EXISTENCE OF INVARIANT TORI FOR NON HAMILTONIAN PERTURBATIONS OF INTEGRABLE SYSTEMS

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Abstract We extend known results on the existence of invariant tori for non hamiltonian perturbations of integrable systems, to the case where the unperturbed system is anisocronus.

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We address here the problem of existence of (attractive) invariant tori for systems of the form

$$\begin{aligned} \dot{I} &= \varepsilon \ f(I, \varphi, \varepsilon) \\ \dot{\varphi} &= \omega(I) + \varepsilon \ g(I, \varphi, \varepsilon) \end{aligned} \tag{1}$$

where $I \in \mathcal{G} \subset \Re^n$ (with $\mathcal{G}$ open) are the slow variables, $\varphi \in \mathbf{T}^m$ are the fast angular variables, and ε is a (small) real parameter. All functions are assumed to be analytic on the closure of $\mathcal{G} \times \mathbf{T}^m$ in the variables I, φ, and in a neighbourhood of the origin in ε. We recall that this kind of systems arise naturally for example as non hamiltonian perturbations of integrable hamiltonian systems. Notice also that if the unperturbed system is resonant, then there are also some angles that play the role of slow variables.

For $m = 1$ the problem is of a different nature, as on the one side we are dealing with periodic solutions, and on the other we have no problems due to resonances. Thus, in the following we will assume we have to deal with a proper torus T^m, i.e. we will assume $m > 1$ (for the case $m = 1$ see e.g. ref. [1]).

We also stress that we focus on the case of attractive tori, which is at the same time the simplest and the most interesting in applications; we expect this approach would apply also for the case of normally hyperbolic tori, in a substantially equivalent way, although technically slightly more complicate.

We will proceed in three steps: (i) we perform a second order averaging: we use average to break the degeneracy, and we go to second order for reasons explained below; (ii) we look for attractive hyperbolic invariant tori of the truncation of this averaged system; (iii) we continue the invariant torus of the averaged system to an invariant torus of (1). To this end we apply a suitable version of the theorem on persistence of normally hyperbolic manifolds (it is in this step that the need for second order averaging arises).

Remark 1. It is simple (although non trivial) to make the above scheme rigorous when the frequency vector is diophantine and constant, i.e. independent of the slow variables I. Indeed, in such a case the averaging transformation which maps (1) into the averaged system is globally defined (on $\mathcal{G}$), and the problem of continuing a torus, namely to achieve step (iii) above can be solved in different ways. Bogoljubov et al. solved it by using a KAM type technique[2], obtaining that there exists a large set of $\Re^m$ such that if the frequency vector belongs to such a set, and if the average F og f with respect to the angles has a zero in $\mathcal{G}$, then the original system has an invariant torus. A different solution was suggested by Yagasaki[3] who realized that, since the zero of F is generically hyperbolic (i.e. all the eigenvalues of the linearization of F at such a point have non vanishing real part), the invariant torus of the averaged system is a normally hyperbolic invariant manifold, and therefore it should persist under small perturbation[4,5]. On the other side, we were not able to find in the literature any result on the general case where $\omega = \omega(I)$ is non constant.

The main difficulties in step (i) are due to the fact that in the case of non constant frequencies the averaging transformation is defined only on the set of slow variables on which the frequency vector is sufficiently non resonant; therefore the system (1) is equivalent to the averaged system only on such a set.

Remark 2. A necessary condition for an invariant torus of the averaged system to be also an invariant torus of (1) is that it is completely contained in such a set. Moreover, it will turn out that the nonresonant set depends on ε, and when $\varepsilon = 0$ it has a dense complement; therefore the question of existence of invariant tori for (1) requires some care.

We proceed as follows. Fix $K := K_* |\ln \varepsilon|$, where K_* is a suitable positive constant depending only on the analyticity strip of the functions f, g, ω, and split f as $f = f^- + f^+$, where

$$f^-(I,\varphi) := \sum_{|k|\leq K} f_k(I)\ e^{ik\cdot\varphi}\ , \tag{2}$$

and $f_k(I)$ is the k-th Fourier coefficient of f. The key remark is that, due to the decay of the Fourier coefficients of an analytic function, one can choose K so large that f^+ is a higher order perturbation. Therefore the construction of the averaging transformation involves only f^-; thus resonances of order higher that K do not matter, and the averaged system is equivalent to (1) on an open set. To characterize it, let us consider the distinguished subset of points $I \in \mathcal{G}$ corresponding to a "sufficiently nonresonant" frequency $\omega(I)$. Formally, we fix three positive real constants γ, τ and N, and consider the set $\Gamma(\gamma,\tau,N)$ defined as

$$\Gamma(\gamma,\tau,N) = \big\{ I \in \mathcal{G}\ :\ |k\cdot\omega(I)| \geq \gamma |k|^{-\tau} \text{ for all } k \in \mathbf{Z}^m \setminus \{0\}\ ,\ |k| \leq N \big\}\ . \tag{3}$$

For positive ρ we define also the extension Γ_ρ of Γ as

$$\Gamma_\rho(\gamma,\tau,N) = \bigcup_{I\in\Gamma(\gamma,\tau,N)} B_\rho(I)\ , \tag{4}$$

Where $B_\rho(I) \subset \Re^n$ is the closed ball of radius ρ and center I. We also write explicitely the averaged system as

$$\dot J = \varepsilon\ \mathcal{F}(J,\varepsilon) + \varepsilon^3 \mathcal{R}(J,\psi,\varepsilon)\ ,\quad \dot\psi = \omega(J) + \varepsilon G(J) + \varepsilon^2 \mathcal{R}_1(J,\psi,\varepsilon)\ . \tag{5}$$

where $\mathcal{F} := F + \varepsilon\mathcal{Z}$, G, and F are respectively the averages of $g(I,\varphi,0)$ and $f(I,\varphi,0)$ over $\mathbf{T}^m$, $\mathcal{Z} = \mathcal{Z}(J,\varepsilon)$ is a normal form term, and $\mathcal{R}$, $\mathcal{R}_1$ are remainder terms.

Then, following the lines of the proof of lemma 4.3 of ref. [6] one can prove the following lemma (for details see [7])

Lemma. *Consider the system of differential equations (1), with f, g and ω analytic on the closure of $\mathcal{G}\times\mathbf{T}^m$, and fix positive τ,γ. Then, provided ε is small enough, there exist positive constants ρ_*, b and an analytic coordinate transformation defined on $\Gamma_\rho(\gamma,\tau,K)$, with $\rho = \rho_*/K^{-(\tau+1)}$, such that in the new variables (1) takes the averaged form (5). Moreover, for any positive integer q there exists a constant α_q such that the C^q norms of $\mathcal{R}$,$\mathcal{Z}$, and $\mathcal{R}_1$ are bounded on $\Gamma_\rho(\gamma,\tau,K)\times\mathbf{T}^m$ by $\alpha_q K^{b+\tau q}$.*

We can now now pass to step (ii). To this end we consider the first order truncation of (5)

$$\dot{J} = \varepsilon\ F(J)\ , \quad \dot{\varphi} = \omega(J)\ . \tag{6}$$

Assume that there exists a $J_0 \in \Gamma(\gamma, \tau, K)$ which is an attractive hyperbolic zero of F. By implicit function theorem one can find an attractive zero J_1 of $\mathcal{F}$ close to J_0. Actually this requires some care since it turns out that the dependence of $\mathcal{Z}$ on ε is not continuous (this is due to the fact that Z is constructed from f^-, which depends on ε through K); however, exploiting the boundedness of Z one can obtain the result. So, one obtains a normally hyperbolic invariant torus $T_1 := J_1 \times \mathbf{T}^m$ of the system

$$\dot{J} = \varepsilon\mathcal{F}\ , \quad \dot{\psi} = \omega + \varepsilon G\ . \tag{7}$$

Since T_1 is $\mathcal{O}(\varepsilon|\ln\varepsilon|^b)$ close to $T_0 := J_0 \times \mathbf{T}^m$, it is contained in the domain $\Gamma_\rho \times \mathbf{T}^m$ whose size is of order $|\ln\varepsilon|^{-(\tau+1)}$.

We come now to step (iii). The difficulty here is related to the fact that the size of the Lyapunov exponents of the invariant torus of the truncated system (7) goes to zero with the size of the perturbation. It follows that in order to ensure applicability of the theorem on persistence of normally hyperbolic manifolds[4], one must explicitly compute the dependence of the threshold for the perturbation on the Lyapunov exponents. By the way, such difficulties are substantially equal in the case of constant frequencies.

We give here the main idea to obtain the proof; due to lack of space the explicit computation will be presented elsewhere[7]. First we recall that by Fenichel's method one constructs an invariant torus of the time T map Φ^T of system (5), where T is a suitable positive time. In particular T has to be large enough to ensure that Φ^T contracts the distances in the directions normal to the unperturbed invariant manifold, but small enough to ensure that the perturbation does not change significantly the dynamics. In our case it turns out that a good choice is $T = \mathcal{O}(\varepsilon^{-p})$, with an arbitrary p satisfying $0 < p < 1$. In this way the normal contraction rate is of order $\varepsilon T = \mathcal{O}(\varepsilon^{1-p})$ which is small, but much greater than ε. This choice, together with some technicalities[7], allows to follow Fenichel's scheme and to prove the following

Theorem. *Assume that there exists an attractive zero J_0 of F, and constants $\gamma > 0, \tau, N$ such that $J_0 \in \Gamma(\gamma, \tau, N)$, then there exists a constant ϵ_*, such that, if $N > K_*\,|\ln\varepsilon_*| = K(\epsilon_*)$, and $\exp(-N/K_*) < \varepsilon < \varepsilon_*$, then (1) has a stable invariant torus close to the torus $\mathbf{T}_0$.*

Remark 3. The condition $J_0 \in \Gamma(\gamma, \tau, N)$ is surely fulfilled if $\omega(J_0)$ is diophantine; but it can be fulfilled also if $\omega(J_0)$ satisfies some very high order resonance relations. To clarify this point assume that there exists $K_\sharp$, such that $\omega(J_0)$ satisfies some resonance relation of order $K_\sharp$ (namely that $\omega(J_0)\cdot k = 0$ for some $k \in Z^m$ with $|k| = K_\sharp$), but there exist τ, γ such that $J_0 \in \Gamma(\gamma, \tau, K_\sharp - 1)$, then our theorem gives no results for very small ε, namely for $0 < \varepsilon < \exp(-K_\sharp/K_*)$, but ensures the existence of an invariant torus for ε satisfying $\exp(-(K_\sharp - 1)/K_*) < \varepsilon < \varepsilon_*$.

Remark 4. The proof of our theorem does not require any nondegeneracy condition for the dependence of ω on the actions I; moreover, the result we obtain are quite different from those that one can expect to obtain by KAM techniques. Indeed, KAM techniques in hamiltonian systems allow to ensure that there exists a large Cantor-like set $\mathcal{S}$ in the action space such that, corresponding to any $I \in \mathcal{S}$ there exists an invariant torus of the system; the set $\mathcal{S}$ usually depends on ε. In the present nonconservative case, one can expect KAM techniques would allow to prove (under nondegeneracy conditions) that, if the solution of an equation of the kind $F_\varepsilon(J) = 0$ is in a suitable Cantor-like set, then correspondingly there exists an invariant torus for (1). If this Cantor set actually depends on ε, then typically an invariant torus exists for a Cantor set of values of ε. On the contrary we obtain existence of the torus for ε in an open set.

Remark 5. Finally, we remark that the condition that T_0 is attractive (obviously, the repulsive case would be equivalent) comes into play through the negativity of nonzero Lyapounov exponents; should we have a generic transversally hyperbolic T_0, we should consider negative and positive Lyapounov exponents, and correspondingly a stable and an unstable manifold; we should then consider their perturbations, and the intersection of the perturbed manifolds, in order to obtain the perturbed invariant torus. Thus, we expect that the present scheme could be extended to the general case along these lines; however, up to now we have not performed the computations.

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