+_{p_0,\mathbf{p},\uparrow} a_{p_0',\mathbf{p}',\uparrow}}{Tr\, e^{-\beta[H_0+I(\{\lambda_l\})]}} \\ &= \beta L^d \delta_{p,p'} \frac{\int F^{\uparrow\uparrow}_{\mathbf{p}}(\phi)\, e^{-\beta L^d V(\phi)} \prod_{\sigma\tau} \prod_{l=0}^{N} du^l_{\sigma\tau} dv^l_{\sigma\tau}}{\int e^{-\beta L^d V(\phi)} \prod_{\sigma\tau} \prod_{l=0}^{N} du^l_{\sigma\tau} dv^l_{\sigma\tau}}\end{aligned} \tag{I.7}$$

Here $\phi^l_{\sigma\tau} = u^l_{\sigma\tau} + iv^l_{\sigma\tau}$, $\sigma\tau \in \{\uparrow\uparrow, \downarrow\downarrow, \uparrow\downarrow\}$,

$$F^{\uparrow\uparrow}_{\mathbf{p}}(\phi) = \frac{(ip_0+e_{\mathbf{p}})[p_0^2+e_{\mathbf{p}}^2+\Phi_{-\mathbf{p}\uparrow\downarrow}\bar\Phi_{-\mathbf{p}\uparrow\downarrow}+\Phi_{\mathbf{p}\downarrow\downarrow}\bar\Phi_{\mathbf{p}\downarrow\downarrow}]}{(p_0^2+e_{\mathbf{p}}^2+\Omega^+_{\mathbf{p}})(p_0^2+e_{\mathbf{p}}^2+\Omega^-_{\mathbf{p}})}$$

$$\Phi_{\mathbf{k}\uparrow\downarrow} = \sum_{l=0}^N \lambda_l^{\frac12}\,\phi^l_{\uparrow\downarrow}\,y_l(\mathbf{k}')\,, \quad \bar\Phi_{\mathbf{k}\uparrow\downarrow} = \sum_{l=0}^N \lambda_l^{\frac12}\,\bar\phi^l_{\uparrow\downarrow}\,\bar y_l(\mathbf{k}')\,,$$

$$\Phi_{\mathbf{k}\sigma\sigma} = \sum_{l=0}^N \lambda_l^{\frac12}\,\phi^l_{\sigma\sigma}\,[y_l(\mathbf{k}')-y_l(-\mathbf{k}')]\,, \quad \bar\Phi_{\mathbf{k}\sigma\sigma} = \sum_{l=0}^N \lambda_l^{\frac12}\,\bar\phi^l_{\sigma\sigma}\,[\bar y_l(\mathbf{k}')-\bar y_l(-\mathbf{k}')]$$

and $\Omega^\pm_{\mathbf{k}}$ are the solutions of the quadratic equation

$$\begin{aligned}\Omega^2 - \Big(\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow} + \bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} + \Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Big)\Omega + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow} \\ + \bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} = 0 \end{aligned} \tag{I.8}$$

Observe that $\bar\Phi_{\mathbf{k}\sigma\tau}$ is not necessarily the complex conjugate of $\Phi_{\mathbf{k}\sigma\tau}$, depending on the signs of the coupling constants λ_l. The effective potential V_β is given by

$$V_\beta(\phi) = \sum_{l=0}^N \left(|\phi^l_{\uparrow\downarrow}|^2 + |\phi^l_{\uparrow\uparrow}|^2 + |\phi^l_{\downarrow\downarrow}|^2\right) - \sum_{\epsilon\in\{+,-\}} \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\Omega^\epsilon_{\mathbf{k}}})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right] \tag{I.9}$$

For a pure even interaction, that is, if $\lambda_\ell = 0$ for all odd angular momentum ℓ, one has $\Phi_{\mathbf{k}\uparrow\uparrow} = \Phi_{\mathbf{k}\downarrow\downarrow} = 0$ and the expectation value simplifies to

$$\langle a^+_{p_0,\mathbf{p},\sigma} a_{p_0',\mathbf{p}',\sigma}\rangle_{\beta,L} = \beta L^d \delta_{p,p'}\,\frac{\int \frac{ip_0+e_{\mathbf{p}}}{p_0^2+e_{\mathbf{p}}^2+\Phi_{\mathbf{p}\uparrow\downarrow}\bar\Phi_{\mathbf{p}\uparrow\downarrow}} e^{-\beta L^d V_\beta(\phi_{\uparrow\downarrow})} \prod_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}}{\int e^{-\beta L^d V_\beta(\phi_{\uparrow\downarrow})} \prod_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}} \tag{I.10}$$

with an effective potential

$$V_\beta(\phi_{\uparrow\downarrow}) = \sum_{l=0}^N |\phi^l_{\uparrow\downarrow}|^2 - \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]^2 \tag{I.11}$$

In particular, for a delta function interaction one obtains the two dimensional integral representation

$$\langle a^+_{p_0,\mathbf{p},\sigma} a_{p_0',\mathbf{p}',\sigma}\rangle_{\beta,L} = \beta L^d \delta_{p,p'}\,\frac{\int \frac{ip_0+e_{\mathbf{p}}}{p_0^2+e_{\mathbf{p}}^2+\lambda(u^2+v^2)}\; e^{-\beta L^d V_\beta(u,v)}dudv}{\int e^{-\beta L^d V_\beta(u,v)}dudv} \tag{I.12}$$

where

$$V_\beta(u,v) = u^2+v^2 - \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda(u^2+v^2)})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]^2 \tag{I.13}$$

A positive λ corresponds to an attractive interaction.

Since the only place where the volume L^d shows up in the integral representations (I.7,10) is the prefactor in the exponential $e^{-\beta L^d V(\phi)}$, the computation of the infinite volume limit comes down to the determination of the global minimum of the real part of the effective potential V as a function of the $\phi^l_{\sigma\tau}$'s.

The expectation values $\langle a^+ a^+\rangle$ and $\langle aa\rangle$ are also computed. To make them nonzero, we introduce a small external field $r = |r|e^{i\alpha}$. That is, we substitute $H(\{\lambda_l\})$ by (we do not consider here $\langle a_\uparrow a_\uparrow\rangle$ expectations)

$$H_r = H(\{\lambda_l\}) + \tfrac{1}{L^d}\sum_{\mathbf{k}}[r\, a_{\mathbf{k}\uparrow} a_{-\mathbf{k}\downarrow} - \bar{r}\, a^+_{\mathbf{k}\uparrow} a^+_{-\mathbf{k}\downarrow}] \tag{I.15}$$

One obtains again a $6(N+1)$ dimensional integral representation. For a delta function interaction, that is for $N = 0$ in (I.6), it reduces to a two dimensional integral:

$$\langle a^+_{p_0,\mathbf{p},\uparrow} a^+_{-p_0',-\mathbf{p}',\downarrow}\rangle_{\beta,L} = \beta L^d \delta_{p,p'} \frac{\int \frac{-i\sqrt{\lambda}\, e^{i\alpha}(u+iv)}{p_0^2+e_{\mathbf{p}}^2+\lambda(u^2+v^2)}\, e^{-\beta L^d V_{\beta,r}(u,v)} du dv}{\int e^{-\beta L^d V_{\beta,r}(u,v)} du dv} \tag{I.16}$$

where

$$V_{\beta,r}(u,v) = u^2 + \left(v + \tfrac{|r|}{\sqrt{\lambda}}\right)^2 - \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\, \tfrac{1}{\beta} \log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda(u^2+v^2)})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]^2 \tag{I.17}$$

For attractive $\lambda > 0$ and sufficiently small $T = \frac{1}{\beta}$, the potential $V_{\beta,r}$ has the form of a mexican hat. For $r = 0$, the global minimum of $V_{\beta,r}$ is degenerated and lies on a circle in the u, v plane. In particular, $V_{\beta,0}$ is an even function of u and v and $\langle a^+_{p\uparrow} a^+_{-p\downarrow}\rangle$ vanishes by symmetry.

For $r \neq 0$, $V_{\beta,r}$ has a unique global minimum at $(u,v) = (0, v_0)$ where v_0 is given by the negative solution of

$$v_0 \left\{\lambda \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d} \frac{\tanh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda v_0^2})}{2\sqrt{e_{\mathbf{k}}^2+\lambda v_0^2}} - 1\right\} = 2|r| \tag{I.18}$$

which, in the limit $r \to 0$, becomes the BCS equation for $|\Delta|^2 = \lambda v_0^2$. Thus

$$\lim_{|r|\to 0}\lim_{L\to\infty} \frac{e^{-\beta L^d V_{\beta,r}(u,v)}}{\int e^{-\beta L^d V_{\beta,r}(u,v)} du dv} = \lim_{|r|\to 0}\lim_{L\to\infty} \frac{e^{-\beta L^d [V_{\beta,r}(u,v) - V_{\beta,r}(0,v_0)]}}{\int e^{-\beta L^d [V_{\beta,r}(u,v) - V_{\beta,r}(0,v_0)]} du dv} = \delta(u)\delta\left(v + \tfrac{|\Delta|}{\sqrt{\lambda}}\right) \tag{I.19}$$

and $\langle a^+_{p\uparrow} a^+_{-p\downarrow}\rangle$ becomes nonzero.

For repulsive $\lambda < 0$, $V_{\beta,r}$ is complex and the real part of $V_{\beta,r}$ has a unique global minimum at $(u,v) = (0,0)$ which results in

$$\lim_{|r|\to 0}\lim_{L\to\infty} \frac{e^{-\beta L^d V_{\beta,r}(u,v)}}{\int e^{-\beta L^d V_{\beta,r}(u,v)} du dv} = \delta(u)\delta(v)$$

and $\lim_{|r|\to 0}\lim_{L\to\infty}\langle a^+_{p\uparrow}a^+_{-p\downarrow}\rangle = 0$.

The starting point for our analysis is the perturbation series for the partition function which in terms of fermionic functional integrals is given by (for $N=0$)

$$
\begin{aligned}
Z(\beta,L,\lambda) &= \frac{Tr\, e^{-\beta\left[\frac{1}{L^d}\sum_{\mathbf{k},\sigma} e_{\mathbf{k}}\, a^+_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}+\frac{\lambda}{L^d}\frac{1}{L^{2d}}\sum_{\mathbf{k},\mathbf{p}} a^+_{\mathbf{k}\uparrow}a^+_{-\mathbf{k}\downarrow}a_{\mathbf{p}\uparrow}a_{-\mathbf{p}\downarrow}\right]}}{Tr\, e^{-\beta\left[\frac{1}{L^d}\sum_{\mathbf{k},\sigma} e_{\mathbf{k}}\, a^+_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\right]}} \\
&= \int e^{-\frac{\lambda}{(\beta L^d)^3}\sum_{k,p,q_0}\bar\psi_{k\uparrow}\bar\psi_{q_0-k\downarrow}\psi_{p\uparrow}\psi_{q_0-p\downarrow}}\,d\mu_C(\psi,\bar\psi)
\end{aligned}
\tag{I.20}
$$

where

$$
d\mu_C(\psi,\bar\psi) = \prod_{k,\sigma}\frac{\beta L^d}{ik_0-e_{\mathbf{k}}}\; e^{-\frac{1}{\beta L^d}\sum_{k,\sigma}(ik_0-e_{\mathbf{k}})\bar\psi_{k,\sigma}\psi_{k,\sigma}}\prod_{k,\sigma} d\psi_{k,\sigma}d\bar\psi_{k,\sigma}
\tag{I.21}
$$

and $k=(k_0,\mathbf{k})\in\frac{\pi}{\beta}(2\mathbf{Z}+1)\times M$. The variables k_0, p_0 and q_0 which do not appear in the first line of (I.20) are the Fourier transform variables of a variable τ which enters the perturbation series because of $\frac{d}{d\lambda}_{|\lambda=0}e^{-\beta(H_0+\lambda I)}=\int_0^\beta e^{-(\beta-\tau)H_0}Ie^{-\tau H_0}d\tau$.

The perturbation series for a δ-function interaction without BCS approximation reads

$$
\int e^{-\lambda\frac{1}{(\beta L^d)^3}\sum_{k,p,q}\bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}\psi_{p\uparrow}\psi_{q-p\downarrow}}\,d\mu_C(\psi,\bar\psi)
\tag{I.22}
$$

Thus the BCS approximation consists in putting the spatial transfer momentum $\mathbf{q}$ equal to zero. For $e_{\mathbf{k}}=\frac{\mathbf{k}^2}{2m}-\mu$, Feldman and Trubowitz [FT] proved that the most singular contributions to the Feynman diagramms come from $q_0=0$. Therefore we put also q_0 equal to zero which (obviously) makes the model explicitly solvable. Thus as our starting point we choose an approximation where $\mathbf{q}=0$ and $q_0=0$.

That is, the model we solve in this paper is not exactly the model defined by (I.1) and (I.2) or (I.3), but is the model which is defined by the perturbation series (for N=0)

$$
Z(\beta,L,\lambda) = \int e^{-\frac{\lambda}{\beta L^d}\frac{1}{(\beta L^d)^2}\sum_{k,p}\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}\psi_{p\uparrow}\psi_{-p\downarrow}}\,d\mu_C(\psi,\bar\psi)
\tag{I.23}
$$

In particular, it does not come from a Hamiltonian.

We start in section II.1 with a solution of the model with δ-function interaction. This is simply done using a standard method. Namely, introduce a bosonic field ϕ, integrate out the fermions and obtain a quotient of determinants which gives the exponential of an effective potential. One just has to observe that for $\mathbf{q}=0$, $q_0=0$ the bosonic field ϕ has to be only two dimensional and that the determinant involving ϕ can be computed explicitely. That is, use the identity ($\phi=u+iv$, $\bar\phi=u-iv\in\mathbf{C}$)

$$
e^{2ab}=\frac{1}{2\pi}\int_{\mathbf{R}^2}e^{a\phi+b\bar\phi}e^{-\frac{1}{2}|\phi|^2}dudv
\tag{I.24}
$$

to obtain

$$\begin{aligned}
Z &= \int e^{-\left(\frac{\lambda}{\beta L^d}\right)^{\frac{1}{2}} \frac{1}{\beta L^d}\sum_k \bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}\left(\frac{\lambda}{\beta L^d}\right)^{\frac{1}{2}} \frac{1}{\beta L^d}\sum_p \psi_{p\uparrow}\psi_{-p\downarrow}} d\mu_C(\psi,\bar\psi)\\
&= \int_{\mathbf{R}^2}\int e^{i\left(\frac{\lambda}{\beta L^d}\right)^{\frac{1}{2}}\phi\,\frac{1}{\beta L^d}\sum_k \psi_{k\uparrow}\psi_{-k\downarrow}+i\left(\frac{\lambda}{\beta L^d}\right)^{\frac{1}{2}}\bar\phi\,\frac{1}{\beta L^d}\sum_k \bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}} d\mu_C(\psi,\bar\psi)\,\tfrac{1}{2\pi}e^{-\frac{1}{2}|\phi|^2}dudv\\
&= \int_{\mathbf{R}^2}\prod_k \tfrac{1}{k_0^2+e_{\mathbf{k}}^2}\det\begin{pmatrix} ik_0-e_{\mathbf{k}} & i\lambda^{\frac{1}{2}}\bar\phi\\ i\lambda^{\frac{1}{2}}\phi & -ik_0-e_{\mathbf{k}}\end{pmatrix}\tfrac{\beta L^d}{\pi}e^{-\beta L^d|\phi|^2}dudv\\
&= \tfrac{\beta L^d}{\pi}\int_{\mathbf{R}^2} e^{-\beta L^d V_\beta(u,v)}dudv
\end{aligned}\tag{I.25}$$

In the same way one obtains integral representations for the two point functions and even for the generating functional of the connected amputated Greens functions.

In section II.2 we show that the partition function may also be computed by a direct summation of the perturbation series. Instead of using (I.24) and integrating out the fermions, the perturbation series (I.23)

$$Z(\beta,L,\lambda)=\sum_{n=0}^\infty \frac{\left(\frac{\lambda}{\kappa^3}\right)^n}{n!}\sum_{k_1,\cdots,k_n}\sum_{p_1,\cdots,p_n}\det\left[\kappa\delta_{k_i,p_j}C(k_i)\right]_{1\le i,j\le n}\det\left[\kappa\delta_{k_i,p_j}C(-k_i)\right]_{1\le i,j\le n}$$

can be summed up explicitly by expanding the determinants and by an appropriate reordering of the resulting diagramms. Here $\kappa=\beta L^d$.

In section III we consider the model for arbitrary N (see I.6). Furthermore, we include a forward and an exchange scattering term in the interaction. That is, we consider the partition function $Z=\int e^{-\mathcal{U}(\psi,\bar\psi)}d\mu_C$ where

$$\mathcal{U}(\psi,\bar\psi)=\tfrac{1}{\kappa^3}\sum_{\sigma,\tau\in\{\uparrow,\downarrow\}}\sum_{k,p,q}U(\mathbf{k}'-\mathbf{p}')\left[\delta_{k,p}+\delta_{k,-p}+\delta_{q,0}\right]\bar\psi_{k,\sigma}\bar\psi_{q-k,\tau}\psi_{p,\sigma}\psi_{q-p,\tau}\tag{I.26}$$

This model can also be solved. The only difference is that more bosonic fields are needed. But still, the dimension of the resulting integral representations does not depend on the volume or on other cuttoffs.

In the fourth section we specialize to the following cases. In IV.1 we consider a δ-function interaction with a forward, an exchange and a BCS term. In section IV.2 we discuss the case of a pure BCS interaction for arbitrary N. That is, the interaction includes higher angular momentum terms. Using the integral representation (I.10), we prove in Theorem IV.2 that, in 3 dimensions, if the interaction is given by a single attractive even ℓ term $\frac{\lambda_\ell}{L^{3d}}\sum_{m=-\ell}^{\ell}\bar Y_{\ell m}(\mathbf{k}')Y_{\ell m}(\mathbf{p}')\,a^+_{\mathbf{k}\uparrow}a^+_{-\mathbf{k}\downarrow}a_{-\mathbf{p}\downarrow}a_{\mathbf{p}\uparrow}$, then, if $e_{\mathbf{k}}$ has $SO(3)$ symmetry, the expectations $\langle a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\rangle$ also have to have $SO(3)$ symmetry. In conjunction with a result of Feldman, Knörrer and Trubowitz [FKT], who proved that the BCS 2×2 matrix gap equation [AB,BW] does not have unitary isotropic solutions for $\ell\ge 2$, this indicates that in 3 dimensions for $\ell\ge 2$ the standard mean field approach may be misleading.

Finally one may hope that the method presented here turns out to be useful also for other models, including bosonic ones. The strategy would be to write down the perturbation expansion in fermionic or bosonic functional integral form and to approximate all quartic terms by a sum of type (I.26). Then one can use (I.24) to transform each quartic term into a quadratic one and the fermionic or bosonic functional integral can be performed. One ends up with a finite dimensional integral, the dimension D being two times the number of quartic terms in the approximation (I.26), where the integrand, an infinite product over momenta, gives the exponential of an effective potential. Then if, as it is the case for the BCS model, the cuttoffs (here volume) only show up as prefactors of the effective potential, then removing the cuttoffs is equivalent to finding the global minimum of the real part of V which is a function of $D/2$ complex variables.

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II. The Model with Delta Function Interaction

II.1 Computation of the Two Point Functions and the Generating Functional

In this section, we compute the partition function

$$Z(\beta,L,\{s_k\},\{r_k\}) = \int \exp\Big\{-\tfrac{\lambda}{\kappa}\tfrac{1}{\kappa^2}\sum_{k,p}\psi_{k,\uparrow}\psi_{-k,\downarrow}\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow} + \tfrac{1}{\kappa}\sum_k[s_{k,\uparrow}\bar\psi_{k,\uparrow}\psi_{k,\uparrow}$$
$$+ s_{k,\downarrow}\bar\psi_{k,\downarrow}\psi_{k,\downarrow} + r_k\psi_{k,\uparrow}\psi_{-k,\downarrow} - \bar r_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}]\Big\}d\mu_C(\psi,\bar\psi) \tag{II.1}$$

for arbitrary numbers $s_{k,\uparrow}, s_{k,\downarrow}, r_k, \bar r_k$ and the two point functions

$$\tfrac{1}{\kappa}\langle\bar\psi_{p,\sigma}\psi_{p,\sigma}\rangle_{\beta,L,r} = \frac{1}{Z(\beta,L,r)}\int \tfrac{1}{\kappa}\bar\psi_{p,\sigma}\psi_{p,\sigma}\ \exp\Big\{-\tfrac{\lambda}{\kappa^3}\sum_{k,p}\psi_{k,\uparrow}\psi_{-k,\downarrow}\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}$$
$$+\tfrac{1}{\kappa}\sum_k[r\psi_{k,\uparrow}\psi_{-k,\downarrow} - \bar r\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}]\Big\}d\mu_C(\psi,\bar\psi)$$
$$= \tfrac{\partial}{\partial s_{k,\sigma}}\log Z(\beta,L,\{s_k\},\{r_k\})\big|_{s_k=0,r_k=r} \tag{II.2}$$

$$\tfrac{1}{\kappa}\langle\psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \frac{1}{Z(\beta,L,r)}\int \tfrac{1}{\kappa}\psi_{p,\uparrow}\psi_{-p,\downarrow}\ \exp\Big\{-\tfrac{\lambda}{\kappa^3}\sum_{k,p}\psi_{k,\uparrow}\psi_{-k,\downarrow}\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}$$
$$+\tfrac{1}{\kappa}\sum_k[r\psi_{k,\uparrow}\psi_{-k,\downarrow} - \bar r\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}]\Big\}d\mu_C(\psi,\bar\psi)$$
$$= \tfrac{\partial}{\partial r_k}\log Z(\beta,L,\{s_k\},\{r_k\})\big|_{s_k=0,r_k=r} \tag{II.3}$$

Here $k=(k_0,\mathbf{k})\in\frac{\pi}{\beta}(2\mathbf{Z}+1)\times M$ where $M=\Big\{\mathbf{k}\in\big(\frac{2\pi}{L}\mathbf{Z}\big)^d\ \big|\ |e(\mathbf{k})|\le\omega_D\Big\}$ and $\kappa=\beta L^d$.

Proposition II.1: Let $\phi=u+iv$, $\bar\phi=u-iv$ where u and v are one dimensional real variables. Let $g=\sqrt{\lambda}\in\mathbf{C}$ and let $\kappa=\beta L^d$. Then, if $a_k=ik_0-e_{\mathbf{k}}$, one has the two dimensional integral representations

a) $$Z(\beta,L,\{s_k\},\{r_k\}) = \int\prod_k \tfrac{(a_k-s_{k,\uparrow})(a_{-k}-s_{-k,\downarrow})+(g\phi-ir_k)(g\bar\phi+i\bar r_k)}{a_k a_{-k}}\ \tfrac{\kappa}{\pi}e^{-\kappa|\phi|^2}dudv$$

In particular, for $r_k=r$ and $s_k=0$ for all k

$$Z(\beta,L,r) = \int\prod_k\Big\{1+\tfrac{(g\phi-ir)(g\bar\phi+i\bar r)}{a_k a_{-k}}\Big\}\tfrac{\kappa}{\pi}e^{-\kappa|\phi|^2}dudv$$

b) The $\bar\psi\psi$ expectation values are given by

$$\tfrac{1}{\kappa}\langle\bar\psi_{p,\sigma}\psi_{p,\sigma}\rangle_{\beta,L,r} = \frac{1}{Z}\int \frac{-a_{-p}}{a_p a_{-p}+(g\phi-ir)\big(g\bar\phi+i\bar r\big)}\ \prod_k\Big\{1+\frac{(g\phi-ir)\big(g\bar\phi+i\bar r\big)}{a_k a_{-k}}\Big\}\tfrac{\kappa}{\pi}e^{-\kappa|\phi|^2}dudv$$

c) The $\psi\psi$ and $\bar\psi\bar\psi$ expectation values are given by

$$\tfrac{1}{\kappa}\langle\psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \frac{1}{Z}\int \frac{-i\big(g\bar\phi+i\bar r\big)}{a_p a_{-p}+(g\phi-ir)\big(g\bar\phi+i\bar r\big)}\ \prod_k\Big\{1+\frac{(g\phi-ir)\big(g\bar\phi+i\bar r\big)}{a_k a_{-k}}\Big\}\tfrac{\kappa}{\pi}e^{-\kappa|\phi|^2}dudv$$

$$\tfrac{1}{\kappa}\langle\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \frac{1}{Z}\int \frac{-i(g\phi-ir)}{a_p a_{-p}+(g\phi-ir)\big(g\bar\phi+i\bar r\big)}\ \prod_k\Big\{1+\frac{(g\phi-ir)\big(g\bar\phi+i\bar r\big)}{a_k a_{-k}}\Big\}\tfrac{\kappa}{\pi}e^{-\kappa|\phi|^2}dudv$$

Proof: a) Define the measure

$$d\nu(\phi,\bar\phi) = \tfrac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)}du\,dv$$

Then, for some commuting elements a, b,

$$\begin{aligned}\int e^{a\phi+b\bar\phi}d\nu(\phi,\bar\phi) &= \tfrac{1}{\sqrt{2\pi}}\int e^{(a+b)u}e^{-\frac{1}{2}u^2}du\tfrac{1}{\sqrt{2\pi}}\int e^{i(a-b)v}e^{-\frac{1}{2}v^2}dv\\ &= e^{\frac{1}{2}(a+b)^2}e^{-\frac{1}{2}(a-b)^2} = e^{2ab}\end{aligned}$$

and we may write

$$\begin{aligned}e^{-\frac{\lambda}{\kappa^3}\sum_{k,p}\psi_{k,\uparrow}\psi_{-k,\downarrow}\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}} &= e^{-2\left(\frac{\lambda}{2\kappa}\right)^{\frac{1}{2}}\frac{1}{\kappa}\sum_k\psi_{k,\uparrow}\psi_{-k,\downarrow}\left(\frac{\lambda}{2\kappa}\right)^{\frac{1}{2}}\frac{1}{\kappa}\sum_p\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}}\\ &= \int e^{i\phi\,\left(\frac{\lambda}{2\kappa}\right)^{\frac{1}{2}}\frac{1}{\kappa}\sum_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+i\bar\phi\,\left(\frac{\lambda}{2\kappa}\right)^{\frac{1}{2}}\frac{1}{\kappa}\sum_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}}d\nu(\phi,\bar\phi)\\ &= \int e^{i\phi\,g\frac{1}{\kappa}\sum_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+i\bar\phi\,g\frac{1}{\kappa}\sum_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}}\tfrac{\kappa}{\pi}\,e^{-\kappa|\phi|^2}dudv\end{aligned}$$

Then the partition function becomes

$$\begin{aligned}Z(\beta,L) = \tfrac{\kappa}{\pi}\int\!\!\int & e^{ig\phi\frac{1}{\kappa}\sum_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+ig\bar\phi\frac{1}{\kappa}\sum_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}}\times\\ & e^{\frac{1}{\kappa}\sum_k[s_{k,\uparrow}\bar\psi_{k,\uparrow}\psi_{k,\uparrow}+s_{k,\downarrow}\bar\psi_{k,\downarrow}\psi_{k,\downarrow}+r_k\psi_{k,\uparrow}\psi_{-k,\downarrow}-\bar r_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}}d\mu_C(\psi,\bar\psi)\;e^{-\kappa|\phi|^2}dudv\\ = \tfrac{\kappa}{\pi}\int\!\!\int & e^{\frac{1}{\kappa}\sum_k i\gamma_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+\frac{1}{\kappa}\sum_k i\bar\gamma_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}}\times\\ & e^{\frac{1}{\kappa}\sum_k[s_{k,\uparrow}\bar\psi_{k,\uparrow}\psi_{k,\uparrow}+s_{k,\downarrow}\bar\psi_{k,\downarrow}\psi_{k,\downarrow}}d\mu_C(\psi,\bar\psi)\;e^{-\kappa|\phi|^2}dudv\end{aligned}$$

where we defined

$$\gamma_k = g\phi - ir_k,\quad \bar\gamma_k = g\bar\phi + i\bar r_k$$

Observe that $\bar\gamma_k$ is not necessarily the complex conjugate of γ_k. Now the Fermionic functional integral can be done. Let

$$a_k = ik_0 - e_{\mathbf{k}}$$

One obtains

$$\begin{aligned}
&\int e^{\frac{1}{\kappa}\sum_k [i\gamma_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+i\bar\gamma_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}+s_{k,\uparrow}\bar\psi_{k,\uparrow}\psi_{k,\uparrow}+s_{k,\downarrow}\bar\psi_{k,\downarrow}\psi_{k,\downarrow}]}d\mu_C(\psi,\bar\psi) \\
&= \prod_k \frac{\kappa^2}{a_k a_{-k}} \int e^{\frac{1}{\kappa}\sum_k [i\gamma_k\psi_{k,\uparrow}\psi_{-k,\downarrow}+i\bar\gamma_k\bar\psi_{k,\uparrow}\bar\psi_{-k,\downarrow}-(a_k-s_{k,\uparrow})\bar\psi_{k,\uparrow}\psi_{k,\uparrow}-(a_k-s_{k,\downarrow})\bar\psi_{k,\downarrow}\psi_{k,\downarrow}]} \prod_{k,\sigma} d\psi_{k,\sigma}d\bar\psi_{k,\sigma} \\
&= \prod_k \frac{\kappa^2}{a_k a_{-k}} \int e^{-\frac{1}{\kappa}\sum_k (\bar\psi_{k,\uparrow},\psi_{-k,\downarrow})\left(\begin{smallmatrix} a_k-s_{k\uparrow} & -i\bar\gamma_k \\ i\gamma_k & -a_{-k}+s_{-k\downarrow}\end{smallmatrix}\right)\left(\begin{smallmatrix}\psi_{k,\uparrow} \\ \bar\psi_{-k,\downarrow}\end{smallmatrix}\right)} \prod_k(-d\psi_{k\uparrow}d\bar\psi_{k\uparrow}d\bar\psi_{-k\downarrow}d\psi_{-k\downarrow}) \\
&= \prod_k \frac{\kappa^2}{a_k a_{-k}} \prod_k \left(\frac{-1}{\kappa^2}\det\begin{bmatrix} a_k-s_{k\uparrow} & -i\bar\gamma_k \\ i\gamma_k & -a_{-k}+s_{-k\downarrow})\end{bmatrix}\right) \\
&= \prod_k \frac{(a_k-s_{k,\uparrow})(a_{-k}-s_{-k,\downarrow})+\gamma_k\bar\gamma_k}{a_k a_{-k}}
\end{aligned}$$

This proves the formula under a). Part b) and c) follow by differentiation with respect to $s_{k,\sigma}$ or r_k, $\bar r_k$ •

Lemma II.2: Let $a_k = ik_0 - e_{\mathbf{k}}$. Using the approximation $\sum_{\mathbf{k}\in M} \approx L^d \int_M \frac{d^d\mathbf{k}}{(2\pi)^d}$ one has

$$\prod_{\mathbf{k}\in M}\ \prod_{k_0\in\frac{\pi}{\beta}(2\mathbf{Z}+1)} \left\{1+\frac{\xi}{a_k a_{-k}}\right\} = e^{\kappa W_\beta(\xi)} \tag{II.4}$$

where $\kappa = \beta L^d$ and

$$W_\beta(\xi) = \int_M \frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{\beta}\log\left[\frac{\cosh\left(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\xi}\right)}{\cosh\left(\frac{\beta}{2}e_{\mathbf{k}}\right)}\right]^2 \tag{II.5}$$

Proof: The product over k_0 may be computed exactly by using the formula [H]

$$\prod_{n=0}^{\infty}\left(1+\frac{\xi}{(2n+1)^2+a^2}\right) = \frac{\cosh\left(\frac{\pi}{2}\sqrt{a^2+\xi}\right)}{\cosh\frac{\pi}{2}a}\,.$$

One obtains

$$\prod_{k_0\in\frac{\pi}{\beta}(2\mathbf{Z}+1)}\left(1+\frac{\xi}{k_0^2+e_{\mathbf{k}}^2}\right) = \left\{\frac{\cosh\left(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\xi}\right)}{\cosh\left(\frac{\beta}{2}e_{\mathbf{k}}\right)}\right\}^2 \tag{II.6}$$

which results in the formula stated above •

Substitution of (II.4) in Proposition II.1 and a transformation of variables gives

Theorem II.3: Let u, v be one dimensional real variables and let $r = |r|e^{i\alpha}$. Define the effective potential

$$V_{\beta,r}(u,v) = u^2 + (v + \tfrac{|r|}{\sqrt{\lambda}})^2 - W_\beta\big(\lambda(u^2+v^2)\big) \tag{II.7}$$

a) There are the two dimensional integral representations

$$Z(\beta,L,r) = \tfrac{\kappa}{\pi} \int e^{-\kappa V_{\beta,r}(u,v)} du dv$$

$$\tfrac{1}{\kappa}\langle \bar\psi_{p,\sigma}\psi_{p,\sigma}\rangle_{\beta,L,r} = \frac{\int \frac{-a_{-p}}{a_p a_{-p} + \lambda(u^2+v^2)}\, e^{-\kappa V_{\beta,r}(u,v)} du dv}{\int e^{-\kappa V_{\beta,r}(u,v)} du dv}$$

$$\tfrac{1}{\kappa}\langle \psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \frac{\int \frac{-i\sqrt{\lambda}\, e^{-i\alpha}(u-iv)}{a_p a_{-p} + \lambda(u^2+v^2)}\, e^{-\kappa V_{\beta,r}(u,v)} du dv}{\int e^{-\kappa V_{\beta,r}(u,v)} du dv}$$

$$\tfrac{1}{\kappa}\langle \bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \frac{\int \frac{-i\sqrt{\lambda}\, e^{i\alpha}(u+iv)}{a_p a_{-p} + \lambda(u^2+v^2)}\, e^{-\kappa V_{\beta,r}(u,v)} du dv}{\int e^{-\kappa V_{\beta,r}(u,v)} du dv}$$

b) Let $V_\beta(\rho) := \rho^2 - W_\beta(\lambda\rho^2)$ and $|\Delta|^2 = \lambda\rho_0^2$ be given by

$$V_\beta(\rho_0) = \min_{\rho\geq 0} V_\beta(\rho) \tag{II.8}$$

Then, if $\Delta = \Delta(\lambda,\beta) = |\Delta|\, e^{-i\alpha}$, one has

$$\lim_{|r|\to 0}\lim_{L\to\infty} \tfrac{1}{\beta L^d} \log Z(\beta,L,r) = -V_\beta(|\Delta|)$$

$$\lim_{|r|\to 0}\lim_{L\to\infty} \tfrac{1}{\kappa}\langle \bar\psi_{p,\sigma}\psi_{p,\sigma}\rangle_{\beta,L,r} = \tfrac{-a_{-p}}{a_p a_{-p} + |\Delta|^2}$$

$$\lim_{|r|\to 0}\lim_{L\to\infty} \tfrac{1}{\kappa}\langle \psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \tfrac{\Delta}{a_p a_{-p} + |\Delta|^2}$$

$$\lim_{|r|\to 0}\lim_{L\to\infty} \tfrac{1}{\kappa}\langle \bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \tfrac{-\bar\Delta}{a_p a_{-p} + |\Delta|^2}$$

but

$$\lim_{L\to\infty}\lim_{|r|\to 0} \tfrac{1}{\kappa}\langle \psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = \lim_{L\to\infty}\lim_{|r|\to 0} \tfrac{1}{\kappa}\langle \bar\psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r} = 0$$

Proof: a) Substituting the product over the momenta by the exponential (II.4) one obtains, if $g = \sqrt{\lambda}$

$$\tfrac{1}{\kappa}\langle \bar\psi_{p,\sigma}\psi_{p,\sigma}\rangle_{\beta,L,r} = \frac{\int \frac{-a_{-p}}{a_p a_{-p} + (g\phi - ir)\big(g\bar\phi + i\bar r\big)}\, e^{-\kappa\left[|\phi|^2 - W_\beta\left((g\phi-ir)(g\bar\phi+i\bar r)\right)\right]} du dv}{\int e^{-\kappa\left[|\phi|^2 - W_\beta\left((g\phi-ir)(g\bar\phi+i\bar r)\right)\right]} du dv}$$

and analog expressions for Z and the $\langle\psi\psi\rangle$ and $\langle\bar\psi\bar\psi\rangle$ expectations. Thus one has to consider integrals of the form

$$\int F\left(g\phi - ir, g\bar\phi + i\bar r\right)\; e^{-\kappa|\phi|^2} dudv$$

By a substitution of variables one finds for both signs of λ

$$\int F\left(g\phi - ir, g\bar\phi + i\bar r\right)\; e^{-\kappa|\phi|^2} dudv = \int F\left(e^{i\alpha}g\phi, e^{-i\alpha}g\bar\phi\right) e^{-\kappa\left(u^2+(v+\frac{|r|}{g})^2\right)} dudv \quad \text{(II.9)}$$

which proves part a). To obtain part b), one has to compute the limit of

$$\delta_{\kappa,r}(u,v) = \frac{e^{-\kappa V_{\beta,r}(u,v)}}{\int e^{-\kappa V_{\beta,r}(u,v)} dudv} \quad \text{(II.10)}$$

where

$$V_{\beta,r}(u,v) = u^2 + (v + \tfrac{|r|}{g})^2 - W_\beta\big(\lambda(u^2+v^2)\big) \quad \text{(II.7)}$$

Recall that

$$W_\beta(\xi) = \tfrac{2}{\beta}\int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\;\log\left[\frac{\cosh\left(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\xi}\right)}{\cosh\left(\frac{\beta}{2}e_{\mathbf{k}}\right)}\right]$$

In particular,

$$\mathrm{sign} W_\beta\big(\lambda(u^2+v^2)\big) = \mathrm{sign}\lambda$$

For positive λ and nonzero r, $V_{\beta,r}(u,v)$ is real and has a unique global minimum determined by

$$\begin{aligned} 2u - 2u\lambda W_\beta'\big(\lambda(u^2+v^2)\big) &= 0 \\ 2(v+|r|) - 2v\lambda W_\beta'\big(\lambda(u^2+v^2)\big) &= 0 \end{aligned}$$

Since $\lambda W_\beta' = 1$ does not solve the second equation, the only solution of the first equation is $u = 0$ and one is left with

$$v\left[\lambda W_\beta'(\lambda v^2) - 1\right] = |r|$$

which has a solution $v = O(|r|)$ which is a local maximum and two nontrivial solutions $\lambda v^2 = |\Delta|^2 + O(|r|)$ where the positive one is only a local minimum and the negative one, v_0, is the global minimum. Therefore

$$\lim_{\kappa\to\infty}\delta_{\kappa,r}(u,v) = \lim_{\kappa\to\infty}\frac{e^{-\kappa[V_{\beta,r}(u,v)-V_{\beta,r}(0,v_0)]}}{\int e^{-\kappa[V_{\beta,r}(u,v)-V_{\beta,r}(0,v_0)]} dudv} = \delta(u)\,\delta(v-v_0)$$

and

$$\lim_{|r|\to 0}\lim_{\kappa\to\infty}\delta_{\kappa,r}(u,v) = \delta(u)\,\delta\left(v + \tfrac{|\Delta|}{g}\right) \quad \text{(II.11)}$$

This proves the formulae under (b) for attractive λ. Since

$$\lim_{|r|\to 0} V_{r,\beta}(u,v) = u^2+v^2 - W_\beta\left(\lambda(u^2+v^2)\right)$$

is an even function in u and v, $\lim_{r\to 0}\langle \psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r}$=$\lim_{r\to 0}\langle \bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}\rangle_{\beta,L,r} = 0$. The limit of the logarithm of the partition function becomes

$$\begin{aligned}\tfrac{1}{\kappa}\log Z &= \tfrac{1}{\kappa}\log\tfrac{\kappa}{\pi}\int e^{-\kappa V_{\beta,r}(u,v)}dudv\\ &= \tfrac{1}{\kappa}\log\tfrac{\kappa}{\pi}\int e^{-\kappa[V_{\beta,r}(u,v)-V_{\beta,r}(0,v_0)]}dudv + \tfrac{1}{\kappa}\log e^{-\kappa V_{\beta,r}(0,v_0)}\end{aligned}$$

The first term on the right hand side may be approximated by ($V_{uv} = \frac{\partial^2 V_{\beta,r}}{\partial u\partial v}(0,v_0) = 0$)

$$\tfrac{1}{\kappa}\log\tfrac{\kappa}{\pi}\int e^{-\frac{\kappa}{2}(V_{uu}u^2+V_{vv}(v-v_0)^2)}dudv = \tfrac{1}{\kappa}\log\tfrac{1}{\pi}\int e^{-\frac{1}{2}(V_{uu}u^2+V_{vv}v^2)}dudv \overset{\kappa\to\infty}{\to} 0$$

which results in

$$\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\log Z = -V_{\beta,r}(0,v_0)$$

Now let λ be negative. In that case the effective potential (II.7) is complex:

$$\begin{aligned}V_{\beta,r}(u,v) &= u^2+v^2-\tfrac{|r|^2}{|\lambda|}-2iv\tfrac{|r|}{\sqrt{|\lambda|}} - W_\beta\left(\lambda(u^2+v^2)\right)\\ &= \rho^2 + |W_\beta\left(\lambda\rho^2\right)| - \tfrac{|r|^2}{|\lambda|} - 2i\rho\cos\varphi\tfrac{|r|}{\sqrt{|\lambda|}}\end{aligned}$$

Since the real part $U_{\beta,r} = \mathrm{Re}V_{\beta,r}$ has a global minimum at $u=v=0$ one has, since $U'' = \frac{\partial^2 U_{\beta,r}}{\partial u^2}(0,0) = \frac{\partial^2 U_{\beta,r}}{\partial v^2}(0,0) > 0$,

$$\begin{aligned}\delta_{\kappa,r}(u,v) &= \frac{e^{-\kappa\left[U_{\beta,r}(u,v)-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}}{\int e^{-\kappa\left[U_{\beta,r}(u,v)-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}dudv} \approx \frac{e^{-\kappa\left[\frac{U''}{2}(u^2+v^2)-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}}{\int e^{-\kappa\left[\frac{U''}{2}(u^2+v^2)-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}dudv}\\ &= \frac{e^{-\kappa\frac{U''}{2}u^2}}{\int e^{-\kappa\frac{U''}{2}u^2}du}\,\frac{e^{-\kappa\left[\frac{U''}{2}v^2-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}}{\int e^{-\kappa\left[\frac{U''}{2}v^2-2iv\frac{|r|}{\sqrt{|\lambda|}}\right]}dv}\\ &= \frac{e^{-\kappa\frac{U''}{2}u^2}}{\int e^{-\kappa\frac{U''}{2}u^2}du}\,\frac{e^{-\kappa\frac{U''}{2}\left[v-i\frac{2|r|}{\sqrt{|\lambda|}U''}\right]^2}}{\int e^{-\kappa\frac{U''}{2}\left[v-i\frac{2|r|}{\sqrt{|\lambda|}U''}\right]^2}dv} \overset{\kappa\to\infty}{\to} \delta(u)\,\delta\Big(v-i\frac{2|r|}{\sqrt{|\lambda|}U''}\Big)\end{aligned}$$

which results in

$$\lim_{|r|\to 0}\lim_{\kappa\to\infty}\delta_{\kappa,r}(u,v) = \delta(u)\,\delta(v) \tag{II.12}$$

and

$$\lim_{|r|\to 0}\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\langle\bar\psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r}=\int\tfrac{-a_{-p}}{a_pa_{-p}+\lambda(u^2+v^2)}\,\delta(u)\delta(v)\,dudv=\tfrac{-a_{-p}}{a_pa_{-p}}=-\tfrac{1}{a_p}$$

$$\lim_{|r|\to 0}\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\langle\psi_{p,\uparrow}\psi_{-p,\downarrow}\rangle_{\beta,L,r}=\lim_{|r|\to 0}\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\langle\bar\psi_{p,\uparrow}\bar\psi_{-p,\downarrow}\rangle_{\beta,L,r}=0 \qquad\bullet$$

In the same way one obtains a two dimensional integral representation for the generating functional for the connected amputated Greens functions. It is defined by

$$G(\eta)=G_r(\beta,L,\eta)=\log\int e^{\mathcal{U}(\psi+\eta,\bar\psi+\bar\eta)}d\mu_C(\psi,\bar\psi) \tag{II.13}$$

where

$$\mathcal{U}(\psi,\bar\psi)=-\tfrac{\lambda}{\kappa}\tfrac{1}{\kappa^2}\sum_{k,p}\psi_{k\uparrow}\psi_{-k\downarrow}\bar\psi_{p\uparrow}\bar\psi_{-p\downarrow}+\tfrac{1}{\kappa}\sum_k[r\psi_{k\uparrow}\psi_{-k\downarrow}-\bar r\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}]$$

and the $\eta_{k\sigma}$ are some Grassmann variables.

Theorem II.4: a) Let u,v be one dimensional real variables, $r=|r|e^{i\alpha}$ and $\gamma=ge^{i\alpha}(u+iv)$, $\bar\gamma=ge^{-i\alpha}(u-iv)$. Let $V_{\beta,r}(u,v)$ be the effective potential (II.7). Then the generating functional for the connected amputated Greens functions is given by

$$G(\eta)-G(0)=-\tfrac{1}{\kappa}\sum_k\Big[a_k\bar\eta_{k\uparrow}\eta_{k\uparrow}+a_{-k}\bar\eta_{-k\downarrow}\eta_{-k\downarrow}\Big]+$$

$$+\log\frac{\int dudv\;e^{-\kappa V_{\beta,r}(u,v)}\exp\Big\{\frac{1}{\kappa}\sum_k(a_k\bar\eta_{k\uparrow},-a_{-k}\eta_{-k\downarrow})\frac{1}{a_ka_{-k}+\gamma\bar\gamma}\begin{pmatrix}a_{-k}&-i\bar\gamma\\ i\gamma&-a_k\end{pmatrix}\begin{pmatrix}a_k\eta_{k\uparrow}\\ -a_{-k}\bar\eta_{-k\downarrow}\end{pmatrix}\Big\}}{\int dudv\;e^{-\kappa V_{\beta,r}(u,v)}}$$

b) For attractive $\lambda>0$ the infinite volume limit of the generating functional is given by

$$\lim_{|r|\to 0}\lim_{L\to\infty}\{G(\beta,L,r,\eta)-G(\beta,L,r,0)\}=$$

$$-\tfrac{1}{\kappa}\sum_k\Big[a_k\tfrac{|\Delta|^2}{a_ka_{-k}+|\Delta|^2}\,\bar\eta_{k\uparrow}\eta_{k\uparrow}+a_{-k}\tfrac{|\Delta|^2}{a_ka_{-k}+|\Delta|^2}\,\bar\eta_{-k\downarrow}\eta_{-k\downarrow}$$

$$+\Delta\tfrac{a_ka_{-k}}{a_ka_{-k}+|\Delta|^2}\,\eta_{-k\downarrow}\eta_{k\uparrow}+\bar\Delta\tfrac{a_ka_{-k}}{a_ka_{-k}+|\Delta|^2}\,\bar\eta_{k\uparrow}\bar\eta_{-k\downarrow}\Big]$$

For repulsive $\lambda<0$ one obtains $\lim_{|r|\to 0}\lim_{L\to\infty}\{G(\beta,L,r,\eta)-G(\beta,L,r,0)\}=0$.

Proof: By a substitution of variables, one has

$$\int e^{-\mathcal{U}(\psi+\eta,\bar\psi+\bar\eta)}d\mu_C(\psi,\bar\psi)=\int e^{-\mathcal{U}(\psi+\eta,\bar\psi+\bar\eta)}\prod_{k\sigma}\tfrac{\kappa}{a_k}\,e^{-\frac{1}{\kappa}\sum_{k\sigma}a_k\bar\psi_{k\sigma}\psi_{k\sigma}}\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}$$

$$=e^{-\frac{1}{\kappa}\sum_{k\sigma}a_k\bar\eta_{k\sigma}\eta_{k\sigma}}\prod_{k\sigma}\tfrac{\kappa}{a_k}\int e^{-\mathcal{U}(\psi,\bar\psi)}e^{\frac{1}{\kappa}\sum_{k\sigma}a_k[\bar\eta_{k\sigma}\psi_{k\sigma}+\bar\psi_{k\sigma}\eta_{k\sigma}-\bar\psi_{k\sigma}\psi_{k\sigma}]}\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}$$

Let $\phi = u + iv,\ \bar\phi = u - iv,\ \gamma_r = \lambda^{\frac12}\phi - ir,\ \bar\gamma_r = \lambda^{\frac12}\bar\phi + i\bar r$. Then

$$e^{-\mathcal{U}(\psi,\bar\psi)} = \tfrac{\kappa}{\pi}\int_{\mathbf{R}^2} e^{\frac{1}{\kappa}\sum_k\left[i\gamma_r\psi_{k\uparrow}\psi_{-k\downarrow}+i\bar\gamma_r\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}\right]}e^{-\kappa|\phi|^2}dudv$$

and

$$\begin{aligned}
&\int e^{-\mathcal{U}(\psi,\bar\psi)}e^{\frac1\kappa\sum_{k\sigma}a_k[\bar\eta_{k\sigma}\psi_{k\sigma}+\bar\psi_{k\sigma}\eta_{k\sigma}-\bar\psi_{k\sigma}\psi_{k\sigma}]}\prod_{k\sigma}\tfrac{\kappa}{a_k}\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}\\
&=\int\int e^{\frac1\kappa\sum_k\left[i\gamma_r\psi_{k\uparrow}\psi_{-k\downarrow}+i\bar\gamma_r\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}\right]}e^{\frac1\kappa\sum_k\left[a_k(\bar\psi_{k\uparrow}\eta_{k\uparrow}+\bar\eta_{k\uparrow}\psi_{k\uparrow})+a_{-k}(\bar\psi_{-k\downarrow}\eta_{-k\downarrow}+\bar\eta_{-k\downarrow}\psi_{-k\downarrow})\right]}\times\\
&\qquad e^{-\frac1\kappa\sum_k\left[a_k\bar\psi_{k\uparrow}\psi_{k\uparrow}+a_{-k}\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\right]}\prod_k\tfrac{\kappa^2}{a_ka_{-k}}\prod_k d\psi_{k\uparrow}d\bar\psi_{k\uparrow}d\psi_{-k\downarrow}d\bar\psi_{k\downarrow}\,\tfrac{\kappa}{\pi}\,e^{-\kappa|\phi|^2}dudv\\
&=\int\int\prod_k\tfrac{a_ka_{-k}+\gamma_r\bar\gamma_r}{a_ka_{-k}}\ e^{\frac{1}{\kappa}\sum_k\left[(\bar\psi_{k\uparrow},\psi_{-k\downarrow})\left(\begin{smallmatrix}a_k\eta_{k\uparrow}\\ -a_{-k}\bar\eta_{-k\downarrow}\end{smallmatrix}\right)+(a_k\bar\eta_{k\uparrow},-a_{-k}\eta_{-k\downarrow})\left(\begin{smallmatrix}\psi_{k\uparrow}\\ \bar\psi_{-k\downarrow}\end{smallmatrix}\right)\right]}\times\\
&\quad\prod_k\tfrac{-\kappa^2}{a_ka_{-k}+\gamma_r\bar\gamma_r}e^{-\frac{1}{\kappa}\sum_k(\bar\psi_{k\uparrow},\psi_{-k\downarrow})\left(\begin{smallmatrix}a_k & -i\bar\gamma_r\\ i\gamma_r & -a_{-k}\end{smallmatrix}\right)\left(\begin{smallmatrix}\psi_{k\uparrow}\\ \bar\psi_{-k\downarrow}\end{smallmatrix}\right)}\prod_k d\psi_{k\uparrow}d\bar\psi_{k\uparrow}d\bar\psi_{-k\downarrow}d\psi_{-k\downarrow}\,\tfrac{\kappa}{\pi}\,e^{-\kappa|\phi|^2}dudv\\
&=\int\prod_k\left[1+\tfrac{\gamma_r\bar\gamma_r}{a_ka_{-k}}\right]e^{\frac{1}{\kappa}\sum_k(a_k\bar\eta_{k\uparrow},-a_{-k}\eta_{-k\downarrow})\left(\begin{smallmatrix}a_k & -i\bar\gamma_r\\ i\gamma_r & -a_{-k}\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}a_k\eta_{k\uparrow}\\ -a_{-k}\bar\eta_{-k\downarrow}\end{smallmatrix}\right)}\tfrac{\kappa}{\pi}\,e^{-\kappa|\phi|^2}dudv
\end{aligned}$$

Therefore the generating functional is given by

$$\begin{aligned}
G(\beta,L,\eta)=\log\tfrac{\kappa}{\pi}\int\prod_k\left[1+\tfrac{\gamma_r\bar\gamma_r}{a_ka_{-k}}\right]e^{-\frac{1}{\kappa}\sum_k[a_k\bar\eta_{k\uparrow}\eta_{k\uparrow}+a_{-k}\bar\eta_{-k\downarrow}\eta_{-k\downarrow}]}\times\\
e^{\frac{1}{\kappa}\sum_k(a_k\bar\eta_{k\uparrow},-a_{-k}\eta_{-k\downarrow})\frac{1}{a_ka_{-k}+\gamma_r\bar\gamma_r}\left(\begin{smallmatrix}a_{-k} & -i\bar\gamma_r\\ i\gamma_r & -a_k\end{smallmatrix}\right)\left(\begin{smallmatrix}a_k\eta_{k\uparrow}\\ -a_{-k}\bar\eta_{-k\downarrow}\end{smallmatrix}\right)}e^{-\kappa|\phi|^2}dudv
\end{aligned}$$

Since $\gamma_r = g\phi - ir,\ \bar\gamma_r = g\bar\phi + i\bar r$ one can apply (II.9)

$$\int F\left(g\phi - ir, g\bar\phi + i\bar r\right)\ e^{-\kappa|\phi|^2}dudv = \int F\left(e^{i\alpha}g\phi, e^{-i\alpha}g\bar\phi\right)e^{-\kappa\left(u^2+(v+\frac{|r|}{g})^2\right)}dudv$$

again to make a substitution of variables which proves part (a).

The infinite volume limit of the generating functional follows from (II.11,12)

$$\lim_{|r|\to0}\lim_{\kappa\to\infty}\frac{e^{-\kappa V_{\beta,r}(u,v)}}{\int e^{-\kappa V_{\beta,r}(u,v)}dudv} = \begin{cases}\delta(u)\delta\left(v+\frac{|\Delta|}{g}\right) & \text{if } \lambda > 0\\ \delta(u)\delta(v) & \text{if } \lambda < 0\ \bullet\end{cases}$$

II.2 Explicit Summation of Diagramms

In this section we compute the partition function ($r_k = s_k = 0$)

$$Z = \int e^{-\frac{\lambda}{\kappa}\frac{1}{\kappa^2}\sum_{k,p}\psi_{k\uparrow}\psi_{-k\downarrow}\bar{\psi}_{p\uparrow}\bar{\psi}_{-p\downarrow}} d\mu_C(\psi,\bar{\psi})$$

by explicit summation of diagramms. That is, we derive the formula

$$Z = 2\kappa \int e^{-\kappa\left(\rho^2 - \frac{1}{L^d}\sum_{\mathbf{k}}\frac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda\rho^2})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]^2\right)} \rho\, d\rho \tag{II.14}$$

by direct summation of the perturbation series without using the identity

$$e^{-\frac{\lambda}{\kappa^3}\sum_{k,p}\psi_{k\uparrow}\psi_{-k\downarrow}\bar{\psi}_{p,\uparrow}\psi_{-p\downarrow}} = \int e^{i\phi\, g\frac{1}{\kappa}\sum_k \psi_{k\uparrow}\psi_{-k\downarrow} + i\bar{\phi}\, g\frac{1}{\kappa}\sum_k \bar{\psi}_{k\uparrow}\bar{\psi}_{-k\downarrow}}\, \frac{\kappa}{\pi}\, e^{-\kappa|\phi|^2} du dv$$

One has

$$\begin{aligned}
Z &= \int e^{-\frac{\lambda}{\kappa^3}\sum_{k,p}\psi_{k\uparrow}\psi_{-k\downarrow}\bar{\psi}_{p\uparrow}\bar{\psi}_{-p\downarrow}} d\mu_C(\psi,\bar{\psi}) \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{\kappa^3}\right)^n}{n!}\sum_{k_1,\cdots,k_n}\sum_{p_1,\cdots,p_n}\int \psi_{k_1\uparrow}\psi_{-k_1\downarrow}\cdots\psi_{k_n\uparrow}\psi_{-k_n\downarrow}\bar{\psi}_{p_1\uparrow}\bar{\psi}_{-p_1\downarrow}\cdots\bar{\psi}_{p_n\uparrow}\bar{\psi}_{-p_n\downarrow}\, d\mu_C \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{\kappa^3}\right)^n}{n!}\sum_{k_1,\cdots,k_n}\sum_{p_1,\cdots,p_n}(-1)^{n^2}\int \psi_{k_1\uparrow}\bar{\psi}_{p_1\uparrow}\cdots\psi_{k_n\uparrow}\bar{\psi}_{p_n\uparrow}\, d\mu_C \times \\
&\qquad\qquad \int \psi_{-k_1\downarrow}\bar{\psi}_{-p_1\downarrow}\cdots\psi_{-k_n\downarrow}\bar{\psi}_{-p_n\downarrow}\, d\mu_C \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{\kappa^3}\right)^n}{n!}\sum_{k_1,\cdots,k_n}\sum_{p_1,\cdots,p_n}(-1)^{n}\int \psi_{k_1\uparrow}\bar{\psi}_{p_1\uparrow}\cdots\psi_{k_n\uparrow}\bar{\psi}_{p_n\uparrow}\, d\mu_C \times \\
&\qquad\qquad \sum_{\pi\in S_n}\epsilon_\pi\, \kappa\delta_{k_1,p_{\pi 1}}C(-k_1)\cdots\kappa\delta_{k_n,p_{\pi n}}C(-k_n) \\
&= \sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{\kappa^2}\right)^n}{n!}\sum_{k_1,\cdots,k_n}(-1)^{n}\int \psi_{k_1\uparrow}\bar{\psi}_{k_1\uparrow}\cdots\psi_{k_n\uparrow}\bar{\psi}_{k_n\uparrow}\, d\mu_C\, n! C(-k_1)\cdots C(-k_n) \\
&= \sum_{n=0}^{\infty}\left(-\tfrac{\lambda}{\kappa^2}\right)^n\sum_{k_1,\cdots,k_n}(-1)^n C(-k_1)\cdots C(-k_n)\sum_{\pi\in S_n}\epsilon_\pi\kappa\delta_{k_1,k_{\pi 1}}C(k_1)\cdots\kappa\delta_{k_n,k_{\pi n}}C(k_n)
\end{aligned}$$

Say that the permutation π is of type $t(\pi) = 1^{b_1}\cdots n^{b_n}$ if the decomposition into disjoint cycles contains b_r r-cycles for $1 \le r \le n$. Necessarily one has $1b_1 + \cdots + nb_n = n$. The number of permutations which have b_r r-cycles for $1 \le r \le n$ is

$$\frac{n!}{b_1!\cdots b_n!\, 1^{b_1}\cdots n^{b_n}}$$

The sign of such a permutation is given by

$$\epsilon_\pi = (-1)^{(1-1)b_1+(2-1)b_2+\cdots+(n-1)b_n} = (-1)^{n-\sum_{r=1}^n b_r}$$

Therefore one obtains

$$\begin{aligned}
Z &= \sum_{n=0}^{\infty} \left(-\tfrac{\lambda}{\kappa}\right)^n \sum_{k_1,\cdots,k_n} (-1)^n C(-k_1)\cdots C(-k_n) \sum_{\pi\in S_n} \epsilon_\pi \delta_{k_1,k_{\pi 1}} C(k_1)\cdots\delta_{k_n,k_{\pi n}} C(k_n) \\
&= \sum_{n=0}^{\infty} \left(-\tfrac{\lambda}{\kappa}\right)^n \sum_{k_1,\cdots,k_n} (-1)^n C(-k_1)\cdots C(-k_n)\times \\
&\qquad\qquad \sum_{\substack{b_1,\cdots,b_n=0 \\ 1b_1+\cdots+nb_n=n}}^{n} \sum_{\substack{\pi\in S_n \\ t(\pi)=1^{b_1}\cdots n^{b_n}}} \epsilon_\pi \delta_{k_1,k_{\pi 1}} C(k_1)\cdots\delta_{k_n,k_{\pi n}} C(k_n) \\
&= \sum_{n=0}^{\infty} \left(-\tfrac{\lambda}{\kappa}\right)^n \sum_{\substack{b_1,\cdots,b_n=0 \\ 1b_1+\cdots+nb_n=n}}^{n} \frac{n!}{b_1!\cdots b_n!\, 1^{b_1}\cdots n^{b_n}} (-1)^n (-1)^{n-\sum_{r=1}^n b_r}\times \\
&\qquad\qquad \left\{\sum_k [C(k)C(-k)]^1\right\}^{b_1} \cdots \left\{\sum_k [C(k)C(-k)]^n\right\}^{b_n} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{b_1,\cdots,b_n=0 \\ 1b_1+\cdots+nb_n=n}}^{n} n! \prod_{r=1}^{n} \frac{1}{b_r!} \left\{-\tfrac{1}{r} \sum_k \left[\left(-\tfrac{\lambda}{\kappa}\right) C(k)C(-k)\right]^r\right\}^{b_r}
\end{aligned}$$

The only factor which prevents us from an explicit summation of the above series is the $n!$. Therefore we substitute

$$n! = \int_0^\infty e^{-x} x^n dx$$

and obtain

$$\begin{aligned}
Z &= \int_0^\infty e^{-x} \sum_{n=0}^{\infty} \sum_{\substack{b_1,\cdots,b_n=0 \\ 1b_1+\cdots+nb_n=n}}^{n} \prod_{r=1}^{n} \frac{1}{b_r!} \left\{-\tfrac{1}{r} \sum_k \left[-\tfrac{\lambda}{\kappa} x\, C(k)C(-k)\right]^r\right\}^{b_r} dx \\
&= \int_0^\infty e^{-x} \prod_{r=1}^{\infty} \sum_{b_r=0}^{\infty} \frac{1}{b_r!} \left\{-\tfrac{1}{r} \sum_k \left[-\tfrac{\lambda}{\kappa} x\, C(k)C(-k)\right]^r\right\}^{b_r} dx \\
&= \int_0^\infty e^{-x} \prod_{r=1}^{\infty} e^{-\frac{1}{r}\sum_k \left[-\frac{\lambda}{\kappa} x\, C(k)C(-k)\right]^r} dx \\
&= \int_0^\infty e^{-x} e^{-\sum_k \sum_{r=1}^\infty \frac{1}{r}\left[-\frac{\lambda}{\kappa} x\, C(k)C(-k)\right]^r} dx \\
&= \int_0^\infty e^{-x} e^{\sum_k \log\left[1+\frac{\lambda x}{\kappa}\, C(k)C(-k)\right]} dx
\end{aligned}$$

$$= 2\kappa \int_0^\infty e^{-\kappa\left(\rho^2 - \frac{1}{\kappa}\sum_k \log\left[1+\frac{\lambda\rho^2}{k_0^2+e_{\mathbf{k}}^2}\right]\right)} \rho\, d\rho$$

$$= 2\kappa \int_0^\infty e^{-\kappa\left(\rho^2 - \frac{1}{L^d}\sum_{\mathbf{k}} \frac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda\rho^2})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]^2\right)} \rho\, d\rho$$

which coincides with (II.14).

III. The Model with Forward, Exchange and BCS Interaction

Let $H = H_0 + H_{int}$ where

$$H_0 = \tfrac{1}{L^d} \sum_{\sigma\in\{\uparrow,\downarrow\}} \sum_{\mathbf{k}\in M} e_{\mathbf{k}}\, a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \tag{III.1}$$

and

$$H_{int} = \tfrac{1}{L^{3d}} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} U(\mathbf{k}-\mathbf{p})\, a^+_{\mathbf{k},\sigma} a^+_{\mathbf{q}-\mathbf{k},\tau} a_{\mathbf{p},\sigma} a_{\mathbf{q}-\mathbf{p},\tau} \tag{III.2}$$

Then

$$Z = \frac{Tr\, e^{-\beta(H_0+H_{int})}}{Tr\, e^{-\beta H_0}} = \int e^{-\mathcal{V}(\psi,\bar\psi)} d\mu_C(\psi,\bar\psi) \tag{III.3}$$

where the exponent in the fermionic integral is given by ($\kappa = \beta L^d$)

$$\mathcal{V}(\psi) = \tfrac{1}{\kappa^3} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{k,p,q} U(\mathbf{k}-\mathbf{p})\, \bar\psi_{k,\sigma}\bar\psi_{q-k,\tau}\psi_{p,\sigma}\psi_{q-p,\tau} \tag{III.4}$$

and $k = (k_0,\mathbf{k}) \in \frac{\pi}{\beta}(2\mathbf{Z}+1)\times M$.

The forward, exchange and BCS approximation is obtained by restricting the above sum to the following terms

$$\text{forward}: \quad \delta_{k,p}\,, \qquad \text{exchange}: \quad \delta_{k,-p}\,, \qquad \text{BCS}: \quad \delta_{q,0}\,. \tag{III.5}$$

We consider the approximation $\mathcal{V}(\psi,\bar\psi) \approx \mathcal{U}(\psi,\bar\psi)$ where

$$\mathcal{U}(\psi,\bar\psi) = \tfrac{1}{\kappa^3} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{k,p,q} U(\mathbf{k}'-\mathbf{p}')\left[\delta_{k,p}+\delta_{k,-p}+\delta_{q,0}\right] \bar\psi_{k,\sigma}\bar\psi_{q-k,\tau}\psi_{p,\sigma}\psi_{q-p,\tau} \tag{III.6}$$

and

$$U(\mathbf{k}'-\mathbf{p}') = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=-n}^{n} \lambda_{|\ell|} e^{i\ell\varphi_{\mathbf{k}}} e^{-i\ell\varphi_{\mathbf{p}}} + \dfrac{\lambda_0}{2} & \text{if } d=2 \\ \displaystyle\sum_{\ell=0}^{n}\sum_{m=-\ell}^{\ell} \lambda_\ell \bar Y_{\ell m}(\mathbf{k}')\, Y_{\ell m}(\mathbf{p}') & \text{if } d=3 \end{cases} =: \sum_{l=0}^{N} \lambda_l\, y_l(\mathbf{k}')\, \bar y_l(\mathbf{p}') \tag{III.7}$$

This model can be solved explicitely.

Theorem III.1: Let $\mathcal{U}(\psi,\bar\psi)$ be given by (III.6) and let

$$Z(\beta,L,\{s_{k,\sigma}\}) = \int e^{-\mathcal{U}(\psi,\bar\psi)+\frac{1}{\kappa}\sum_{k,\sigma} s_{k,\sigma}\bar\psi_{k,\sigma}\psi_{k,\sigma}}\, d\mu_C(\psi,\bar\psi)$$

Let $w\in\mathbf{R}$, $\xi^l_{\sigma\tau}=a^l_{\sigma\tau}+ib^l_{\sigma\tau}$, $\phi^l_{\sigma\tau}=u^l_{\sigma\tau}+iv^l_{\sigma\tau}\in\mathbf{C}$ for $0\le l\le N$, $(\sigma\tau)\in\{\uparrow\uparrow,\downarrow\downarrow,\uparrow\downarrow\}$ and define the fields

$$\Xi_{\mathbf{k}\uparrow\downarrow}=\sum_{l=0}^N(2\lambda_l)^{\frac12}\xi^l_{\uparrow\downarrow}\,y_l(\mathbf{k}')\,,\quad \bar\Xi_{\mathbf{k}\uparrow\downarrow}=\sum_{l=0}^N(2\lambda_l)^{\frac12}\bar\xi^l_{\uparrow\downarrow}\,\bar y_l(\mathbf{k}')$$

$$\Xi_{\mathbf{k}\sigma\sigma}=\sum_{l=0}^N\lambda_l^{\frac12}\xi^l_{\sigma\sigma}\,y_l(\mathbf{k}')\,,\quad \bar\Xi_{\mathbf{k}\sigma\sigma}=\sum_{l=0}^N\lambda_l^{\frac12}\bar\xi^l_{\sigma\sigma}\,\bar y_l(\mathbf{k}')\,,$$

$$\Gamma_{\mathbf{k}\uparrow}=U_0^{\frac12}w+i\Xi_{\mathbf{k}\uparrow\uparrow}+i\bar\Xi_{\mathbf{k}\uparrow\uparrow}\,,\quad \Gamma_{\mathbf{k}\downarrow}=U_0^{\frac12}w+i\Xi_{\mathbf{k}\downarrow\downarrow}+i\bar\Xi_{\mathbf{k}\downarrow\downarrow}\,,$$

$$\Phi_{\mathbf{k}\uparrow\downarrow}=\sum_{l=0}^N(2\lambda_l)^{\frac12}\phi^l_{\uparrow\downarrow}\,y_l(\mathbf{k}')\,,\quad \bar\Phi_{\mathbf{k}\uparrow\downarrow}=\sum_{l=0}^N(2\lambda_l)^{\frac12}\bar\phi^l_{\uparrow\downarrow}\,\bar y_l(\mathbf{k}')$$

$$\Phi_{\mathbf{k}\sigma\sigma}=\sum_{l=0}^N\lambda_l^{\frac12}\phi^l_{\sigma\sigma}\,[y_l(\mathbf{k}')-y_l(-\mathbf{k}')]\,,\quad \bar\Phi_{\mathbf{k}\sigma\sigma}=\sum_{l=0}^N\lambda_l^{\frac12}\bar\phi^l_{\sigma\sigma}\,[\bar y_l(\mathbf{k}')-\bar y_l(-\mathbf{k}')]\,.$$

where $U_0=U(\mathbf{k}=\mathbf{0})=\int d^d\mathbf{x}\,U(\mathbf{x})$. For each $k=(k_0,\mathbf{k})$, let S_k be the 8×8 skew symmetric matrix

$$S_k=-i\begin{pmatrix}
0 & -\frac1i A_{k\uparrow} & \bar\Phi_{\mathbf{k}\uparrow\downarrow} & 0 & \bar\Xi_{\mathbf{k}\uparrow\downarrow} & 0 & 0 & \bar\Phi_{\mathbf{k}\uparrow\uparrow}\\
\frac1i A_{k\uparrow} & 0 & 0 & \Phi_{\mathbf{k}\uparrow\downarrow} & 0 & -\Xi_{\mathbf{k}\uparrow\downarrow} & \Phi_{\mathbf{k}\uparrow\uparrow} & 0\\
-\bar\Phi_{\mathbf{k}\uparrow\downarrow} & 0 & 0 & -\frac1i A_{-k\downarrow} & 0 & -\bar\Phi_{\mathbf{k}\downarrow\downarrow} & \Xi_{-\mathbf{k}\uparrow\downarrow} & 0\\
0 & -\Phi_{\mathbf{k}\uparrow\downarrow} & \frac1i A_{-k\downarrow} & 0 & -\Phi_{\mathbf{k}\downarrow\downarrow} & 0 & 0 & -\bar\Xi_{-\mathbf{k}\uparrow\downarrow}\\
-\bar\Xi_{\mathbf{k}\uparrow\downarrow} & 0 & 0 & \Phi_{\mathbf{k}\downarrow\downarrow} & 0 & \frac1i A_{k\downarrow} & -\Phi_{-\mathbf{k}\uparrow\downarrow} & 0\\
0 & \Xi_{\mathbf{k}\uparrow\downarrow} & \bar\Phi_{\mathbf{k}\downarrow\downarrow} & 0 & -\frac1i A_{k\downarrow} & 0 & 0 & -\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\\
0 & -\Phi_{\mathbf{k}\uparrow\uparrow} & -\Xi_{-\mathbf{k}\uparrow\downarrow} & 0 & \Phi_{-\mathbf{k}\uparrow\downarrow} & 0 & 0 & \frac1i A_{-k\uparrow}\\
-\bar\Phi_{\mathbf{k}\uparrow\uparrow} & 0 & 0 & \bar\Xi_{-\mathbf{k}\uparrow\downarrow} & 0 & \bar\Phi_{-\mathbf{k}\uparrow\downarrow} & -\frac1i A_{-k\uparrow} & 0
\end{pmatrix}$$

where $A_{k\sigma}=a_k-s_{k\sigma}-\Gamma_{k\sigma}=ik_0-e_{\mathbf{k}}-s_{k\sigma}-\Gamma_{k\sigma}$ and let $\mathrm{Pf}S_k$ be the Pfaffian of S_k given by Lemma III.2 below. Then:

a)
$$Z(\beta,L,\{s_{k,\sigma}\})=\int\prod_{\substack{k_0>0\\ \mathbf{k}\in M}}\Big\{\Big(\tfrac{1}{a_ka_{-k}}\Big)^2\mathrm{Pf}S_k\Big\}d\nu_\kappa(w,\xi,\phi)$$

where

$$d\nu_\kappa(w,\xi,\phi)=\big(\tfrac{\kappa}{4\pi}\big)^{\frac12}e^{-\frac{\kappa}{4}w^2}dw\prod_{l=0}^N\prod_{\sigma\tau}\Big\{\big(\tfrac{\kappa}{\pi}\big)^2\,e^{-\kappa(|\phi^l_{\sigma\tau}|^2+|\xi^l_{\sigma\tau}|^2)}du^l_{\sigma\tau}dv^l_{\sigma\tau}da^l_{\sigma\tau}db^l_{\sigma\tau}\Big\}$$

b)
$$\langle\bar\psi_{p\sigma}\psi_{p'\sigma}\rangle=\beta L^d\delta_{p,p'}\,\frac{\int\frac{\frac{\partial}{\partial s_{p\sigma}}\mathrm{Pf}S_p}{\mathrm{Pf}S_p}e^{-\kappa V(w,\xi,\phi)}dw\,\Pi\,d\xi d\phi}{\int\ e^{-\kappa V(w,\xi,\phi)}dw\,\Pi\,d\xi d\phi}$$

where the effective potential V is given by

$$V(w,\xi,\phi)=\tfrac14 w^2+\sum_{\sigma\tau}\sum_{l=0}^{N}(|\phi^l_{\sigma\tau}|^2+|\xi^l_{\sigma\tau}|^2)-\tfrac{1}{\beta L^d}\sum_{\mathbf{k}\in M}\log\prod_{k_0>0}\tfrac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}\tag{III.8}$$

and $\Pi\,d\xi d\phi=\prod_{\sigma\tau}\prod_{l=0}^{N}d\xi^l_{\sigma\tau}d\phi^l_{\sigma\tau}$. Here $\sigma\tau\in\{\uparrow\uparrow,\downarrow\downarrow,\uparrow\downarrow\}$.

Remark: The product $\prod_{k_0>0}\frac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}$ in the effective potential (III.8) where $\mathrm{Pf}S_k=\mathrm{Pf}S_k(a_k,a_{-k})$ has to be computed according to the rule

$$\prod_{k_0>0}\frac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}=\lim_{\substack{\epsilon\to0\\ \epsilon<0}}\prod_{k_0>0}\frac{\mathrm{Pf}S_k(e^{ik_0\epsilon}a_k,e^{-ik_0\epsilon}a_{-k})}{a_k^2a_{-k}^2}$$

That there are cases where it is necessary to make the phase factors explicit can be seen in looking at the example given in the proof of Lemma IV.1.

Proof: Since we assume $U(\mathbf{p}-\mathbf{k})=U(\mathbf{k}-\mathbf{p})$, one has

$$\begin{aligned}\mathcal{U}(\psi,\bar\psi)&=\tfrac{1}{\kappa^3}\sum_{\sigma,\tau}\sum_{k,p}U(\mathbf{0})\,\bar\psi_{k,\sigma}\bar\psi_{p,\tau}\psi_{k,\sigma}\psi_{p,\tau}+\tfrac{1}{\kappa^3}\sum_{\sigma,\tau}\sum_{k,p}U(\mathbf{k}'-\mathbf{p}')\,\bar\psi_{p,\sigma}\bar\psi_{k,\tau}\psi_{k,\sigma}\psi_{p,\tau}\\&\quad+\tfrac{1}{\kappa^3}\sum_{\sigma,\tau}\sum_{k,p}U(\mathbf{k}'-\mathbf{p}')\,\bar\psi_{p,\sigma}\bar\psi_{-p,\tau}\psi_{k,\sigma}\psi_{-k,\tau}\\&=-\tfrac{1}{\kappa^2}\tfrac{U_0}{\kappa}\sum_{k,p}\Big(\bar\psi_{k\uparrow}\psi_{k\uparrow}+\bar\psi_{k\downarrow}\psi_{k\downarrow}\Big)\Big(\bar\psi_{p\uparrow}\psi_{p\uparrow}+\bar\psi_{p\downarrow}\psi_{p\downarrow}\Big)\\&\quad+\tfrac{1}{\kappa^3}\sum_{k,p}U(\mathbf{k}'-\mathbf{p}')\Big\{\bar\psi_{p\uparrow}\psi_{p\uparrow}\bar\psi_{k\uparrow}\psi_{k\uparrow}+\bar\psi_{p\downarrow}\psi_{p\downarrow}\bar\psi_{k\downarrow}\psi_{k\downarrow}+2\bar\psi_{p\uparrow}\psi_{p\downarrow}\bar\psi_{k\downarrow}\psi_{k\uparrow}\Big\}\\&\quad+\tfrac{1}{\kappa^3}\sum_{k,p}U(\mathbf{k}'-\mathbf{p}')\Big\{\bar\psi_{p\uparrow}\bar\psi_{-p\uparrow}\psi_{k\uparrow}\psi_{-k\uparrow}+\bar\psi_{p\downarrow}\bar\psi_{-p\downarrow}\psi_{k\downarrow}\psi_{-k\downarrow}+2\bar\psi_{p\uparrow}\bar\psi_{-p\downarrow}\psi_{k\uparrow}\psi_{-k\downarrow}\Big\}\end{aligned}$$

We substitute $U(\mathbf{k}'-\mathbf{p}')=\sum_{l=0}^N\lambda_l y_l(\mathbf{k}')\bar y_l(\mathbf{p}')$ and use the identities

$$e^{\frac12X^2}=\tfrac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{Xw}e^{-\frac12w^2}dw$$

$$e^{-2XY}=\tfrac{1}{2\pi}\int_{\mathbf{R}^2}e^{iX\phi+iY\bar\phi}e^{-\frac12|\phi|^2}dudv$$

to obtain

$$
\begin{aligned}
&e^{-\mathcal{U}(\psi,\bar\psi)} = \\
&\quad\int \exp\Big\{ \big(\tfrac{2U_0}{\kappa}\big)^{\frac12}\, w\, \tfrac{1}{\kappa} \sum_k \Big[\bar\psi_{k\uparrow}\psi_{k\uparrow} + \bar\psi_{k\downarrow}\psi_{k\downarrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \bar\xi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\psi_{k\uparrow} + i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \xi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\psi_{k\uparrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \bar\xi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\downarrow}\psi_{k\downarrow} + i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \xi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\bar\psi_{k\downarrow}\psi_{k\downarrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{\kappa}\big)^{\frac12}\, \bar\xi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\psi_{k\downarrow} + i\,\big(\tfrac{\lambda_l}{\kappa}\big)^{\frac12}\, \xi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\bar\psi_{k\downarrow}\psi_{k\uparrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \bar\phi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\bar\psi_{-k\uparrow} + i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \phi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\psi_{k\uparrow}\psi_{-k\uparrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \bar\phi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\downarrow}\bar\psi_{-k\downarrow} + i\,\big(\tfrac{\lambda_l}{2\kappa}\big)^{\frac12}\, \phi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\psi_{k\downarrow}\psi_{-k\downarrow}\Big]\Big\}\times \\
&\qquad \exp\Big\{ \sum_l \Big[i\,\big(\tfrac{\lambda_l}{\kappa}\big)^{\frac12}\, \bar\phi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + i\,\big(\tfrac{\lambda_l}{\kappa}\big)^{\frac12}\, \phi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\psi_{k\uparrow}\psi_{-k\downarrow}\Big]\Big\} d\nu(w,\xi,\phi)
\end{aligned}
$$

where

$$
d\nu(w,\xi,\phi) = \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac12 w^2} dw\, \prod_l \prod_{\substack{(\sigma\tau)\in\\ \{\uparrow\uparrow,\downarrow\downarrow,\uparrow\downarrow\}}} \tfrac{1}{(2\pi)^2}\, e^{-\frac12(|\phi^l_{\sigma\tau}|^2+|\xi^l_{\sigma\tau}|^2)}\, du^l_{\sigma\tau} dv^l_{\sigma\tau} da^l_{\sigma\tau} db^l_{\sigma\tau}
$$

By a substitution of variables and collecting terms, one obtains

$$
\begin{aligned}
&e^{-\mathcal{U}(\psi,\bar\psi)} = \\
&\int \exp\Big\{ \tfrac{1}{\kappa}\sum_k \Big[U_0^{\frac12} w + i\sum_l \lambda_l^{\frac12}\Big(\bar\xi^l_{\uparrow\uparrow}\,\bar y_l(\mathbf{k}') + \xi^l_{\uparrow\uparrow}\, y_l(\mathbf{k}')\Big)\Big]\bar\psi_{k\uparrow}\psi_{k\uparrow}\Big\}\times \\
&\quad \exp\Big\{ \tfrac{1}{\kappa}\sum_k \Big[U_0^{\frac12} w + i\sum_l \lambda_l^{\frac12}\Big(\bar\xi^l_{\downarrow\downarrow}\,\bar y_l(\mathbf{k}') + \xi^l_{\downarrow\downarrow}\, y_l(\mathbf{k}')\Big)\Big]\bar\psi_{k\downarrow}\psi_{k\downarrow}\Big\}\times \\
&\quad \exp\Big\{ \sum_l \Big[i(2\lambda_l)^{\frac12}\bar\xi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\psi_{k\downarrow} + i(2\lambda_l)^{\frac12}\xi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\bar\psi_{k\downarrow}\psi_{k\uparrow}\Big]\Big\}\times \\
&\quad \exp\Big\{ \sum_l \Big[i\lambda_l^{\frac12}\,\bar\phi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k \tfrac{\bar y_l(\mathbf{k}')-\bar y_l(-\mathbf{k}')}{2}\,\bar\psi_{k\uparrow}\bar\psi_{-k\uparrow} + i\lambda_l^{\frac12}\,\phi^l_{\uparrow\uparrow}\, \tfrac{1}{\kappa}\sum_k \tfrac{y_l(\mathbf{k}')-y_l(-\mathbf{k}')}{2}\,\psi_{k\uparrow}\psi_{-k\uparrow}\Big]\Big\}\times \\
&\quad \exp\Big\{ \sum_l \Big[i\lambda_l^{\frac12}\,\bar\phi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \tfrac{\bar y_l(\mathbf{k}')-\bar y_l(-\mathbf{k}')}{2}\,\bar\psi_{k\downarrow}\bar\psi_{-k\downarrow} + i\lambda_l^{\frac12}\,\phi^l_{\downarrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \tfrac{y_l(\mathbf{k}')-y_l(-\mathbf{k}')}{2}\,\psi_{k\downarrow}\psi_{-k\downarrow}\Big]\Big\}\times \\
&\quad \exp\Big\{ \sum_l \Big[i(2\lambda_l)^{\frac12}\bar\phi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k \bar y_l(\mathbf{k}')\,\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + i(2\lambda_l)^{\frac12}\phi^l_{\uparrow\downarrow}\, \tfrac{1}{\kappa}\sum_k y_l(\mathbf{k}')\,\psi_{k\uparrow}\psi_{-k\downarrow}\Big]\Big\} d\nu_\kappa(w,\xi,\phi)
\end{aligned}
$$

where $d\nu_\kappa(w,\xi,\phi)$ is defined in the statement of the theorem. Using the definition of the fields Ξ, Γ and Φ, the above expression reads

$$e^{-\mathcal{U}(\psi,\bar\psi)} = \int \exp\Big\{ \tfrac{1}{\kappa}\sum_k \Big[\Gamma_{\mathbf{k}\uparrow}\,\bar\psi_{k\uparrow}\psi_{k\uparrow} + \Gamma_{\mathbf{k}\downarrow}\,\bar\psi_{k\downarrow}\psi_{k\downarrow} + i\bar\Xi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\uparrow}\psi_{k\downarrow} + i\Xi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\downarrow}\psi_{k\uparrow}$$

$$+ i\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + i\Phi_{\mathbf{k}\uparrow\downarrow}\psi_{k\uparrow}\psi_{-k\downarrow} + \tfrac{i}{2}\sum_\sigma \big(\bar\Phi_{\mathbf{k}\sigma\sigma}\bar\psi_{k\sigma}\bar\psi_{-k\sigma} + \Phi_{\mathbf{k}\sigma\sigma}\psi_{k\sigma}\psi_{-k\sigma}\big)\Big]\Big\} d\nu_\kappa$$

We now rewrite the exponent in order to perform the fermionic functional integral. Since, if the set of spatial momenta satisfy $\mathbf{k}\in M = -M$,

$$\sum_k \Gamma_{k\uparrow}\,\bar\psi_{k\uparrow}\psi_{k\uparrow} = \sum_{k_0\in\frac{\pi}{\beta}(2\mathbf{Z}+1)}\sum_{\mathbf{k}\in M}\Gamma_{k\uparrow}\,\bar\psi_{k\uparrow}\psi_{k\uparrow} = \sum_{k_0>0}\sum_{\mathbf{k}\in M}\Big[\Gamma_{k\uparrow}\,\bar\psi_{k\uparrow}\psi_{k\uparrow} + \Gamma_{-k\uparrow}\,\bar\psi_{-k\uparrow}\psi_{-k\uparrow}\Big]$$

$$= \tfrac{1}{2}\sum_{\substack{k\\ k_0>0}}\Big[\Gamma_{k\uparrow}\big(\bar\psi_{k\uparrow}\psi_{k\uparrow} - \psi_{k\uparrow}\bar\psi_{k\uparrow}\big) + \Gamma_{-k\uparrow}\big(\bar\psi_{-k\uparrow}\psi_{-k\uparrow} - \psi_{-k\uparrow}\bar\psi_{-k\uparrow}\big)\Big]$$

one obtains, using the antisymmetry of the $\Phi_{k\sigma\sigma}$, $\bar\Phi_{k\sigma\sigma}$

$$\sum_k\Big\{\textstyle\sum_\sigma \Gamma_{\mathbf{k}\sigma}\,\bar\psi_{k\sigma}\psi_{k\sigma} + i\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} + i\Phi_{\mathbf{k}\uparrow\downarrow}\psi_{k\uparrow}\psi_{-k\downarrow}$$

$$+ i\bar\Xi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\uparrow}\psi_{k\downarrow} + i\Xi_{\mathbf{k}\uparrow\downarrow}\bar\psi_{k\downarrow}\psi_{k\uparrow} + \tfrac{i}{2}\textstyle\sum_\sigma\big(\bar\Phi_{\mathbf{k}\sigma\sigma}\bar\psi_{k\sigma}\bar\psi_{-k\sigma} + \Phi_{\mathbf{k}\sigma\sigma}\psi_{k\sigma}\psi_{-k\sigma}\big)\Big\}$$

$$= \tfrac{1}{2}\sum_{\substack{k\\ k_0>0}}\Big\{\textstyle\sum_\sigma\Big[\Gamma_{\mathbf{k}\sigma}\big(\bar\psi_{k\sigma}\psi_{k\sigma} - \psi_{k\sigma}\bar\psi_{k\sigma}\big) + \Gamma_{-\mathbf{k}\sigma}\big(\bar\psi_{-k\sigma}\psi_{-k\sigma} - \psi_{-k\sigma}\bar\psi_{-k\sigma}\big)\Big]$$

$$+ i\bar\Phi_{\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow} - \bar\psi_{-k\downarrow}\bar\psi_{k\uparrow}\big) + i\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{-k\uparrow}\bar\psi_{k\downarrow} - \bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\big)$$
$$+ i\Phi_{\mathbf{k}\uparrow\downarrow}\big(\psi_{k\uparrow}\psi_{-k\downarrow} - \psi_{-k\downarrow}\psi_{k\uparrow}\big) + i\Phi_{-\mathbf{k}\uparrow\downarrow}\big(\psi_{-k\uparrow}\psi_{k\downarrow} - \psi_{k\downarrow}\psi_{-k\uparrow}\big)$$
$$+ i\bar\Xi_{\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{k\uparrow}\psi_{k\downarrow} - \psi_{k\downarrow}\bar\psi_{k\uparrow}\big) + i\bar\Xi_{-\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{-k\uparrow}\psi_{-k\downarrow} - \psi_{-k\downarrow}\bar\psi_{-k\uparrow}\big)$$
$$+ i\Xi_{\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{k\downarrow}\psi_{k\uparrow} - \psi_{k\uparrow}\bar\psi_{k\downarrow}\big) + i\Xi_{-\mathbf{k}\uparrow\downarrow}\big(\bar\psi_{-k\downarrow}\psi_{-k\uparrow} - \psi_{-k\uparrow}\bar\psi_{-k\downarrow}\big)$$
$$+ i\textstyle\sum_\sigma\Big[\bar\Phi_{\mathbf{k}\sigma\sigma}\big(\bar\psi_{k\sigma}\bar\psi_{-k\sigma} - \bar\psi_{-k\sigma}\bar\psi_{k\sigma}\big) + \Phi_{\mathbf{k}\sigma\sigma}\big(\psi_{k\sigma}\psi_{-k\sigma} - \psi_{-k\sigma}\psi_{k\sigma}\big)\Big]\Big\}$$

Since, if $a_k = ik_0 - e_{\mathbf{k}}$

$$d\mu_C(\psi,\bar\psi) = \prod_{k,\sigma}\tfrac{\kappa}{a_k}\; e^{-\frac{1}{\kappa}\sum_{k,\sigma} a_k\bar\psi_{k,\sigma}\psi_{k,\sigma}} \prod_{k,\sigma} d\psi_{k,\sigma}d\bar\psi_{k,\sigma}$$

one obtains

$$Z(\beta,L,\{s_{k,\sigma}\}) = \int e^{-\mathcal{U}(\psi,\bar\psi)+\frac{1}{\kappa}\sum_{k,\sigma}s_{k,\sigma}\bar\psi_{k,\sigma}\psi_{k,\sigma}}\,d\mu_C(\psi,\bar\psi)$$

$$= \prod_{k,\sigma}\tfrac{\kappa}{a_k}\int\!\!\int e^{-\frac{1}{2}\frac{1}{\kappa}\sum_{k_0>0}\sum_{\mathbf{k}}\langle\Psi_k,S_k\Psi_k\rangle}\prod_{k,\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}\;d\nu_\kappa(w,\xi,\phi)$$

$$= \int \prod_{\substack{k_0>0 \\ \mathbf{k}\in M}} \left\{ \left(\tfrac{\kappa^2}{a_k a_{-k}}\right)^2 \int e^{-\frac{1}{2}\frac{1}{\kappa}\langle \Psi_k, S_k \Psi_k\rangle} d\bar\psi_{k\uparrow} d\psi_{k\uparrow} d\bar\psi_{-k\downarrow} d\psi_{-k\downarrow} \times \right.$$

$$\left. d\psi_{k\downarrow} d\bar\psi_{k\downarrow} d\psi_{-k\uparrow} d\bar\psi_{-k\uparrow} \right\} d\nu_\kappa(w,\xi,\phi)$$

$$= \int \prod_{\substack{k_0>0 \\ \mathbf{k}\in M}} \left\{ \left(\tfrac{1}{a_k a_{-k}}\right)^2 \mathrm{Pf} S_k \right\} d\nu_\kappa(w,\xi,\phi)$$

where

$$\Psi_k = \left(\bar\psi_{k\uparrow}, \psi_{k\uparrow}, \bar\psi_{-k\downarrow}, \psi_{-k\downarrow}, \psi_{k\downarrow}, \bar\psi_{k\downarrow}, \psi_{-k\uparrow}, \bar\psi_{-k\uparrow}\right)$$

and $\mathrm{Pf}S_k$ is the Pfaffian of the 8×8 skew symmetric matrix S_k defined in the statement of the theorem. We used that

$$\prod_{k,\sigma} d\psi_{k\sigma} d\bar\psi_{k\sigma} = \prod_{k_0>0} \prod_{\mathbf{k}} \left\{ + d\bar\psi_{k\uparrow} d\psi_{k\uparrow} d\bar\psi_{-k\downarrow} d\psi_{-k\downarrow} d\psi_{k\downarrow} d\bar\psi_{k\downarrow} d\psi_{-k\uparrow} d\bar\psi_{-k\uparrow} \right\}$$

Part b) of the theorem follows from

$$\tfrac{1}{\kappa}\langle \bar\psi_{p\sigma}\psi_{p\sigma}\rangle = \tfrac{\partial}{\partial s_{p\sigma}}_{|s=0} \log Z(\beta, L, \{s_{k\sigma}\}) \qquad \bullet$$

Lemma III.2: Let S_k be the skew symmetric 8×8 matrix of Theorem III.1. Then the Pfaffian of S_k is given by

$$\begin{aligned}
\mathrm{Pf}S_k = & \left(A_{k\uparrow}A_{-k\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\right)\left(A_{-k\uparrow}A_{k\downarrow} + \bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\right) \\
& + A_{k\uparrow}A_{-k\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} + A_{k\downarrow}A_{-k\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} \\
& + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} \\
& + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} \\
& + i\Xi_{\mathbf{k}\uparrow\downarrow}A_{-k\uparrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\downarrow\downarrow} + i\bar\Xi_{\mathbf{k}\uparrow\downarrow}A_{-k\uparrow}\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} \\
& - i\Xi_{\mathbf{k}\uparrow\downarrow}A_{-k\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} - i\bar\Xi_{\mathbf{k}\uparrow\downarrow}A_{-k\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow} \\
& + i\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} + i\bar\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow} \\
& - i\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\uparrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\downarrow\downarrow} - i\bar\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\uparrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} \\
& + \Xi_{\mathbf{k}\uparrow\downarrow}\bar\Xi_{\mathbf{k}\uparrow\downarrow}A_{-k\uparrow}A_{-k\downarrow} + \Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\uparrow}A_{k\downarrow} \\
& + \Xi_{\mathbf{k}\uparrow\downarrow}\bar\Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Xi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow} \\
& - \Xi_{\mathbf{k}\uparrow\downarrow}\Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow} - \bar\Xi_{\mathbf{k}\uparrow\downarrow}\bar\Xi_{-\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} \\
& + \Xi_{\mathbf{k}\uparrow\downarrow}\bar\Xi_{\mathbf{k}\uparrow\downarrow}\Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Xi_{-\mathbf{k}\uparrow\downarrow}
\end{aligned}$$

Proof: The Pfaffian of S_k is given by the sum of all contractions $\sum\prod\langle\psi\psi\rangle$ of the fields

$$\bar\psi_{k\uparrow}, \psi_{k\uparrow}, \bar\psi_{-k\downarrow}, \psi_{-k\downarrow}, \psi_{k\downarrow}, \bar\psi_{k\downarrow}, \psi_{-k\uparrow}, \bar\psi_{-k\uparrow}$$

where the value $\langle\psi\psi\rangle$ is given by the corresponding matrix element. That is, the Pfaffian can be evaluated by using Wick's Theorem or integration by parts. Since S_k is an 8×8 matrix, one has $\mathrm{Pf}[-S_k]=\mathrm{Pf}S_k$ and

$$
\begin{aligned}
\mathrm{Pf}S_k =&\langle\bar\psi_{k\uparrow}\psi_{k\uparrow}\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
=&-A_{k\uparrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}-i\bar\Phi_{\mathbf{k}\uparrow\downarrow}\langle\psi_{k\uparrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-i\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\langle\psi_{k\uparrow}\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}+i\bar\Phi_{\mathbf{k}\uparrow\uparrow}\langle\psi_{k\uparrow}\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\rangle_{S_k}\\
=&+A_{k\uparrow}A_{-k\downarrow}\langle\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}+iA_{k\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\langle\psi_{-k\downarrow}\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&+iA_{k\uparrow}\Xi_{-\mathbf{k}\uparrow\downarrow}\langle\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&+\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\langle\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\langle\psi_{-k\downarrow}\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\langle\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&+\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\rangle_{S_k}-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Xi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\uparrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\rangle_{S_k}\\
=&+\big(A_{k\uparrow}A_{-k\downarrow}+\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\big)\langle\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&+\big(A_{k\uparrow}i\Xi_{-\mathbf{k}\uparrow\downarrow}-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\big)\langle\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&+\big(A_{k\uparrow}i\bar\Phi_{\mathbf{k}\downarrow\downarrow}-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\big)\langle\psi_{-k\downarrow}\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{-k\uparrow}\bar\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\bar\psi_{k\downarrow}\bar\psi_{-k\uparrow}\rangle_{S_k}+\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\psi_{-k\uparrow}\rangle_{S_k}\\
&-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Xi_{\mathbf{k}\uparrow\downarrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\psi_{-k\uparrow}\rangle_{S_k}-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\uparrow}\langle\bar\psi_{-k\downarrow}\psi_{-k\downarrow}\psi_{k\downarrow}\bar\psi_{k\downarrow}\rangle_{S_k}\\
=&+\big(A_{k\uparrow}A_{-k\downarrow}+\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\big)\big(A_{k\downarrow}A_{-k\uparrow}+\Phi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\big)\\
&+\big(A_{k\uparrow}i\Xi_{-\mathbf{k}\uparrow\downarrow}-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\big)\big(-\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}-i\bar{\Xi}_{-k\uparrow\downarrow}A_{k\downarrow}\big)\\
&+\big(A_{k\uparrow}i\bar\Phi_{\mathbf{k}\downarrow\downarrow}-\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\big)\Big(-i\Phi_{\mathbf{k}\downarrow\downarrow}A_{-k\uparrow}-\bar{\Xi}_{-k\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\Big)\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\big(-i\bar\Phi_{\mathbf{k}\downarrow\downarrow}A_{-k\uparrow}-\Xi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow}\big)\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Xi_{\mathbf{k}\uparrow\downarrow}\big(-A_{-k\downarrow}A_{-k\uparrow}-\Xi_{-\mathbf{k}\uparrow\downarrow}\bar{\Xi}_{-\mathbf{k}\uparrow\downarrow}\big)\\
&-\bar{\Xi}_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\big(A_{-k\downarrow}i\bar\Phi_{-\mathbf{k}\uparrow\downarrow}+\bar\Phi_{\mathbf{k}\downarrow\downarrow}\bar{\Xi}_{-k\uparrow\downarrow}\big)\\
&+\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\downarrow}\big(\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}+i\Xi_{-\mathbf{k}\uparrow\downarrow}A_{k\downarrow}\big)\\
&-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Xi_{\mathbf{k}\uparrow\downarrow}\big(A_{-k\downarrow}i\Phi_{-\mathbf{k}\uparrow\downarrow}+\Xi_{-\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\downarrow\downarrow}\big)\\
&-\bar\Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\uparrow\uparrow}\big(-A_{-k\downarrow}A_{k\downarrow}-\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{\mathbf{k}\downarrow\downarrow}\big)
\end{aligned}
$$

By multiplying out the brackets one obtains the stated formula •

Before we consider some special cases in section IV where the effective potential and the two point functions are computed more explicitely, in the following theorem we write down the integral representation for the generating functional of the connected amputated Greens functions.

Theorem III.3: Let $\mathcal{U}(\psi,\bar\psi)$ be given by (III.6) and let V be the effective potential (III.8) Let

$$G(\eta)=\log\int e^{-\mathcal{U}(\psi+\eta,\bar\psi+\bar\eta)}d\mu_C(\psi,\bar\psi)$$

be the generating functional for the connected amputated Greens functions. Then, if $\underline{\zeta}_k$ denotes the eight component vector

$$\underline{\zeta}_k=(a_k\eta_{k\uparrow},-a_k\bar\eta_{k\uparrow},a_{-k}\eta_{-k\downarrow},-a_{-k}\bar\eta_{-k\downarrow},-a_k\bar\eta_{k\downarrow},a_k\eta_{k\downarrow},-a_{-k}\bar\eta_{-k\uparrow},a_{-k}\eta_{-k\uparrow})$$

and $S_k=S_k(w,\xi,\phi)$ is the 8×8 matrix of Theorem III.1, one has the following integral representation

$$G(\eta)-G(0)=-\tfrac{1}{\kappa}\sum_{k\sigma}a_k\bar\eta_{k\sigma}\eta_{k\sigma}\;+\;\log\frac{\int e^{-\frac12\frac1\kappa\sum_k\langle\underline{\zeta}_k,S_k^{-1}\underline{\zeta}_k\rangle}e^{-\kappa V(w,\xi,\phi)}dw\,\Pi\,d\xi d\phi}{\int\;\;e^{-\kappa V(w,\xi,\phi)}dw\,\Pi\,d\xi d\phi}$$

Proof: By a substitution of Grassmann variables,

$$\int e^{-\mathcal{U}(\psi+\eta,\bar\psi+\bar\eta)}d\mu_C=$$

$$e^{-\frac1\kappa\sum_{k\sigma}a_k\bar\eta_{k\sigma}\eta_{k\sigma}}\int e^{-\mathcal{U}(\psi,\bar\psi)}\prod_{k\sigma}\tfrac{\kappa}{a_k}\,e^{\frac1\kappa\sum_{k\sigma}a_k[\bar\psi_{k\sigma}\eta_{k\sigma}+\bar\eta_{k\sigma}\psi_{k\sigma}-\bar\psi_{k\sigma}\psi_{k\sigma}]}\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}$$

As in the proof of Theorem III.1, one has

$$e^{-\mathcal{U}(\psi,\bar\psi)-\frac1\kappa\sum_{k\sigma}a_k\bar\psi_{k\sigma}\psi_{k\sigma}}=\int e^{-\frac12\frac1\kappa\sum_{k_0>0}\sum_{\mathbf{k}}\langle\Psi_k,S_k\Psi_k\rangle}d\nu_\kappa(w,\xi,\phi)$$

such that

$$\begin{aligned}
\int e^{-\mathcal{U}(\psi,\bar\psi)}&\prod_{k\sigma}\tfrac{\kappa}{a_k}\,e^{\frac1\kappa\sum_{k\sigma}a_k[\bar\psi_{k\sigma}\eta_{k\sigma}+\bar\eta_{k\sigma}\psi_{k\sigma}-\bar\psi_{k\sigma}\psi_{k\sigma}]}\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}\\
&=\int\int\prod_{k_0>0,\mathbf{k}}\left(\tfrac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}\right)e^{\frac1\kappa\sum_{k\sigma}a_k[\bar\psi_{k\sigma}\eta_{k\sigma}+\bar\eta_{k\sigma}\psi_{k\sigma}]}\times\\
&\qquad\prod_{k_0>0,\mathbf{k}}\tfrac{\kappa^4}{\mathrm{Pf}S_k}\,e^{-\frac12\frac1\kappa\sum_{k_0>0}\sum_{\mathbf{k}}\langle\Psi_k,S_k\Psi_k\rangle}d\nu_\kappa(w,\xi,\phi)\prod_{k\sigma}d\psi_{k\sigma}d\bar\psi_{k\sigma}\\
&=\int\int\prod_{k_0>0,\mathbf{k}}\left(\tfrac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}\right)e^{\frac1\kappa\sum_{k_0>0,\mathbf{k}}\langle\Psi_k,\underline{\zeta}_k\rangle}\times\\
&\qquad\prod_{k_0>0,\mathbf{k}}\tfrac{\kappa^4}{\mathrm{Pf}S_k}\,e^{-\frac12\frac1\kappa\sum_{k_0>0}\sum_{\mathbf{k}}\langle\Psi_k,S_k\Psi_k\rangle}\prod_{k_0>0,\mathbf{k}}d\Psi_k\;d\nu_\kappa(w,\xi,\phi)\\
&=\int\prod_{k_0>0,\mathbf{k}}\left(\tfrac{\mathrm{Pf}S_k}{a_k^2a_{-k}^2}\right)e^{-\frac12\frac1\kappa\sum_{k_0>0,\mathbf{k}}\langle\underline{\zeta}_k,S_k^{-1}\underline{\zeta}_k\rangle}d\nu_\kappa(w,\xi,\phi)
\end{aligned}$$

where

$$d\Psi_k=d\bar\psi_{k\uparrow}d\psi_{k\uparrow}d\bar\psi_{-k\downarrow}d\psi_{-k\downarrow}d\psi_{k\downarrow}d\bar\psi_{k\downarrow}d\psi_{-k\uparrow}d\bar\psi_{-k\uparrow}$$

By definition of V

$$\prod_{k_0>0,\mathbf{k}} \left(\frac{\mathrm{Pf} S_k}{a_k^2 a_{-k}^2}\right) d\nu_\kappa(w,\xi,\phi) = e^{-\kappa V(w,\xi,\phi)} dw \prod d\xi d\phi$$

which proves the theorem •

IV. Some Special Cases

IV.1 A Delta Function Interaction with Forward, Exchange and BCS Term

We consider the model

$$Z(\beta, L, \{s_{k,\sigma}\}) = \int e^{-\mathcal{U}(\psi,\bar\psi)+\frac{1}{\kappa}\sum_{k,\sigma} s_{k,\sigma}\bar\psi_{k,\sigma}\psi_{k,\sigma}} d\mu_C(\psi,\bar\psi) \tag{IV.1}$$

where

$$\mathcal{U}(\psi,\bar\psi) = \tfrac{\lambda}{\kappa^3} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{k,p,q} \left[\delta_{k,p} + \delta_{k,-p} + \delta_{q,0}\right] \bar\psi_{k,\sigma}\bar\psi_{q-k,\tau}\psi_{p,\sigma}\psi_{q-p,\tau} \tag{IV.2}$$

In that case one has $N=0$ in Theorem III.1 and all fields are independent of $\mathbf{k}$:

$$\Xi_{\mathbf{k}\uparrow\downarrow} = (2\lambda)^{\frac12}\xi_{\uparrow\downarrow}\,, \quad \bar\Xi_{\mathbf{k}\uparrow\downarrow} = (2\lambda)^{\frac12}\bar\xi_{\uparrow\downarrow}\,, \qquad \Xi_{\mathbf{k}\sigma\sigma} = \lambda^{\frac12}\xi_{\sigma\sigma}\,, \quad \bar\Xi_{\mathbf{k}\sigma\sigma} = \lambda^{\frac12}\bar\xi_{\sigma\sigma}\,,$$

$$\Gamma_{\mathbf{k}\sigma} = \Gamma_\sigma = \lambda^{\frac12}(w + i\xi_{\sigma\sigma} + i\bar\xi_{\sigma\sigma}) = (2\lambda)^{\frac12}\tfrac{w+2ia_{\sigma\sigma}}{\sqrt2}$$

if $\xi_{\sigma\sigma} = a_{\sigma\sigma} + ib_{\sigma\sigma}$ and

$$\Phi_{\mathbf{k}\uparrow\downarrow} = (2\lambda)^{\frac12}\phi_{\uparrow\downarrow}\,, \quad \bar\Phi_{\mathbf{k}\uparrow\downarrow} = (2\lambda)^{\frac12}\bar\phi$$

and $\Phi_{\mathbf{k}\uparrow\uparrow} = \Phi_{\mathbf{k}\downarrow\downarrow} = 0$. Substituting 2λ by λ, the Pfaffian of S_k becomes

$$\begin{aligned}\mathrm{Pf} S_k =& (A_{k\uparrow}A_{-k\downarrow} + \Phi_{\uparrow\downarrow}\bar\Phi_{\uparrow\downarrow})(A_{-k\uparrow}A_{k\downarrow} + \Phi_{\uparrow\downarrow}\bar\Phi_{\uparrow\downarrow}) + (A_{k\uparrow}A_{k\downarrow} + A_{-k\uparrow}A_{-k\downarrow})\Xi_{\uparrow\downarrow}\bar\Xi_{\uparrow\downarrow}\\ &+ 2\Xi_{\uparrow\downarrow}\bar\Xi_{\uparrow\downarrow}\Phi_{\uparrow\downarrow}\bar\Phi_{\uparrow\downarrow} + (\Xi_{\uparrow\downarrow}\bar\Xi_{\uparrow\downarrow})^2\\ =& (A_{k\uparrow}A_{-k\downarrow} + \lambda\rho^2)(A_{-k\uparrow}A_{k\downarrow} + \lambda\rho^2) + (A_{k\uparrow}A_{k\downarrow} + A_{-k\uparrow}A_{-k\downarrow})\lambda x^2\\ &+ 2\lambda^2 x^2\rho^2 + \lambda^2 x^4\end{aligned} \tag{IV.3}$$

where $\rho = |\phi|$, $x = |\xi_{\uparrow\downarrow}|$ and $A_{k\sigma} = ik_0 - e_{\mathbf{k}} - s_{k\sigma} - \Gamma_\sigma$. We assume $e_{\mathbf{k}} = e_{-\mathbf{k}}$. The two point function $\langle\bar\psi_{p\uparrow}\psi_{p\uparrow}\rangle$ is given by

$$\langle\bar\psi_{p\uparrow}\psi_{p'\uparrow}\rangle = \beta L^d \delta_{p,p'} \int \frac{\frac{\partial}{\partial s_{p\uparrow}}\mathrm{Pf} S_p}{\mathrm{Pf} S_p}\, \delta_\kappa(\Gamma_\sigma, x, \rho)\, dw da_{\uparrow\uparrow} da_{\downarrow\downarrow} x dx \rho d\rho \tag{IV.4}$$

where

$$\delta_\kappa(\Gamma_\sigma, x, \rho) = \frac{e^{-\kappa V(\Gamma_\sigma, x, \rho)}}{\int e^{-\kappa V(\Gamma_\sigma, x, \rho)} dw da_{\uparrow\uparrow} da_{\downarrow\downarrow} x dx \rho d\rho} \tag{IV.5}$$

is, if the volume or $\kappa = \beta L^d$ goes to infinity, a δ-sequence which forces the variables $w, a_{\sigma\sigma}, x$ and ρ to take values where the real part of the effective potential

$$V(\Gamma_\sigma, x, \rho) = \tfrac{1}{4}w^2 + a_{\uparrow\uparrow}^2 + a_{\downarrow\downarrow}^2 + x^2 + \rho^2 - \tfrac{1}{\beta L^d} \sum_{\mathbf{k}\in M} \log \prod_{k_0>0} \tfrac{\mathrm{Pf} S_k}{a_k^2 a_{-k}^2}$$

has its global minimum.

A detailed analysis to find the global minimum, in particular for the case discussed in section IV.2, we plan to give in a forthcoming paper. Here we simply write down the two point function under certain assumptions of what the minimum may be.

Since V is symmetric in $a_{\uparrow\uparrow}$ and $a_{\downarrow\downarrow}$, we substitute (IV.4) by

$$\langle \bar\psi_{p\uparrow} \psi_{p'\uparrow} \rangle = \beta L^d \delta_{p,p'} \frac{\int F_p^{\uparrow\uparrow}(w,a,x,\rho)\, e^{-\kappa V(w,a,x,\rho)} dw da x dx \rho d\rho}{\int \; e^{-\kappa V(w,a,x,\rho)} dw da x dx \rho d\rho} \tag{IV.6}$$

where

$$V(w,a,x,\rho) = \tfrac{1}{4}w^2 + a^2 + x^2 + \rho^2 - \tfrac{1}{\beta L^d} \sum_{\mathbf{k}\in M} \log \prod_{k_0>0} \tfrac{\mathrm{Pf} S_k}{a_k^2 a_{-k}^2} \tag{IV.7}$$

and, for $s_{k\sigma} = 0$, $A_{k\uparrow} = A_{k\downarrow} = ik_0 - e_{\mathbf{k}} - \gamma$ where $\gamma = \lambda^{\frac{1}{2}}(w + \sqrt{2}\, a)$. Then the two point function is given by

$$F_p^{\uparrow\uparrow}(w,a,x,\rho) = \frac{\frac{\partial}{\partial s_{p\uparrow}} \mathrm{Pf} S_p}{\mathrm{Pf} S_p}\Big|_{s_{p\sigma}=0} = -\frac{A_{-p}(A_p A_{-p} + \lambda\rho^2) + A_p \lambda x^2}{(A_p A_{-p} + \lambda\rho^2)^2 + (A_p^2 + A_{-p}^2 + 2\lambda\rho^2)\lambda x^2 + \lambda^2 x^4} \tag{IV.8}$$

To proceed one has to compute the effective potential, that is, the product in (IV.7).

Lemma IV.1: a) Let $a_k = ik_0 - e_{\mathbf{k}}$, $e_{-\mathbf{k}} = e_{\mathbf{k}}$ and let b, c, d be some complex numbers. Then

$$\begin{aligned} \prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)} \tfrac{a_k a_{-k} + b a_k + c a_{-k} + d}{a_k a_{-k}} &:= \lim_{\substack{\epsilon\to 0 \\ \epsilon<0}} \prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)} \tfrac{a_k a_{-k} + b e^{i\epsilon k_0} a_k + c e^{-i\epsilon k_0} a_{-k} + d}{a_k a_{-k}} \\ &= \tfrac{\cosh \frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2 + \omega_+}}{\cosh \frac{\beta}{2} e_{\mathbf{k}}} \; \tfrac{\cosh \frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2 + \omega_-}}{\cosh \frac{\beta}{2} e_{\mathbf{k}}} \; e^{\frac{\beta}{2}(b+c)} \end{aligned} \tag{IV.9}$$

where $\omega_\pm = \frac{p \pm \sqrt{p^2 - 4q}}{2}$ with $p = (b+c-e_{\mathbf{k}})^2 - e_k^2 - 4bc + 2d$ and $q = 4bce_{\mathbf{k}}^2 - 2(b+c)e_{\mathbf{k}} d + d^2$. In particular, for $b = c$ and $d = c^2$

$$\prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)} \left(1 + \tfrac{c}{a_k}\right) := \lim_{\substack{\epsilon\to 0 \\ \epsilon<0}} \prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)} \left(1 + e^{-i\epsilon k_0} \tfrac{c}{a_k}\right) = \tfrac{1 + e^{-\beta(e_{\mathbf{k}} - c)}}{1 + e^{-\beta e_{\mathbf{k}}}}$$

b) Let $A_k = a_k - \gamma$, γ a complex number, and let

$$\mathrm{Pf}S_k = \mathrm{Pf}S_k(a_k, a_{-k}) = (A_k A_{-k} + \lambda\rho^2)^2 + (A_k^2 + A_{-k}^2 + 2\lambda\rho^2)\lambda x^2 + \lambda^2 x^4$$

Then

$$\begin{aligned}\prod_{\substack{k_0\in\frac{\pi}{\beta}(2\mathbf{Z}+1)\\ k_0>0}} \frac{\mathrm{Pf}S_k(a_k,a_{-k})}{a_k^2 a_{-k}^2} &:= \lim_{\substack{\epsilon\to 0\\ \epsilon<0}} \prod_{\substack{k_0\in\frac{\pi}{\beta}(2\mathbf{Z}+1)\\ k_0>0}} \frac{\mathrm{Pf}S_k(e^{i\epsilon k_0}a_k, e^{-i\epsilon k_0}a_{-k})}{a_k^2 a_{-k}^2} \\ &= \frac{\sinh^2(\frac{\beta}{2} i\sqrt{\lambda}\, x) + \cosh^2(\frac{\beta}{2}\sqrt{(e_{\mathbf{k}}+\gamma)^2+\lambda\rho^2}\,)}{\cosh^2\frac{\beta}{2}e_{\mathbf{k}}}\, e^{-\beta\gamma} \qquad \text{(IV.10)}\end{aligned}$$

Proof: We first explain why the definition of the product given in the lemma is the right one.

Consider the quadratic perturbation

$$H = H_0 + \lambda N = \tfrac{1}{L^d}\sum_{\mathbf{k}\sigma} e_{\mathbf{k}}\, a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \tfrac{\lambda}{L^d}\sum_{\mathbf{k}\sigma} a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}$$

One finds by explicit computation of the trace

$$Z(\lambda,\beta) = \frac{Tr\, e^{-\beta(H_0+\lambda N)}}{Tr\, e^{-\beta H_0}} = \prod_{\mathbf{k}\sigma} \frac{1+e^{-\beta(e_{\mathbf{k}}+\lambda)}}{1+e^{-\beta e_{\mathbf{k}}}}$$

On the other hand, by explicit summation of the perturbation series or equivalently, by performing the fermionic functional integral, one obtains

$$\begin{aligned} Z(\lambda,\beta) &= \int e^{\frac{\lambda}{\beta L^d}\sum_{k\sigma}\bar\psi_{k\sigma}\psi_{k\sigma}}\, d\mu_C \\ &= \prod_{k\sigma} \frac{\beta L^d}{ik_0 - e_{\mathbf{k}}} \int e^{\frac{\lambda}{\beta L^d}\sum_{k\sigma}\bar\psi_{k\sigma}\psi_{k\sigma}}\, e^{-\frac{1}{\beta L^d}\sum_{k\sigma}(ik_0-e_{\mathbf{k}})\bar\psi_{k\sigma}\psi_{k\sigma}} \prod_{k\sigma} d\psi_{k\sigma} d\bar\psi_{k\sigma} \\ &= \prod_{k\sigma} \frac{ik_0-e_{\mathbf{k}}-\lambda}{ik_0-e_{\mathbf{k}}} = \prod_{\mathbf{k}\sigma}\prod_{k_0}\left(1-\frac{\lambda}{a_k}\right) \qquad \text{(IV.11)}\end{aligned}$$

In the perturbation expansion the product in the last equation results from

$$\prod_{k\sigma}\left(1-\frac{\lambda}{a_k}\right) = e^{\sum_{k\sigma}\log\left(1-\frac{\lambda}{a_k}\right)} = e^{-\sum_{r=1}^{\infty}\frac{\lambda^r}{r}\sum_{k\sigma}\frac{1}{a_k{}^r}}$$

The term for $r = 1$ requires special attention since $\sum_{k_0}\frac{1}{ik_0 - e_{\mathbf{k}}}$ is not absolutely convergent. This sum enters the perturbation series as the propagator

$$C(x_0 - x_0', \mathbf{x}-\mathbf{x}') = \frac{Tr\, e^{-\beta H_0} T\psi(x_0\mathbf{x})\psi^+(x_0'\mathbf{x}')}{Tr\, e^{-\beta H_0}}$$

evaluated at $x_0 - x_0' = 0$. Here $T\psi(x_0\mathbf{x})\psi^+(x_0'\mathbf{x}')$ is defined to be $\psi(x_0\mathbf{x})\psi^+(x_0'\mathbf{x}')$ if $x_0 \geq x_0'$ and $-\psi^+(x_0'\mathbf{x}')\psi(x_0\mathbf{x})$ if $x_0 < x_0'$. By explicit computation one finds that for $x_0 \neq x_0'$

$$\begin{aligned}
C(x_0 - x_0', \mathbf{x} - \mathbf{x}') &= \tfrac{1}{L^d} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{-e_{\mathbf{k}}(x_0 - x_0')} \left[\theta_\beta(-e_{\mathbf{k}})\theta(x_0' - x_0) - \theta_\beta(e_{\mathbf{k}})\theta(x_0 - x_0')\right] \\
&= \tfrac{1}{\beta L^d} \sum_{\mathbf{k},k_0} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}') - ik_0(x_0 - x_0')} \tfrac{1}{ik_0 - e_{\mathbf{k}}}
\end{aligned}$$

and for $x_0 = x_0'$

$$C(0, \mathbf{x} - \mathbf{x}') = \tfrac{1}{L^d} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \theta_\beta(-e_{\mathbf{k}}) = \lim_{\substack{\epsilon \to 0 \\ \epsilon < 0}} \tfrac{1}{\beta L^d} \sum_{\mathbf{k},k_0} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}') - ik_0\epsilon} \tfrac{1}{ik_0 - e_{\mathbf{k}}}$$

Here θ is a step function being 1 for positive arguments and $\theta_\beta(-v) = \frac{e^{-\beta v}}{1+e^{-\beta v}}$ is an approximate step function. Therefore the k_0-sums have to be evaluated according to

$$\sum_{k_0} \tfrac{1}{ik_0 - e_{\mathbf{k}}} = \sum_{k_0}^{\epsilon} \tfrac{1}{ik_0 - e_{\mathbf{k}}} := \lim_{\substack{\epsilon \to 0 \\ \epsilon < 0}} \sum_{k_0} e^{-ik_0\epsilon} \tfrac{1}{ik_0 - e_{\mathbf{k}}} = \beta\theta_\beta(-e_{\mathbf{k}}) = \beta \, \tfrac{e^{-\beta e_{\mathbf{k}}}}{1+e^{-\beta e_{\mathbf{k}}}}$$

We emphasize this point because the following computation leads to a wrong result

$$\sum_{k_0} \tfrac{1}{ik_0 - e_{\mathbf{k}}} = \sum_{k_0}^{sym} \tfrac{1}{ik_0 - e_{\mathbf{k}}} := \tfrac{1}{2} \sum_{k_0} \left[\tfrac{1}{ik_0 - e_{\mathbf{k}}} + \tfrac{1}{-ik_0 - e_{\mathbf{k}}}\right] = \sum_{k_0} \tfrac{-e_{\mathbf{k}}}{k_0^2 + e_{\mathbf{k}}^2} = -\tfrac{\beta}{2} \, \tfrac{1-e^{-\beta e_{\mathbf{k}}}}{1+e^{-\beta e_{\mathbf{k}}}}$$

Therefore the product in (IV.11) is found to be

$$\begin{aligned}
\prod_{k_0} \left(1 - \tfrac{\lambda}{ik_0 - e_{\mathbf{k}}}\right) &= \prod_{k_0}^{\epsilon} \left(1 - \tfrac{\lambda}{ik_0 - e_{\mathbf{k}}}\right) = e^{-\sum_{r=1}^{\infty} \frac{\lambda^r}{r} \sum_{k_0}^{\epsilon} \frac{1}{(ik_0 - e_{\mathbf{k}})^r}} \\
&= e^{-\sum_{r=1}^{\infty} \frac{\lambda^r}{r} \sum_{k_0}^{sym} \frac{1}{(ik_0 - e_{\mathbf{k}})^r}} \, e^{\sum_{k_0}^{sym} \frac{\lambda}{ik_0 - e_{\mathbf{k}}} - \sum_{k_0}^{\epsilon} \frac{\lambda}{ik_0 - e_{\mathbf{k}}}} \\
&= \prod_{k_0}^{sym} \left(1 - \tfrac{\lambda}{ik_0 - e_{\mathbf{k}}}\right) e^{-\frac{\beta}{2}\lambda} = \prod_{k_0 > 0} \tfrac{k_0^2 + (e_{\mathbf{k}} + \lambda)^2}{k_0^2 + e_{\mathbf{k}}^2} \, e^{-\frac{\beta}{2}\lambda} \\
&\overset{(\mathrm{II.6})}{=} \tfrac{\cosh \frac{\beta}{2}(e_{\mathbf{k}} + \lambda)}{\cosh \frac{\beta}{2} e_{\mathbf{k}}} \, e^{-\frac{\beta}{2}\lambda} = \tfrac{1+e^{-\beta(e_{\mathbf{k}} + \lambda)}}{1+e^{-\beta e_{\mathbf{k}}}}
\end{aligned} \tag{IV.12}$$

which proves (IV.9) for $b = c$ and $d = c^2$.

In the general case one may proceed similarly to obtain (using $a_k + a_{-k} = -2e_{\mathbf{k}}$)

$$\begin{aligned}
\prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)} \tfrac{a_k a_{-k} + b a_k + c a_{-k} + d}{a_k a_{-k}} &= \lim_{\substack{\epsilon \to 0 \\ \epsilon < 0}} \prod_{k_0} \left(1 + \tfrac{b e^{i\epsilon k_0} a_k + c e^{-i\epsilon k_0} a_{-k} + d}{a_k a_{-k}}\right) =: \prod_{k_0}^{\epsilon} \cdots \\
&= \prod_{k_0}^{sym} \left(1 + \tfrac{b a_k + c a_{-k} + d}{a_k a_{-k}}\right) e^{b(\sum_{k_0}^{\epsilon} - \sum_{k_0}^{sym}) \frac{1}{a_{-k}} + c(\sum_{k_0}^{\epsilon} - \sum_{k_0}^{sym}) \frac{1}{a_k}} \\
&= \prod_{k_0 > 0} \tfrac{(a_k a_{-k} + b a_k + c a_{-k} + d)(a_k a_{-k} + b a_{-k} + c a_k + d)}{(a_k a_{-k})^2} \, e^{\frac{\beta}{2}(b+c)} \\
&= \prod_{k_0 > 0} \tfrac{(a_k a_{-k})^2 + a_k a_{-k}(-2e_{\mathbf{k}}(b+c) + 2d + b^2 + c^2) + bc(a_k^2 + a_{-k}^2) - 2e_{\mathbf{k}} d(b+c) + d^2}{(a_k a_{-k})^2} \, e^{\frac{\beta}{2}(b+c)}
\end{aligned}$$

Since $a_k^2 + a_{-k}^2 = -2k_0^2 + 2e_{\mathbf{k}}^2 = -2a_k a_{-k} + 4e_{\mathbf{k}}^2$ one obtains

$$\begin{aligned}
\prod_{k_0 \in \frac{\pi}{\beta}(2\mathbf{Z}+1)}^{\epsilon} \frac{a_k a_{-k} + b a_k + c a_{-k} + d}{a_k a_{-k}} &= \prod_{k_0>0} \frac{(a_k a_{-k})^2 + p a_k a_{-k} + q}{(a_k a_{-k})^2}\, e^{\frac{\beta}{2}(b+c)} \\
&= \prod_{k_0>0} \left(1 + \tfrac{\omega_+}{a_k a_{-k}}\right)\left(1 + \tfrac{\omega_-}{a_k a_{-k}}\right)\, e^{\frac{\beta}{2}(b+c)}
\end{aligned}$$

where $p = -2e_{\mathbf{k}}(b+c) + 2d + (b+c)^2 - 4bc$ and $q = 4bce_{\mathbf{k}}^2 - 2e_{\mathbf{k}}d(b+c) + d^2$. Application of Lemma II.2 proves part (a) of the lemma.

Part b) One has $A_k A_{-k} = a_k a_{-k} - \gamma(a_k + a_{-k}) + \gamma^2$ and

$$A_k^2 + A_{-k}^2 = a_k^2 + a_{-k}^2 - 2\gamma(a_k + a_{-k}) + 2\gamma^2 = 2A_k A_{-k} + (a_k - a_{-k})^2$$

which gives

$$\begin{aligned}
\mathrm{Pf} S_k &= (A_k A_{-k} + \lambda\rho^2 + \lambda x^2)^2 + (a_k - a_{-k})^2 \lambda x^2 \\
&= [A_k A_{-k} + \lambda\rho^2 + \lambda x^2 + i\lambda^{\frac{1}{2}} x(a_k - a_{-k})]\,[A_k A_{-k} + \lambda\rho^2 + \lambda x^2 - i\lambda^{\frac{1}{2}} x(a_k - a_{-k})] \\
&= \left[a_k a_{-k} - (\gamma - i\lambda^{\frac{1}{2}} x) a_k - (\gamma + i\lambda^{\frac{1}{2}} x) a_{-k} + \gamma^2 + \lambda\rho^2 + \lambda x^2\right] \times \\
&\qquad \left[a_k a_{-k} - (\gamma + i\lambda^{\frac{1}{2}} x) a_k - (\gamma - i\lambda^{\frac{1}{2}} x) a_{-k} + \gamma^2 + \lambda\rho^2 + \lambda x^2\right]
\end{aligned}$$

Therefore one obtains

$$\begin{aligned}
\prod_{k_0>0}^{\epsilon} \frac{\mathrm{Pf} S_k(a_k, a_{-k})}{a_k^2 a_{-k}^2} &= \lim_{\substack{\epsilon\to 0\\ \epsilon<0}} \prod_{k_0} \frac{a_k a_{-k} - (\gamma - i\lambda^{\frac{1}{2}} x) e^{ik_0\epsilon} a_k - (\gamma + i\lambda^{\frac{1}{2}} x) e^{-ik_0\epsilon} a_{-k} + \gamma^2 + \lambda\rho^2 + \lambda x^2}{a_k a_{-k}} \\
&\overset{(a)}{=} \frac{\cosh \frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2 + \omega_+}}{\cosh \frac{\beta}{2} e_{\mathbf{k}}} \; \frac{\cosh \frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2 + \omega_-}}{\cosh \frac{\beta}{2} e_{\mathbf{k}}}\; e^{-\beta\gamma}
\end{aligned}$$

where

$$p = (e_{\mathbf{k}} + 2\gamma)^2 - e_{\mathbf{k}}^2 - 4(\gamma^2 + \lambda x^2) + 2\gamma^2 + 2\lambda\rho^2 + 2\lambda x^2 = 4\gamma e_{\mathbf{k}} + 2\gamma^2 + 2\lambda\rho^2 - 2\lambda x^2$$

$$q = 4(\gamma^2 + \lambda x^2) e_{\mathbf{k}}^2 + 4\gamma e_{\mathbf{k}}(\gamma^2 + \lambda\rho^2 + \lambda x^2) + (\gamma^2 + \lambda\rho^2 + \lambda x^2)^2 = (2\gamma e_{\mathbf{k}} + \gamma^2 + \lambda\rho^2 + \lambda x^2)^2 + 4\lambda x^2 e_{\mathbf{k}}^2$$

such that

$$\tfrac{p^2}{4} - q = -4(2\gamma e_{\mathbf{k}} + \gamma^2 + \lambda\rho^2)\lambda x^2 - 4\lambda x^2 e_{\mathbf{k}}^2 = -4\left[(e_{\mathbf{k}} + \gamma)^2 + \lambda\rho^2\right]\lambda x^2$$

and therefore

$$\begin{aligned}
e_{\mathbf{k}}^2 + \omega_\pm &= (e_{\mathbf{k}} + \gamma)^2 + \lambda\rho^2 - \lambda x^2 \pm 2i\lambda^{\frac{1}{2}} x\left[(e_{\mathbf{k}} + \gamma)^2 + \lambda\rho^2\right]^{\frac{1}{2}} \\
&= \left\{\left[(e_{\mathbf{k}} + \gamma)^2 + \lambda\rho^2\right]^{\frac{1}{2}} \pm i\lambda^{\frac{1}{2}} x\right\}^2
\end{aligned}$$

which results in

$$\prod_{k_0>0}^{\epsilon} \frac{\mathrm{Pf}S_k(a_k,a_{-k})}{a_k^2 a_{-k}^2} = \frac{\cosh\frac{\beta}{2}\left\{[(e_{\mathbf{k}}+\gamma)^2+\lambda\rho^2]^{\frac12}+i\lambda^{\frac12}x\right\}}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\,\frac{\cosh\frac{\beta}{2}\left\{[(e_{\mathbf{k}}+\gamma)^2+\lambda\rho^2]^{\frac12}-i\lambda^{\frac12}x\right\}}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\,e^{-\beta\gamma}$$
$$= \frac{\sinh^2(\frac{\beta}{2}i\sqrt{\lambda}\,x)+\cosh^2(\frac{\beta}{2}\sqrt{(e_{\mathbf{k}}+\gamma)^2+\lambda\rho^2}\,)}{\cosh^2\frac{\beta}{2}e_{\mathbf{k}}}\,e^{-\beta\gamma} \qquad \bullet$$

Using the lemma above the effective potential (IV.7) becomes

$$V(w,a,x,\rho) = \tfrac14 w^2+a^2+x^2+\rho^2-\int_M \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{\beta}\log\left[\frac{\sinh^2(\frac{\beta}{2}i\sqrt{\lambda}\,x)+\cosh^2(\frac{\beta}{2}\sqrt{(e_{\mathbf{k}}+\gamma)^2+\lambda\rho^2}\,)}{\cosh^2\frac{\beta}{2}e_{\mathbf{k}}}\,e^{-\beta\gamma}\right] \tag{IV.13}$$

To compute the two point function, one has to find the global minimum of the real part of (IV.13).

For attractive $\lambda = g^2 > 0$ one has $\sinh^2(\frac{\beta}{2}i\sqrt{\lambda}\,x) = -\sin^2(\frac{\beta}{2}gx)$ and it seems likely that $\rho > 0$ and $x = 0$ at the global minimum. Furthermore it seems favorable to put the imaginary part of $\gamma = g(w+\sqrt{2}\,ia)$ to 0. That is $a = 0$ and w may be non zero to shift the Fermi surface to $e_{\mathbf{k}g} = e_{\mathbf{k}} + gw$. The two point function becomes in that case

$$\lim_{\kappa\to\infty}\frac{1}{\kappa}\langle\bar\psi_{k\uparrow}\psi_{k'\uparrow}\rangle = -\frac{A_{-k}(A_kA_{-k}+\lambda\rho^2)}{(A_kA_{-k}+\lambda\rho^2)^2} = \frac{ik_0+e_{\mathbf{k}g}}{k_0^2+e_{\mathbf{k}g}^2+g^2\rho^2} \tag{IV.14}$$

For repulsive $\lambda = -g^2 < 0$ $\sinh^2(\frac{\beta}{2}i\sqrt{\lambda}\,x) = \sinh^2(\frac{\beta}{2}g\,x)$ and the values $\rho = 0$ and $x > 0$ seem favourable. The imaginary part of γ is now given by w which again we put equal to 0. The value of a may be non zero to shift $e_{\mathbf{k}}$ to $e_{\mathbf{k}g} = e_{\mathbf{k}} + 2^{\frac12}ga \equiv e_{\mathbf{k}} + \delta_g$. Under these assumptions the two point function becomes

$$\lim_{\kappa\to\infty}\frac{1}{\kappa}\langle\bar\psi_{k\uparrow}\psi_{k\uparrow}\rangle = -\frac{A_kA_{-k}^2+A_k\lambda x^2}{A_k^2A_{-k}^2+(A_k^2+A_{-k}^2)\lambda x^2+\lambda^2x^4} = -\frac{A_k}{A_k^2-g^2x^2}$$
$$= -\frac12\left[\frac{1}{ik_0-e_{\mathbf{k}g}-gx}+\frac{1}{ik_0-e_{\mathbf{k}g}+gx}\right] \tag{IV.15}$$

which, at zero temperature, results in a momentum distribution

$$n_{\mathbf{k}} = \lim_{L\to\infty}\langle a_{\mathbf{k}}^+a_{\mathbf{k}}\rangle_{\beta,L} = \lim_{\substack{\epsilon\to0\\ \epsilon<0}}\int_{\mathbf{R}}\frac{dk_0}{2\pi}\frac12\left[\frac{1}{e^{ik_0\epsilon}(ik_0-e_{\mathbf{k}})-\delta_g-gx}+\frac{1}{e^{ik_0\epsilon}(ik_0-e_{\mathbf{k}})-\delta_g+gx}\right]$$
$$= \frac12\Big[\chi(e_{\mathbf{k}g}-gx<0)+\chi(e_{\mathbf{k}g}+gx<0)\Big] = \begin{cases}1 & \text{if } e_{\mathbf{k}g} < -gx\\ \frac12 & \text{if } -gx < e_{\mathbf{k}g} < gx\\ 0 & \text{if } gx < e_{\mathbf{k}g}\end{cases} \tag{IV.16}$$

IV.2 BCS with Non Zero Angular Momentum Terms

We consider the model

$$Z(\beta, L, \{s_{k,\sigma}\}) = \int e^{-\mathcal{U}(\psi,\bar\psi)+\frac{1}{\kappa}\sum_{k,\sigma} s_{k,\sigma}\bar\psi_{k,\sigma}\psi_{k,\sigma}} d\mu_C(\psi,\bar\psi) \tag{IV.17}$$

with $e_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu$, $C(k) = \frac{1}{ik_0 - e_{\mathbf{k}}}$ and

$$\mathcal{U}(\psi,\bar\psi) = \tfrac{1}{\kappa^2} \sum_{\sigma,\tau\in\{\uparrow,\downarrow\}} \sum_{k,p} \tfrac{1}{\kappa} U(\mathbf{k}'-\mathbf{p}')\, \bar\psi_{k,\sigma}\bar\psi_{-k,\tau}\psi_{p,\sigma}\psi_{-p,\tau} \tag{IV.18}$$

The electron electron interaction is given by (III.7)

$$U(\mathbf{k}'-\mathbf{p}') = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=-n}^{n} \lambda_{|\ell|} e^{i\ell\varphi_{\mathbf{k}}} e^{-i\ell\varphi_{\mathbf{p}}} + \dfrac{\lambda_0}{2} & \text{if } d=2 \\ \displaystyle\sum_{\ell=0}^{n}\sum_{m=-\ell}^{\ell} \lambda_\ell Y_{\ell m}(\mathbf{k}')\, \bar Y_{\ell m}(\mathbf{p}') & \text{if } d=3 \end{cases} =: \sum_{l=0}^{N} \lambda_l\, y_l(\mathbf{k}')\, \bar y_l(\mathbf{p}')$$

where $\mathbf{k}' = \sqrt{2m\mu}\frac{\mathbf{k}}{|\mathbf{k}|}$ is the projection of $\mathbf{k}$ onto the Fermi surface $e_{\mathbf{k}} = 0$. For simplicity, we write $Y_{\ell m}(\mathbf{k}')$ instead of $Y_{\ell m}\left(\frac{\mathbf{k}'}{k_F}\right)$.

To write down the effective potential and the two point functions in this case, we first have to compute the Pfaffian of the matrix S_k of Theorem III.1. Since we consider only a BCS interaction, the Ξ and Γ fields are zero. With Lemma III.2 one obtains

$$\begin{aligned} \mathrm{Pf} S_k =& \big(a_{k\uparrow}a_{-k\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\big)\big(a_{-k\uparrow}a_{k\downarrow} + \bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\big) \\ &+ a_{k\uparrow}a_{-k\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} + a_{k\downarrow}a_{-k\downarrow}\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} \\ &+ \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} \end{aligned}$$

where $a_{k\sigma} = a_k - s_{k\sigma}$. In particular, for $s_{k\sigma} = 0$

$$\mathrm{Pf} S_k = (a_k a_{-k} + \Omega^+_{\mathbf{k}})(a_k a_{-k} + \Omega^-_{\mathbf{k}})$$

where $\Omega^\pm_{\mathbf{k}}$ are the solutions of the quadratic equation

$$\begin{aligned} \Omega^2 - \Big(&\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow} + \bar\Phi_{-\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow} + \Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Big)\Omega + \Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow} \\ &+ \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\downarrow}\bar\Phi_{-\mathbf{k}\uparrow\downarrow} + \bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{-\mathbf{k}\uparrow\downarrow} + \Phi_{\mathbf{k}\uparrow\uparrow}\Phi_{\mathbf{k}\downarrow\downarrow}\bar\Phi_{\mathbf{k}\uparrow\uparrow}\bar\Phi_{\mathbf{k}\downarrow\downarrow} = 0 \end{aligned} \tag{I.8}$$

Using Lemma II.2 again to compute the product over k_0, the effective potential becomes

$$\begin{aligned} V(\phi) &= \sum_{\sigma\tau\in\{\uparrow\uparrow,\downarrow\downarrow,\uparrow\downarrow\}} \sum_{l=0}^{N} |\phi^l_{\sigma\tau}|^2 - \tfrac{1}{\beta L^d}\sum_{\mathbf{k}\in M} \log \prod_{k_0>0} \left\{ \tfrac{a_k a_{-k} + \Omega^+_{\mathbf{k}}}{a_k a_{-k}} \tfrac{a_k a_{-k} + \Omega^-_{\mathbf{k}}}{a_k a_{-k}} \right\} \\ &= \sum_{l=0}^{N} \big(|\phi^l_{\uparrow\downarrow}|^2 + |\phi^l_{\uparrow\uparrow}|^2 + |\phi^l_{\downarrow\downarrow}|^2\big) - \sum_{\epsilon\in\{+,-\}} \int_M \tfrac{d^d\mathbf{k}}{(2\pi)^d}\, \tfrac{1}{\beta} \log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e^2_{\mathbf{k}} + \Omega^\epsilon_{\mathbf{k}}})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}} \right] \end{aligned} \tag{IV.19}$$

In the following we consider the case of an even and an odd interaction $U(\mathbf{k}'-\mathbf{p}')$. For even U, the sum in (III.7) goes only over even angular momenta and one has $y_l(-\mathbf{k}')=y_l(\mathbf{k}')$. That is, $\Phi_{\mathbf{k}\uparrow\uparrow}=\Phi_{\mathbf{k}\downarrow\downarrow}=0$ and $\Phi_{-\mathbf{k}\uparrow\downarrow}=\Phi_{\mathbf{k}\uparrow\downarrow}$. The quadratic equation becomes

$$\Omega^2-2\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\Omega+\left(\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}\right)^2=0$$

which gives $\Omega^\pm_{\mathbf{k}}=\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}$ and

$$V(\phi_{\uparrow\downarrow})=\sum_{l=0}^N|\phi^l_{\uparrow\downarrow}|^2-\int_M\tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\tfrac{2}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e^2_{\mathbf{k}}+\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}})}{\cosh\frac{\beta}{2}e_{\mathbf{k}}}\right]\tag{IV.20}$$

The two point function is given by

$$\begin{aligned}\tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle&=\int\tfrac{1}{\kappa}\bar\psi_{p\sigma}\psi_{p\sigma}\,e^{-\mathcal{U}(\psi,\bar\psi)}d\mu_C(\psi,\bar\psi)\\&=-\frac{\int\frac{a_{-p}}{a_pa_{-p}+\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}}\,e^{-\kappa V(\phi_{\uparrow\downarrow})}\prod\limits_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}}{\int e^{-\kappa V(\phi_{\uparrow\downarrow})}\prod\limits_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}}\end{aligned}\tag{IV.21}$$

For a pure odd interaction the sum in (III.7) contains only odd values of l and one has $\Phi_{-\mathbf{k}\sigma\tau}=-\Phi_{\mathbf{k}\sigma\tau}$. Furthermore, since V is symmetric with respect to $\phi^l_{\uparrow\uparrow}\leftrightarrow\phi^l_{\downarrow\downarrow}$, one may substitute $\phi^l_{\uparrow\uparrow}=\phi^l_{\downarrow\downarrow}=\phi^l$, $\Phi_{\mathbf{k}\uparrow\uparrow}=\Phi_{\mathbf{k}\downarrow\downarrow}=\Phi_{\mathbf{k}}=\sum_l\lambda_l^{\frac12}y_l(\mathbf{k}')\phi^l$ and the integral over $du^l_{\uparrow\uparrow}dv^l_{\uparrow\uparrow}du^l_{\downarrow\downarrow}dv^l_{\downarrow\downarrow}$ by an integral over du^ldv^l. The quadratic equation becomes in that case

$$\Omega^2-2\Big(\bar\Phi_{\mathbf{k}\uparrow\downarrow}\Phi_{\mathbf{k}\uparrow\downarrow}+\Phi_{\mathbf{k}}\bar\Phi_{\mathbf{k}}\Big)\Omega+\bar\Phi^2_{\mathbf{k}\uparrow\downarrow}\Phi^2_{\mathbf{k}\uparrow\downarrow}-\Phi^2_{\mathbf{k}}\bar\Phi^2_{\mathbf{k}\uparrow\downarrow}-\bar\Phi^2_{\mathbf{k}}\Phi^2_{\mathbf{k}\uparrow\downarrow}+\Phi^2_{\mathbf{k}}\bar\Phi^2_{\mathbf{k}}=0$$

which gives $\Omega^\pm_{\mathbf{k}}=(\Phi_{\mathbf{k}\uparrow\downarrow}\pm\Phi_{\mathbf{k}})(\bar\Phi_{\mathbf{k}\uparrow\downarrow}\pm\bar\Phi_{\mathbf{k}})$ The two point function reads in this case

$$\tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle=\frac{\int\;F_{\mathbf{p}}(\phi)\,e^{-\kappa V(\phi)}\prod\limits_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}du^ldv^l}{\int e^{-\kappa V(\phi)}\prod\limits_{l=0}^N du^l_{\uparrow\downarrow}dv^l_{\uparrow\downarrow}du^ldv^l}\tag{IV.22}$$

where

$$F_{\mathbf{p}}(\phi)=-\tfrac{a_{-p}[a_pa_{-p}+\Phi_{\mathbf{p}\uparrow\downarrow}\bar\Phi_{\mathbf{p}\uparrow\downarrow}+\Phi_{\mathbf{p}}\bar\Phi_{\mathbf{p}}]}{(a_pa_{-p}+\Omega^+_{\mathbf{p}})(a_pa_{-p}+\Omega^-_{\mathbf{p}})}\tag{IV.23}$$

A detailed analysis to compute the infinite volume limit of (IV.21,22) we plan to give in a forthcoming paper. In the following, concerning the two dimensional case, we only report on a result which is still under investigation of A. Schuette. In three dimensions, we argue in Theorem IV.2 below using only symmetry considerations that, if $e(\mathbf{k})$ has $SO(3)$ symmetry, then the $\langle a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\rangle$ expectations also have to have $SO(3)$ symmetry.

Consider first the two dimensional case. An analysis done by Albrecht Schuette from ETH Zürich indicates the following result:

Let $U(\mathbf{k}'-\mathbf{p}')=\sum_{\ell=-n}^{n}\lambda_\ell e^{i\ell(\varphi_{\mathbf{k}}-\varphi_{\mathbf{p}})}$, suppose that $\lambda_m>0$ is attractive and $\lambda_m>\lambda_\ell$ for all $\ell\neq m$. Then

$$\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle_{\beta,L}=\tfrac{ik_0+e_{\mathbf{p}}}{p_0^2+e_{\mathbf{p}}^2+\lambda_m\rho_m^2}\tag{IV.24}$$

where ρ_m is determined by the BCS equation

$$1-\tfrac{\lambda_m}{4\pi}\int d|\mathbf{k}||\mathbf{k}|\tfrac{\tanh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\lambda_m\rho_m^2})}{\sqrt{e_{\mathbf{k}}^2+\lambda\rho_m^2}}=0$$

The form of the two point function (IV.24) can still be obtained by applying the standard mean field formalism [AB], [BW]. However, the situation is different in 3 dimensions. Before we state the corresponding theorem, we shortly recall the mean field equations [AB], [BW]:

The $\langle a^+a\rangle$ expectations are given by

$$\langle a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\rangle=\tfrac12\left(1-e_{\mathbf{k}}\left[\tfrac{\tanh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}})}{\sqrt{e_{\mathbf{k}}^2+\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}}}\right]_{\sigma\sigma}\right)\tag{IV.25}$$

where the 2×2 matrix $\Delta_{\mathbf{k}}$, $\Delta_{\mathbf{k}}^T=-\Delta_{-\mathbf{k}}$, is a solution of the gap equation

$$\Delta_{\mathbf{p}}=\int_M\tfrac{d^d\mathbf{k}}{(2\pi)^d}\,U(\mathbf{p}'-\mathbf{k}')\,\Delta_{\mathbf{k}}\tfrac{\tanh(\frac{\beta}{2}\sqrt{e_{\mathbf{k}}^2+\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}})}{2\sqrt{e_{\mathbf{k}}^2+\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}}}\tag{IV.26}$$

In 2 dimensions, if one substitutes $U(\mathbf{p}'-\mathbf{k}')$ by a single attractive term $\lambda_\ell e^{i\ell(\varphi_{\mathbf{p}}-\varphi_{\mathbf{k}})}$, then (IV.26) has the unitary isotropic solution $\Delta_{\mathbf{k}}=\Delta\left(\begin{smallmatrix}0 & e^{i\ell\varphi_{\mathbf{k}}}\\ -e^{i\ell\varphi_{\mathbf{k}}} & 0\end{smallmatrix}\right)$ for even ℓ and $\Delta_{\mathbf{k}}=\Delta\left(\begin{smallmatrix}\cos\ell\varphi_{\mathbf{k}} & \sin\ell\varphi_{\mathbf{k}}\\ \sin\ell\varphi_{\mathbf{k}} & -\cos\ell\varphi_{\mathbf{k}}\end{smallmatrix}\right)$ if ℓ is odd. In that case $\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}=|\Delta|^2 Id$ and (IV.25) coincides with (IV.24).

In 3 dimensions, it has been proven by Feldman, Knörrer and Trubowitz [FKT] that for all $\ell\geq 2$ (IV.26) does not have unitary isotropic ($\Delta_{\mathbf{k}}^*\Delta_{\mathbf{k}}=const\,Id$) solutions. In view of that result, the symmetry considerations below indicate that in 3 dimensions for $\ell\geq 2$ the standard mean field approach may be misleading since one would no longer expect $SO(3)$ invariance for the $\langle a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\rangle$ expectations according to (IV.25). But this is indeed the case.

Theorem IV.2: Let ℓ be even, $\lambda_\ell>0$ be attractive and let

$$\mathcal{U}(\psi,\bar\psi)=\tfrac{\lambda_\ell}{L^{3d}}\sum_{k,p}\sum_{m=-\ell}^{\ell}\bar Y_{\ell m}(\mathbf{k}')Y_{\ell m}(\mathbf{p}')\,\bar\psi_{k\uparrow}\bar\psi_{-k\downarrow}\psi_{p\uparrow}\psi_{-p\downarrow}$$

Then, if $e_{R\mathbf{k}}=e_{\mathbf{k}}$ for all $R\in SO(3)$, one has

$$\begin{aligned}\lim_{\kappa\to\infty}\tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle_{\beta,L}&=\lim_{\kappa\to\infty}\int\tfrac{1}{\kappa}\bar\psi_{p\sigma}\psi_{p\sigma}\,e^{-\mathcal{U}(\psi,\bar\psi)}d\mu_C(\psi,\bar\psi)\\&=\int_{S^2}\tfrac{ip_0+e_{\mathbf{p}}}{p_0^2+e_{\mathbf{p}}^2+\lambda_\ell\rho_0^2|\,\Sigma_m\,\alpha_m^0\,Y_{\ell m}(\mathbf{x})|^2}\,\tfrac{d\Omega(\mathbf{x})}{4\pi}\end{aligned}$$

where $\rho_0 \geq 0$ and $\alpha^0 \in \mathbf{C}^{2\ell+1}$, $\Sigma_m |\alpha^0_m|^2 = 1$, are values at the global minimum (which is degenerated) of

$$V(\rho,\alpha) = \rho^2 - \int_M \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_\mathbf{k}^2+\lambda_\ell\rho^2|\,\Sigma_m\,\alpha_m Y_{\ell m}(\mathbf{k}')|^2})}{\cosh\frac{\beta}{2}e_\mathbf{k}}\right]^2$$

In particular, at zero temperature the momentum distribution $n_\mathbf{p}$ is given by

$$n_\mathbf{p} = \lim_{L\to\infty}\langle a^+_{\mathbf{p}\sigma}a_{\mathbf{p}\sigma}\rangle_{\beta,L} = \int_{S^2}\tfrac{1}{2}\left(1 - \frac{e_\mathbf{p}}{\sqrt{e_\mathbf{p}^2+|\Delta(\mathbf{x})|^2}}\right)\tfrac{d\Omega(\mathbf{x})}{4\pi}$$

and has $SO(3)$ symmetry. Here $\Delta(\mathbf{x}) = \lambda_\ell^{\frac{1}{2}}\rho_0\,\Sigma_m\,\alpha^0_m Y_{\ell m}(\mathbf{x})$.

Proof: Substituting $u^l_{\uparrow\downarrow}$ by u_m and $u^l_{\uparrow\downarrow}$ by $-v_m$ in (IV.21), one has to compute the infinite volume limit of

$$\tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle = -\frac{\int_{\mathbf{R}^{4\ell+2}}\frac{a_{-p}}{a_p a_{-p}+|\Phi_\mathbf{p}|^2}\,e^{-\kappa V(\phi)}\,\Pi_{m=-\ell}^{\ell}\,du_m dv_m}{\int_{\mathbf{R}^{4\ell+2}}e^{-\kappa V(\phi)}\,\Pi_{m=-\ell}^{\ell}\,du_m dv_m} \tag{IV.27}$$

where

$$V(\phi) = \sum_{m=-\ell}^{\ell}|\phi_m|^2 - \int_M \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_\mathbf{k}^2+|\Phi_\mathbf{k}|^2})}{\cosh\frac{\beta}{2}e_\mathbf{k}}\right]^2$$

and

$$\Phi_\mathbf{k} = \lambda_\ell^{\frac{1}{2}}\sum_{m=-\ell}^{\ell}\bar\phi_m Y_{\ell m}(\mathbf{k}')$$

Let $U(R)$ be the unitary representation of $SO(3)$ given by

$$Y_{\ell m}(R\mathbf{k}') = \sum_{m'}U(R)_{mm'}Y_{\ell m'}(\mathbf{k}')$$

and let $\sum_m(\overline{U\phi})_m Y_{\ell m}(\mathbf{k}') =: (U\Phi)_\mathbf{k}$. Then for all $R\in SO(3)$ one has $\big(U(R)\Phi\big)_\mathbf{k} = \Phi_{R^{-1}\mathbf{k}}$ and

$$\begin{aligned} V\big(U(R)\phi\big) &= \sum_m |[U(R)\phi]_m|^2 - \int_M \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_\mathbf{k}^2+|[U\Phi]_\mathbf{k}|^2})}{\cosh\frac{\beta}{2}e_\mathbf{k}}\right]^2 \\ &= \sum_m|\phi_m|^2 - \int_M \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_\mathbf{k}^2+|\Phi_{R^{-1}\mathbf{k}}|^2})}{\cosh\frac{\beta}{2}e_\mathbf{k}}\right]^2 \\ &= \sum_m|\phi_m|^2 - \int_M \tfrac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{\beta}\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_\mathbf{k}^2+|\Phi_\mathbf{k}|^2})}{\cosh\frac{\beta}{2}e_\mathbf{k}}\right]^2 = V(\phi) \end{aligned} \tag{IV.28}$$

Let $S^{4\ell+1} = \{\phi \in \mathbf{C}^{2\ell+1} \mid \sum_m |\phi_m|^2 = 1\}$. Since $U(R)$ leaves $S^{4\ell+1}$ invariant, $S^{4\ell+1}$ can be written as the union of disjoint orbits,

$$S^{4\ell+1} = \cup_{[\alpha]\in O}[\alpha]$$

where $[\alpha] = \{U(R)\alpha \mid R \in SO(3)\}$ is the orbit of $\alpha \in S^{4\ell+1}$ under the action of $U(R)$ and O is the set of all orbits. If one chooses a fixed representant α in each orbit $[\alpha]$, that is, if one chooses a fixed section $\sigma : O \to S^{4\ell+1}$, $[\alpha] \to \sigma_{[\alpha]}$ with $[\sigma_{[\alpha]}] = [\alpha]$, every $\phi \in \mathbf{C}^{2\ell+1}$ can be uniquely written as

$$\phi = \rho\, U(R)\sigma_{[\alpha]}\,, \quad \rho = \|\phi\| \geq 0, \quad \alpha = \tfrac{\phi}{\|\phi\|}, \quad [\alpha] \in O \text{ and } R \in SO(3)/I_{[\alpha]}$$

where $I_{[\alpha]} = I^{\sigma}_{[\alpha]} = \{S \in SO(3) \mid U(S)\sigma_{[\alpha]} = \sigma_{[\alpha]}\}$ is the isotropy subgroup of $\sigma_{[\alpha]}$. Let

$$\int_{\mathbf{R}^{4\ell+2}} \Pi_m\, du_m dv_m\; g(\phi) = \int_{\mathbf{R}^+} D\rho \int_O D[\alpha] \int_{[\alpha]} DR\; g\big(\rho\, U(R)\sigma_{[\alpha]}\big)$$

be the integral in (IV.27) over $\mathbf{R}^{4\ell+2}$ in the new coordinates. That is, for example, $D\rho = \rho^{4\ell+1} d\rho$. In the new coordinates

$$|\Phi_{\mathbf{p}}|^2 = \lambda_\ell \rho^2 \Big| \sum_m \big(\overline{U(R)\sigma_{[\alpha]}}\,\big)_m Y_{\ell m}(\mathbf{p}') \Big|^2 = \lambda_\ell \rho^2 \Big| \sum_m \bar\sigma_{[\alpha],m} Y_{\ell m}\left(R^{-1}\mathbf{p}'\right) \Big|^2$$

such that

$$\tfrac{-a_{-p}}{a_p a_{-p} + |\Phi_{\mathbf{p}}|^2} = \tfrac{-a_{-p}}{a_p a_{-p} + \lambda_\ell \rho^2 |\sum_m \bar\sigma_{[\alpha],m} Y_{\ell m}(R^{-1}\mathbf{p}')|^2} \equiv f(\rho, [\alpha], R^{-1}\mathbf{p})$$

Since $V(\phi) = V(\rho, [\alpha])$ is independent of R, one obtains

$$\begin{aligned}
\tfrac{1}{\kappa}\langle \bar\psi_{p\sigma}\psi_{p\sigma}\rangle &= \frac{\int \frac{-a_{-p}}{a_p a_{-p} + |\Phi_{\mathbf{p}}|^2}\, e^{-\kappa V(\phi)} \prod_{m=-\ell}^{\ell} du_m dv_m}{\int e^{-\kappa V(\phi)} \prod_{m=-\ell}^{\ell} du_m dv_m} \\
&= \frac{\int_{\mathbf{R}^+} D\rho \int_O D[\alpha] \int_{[\alpha]} DR\; f(\rho,[\alpha],R^{-1}\mathbf{p})\; e^{-\kappa V(\rho,[\alpha])}}{\int_{\mathbf{R}^+} D\rho \int_O D[\alpha] \int_{[\alpha]} DR\; e^{-\kappa V(\rho,[\alpha])}} \\
&= \frac{\int_{\mathbf{R}^+} D\rho \int_O D[\alpha]\, \mathrm{vol}([\alpha])\, \frac{\int_{[\alpha]} DR\, f(\rho,[\alpha],R^{-1}\mathbf{p})}{\int_{[\alpha]} DR}\; e^{-\kappa V(\rho,[\alpha])}}{\int_{\mathbf{R}^+} D\rho \int_O D[\alpha]\, \mathrm{vol}([\alpha])\; e^{-\kappa V(\rho,[\alpha])}}
\end{aligned} \tag{IV.29}$$

It is plausible to assume that at the global minimum of $V(\rho,[\alpha])$ ρ is uniquely determined, say ρ_0. Let $O_{\min} \subset O$ be the set of all orbits at which $V(\rho_0,[\alpha])$ takes its global minimum. Then in the infinite volume limit (IV.29) becomes

$$\lim_{\kappa\to\infty} \tfrac{1}{\kappa}\langle \bar\psi_{p\sigma}\psi_{p\sigma}\rangle = \frac{\int_{O_{\min}} D[\alpha]\, \mathrm{vol}([\alpha])\, \frac{\int_{[\alpha]} DR\, f(\rho_0,[\alpha],R^{-1}\mathbf{p})}{\int_{[\alpha]} DR}}{\int_{O_{\min}} D[\alpha]\, \mathrm{vol}([\alpha])} \tag{IV.30}$$

Consider the quotient of integrals in the numerator of (IV.30). Since

$$
\begin{aligned}
f\left(\rho,[\alpha],R^{-1}\mathbf{p}\right) &= f\Big(\rho^2\big|\sum_m \bar\sigma_{[\alpha],m} Y_{\ell m}\left(R^{-1}\mathbf{p}\right)\big|^2\Big)\\
&= f\Big(\rho^2\big|\sum_m \big(\overline{U(S)\sigma_{[\alpha]}}\big)_m Y_{\ell m}\left(R^{-1}\mathbf{p}\right)\big|^2\Big)\\
&= f\Big(\rho^2\big|\sum_m \bar\sigma_{[\alpha],m} Y_{\ell m}\left((RS)^{-1}\mathbf{p}\right)\big|^2\Big) = f\left(\rho,[\alpha],(RS)^{-1}\mathbf{p}\right)
\end{aligned}
$$

for all $S\in I_{[\alpha]}$, one has, since $[\alpha]\simeq SO(3)/I_{[\alpha]}$

$$
\begin{aligned}
\frac{\int_{[\alpha]} DR\, f(\rho_0,[\alpha],R^{-1}\mathbf{p})}{\int_{[\alpha]} DR} &= \frac{\int_{SO(3)/I_{[\alpha]}} DR\, f(\rho_0,[\alpha],R^{-1}\mathbf{p}) \int_{I_{[\alpha]}} DS}{\int_{SO(3)/I_{[\alpha]}} DR \int_{I_{[\alpha]}} DS}\\
&= \frac{\int_{SO(3)/I_{[\alpha]}} DR \int_{I_{[\alpha]}} DS\, f(\rho_0,[\alpha],(RS)^{-1}\mathbf{p})}{\int_{SO(3)/I_{[\alpha]}} \int_{I_{[\alpha]}} DR\, DS}\\
&= \frac{\int_{SO(3)} DR\, f(\rho_0,[\alpha],R^{-1}\mathbf{p})}{\int_{SO(3)} DR}\\
&= \frac{\int_{S^2} d\Omega(\mathbf{x}) \int_{SO(3)_{\mathbf{x}\to\mathbf{p}}} DR\, f(\rho_0,[\alpha],R^{-1}\mathbf{p})}{\int_{S^2} d\Omega(\mathbf{x}) \int_{SO(3)_{\mathbf{x}\to\mathbf{p}}} DR}\\
&= \frac{\int_{S^2} d\Omega(\mathbf{x})\, f(\rho_0,[\alpha],\mathbf{x}) \int_{SO(3)_{\mathbf{x}\to\mathbf{p}}} DR}{\int_{S^2} d\Omega(\mathbf{x}) \int_{SO(3)_{\mathbf{x}\to\mathbf{p}}} DR}
\end{aligned}
\qquad \text{(IV.31)}
$$

where $SO(3)_{\mathbf{x}\to\mathbf{p}} = \{R\in SO(3)\,|\,R\mathbf{x}=\mathbf{p}\}$. If one assumes that DR has the usual invariance properties of the Haar measure, then $\int_{SO(3)_{\mathbf{x}\to\mathbf{p}}} DR$ does not depend on $\mathbf{x}$ and it cancels out in (IV.31). Then (IV.30) gives

$$
\lim_{\kappa\to\infty} \tfrac{1}{\kappa}\langle\bar\psi_{p\sigma}\psi_{p\sigma}\rangle = \frac{\int_{O_{\min}} D[\alpha]\,\mathrm{vol}([\alpha])\, \frac{\int_{S^2} d\Omega(\mathbf{x})\, f(\rho,[\alpha],\mathbf{x})}{\int_{S^2} d\Omega(\mathbf{x})}}{\int_{O_{\min}} D[\alpha]\,\mathrm{vol}([\alpha])} \qquad \text{(IV.32)}
$$

Now, since the effective potential, which is constant on $O_{\min}$, may be written as

$$
V(\rho,[\alpha]) = \int_{S^2} \tfrac{d\Omega(\mathbf{x})}{4\pi}\, G\Big(\rho^2\big|\Sigma_m \bar\sigma_{[\alpha],m} Y_{\ell m}(\mathbf{x})\big|^2\Big)
$$

with $G(X) = \rho^2 - \int \frac{dk\,k^2}{2\pi^2}\,\log\left[\frac{\cosh(\frac{\beta}{2}\sqrt{e_k^2+\lambda_\ell X})}{\cosh\frac{\beta}{2}e_k}\right]$, it is plausible to assume that also

$$
\frac{\int_{S^2} d\Omega(\mathbf{x})\, f(\rho_0,[\alpha],\mathbf{x})}{\int_{S^2} d\Omega(\mathbf{x})} = \int_{S^2} \frac{d\Omega(\mathbf{x})}{4\pi}\, \frac{ip_0+e_{\mathbf{p}}}{p_0^2+e_{\mathbf{p}}^2+\lambda_\ell\rho_0^2|\,\Sigma_m \bar\sigma_{[\alpha],m} Y_{\ell m}(x)|^2}
$$

is constant on $O_{\min}$. In that case also the integrals over $O_{\min}$ in (IV.32) cancel out and the theorem is proven •

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