KAM-Renormalization Group Analysis of Stability in Hamiltonian Flows

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Abstract

We study the stability and break-up of invariant tori in Hamiltonian flows using a combination of KAM theory and renormalization group techniques. We implement the scheme numerically for a family of Hamiltonians quadratic in the actions to analyse the strong coupling regime. We show that the KAM iteration converges up to the critical coupling at which the torus breaks up. Adding a renormalization scheme consisting of a rescaling of phase space and a shift of the relevant resonances, we obtain a much more efficient method that allows to determine the critical coupling with high accuracy. The results indicate that this approach captures the essential physical mechanism of the break-up of invariant tori. We determine a non-trivial fixed point of the renormalization transformation, and discuss the associated universality properties.

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The KAM technique was developped originally to prove the stability of invariant tori of an integrable Hamiltonian under small perturbations. It has been later applied to a very large spectrum of phenomena both in classical and in quantum mechanics. The KAM theorem [1–3] states that, if the frequency satisfies a diophantine condition, and the size of the perturbation ε is smaller than an ε_0, then a torus of that frequency will be stable. However, it was remarked early that the value ε_0 obtained in the theorem is much smaller than the critical coupling ε_c at which the torus is known [4] to become unstable and to break up into a Cantor set (Aubry-Mather set) [5,6]. The best values of ε_0 were obtained by Celletti and Chierchia [7]. Since there is no physical phenomenon associated with ε_0, the difficulty to prove the convergence of the KAM iteration for $\varepsilon \in [\varepsilon_0, \varepsilon_c)$ was interpreted as a shortcoming of the iterative approach. A first goal of the present work is to provide numerical evidence that the KAM iteration can be expected to converge all the way to the critical coupling ε_c.

Renormalization group ideas were proposed for the analysis of the break-up of invariant tori [8–11] based on the observation of self-similar scaling properties [12–14], and inspired on the renormalization theory of phase transitions [15]. For Hamiltonian flows, D. Escande and F. Doveil set up an approximate renormalization scheme [8,9] that consisted of a combination of KAM transformations with a rescaling and shift of resonances. The approximation consisted in retaining at each step only two Fourier modes, corresponding to the main resonances, and neglecting all the other modes as well as all terms of order higher than two in the momentum. Although the approximation seems very drastic, since the effect of the neglected terms could not be controlled, the general ideas of the renormalization group approaches suggest that if the essential features are kept in the approximate scheme, the neglected parts may turn out to be *irrelevant*. This can of course only be firmly justified by treating the problem without the approximation, and establishing which degrees of freedom are relevant and which ones are irrelevant.

An outstanding question in this context has been whether this approximate renormalization scheme can be improved in a systematic way, in order to understand the properties of an exact renormalization scheme. The idea of the renormalization approach is to iterate a transformation in the space of Hamiltonians. For couplings $\varepsilon < \varepsilon_c$ the iteration should converge to a *trivial fixed point*. This can be considered as an alternative version of the KAM theorem [16]. The set of Hamiltonians with critical coupling ε_c form a surface in the space of Hamiltonians. The renormalization transformation maps this surface into itself. The main hypothesis of the renormalization group approach is that there should be another *nontrivial fixed point* on the critical surface that is attractive for Hamiltonians on that surface. Its existence has strong implications concerning universal properties in the mechanism of the break-up of invariant tori.

The analysis of the approximate scheme of Escande and Doveil [8,9], and conceptually similar studies on the dynamics of area-preserving maps [11], give support to the validity of the general picture. In order to give a firm basis to this approach, some essential points need to be established: since the renormalization transformation contains at each step a KAM transformation, it is indispensable to rely on the existence of a well defined KAM transformation for ε all the way to ε_c. Furthermore the KAM iteration should either converge for all $\varepsilon < \varepsilon_c$ or at least it should do so when combined with the rescaling and resonance shift.

Our approach has been to implement the KAM algorithm numerically for Hamiltonians with two degrees of freedom, quadratic in the action variables $\boldsymbol{A} = (A_1, A_2)$, of the form

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2} m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + [\boldsymbol{\omega}_0 + \varepsilon g(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A} + \varepsilon f(\boldsymbol{\varphi}) \ , \tag{1}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ are two angles, $\boldsymbol{\omega}_0$ is the frequency vector of the considered torus, and $\boldsymbol{\Omega}$ is some other constant vector. This type of model can describe e.g. a charged particle interacting with two waves in a plasma [9], and has a wide range of applications.

First we study the convergence properties of the KAM iteration. Then we define a renormalization transformation (KAM-RG) by combining it with a shift of resonances, and a rescaling of momentum and energy.

We show that the KAM algorithm can be used to determine the critical coupling ε_c. We then show that this can also be achieved with the KAM-RG transformation, and that it is much more efficient, in that very high precision can be obtained already with few Fourier coefficients.

We establish the existence of a nontrivial fixed point by iterating the KAM-RG transformation on the critical surface. We characterize the nontrivial fixed point Hamiltonian by its Fourier representation with a very large number of coefficients (around 4000).

We calculate the critical exponents by linearizing the renormalization map around the fixed point. The result shows that there is only one unstable direction, transverse to the critical surface.

KAM iteration.– We consider a Hamiltonian of the form (1), where $\varepsilon g(\boldsymbol{\varphi})\boldsymbol{\Omega}$ and $\varepsilon f(\boldsymbol{\varphi})$ are the perturbations. The KAM procedure consists of eliminating the perturbation of order ε by a canonical transformation, which in turn produces terms of order ε^2. In the approach we take, following W. Thirring [17,18], $m(\boldsymbol{\varphi})$ is of order one and does not need to be eliminated. The point is that we want to establish the existence of a torus of frequency $\boldsymbol{\omega}_0$. If (1) can be transformed into a Hamiltonian of the same form but with $f=0$, $g=0$, with arbitrary $m(\boldsymbol{\varphi})$, the equations of motion show immediately that at $\boldsymbol{A}=0$ there is an invariant torus of frequency $\boldsymbol{\omega}_0$. The great advantage of this approach is that the KAM transformation can be taken such that the transformed Hamiltonian is also quadratic in the actions, i.e. the KAM transformation is mapping among Hamiltonians of the form (1) ($\boldsymbol{\Omega}$ is not changed). This is very convenient for a numerical implementation, since at each step of the iteration the Hamiltonian is completely determined by three functions f, g, m of the angles $\boldsymbol{\varphi}$, that can be represented by their Fourier coefficients. Thus the only approximation involved is the representation of the functions by a finite number of coefficients. In our implementation we take all the Fourier modes with wavenumber $\boldsymbol{k} \in \mathbb{Z}^2$ in the square $\mathcal{C}_L$ of length $2L+1$ centered at $(0,0)$.

We perform the KAM transformation as a canonical change of coordinates $(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (\boldsymbol{A}',\boldsymbol{\varphi}')$ in the Lie representation [19]:

$$
\begin{aligned}
H'(\boldsymbol{A}',\boldsymbol{\varphi}') &= e^{+\varepsilon\hat{S}(A,\varphi)} H(\boldsymbol{A},\boldsymbol{\varphi})\,|_{(A',\varphi')} \\
&= H + \varepsilon\{S,H\} + \frac{\varepsilon^2}{2!}\{S,\{S,H\}\}\ldots
\end{aligned}
\tag{2}
$$

generated by a function S, linear in the actions $\boldsymbol{A}$, of the form

$$
S(\boldsymbol{A},\boldsymbol{\varphi}) = Y(\boldsymbol{\varphi})\,\boldsymbol{\Omega}\cdot\boldsymbol{A} + Z(\boldsymbol{\varphi}) + a\,\boldsymbol{\Omega}\cdot\boldsymbol{\varphi}\ . \tag{3}
$$

The functions $Y(\boldsymbol{\varphi})$, $Z(\boldsymbol{\varphi})$ and the constant a are chosen such that the terms of order ε cancel out. We start with the same initial Hamiltonian as in Refs. [7,9]

$$
H(\boldsymbol{A},\boldsymbol{\varphi}) = \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \varepsilon f(\boldsymbol{\varphi})\ , \tag{4}
$$

where $\boldsymbol{\Omega} = (1,0)$, $\boldsymbol{\omega}_0 = (1/\gamma,-1)$, $\gamma = (1+\sqrt{5})/2$, $f(\boldsymbol{\varphi}) = \cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi}) + \cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi})$, where $\boldsymbol{\nu}_1 = (1,0)$ and $\boldsymbol{\nu}_2 = (1,1)$. We perform an iteration of the above KAM transformation, representing all the functions by their Fourier series truncated by retaining only the coefficients in the square $\mathcal{C}_L$. For fixed L we take successively larger couplings ε and determine whether the iteration converges to a Hamiltonian with $f=0$, $g=0$ or diverges. By a bisection procedure we determine the critical coupling $\varepsilon_c(L)$. We repeat the calculation with larger numbers of Fourier coefficients, to obtain a more accurate approximation. In Figure 1 we show $\varepsilon_c(L)$ i.e. the dependence of the critical coupling on the number of Fourier coefficients retained. We observe that $\varepsilon_c(L)$ decreases with L in a stepwise manner. It stays essentially constant except at the points where a new rational approximant of the frequency $\boldsymbol{\omega}_0$ is included, corresponding to a resonance at the next smaller scale. The size of the jumps diminishes approximately geometrically, and we can extrapolate to obtain the value $\varepsilon_c(L)\to 0.0276$. This value is close to the critical $\varepsilon_c = 0.0275856$ obtained by the Greene criterion [7,20] which is surmised to yield the exact value.

The calculation provides an explanation of the fact that this global convergence is not easy to establish analytically. The KAM theorem is proven by showing that for $\varepsilon < \varepsilon_0$ the iteration is contractive. We observe in the numerical implementation of the iteration that for larger ε the convergence is not monotonic: in the first KAM steps the norm of the perturbation can actually grow and oscillate, until it reaches a region where it is contractive.

KAM-renormalization group transformation.– In this section we show that by combining the KAM transformation with a rescaling and a shift of resonances we can greatly improve the determination of the critical coupling. The intuition is that the renormalization treats specifically the effect of the main resonances at each scale. Their Fourier modes correspond to the rational approximants of $\boldsymbol{\omega}_0$, which produce the small denominators in the perturbation expansion and are responsible of the eventual break-up of the torus. Our results show that renormalization flattens out the step structure of the jumps in $\varepsilon_c(L)$, and gives a very accurate value for the critical coupling, already for a representation with a small number of Fourier coefficients.

The KAM-RG transformation is composed of four steps:

1) a KAM transformation as described above,

2) a shift of the resonances: a canonical change of coordinates that maps the next pair of daughter resonances of the sequence of rational approximants into the two main resonances,

3) a rescaling of energy (or equivalently of time),

4) a rescaling of the action variables (which is a generalized canonical transformation).

The aim of this transformation is to go from one scale to a smaller scale. The last three steps are implemented as follows: The two main resonances $(1,0)$ and $(1,1)$ are replaced by the next pair of daughter resonances $(2,1)$ and $(3,2)$. This change is done via a canonical transformation $(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (N^{-2}\boldsymbol{A}, N^2\boldsymbol{\varphi})$ with $N^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This linear transformation multiplies $\boldsymbol{\omega}_0$ by γ^{-2}; therefore we rescale the energy by a factor γ^2 in order to keep the frequency at $\boldsymbol{\omega}_0$. Then we perform a rescaling of the action variables: we change the Hamiltonian H into $H'(\boldsymbol{A},\boldsymbol{\varphi}) = \lambda H\left(\frac{\boldsymbol{A}}{\lambda},\boldsymbol{\varphi}\right)$ with λ such that the mean value of m remains equal to 1. This magnifies the size of the daughter resonances, and places them at the location of the original ones.

The functions m, g, f are rescaled and $\boldsymbol{\Omega} = (1,\alpha)$ is transformed as

$$\alpha \mapsto \alpha' = \frac{1+\alpha}{2+\alpha}. \tag{5}$$

An effect of the shifting and rescaling is thus to select $\alpha = \gamma^{-1}$: $\boldsymbol{\Omega}$ converges under successive iterations to $\boldsymbol{\Omega}_*$, which is orthogonal to $\boldsymbol{\omega}_0$ and is the unstable eigenvector of N^2 with the largest eigenvalue γ^2.

In Figure 1 we show the values of the critical coupling $\varepsilon_c(L)$, calculated by this KAM-RG transformation. We obtain $\varepsilon_c \in [0.02758, 0.02760]$, which is in very good agreement with the value $\varepsilon_c = 0.0275856$ obtained with the Greene criterion.

The improvement with respect to the KAM iteration is not only quantitative; the disappearence of the steps is a strong evidence that the KAM-RG transformation we have constructed captures the essential physical mechanism of the break-up of the tori.

Nontrivial fixed point.– By iterating the KAM-RG transformation starting from a point on the critical surface, we observe that the process converges to a nontrivial critical point H_*, which we characterize by the Fourier coefficients of the three functions f_*, g_*, m_* and $\boldsymbol{\Omega}_* = (1, \gamma^{-1})$. Figure 2 shows the weight of the Fourier coefficients of f_*. We observe that the nonzero coefficients are strongly concentrated on a band around the direction $D_\perp$ perpendicular to the line of resonances, i.e. to the frequency vector $\boldsymbol{\omega}_0$. The decrease of the size of the coefficients along $D_\perp$ is quite slow, reflecting the fact that H_* has a self-similar structure. The Fourier coefficients of g_* and m_* have a similar overall behavior, but they decay faster in the $D_\perp$ direction. By linearizing the KAM-RG transformation around H_*, we calculate the critical exponents. There is only one with $|\delta| > 1$. This implies that the critical surface, which is the stable manifold of H_*, is of codimension one. The value we obtain for the relevant critical exponent is $\delta \in [2.67, 2.68]$ which is quite close to the one obtained by MacKay for area-preserving maps [11] ($\delta = 2.65$), and to the one obtained by Escande *et al.* with the approximate scheme ($\delta = 2.75$) [21].
We also obtain an attractive cycle of period three, as it had also been encountered in area-preserving maps [22]. If odd perturbations are included, higher period can appear. In the Hamiltonian case these cycles are simply related to the fixed point H_* by symmetry.

Conclusion.– We have shown that the KAM-RG technique can be implemented numerically with high accuracy. The results show that the KAM-RG transformation follows closely the mechanism of the break-up of the invariant tori by a sequence of resonances. Their effect at all scales leads to the universal self-similar structure of the critical tori. We have presented results for two degrees of freedom, but the extension to three [23] or higher dimensional systems should be accessible. The extension of the scheme to other frequencies that are quadratic irrationals is relatively clear, but the case of a general irrational frequency will involve some qualitatively new features.

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FIGURES

FIG. 1. Critical coupling $\varepsilon_c(L)$ as a function of L, the size of the cell $\mathcal{C}_L$ containing $(2L+1)^2$ Fourier coefficients. The upper curve corresponds to the KAM transformation, and the lower one (enlarged in the inset) to the KAM-RG transformation.

FIG. 2. Weight of the Fourier coefficients of f_* indicated by grey-levels: a) white: $< 10^{-10}$, b) $[10^{-10}, 10^{-7}]$, c) $[10^{-7}, 10^{-5}]$, d) $[10^{-5}, 10^{-3}]$, e) black: $[10^{-3}, 10^{-2}]$.

FIG. 1, Govin, PRL

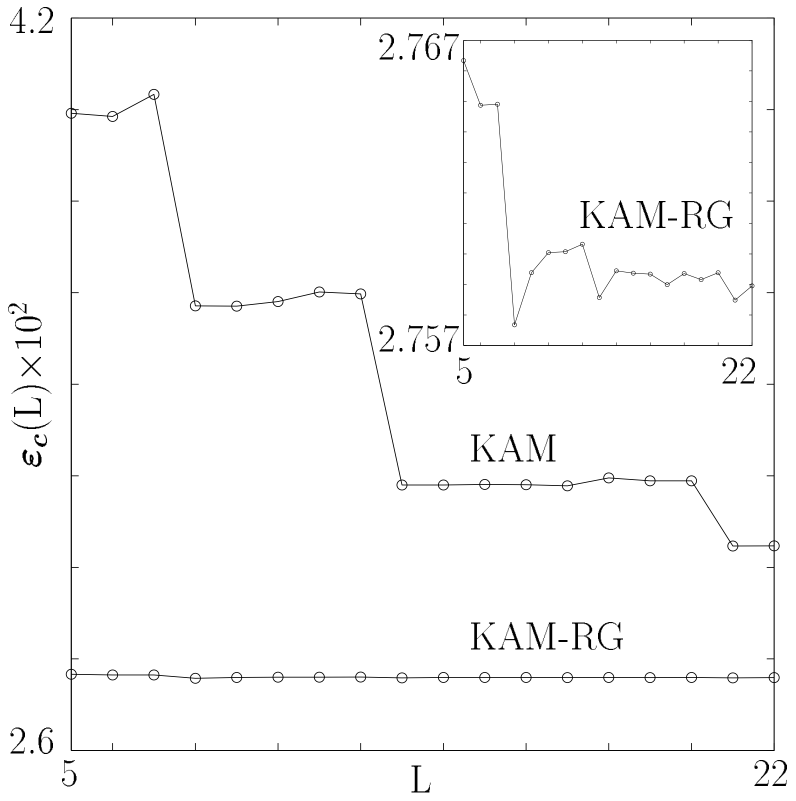
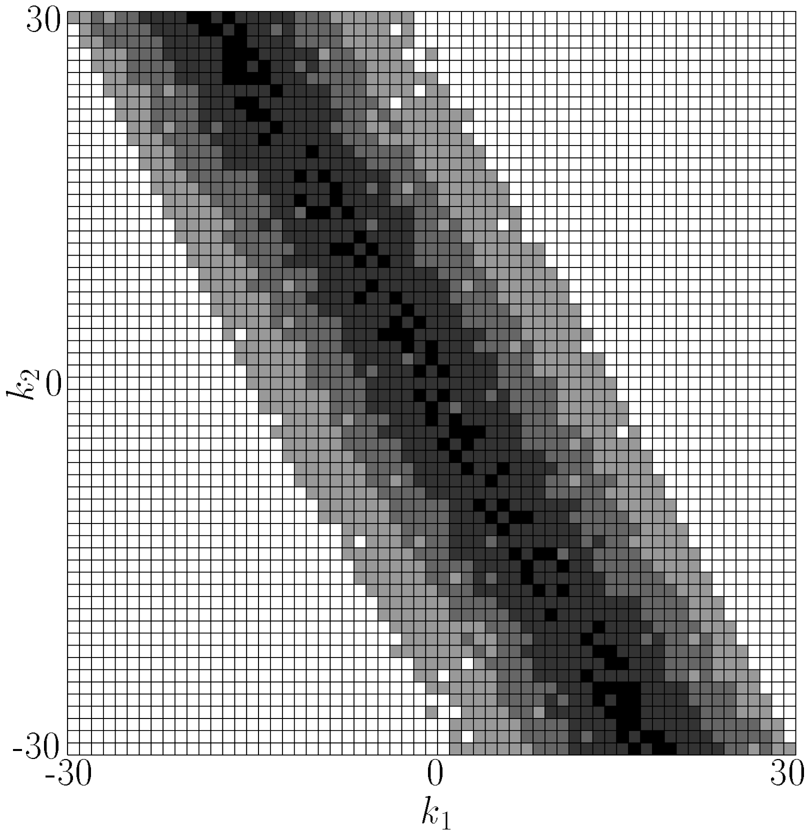


FIG. 2, Govin, PRL



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