ABSTRACT. We generalize Aubry-Mather theory for configurations on the line to general sets with a group action. Cocycles on the group play the role of rotation numbers. The notion of Birkhoff configuration can be generalized to this setting. Under mild conditions on the group, we show how to find Birkhoff ground states for many-body interactions which are ferromagnetic, invariant under the group action and having periodic phase space.

ON THE AUBRY–MATHER THEORY IN STATISTICAL MECHANICS

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1. INTRODUCTION

In the motivation from Solid State Physics, Aubry-Mather theory describes the structure of solutions to the following problem. Frenkel and Kontorova proposed a very simple model for a one-dimensional crystal to describe the structure of dislocations. They considered a one-dimensional chain of identical atoms, connected by springs, placed in a periodic substrate potential V with period p. The potential energy of such system or configuration u (indexed by the integers Z) describes the interaction of a particle $u(j)$ with its neighbors, and is given by the formal expression

$$S(u) = \sum_{j \in Z} \frac{1}{2}(u(j) - u(j+1) - a)^2 + V(u(j)).$$

The parameter a is the length of the connecting springs. The problem is then to find configurations u which minimize this potential, and to describe their structure.

The first part of the sum above is the energy of the internal interaction between particles, the second is the external energy. In the absence of substrate potential, that is, when $V = 0$, the configurations of minimal energy are given by $u(j) = aj+k$. The mean spacing is the rest distance of the spring. (This situation corresponds to the integrable case in the dynamical systems interpretation.)

When V is large, it is reasonable to expect – and indeed it can be proved – that the particles settle around the minima of the potential V, hence that the configurations that minimize the energy are spaced with period p. Thus ground states appear as a compromise between two different periods, and one is then interested in describing their structure.

The various possible configurations of the atoms in the chain are characterized by trajectories in phase space, which turns out to be orbits of a twist map of an annulus. This is obtained by looking at the variational equations that define the map. In general one finds periodic orbits, to which one can assign a rational rotation number. Other orbits have irrational rotation number, corresponding to incommensurate structures.

Several authors (see [2] , [6], [9]) have considered extensions of the Aubry-Mather theory to higher dimensions, that is, configurations on lattices Z^n.

R. Ll. partially supported by research grants from the NSF

In [2], a study is made of minimal configurations of the variational problem with nearest neighbor potentials; while in [6] the authors study variational solutions which are well-ordered (Birkhoff configurations). In fact, it is shown in [2] that in higher dimensions, there may be minimal configurations that are not well ordered.

In [9] the Aubry-Mather theory is discussed for nearest neighbor potential for lattices in the plane. One can find there some interesting physical applications of this theory.

For a more detailed survey of the physical significance of the theory of incommensurate crystals in several dimensions, the reader may consult [8].

The main goal of this paper is to generalize the setting of Aubry-Mather theory to configurations on sets more general than Z^d and for general many-body interactions. We consider a set Λ, together with potentials H_B associated to finite subsets B of Λ. We will need to assume that Λ admits a group action by a group G satisfying some mild assumptions. The potentials H_B describe the interaction of a particle lying in Λ with its neighbors. The potential energy of such a system can be modeled by a formal sum

$$S(u) = \sum_{B \subset \Lambda} H_B(u),$$

where $u : \Lambda \to R$ is a configuration on Λ. We will require that H_B satisfy a ferromagnetic or twist condition, that they are invariant under the action of G and also satisfy some other periodicity assumption.

Physically, we could think of the points of Λ as atoms, whose state is characterized by one number. The configurations describe what is the situation of all the atoms.

We look for configurations u on Λ which are stationary points or minimizers for S and which have a prescribed rotation number. In our situation, the rotation number is a cocycle for the group G.

Solutions to the variational equations with given rotation number always exists. But in order to find ground state configurations we have to require that the group G satisfies a (very natural) property. Essentially, this property says that we can exhaust Λ by fundamental domains for subgroups of G of finite index.

We will discuss the well-order properties of the solutions that we find. These properties are the exact analog of the non-crossing properties of codimension one minimal surfaces of Riemannian manifolds, which obviously is the natural generalization of the relation between solutions to the one-dimensional Frenkel-Kontorova model and geodesics on a torus.

We think that the generality emphasizes the relevant features of the theory and that it would be of interest even if we were interested only in the one dimensional case. Nevertheless, we had several concrete examples that we wanted to consider.

One of them was the following. Consider a completely homogeneous discrete subset of the hyperbolic plane, something we may call a hyperbolic crystal. This set, say Λ, is assumed to be left invariant by a discrete cocompact group G_0 of hyperbolic isometries, that is $G_0 \subset \mathrm{PSl}(2, R)$ is its isotropy group. There is a natural potential in this situation, namely the one whose solutions are discrete harmonic functions. Our theory applies to this case, and to perturbations of it.

Although hyperbolic crystals may not seem very realistic, let us mention the work of Kleiman and Saduc [5]. They suggest that amorphous structures could be regarded as projections of ordered patterns in hyperbolic space. They could be seen as models for disclinations in ordered Euclidean structures. Imagine we add wedges to a crystal. This operation decreases the curvature, thus the new crystal resembles a periodic pattern in hyperbolic space.

Finally let us mention that the theory presented here applies to the Bethe lattice. Besides its applications in statistical mechanics, this seems a reasonable model for hierarchical networks, for example computer networks. It is not hard to imagine situations in which there is a penalty for computers to be out of phase and also for not being in phase with the environment that they are occupying.

In the next section, we will describe the properties of the geometry of the lattice that we will need, in the following one, we will describe the properties of interactions. Then, we will state and prove the main theorem that states that there are solutions for the variational equations satisfying extra properties. Finally, we will briefly discuss some other problems.

2. Generalities on groups and cocycles

The situation we will consider is as follows. There is a countable space Λ on which a finitely generated group G which acts. We impose some conditions on the action of G on Λ. First, it is effective, that is only the identity element of G acts trivially (this is no restriction because we can always pass to a quotient of G satisfying this property). Second, there is a finite fundamental domain for the action, that is, a finite subset F of Λ which intersects each orbit of G in exactly one point.

For instance, Λ could be the set of vertices of some graph (we would like to keep in mind the edges as well, as we may use them for the path metric). Perhaps to avoid unnecessary complications we should assume that the graph is locally finite or uniformly locally finite (it does not became more and more populated as one goes to infinity).

Now this more general situation includes the Cayley graph of finitely generated group, graphs which are lifts of the 1-skeleton of a compact manifold to its universal covering, graphs which are quasi-isometric to manifolds of bounded geometry, etc. For example, it includes the Bethe lattice as the universal cover of the figure 8. Consult [3] for the thermodynamic properties of the Bethe lattice.

In this non-commutative situation that we will consider, the role of rotation numbers in Aubry-Mather theory would be played by cocycles on the group G, that is maps

$$\sigma : G \to R$$

such that $\sigma(\gamma\gamma') = \sigma(\gamma) + \sigma(\gamma')$. The space of cocycles on G forms a real vector space, which will be denoted by $H^1(G; R)$.

We will assume that G acts on Λ with small stabilizers, meaning that if an element γ of G fixes a point of Λ, then $\sigma(\gamma) = 0$ for any cocycle σ on G. For this to hold it is sufficient that the stabilizer of each point of Λ is a torsion group, or, more generally, that it is a group with trivial first cohomology).

One technical result that we will need is the possibility of approximating arbitrary cocycles by simpler ones, namely

Proposition 1. *Let $\sigma : G \to R$ be a cocycle. Then there is a sequence of cocycles σ_n which converges point-wise to σ and such that σ_n has integral values when restricted to some subgroup G_n of finite index in G.*

Proof. Since G is finitely generated, $H^1(G, R)$ is a finite dimensional real vector space. Take a basis for it formed by cocycles with integral values, say $\{\tau_1, \dots, \tau_e\}$. Then we can write $\sigma = a_1\tau_1 + \dots + a_n\tau_n$, some real numbers a_j. For each $k = 1, 2, \dots$, choose rational numbers $b_{i,k}$, $i = 1, \dots, e$, such that $|a_i - b_{i,k}| < 1/k^2$. Then:

$$|\sigma(\gamma) - b_{1,k}\tau_1(\gamma) - \dots - b_{e,k}\tau_e(\gamma)| < |\gamma|_1/k^2,$$

where $|\gamma|_1$ is the l_1-norm of γ in $H^1(G; R)$ with respect to the basis $\{\tau_i\}$. Furthermore, this cocycle $\sigma_k = \sum_i b_{i,k}\tau_i$ takes on rational values only. Since G is finitely generated, its image by η, say P_k, is a finitely generated subgroup of $\mathbb{Q}$. Thus there is an integer m such that $mP_k = Z$, and P_k/mP_k is a finite group. In conclusion, there is a finite index subgroup of G such that σ_k has integer values when restricted to it.

Finally, if $\gamma \in G$ has $|\gamma|_1 < k$, then for $n > k$ we get

$$|\sigma(\gamma) - \sigma_n(\gamma)| < |\gamma|_1/k^2 < 1/k,$$

so that σ_n converges point-wise to σ as required.

Also,

$$\|\sigma - \eta\| = \sup_{\gamma \in G} |\sigma(\gamma) - \sigma_n(\gamma)|/|\gamma| \leq \sup_{\gamma \in G} |\sigma(\gamma) - \sigma_n(\gamma)|/|\gamma|_1 < 1/k^2,$$

because $|\gamma|_1 \leq |\gamma|$ as one easily verifies.

There is one property of the group that we will need to consider when proving existence of ground states, and it is that of being residually finite. This means that for each element of G, other than the identity, there is a finite index subgroup of G which does not contain it. Most familiar groups are residually finite, for instance, a theorem of Selberg asserts that, if R is a field and $n \geq 1$, then every finitely generated subgroup of $SL(n, R)$ is residually finite.

The reason for requiring this property is that it would allow us to do a kind of renormalization needed later in our discussion. More specifically we need the following:

Proposition 2. *Let G, Λ be as above with G residually finite. Then there is a sequence $G_1 \supset G_2 \supset \dots$ of finite index subgroups of G whose fundamental domains exhaust Λ.*

Proof. If F is a fundamental domain for G, we can write

$$\Lambda = F \cup g_1 F \cup \dots$$

where g_i are the nontrivial elements of G. By induction, choose a subgroup G_n of finite index in G_{n-1} which does not contain g_n. (Recall that residually finite is a property inherited by finite index subgroups.) Since a fundamental domain for G_n may be constructed by adding the translates of a fundamental domain of G_{n-1} by representatives of elements of G_{n-1}/G_n, the proof is complete.

3. Configurations, the Birkhoff property and rotation cocycles

A configuration on Λ is a map $u : \Lambda \to R$. The set of all configurations on Λ has the structure of a real vector space with the obviously defined operations, and we denote it by $\mathcal{C}(\Lambda)$. The space of configurations is partially ordered in the following sense. We say that two configurations u, v are such that $u \le v$ if $u(p) \le v(p)$ for all p in Λ. This makes $\mathcal{C}(\Lambda)$ into a partially ordered vector space.

One metric we want to consider on the space of configurations is the one induced by the family of semi-norms $|u(p)|$, $p \in \Lambda$, which gives rise to the pointwise convergence topology. An explicit form for this metric is the following: if we enumerate the points of Λ as $p_1, p_2, \cdots, p_n, \cdots$, an explicit form is

$$d(u,v) = \sum_{p_n \in \Lambda} \frac{1}{2^n} \frac{|u(p_n) - v(p_n)|}{1 + |u(p_n) - v(p_n)|}.$$

We denote $\|u\| = d(0,u)$ and note that $\|tu\| \le (1+|t|)\|u\|$ for any real number t.

The action of G on Λ extends to one on $\mathcal{C}(\Lambda)$, defined in the following manner: if $\gamma \in G$ and $u \in \mathcal{C}(\Lambda)$, then

$$\mathcal{T}_\gamma u(p) = u(\gamma(p))$$

for any element p of Λ. It is evident that this action of G on $\mathcal{C}\Lambda$ preserves the partial order on configurations.

We will denote by χ_A the characteristic configuration associated with a subset A of Λ, and which is defined as $\chi_A(p) = 1$ if $p \in A$, $\chi_A(p) = 0$ otherwise.

We define operators $\mathcal{R}_A : \mathcal{C}(\Lambda) \longrightarrow \mathcal{C}(\Lambda)$ by

$$\mathcal{R}_A(u) = u - \chi_A.$$

It is clear that they are invertible and furthermore that if A, B are subsets of Λ, then the operators $\mathcal{R}_A$ and $\mathcal{R}_B$ commute.

We denote by Λ_α, $\alpha = 1, \ldots, l$ the equivalence classes of Λ modulo G, and denote by $\mathcal{R}_\alpha$ the operators corresponding to the sets Λ_α. Given any s in Z^l, $s = (s_1, \ldots, s_l)$, we denote by $\mathcal{R}_s$ the operator $\mathcal{R}_1^{s_1} \ldots \mathcal{R}_l^{s_l}$.

A cocycle σ defines a configuration u_σ on Λ as follows. Let F be a fundamental domain for the action of G on Λ. Any p of Λ can be written as $p = \gamma q$ for a unique q in F. Then define $u_\sigma(p) = \sigma(\gamma)$. It is elementary that u_σ is well-defined, for if also $p = \gamma_1 q$, then $\gamma^{-1}\gamma_1$ stabilizes q, so $\sigma(\gamma) = \sigma(\gamma_1)$ by our hypothesis. Note that $u_\sigma(q) = 0$ for all the elements of F.

Then, with these conventions, we say that a configuration u is of type σ if

$$\sup_{p \in \Lambda} |u(p) - u_\sigma(p)| < \infty.$$

The set of configurations of type σ will be denoted by $\mathcal{O}_\sigma$.

Put in another way, once a fundamental domain $F \subset \Lambda$ is fixed, any cocycle σ on G defines a configuration and $\mathcal{O}_\sigma$ is the subspace of $\mathcal{C}(\Lambda)$ formed by those configurations u at bounded distance from u_σ. Thus $\mathcal{O}_\sigma$ is an affine space modeled on $\ell^\infty(\Lambda, R)$. The induced metric in $\mathcal{O}_\sigma$ is

$$|u(\gamma) - v(\gamma)|_\infty = \sup_{\gamma \in G} |u(\gamma) - v(\gamma)|,$$

which makes it into a Banach space.

Proposition 3. *The space $\mathcal{O}_\sigma$ is independent of the fundamental domain.*

This is an elementary consequence of the cocycle property. Also, the same property implies:

Proposition 4. *The operators $\mathcal{T}_\gamma$, $\gamma \in G$, and $\mathcal{R}$ leave $\mathcal{O}_\sigma$ invariant.*

Next we introduce the notion of Birkhoff configuration. A configuration u of type σ is called a Birkhoff configuration for the group G if $\sigma(\gamma) \geq (\leq) s_\alpha$ for $\gamma \in G$, $s \in N^l$ only when $\mathcal{T}_\gamma(u) \geq (\leq) \mathcal{R}_s(u)$. The set of Birkhoff configurations in $\mathcal{O}_\sigma$ is denoted by $\mathcal{B}_\sigma$.

Since $\mathcal{R}_s(u)(p) = u(p) - s_\alpha$ if p belongs to the equivalence class Λ_α of Λ/G, we see that u is a Birkhoff configuration if either $\mathcal{T}_\gamma u + \sigma(\gamma) \geq u$ or $\mathcal{T}_\gamma u + \sigma(\gamma) \leq u$ for the partial order of configurations. It also follows immediately that:

Proposition 5. *For every γ in G there exists s^+, s^- such that*

$$\mathcal{R}_{s^-}(u) \leq \mathcal{T}_\gamma(u) \leq \mathcal{R}_{s^+}(u)$$

for any u in $\mathcal{B}_\sigma$.

Although we have defined the Birkhoff property of a configuration by referring to a given cocycle, we could have avoided it. We show next that the non-intersecting property of a configuration implies that there is a cocycle for which it is Birkhoff. That is, suppose that $u : \Lambda \to R$ is a configuration which satisfies the following property: for each $\gamma \in G$ and any integer $k \in Z$, we have

$$u(\gamma p) + k \leq u(p)$$

for all $p \in \Lambda$, or

$$u(\gamma p) + k \geq u(p)$$

for all $p \in \Lambda$. We are going to define a cocycle $\sigma : G \to R$ such that $u \in \mathcal{O}_\sigma$.

First, let

$$\tau^+(\gamma) = \sup_p (u(\gamma p) - u(p))$$

and

$$\tau^-(\gamma) = \inf_p (u(\gamma p) - u(p)).$$

Note that they are finite numbers. Furthermore, τ^+ is a sub-cocycle, that is,

$$\tau^+(\gamma_1\gamma_2) \leq \tau^+(\gamma_1) + \tau^+(\gamma_2),$$

and τ^- is a super-cocycle. This implies that we can define

$$\sigma(\gamma) = \lim_{n\to\infty} \frac{\tau^+(\gamma^n)}{n} = \lim_{n\to\infty} \frac{\tau^-(\gamma^n)}{n}.$$

because both limits exist and they are equal. We also see that

$$\sigma(\gamma_1\gamma_2) = \lim_{n\to\infty} \frac{\tau^+((\gamma_1\gamma_2)^n)}{n} = \lim_{n\to\infty} \frac{\tau^+(\gamma_1(\gamma_2\gamma_1)^{n-1}\gamma_2)}{n} \leq \sigma(\gamma_2\gamma_1)$$

by elementary use of the sub-cocycle and super-cocycle properties and of the definition of σ. Therefore $\sigma(\gamma_1\gamma_2) = \sigma(\gamma_2\gamma_1)$.

It is also elementary to check that $\sigma(\gamma^{-1}) = -\sigma(\gamma)$, and that, if γ_1 and γ_2 commute, then
$$\sigma(\gamma_1\gamma_2) = \sigma(\gamma_1) + \sigma(\gamma_2).$$
From all these properties it follows that σ is a cocycle.

Finally we show that the configuration u has σ as rotation cocycle, that is, that
$$\sup_{p\in\Lambda} |u(p) - \sigma(\tau_p)| < \infty,$$
which is the same as to show that
$$\sup_{\gamma\in G, q\in F} |u(\gamma q) - \sigma(\gamma)| < \infty.$$
Here $F \subset \Lambda$ denotes a finite fundamental domain for the action of G.

Fix $q \in F$ and $\gamma \in G$, and consider the configuration $n \in Z \mapsto u(\gamma^n q)$. Since
$$\tau^-(\gamma^n) \leq u(\gamma^n q) - u(q) \leq \tau^+(\gamma^n),$$
it follows from the definitions that it has rotation number $\sigma(\gamma)$. As is well-known, this implies that
$$u(q) + n\sigma(\gamma) \leq u(\gamma^n q) < u(q) + n\sigma(\gamma) + 1$$
from what it follows that
$$|u(\gamma q) - \sigma(\gamma)| \leq 1 + |u(q)|.$$
Since the fundamental domain F is finite, this implies that the rotation cocycle of u is σ, as claimed above.

Proposition 6. *If u is a configuration satisfying the well-ordering property above, then there is a cocycle σ on G for which $u \in \mathcal{B}_\sigma$.*

In particular,

Corollary 1. *If the group has no nontrivial cocycles, then every Birkhoff configuration is bounded.*

A simple example of a group without cocycles is
$$G =< a, b; a^p = b^q = (a^{-1}b)^r = 1 >$$
with $1/p + 1/q + 1/r < 1$. This group can be realized faithfully as a discrete group of isometries of the hyperbolic plane. However, this groups have finite index subgroups which do have nontrivial cocycles.

We also mention the following elementary property of Birkhoff configurations.

Proposition 7. *The space of Birkhoff configurations is closed for the product topology on R^Λ. Furthermore, for any cocycle σ, the set $\mathcal{B}_\sigma$ is non-empty and convex.*

Proof. For the first part, simply note that by the previous proposition the space of Birkhoff configurations is characterized by the inequalities $T_\gamma u+n \geq u$ or $T_\gamma u+n \geq u$, so it can be written as an intersection of closed sets.

The configuration u_σ associated to the cocycle is Birkhoff, so $\mathcal{B}_\sigma$ is non-empty. The convexity property is obvious.

Similarly, we also have

Proposition 8. *Let σ be a cocycle in G, and consider $\mathcal{O}_\sigma$ with the Banach space topology given by uniform convergence. Then the space $\mathcal{B}_\sigma$ of Birkhoff configurations in $\mathcal{O}_\sigma$ is closed.*

In analogy with rational rotation numbers, we say that a configuration u has rational rotation cocycle if there is a normal finite index subgroup G' of G and an integer valued cocycle $\sigma : G' \to Z$ such that

$$u(\gamma(p)) = u(p) + \sigma(\gamma)$$

for all $p \in \Lambda$.

This definition could have been formulated in a slightly different way by using the following two propositions.

Proposition 9. *Let $\sigma : G' \to Z$, where G' is a finite index normal subgroup of G with G/G' commutative. Then σ extends to a rational cocycle on G.*

Proof. For γ in G, let n be any nonzero integer such that $\gamma^n \in G'$. Then define $\sigma(\gamma) = \sigma(\gamma^n)/n$. It is immediate to check that $\sigma : G \to \mathbb{Q}$ is well defined and, because G/G' is commutative, it is a homomorphism.

Proposition 10. *Let $u \in \mathcal{C}(\Lambda)$ be a configuration such that $u(\gamma(p)) = u(p) + \sigma(\gamma)$ for all p in Λ and all elements γ of a finite index normal subgroup G' of G, and for $\sigma : G' \to Z$. Suppose that σ extends to a cocycle on G. Then $u \in \mathcal{O}_\sigma$, σ the extension of the cocycle to G.*

Proof. Let $p_0, \dots, p_r$ be a finite set such that the translates $\{\gamma(p_i); \gamma \in G'\}$ fill up Λ. Choose an upper bound M for the finite sets of numbers $\{|u(p_0)|, \dots, |u(p_r)|\}$ and $\{|\sigma(\tau_{p_0})|, \dots, |\sigma(\tau_{p_r})|\}$. Then, for any $p \in \Lambda$ we have:

$$|u(p) - \sigma(\tau_p)| = |u(\gamma(p_k)) - \sigma(\tau_p)| = |u(p_k) - \sigma(\tau_{p_k})| \leq 2M.$$

We will use another property of Birkhoff configurations that we now explain.

Proposition 11. *Suppose that u is a Birkhoff configuration that has integer valued rotation cocycle $\sigma : G \to \mathbb{Z}$. Suppose that u is periodic with respect to some finite index subgroup G' of G, that is,*

$$u(\gamma(p)) = u(p) + \sigma(\gamma)$$

for all p in Λ and all γ in G'. Then u is also G-periodic.

The proof of this fact goes as follows. Let γ be an element of G. Then $\gamma^k \in G'$ for some integer k because G/G' is finite. If u is not γ-periodic, then let p be such

$$u(\gamma(p)) > u(p) + \sigma(\gamma)$$

Using the Birkhoff property of u this implies that

$$u(\gamma^k(p)) > u(p) + k\sigma(\gamma)$$

which contradicts the G'-periodicity of u.

4. Interaction potentials and the variational problem

Let $\mathcal{S}$ denote the collection of finite nonempty subsets of Λ. An interaction potential is a collection of maps $H = \{H_B; B \in \mathcal{S}\}$, where each $H_B : \mathcal{C}(\Lambda) \to R$, such that $H_B(u) = H_B(v)$ whenever u and v agree on B (so H_B is interpreted as a function $H_B : R^B \to R$), and for any finite subset X of Λ, the series

$$H(u, X) = \sum_{B \cap X \neq \emptyset} H_B(u)$$

converges. We call $H(u, X)$ the (total) energy of u in X.

A potential is absolutely summable if the series

$$\sum_{B, p \in B} |H_B|_\infty$$

converges for all $p \in \Lambda$. Denoting by $|H|_p$ be the value of the series above, we have a family of semi-norms in the space $\mathcal{P}$ of absolutely summable potentials, making it into a Frechet space.

A potential H is said to be of finite range if for each $p \in \Lambda$ there is some finite set $B_p \subset \Lambda$ such that $H_B = 0$ unless $B \subset B_p$ for all $p \in B$. We say that H is of bounded range if there is a number M such that $H_B = 0$ if $\operatorname{diam}(B) > M$. Clearly, if H is finite range and all H_B are bounded, then $H \in \mathcal{P}$. Moreover, absolutely summable finite range potentials are dense in $\mathcal{P}$.

A configuration u is a ground state for the interaction potential $H = \{H_B\}$ if

$$H(u, X) \leq H(v, X)$$

for any finite set X and any configuration v such that $u = v$ on $\Lambda \setminus X$. Thus u is a ground state if the energy of any finite perturbation of u exceeds that of u.

Our goal is to seek minimal configurations which belong to the spaces $\mathcal{O}_\sigma$. Note that if u is a minimal configuration for the energy problem in $\mathcal{O}_\sigma$, it is also a minimal configuration for the global problem in $\mathcal{C}(\Lambda)$, simply because if $u \in \mathcal{O}_\sigma$ and $u = v$ outside some finite set X, then also $v \in \mathcal{O}_\sigma$. On the other hand, note that a configuration that minimizes $H(\cdot, X)$ in $\mathcal{B}_\sigma$ may not necessarily be a ground state configuration.

Some other properties of ground state configurations that we will use later are collected in the following proposition.

Proposition 12. *The set of all ground state configurations for a continuous potential H is closed in R^Λ with the product topology.*

If $\mathcal{O}$ denotes the space of configurations at bounded distance from a fixed one, the space of ground states in $\mathcal{O}$ is closed in the Banach space topology.

Proof. Suppose that a sequence u_n of ground state configurations converges pointwise to the configuration u. If u is not a ground state configuration, then there is a finite set X and a configuration v which equals 0 outside X and such that

$$H(u, X) - H(u + v, X) \geq a > 0.$$

On the finite set X, the convergence $u_n \to u$ is uniform, and the continuity of the potential implies that, for large n we have

$$H(u_n, X) - H(u_n + v, X) > 0$$

contradicting the minimality of the u_n's.

The same argument also proves the second part, because if $u_n \in \mathcal{O}$ converges to u in the Banach space topology then it also converges uniformly on each finite subset X of Λ.

We introduce some more definitions in order to restrict the type of potentials to be considered. An interaction potential $H = \{H_B\}$ is G-invariant if

$$H_{\gamma B}(u) = H_B(T_\gamma u)$$

for all $\gamma \in G$, all B and all configurations u on Λ.

If the interaction potential H satisfies

$$H_B(\mathcal{R}_s u) = H_B(u)$$

for all configuration u, all B and all $s \in Z^l$, then we say that H has periodic phase. This periodic phase property allows us to consider the H_B's as maps $\mathcal{C}(\Lambda)/\mathcal{R} \to R$.

The Banach space structure of $\mathcal{O}_\sigma$ permits us to write down the variational equations if the potential is differentiable. Thus, with the assumption of differentiability of potentials, a necessary condition for u to be a minimal configuration is that it satisfies the variational equations

$$\nabla H(u, X) = \sum_{X \cap B \neq \emptyset, p \in B} \frac{\partial}{\partial p} H_B(u) = 0.$$

This sum converges for finite range interaction. In general, we have to make some further restrictions.

Let $\mathcal{O}$ be a convex subset of the set of configurations. We say that an interaction H is C^r-bounded on $\mathcal{O}$ if

$$\|H\|_{r,\mathcal{O}} = \sup_{u \in \mathcal{O}} \sum_{|j| \leq r} |D^j H_B(u)| < \infty$$

Note that the space $\mathcal{O}$ in the definition above maybe a proper subset of one of the $\mathcal{O}_\sigma$'s. Examples show that interaction may not be bounded in $\mathcal{O}_\sigma$ but they are in subspaces of the form $\mathcal{O} = \{u;\, |u - u_\sigma|_\infty \leq K\}$.

If H is a C^1-bounded potential on $\mathcal{O}$, we define a map $u \mapsto V(u)$ on configurations $u \in \mathcal{O}$ with values in $\ell^\infty(\Lambda)$ by

$$V(u)(p) = -\sum_{B \ni p} \frac{\partial}{\partial p} H_B(u)$$

and therefore the variational equations can be written as

$$V(u) = 0$$

In the cases that we will consider $\mathcal{O}$ will be an affine space over $\ell^\infty(\Lambda)$, in particular one of the $\mathcal{O}_\sigma$, and the hypothesis of C^r boundedness of H will ensure that the $r-1$ derivative of V exists and is uniformly bounded in the sense of derivatives of Banach spaces.

Since $\mathcal{O}_\sigma$ is a Banach space isomorphic to $\ell^\infty(\Lambda)$ the natural interpretation of V is as a vector field on $\mathcal{O}_\sigma$.

In any case, the above definitions show that the function V associated to a C^r-bounded potential H on $\mathcal{O}$ is globally Lipschitz, that is:

Corollary 2. *If V is defined as above, then*

$$|V(u) - V(v)|_\infty \leq \|H\|_2 |u - v|_\infty.$$

Before we state the main theorem on the existence of minimal configurations we still need new definition concerning the potentials to be considered. This is the twist condition of Hamiltonian mechanics or the ferromagnetic property in statistical mechanics. More precisely, we say that an interaction potential H which is C^2-bounded on $\mathcal{O}$ satisfies the twist condition if

$$\sum_{B \ni q} \frac{\partial^2}{\partial p \partial q} H_B(u) \leq 0$$

for all configurations u in $\mathcal{O}$ and all p, q in Λ with $p \neq q$.

This property is obviously implied by the strong twist condition, that is

$$\frac{\partial^2}{\partial p \partial q} H_B(u) \leq 0$$

for all u and all $p \neq q$.

5. Existence of solutions to variational equations

In this section we prove the existence of solution to the variational equation of a twist potential. The idea is inspired by [4] and [6]. There the strategy of the proof is to consider the flow $\phi_t : \mathcal{O}_\sigma \to \mathcal{O}_\sigma$ defined by the differential equation

$$\frac{d}{dt}\phi_t(u) = -V(\phi_t(u)).$$

Since the right-hand side is globally Lipschitz, solutions exist for all times. Then we will argue that this flow preserves the Birkhoff condition on configurations, so it is actually a flow on $\mathcal{B}_\sigma$. Furthermore, it commutes with the operators $\mathcal{R}$ so that the end result is a flow on $\mathcal{B}_\sigma/\mathcal{R}$. Once here we will be able to use compactness and therefore fixed points for the flow.

Unfortunately, for this outline to work we need to add a further condition on the space Λ, namely that it has polynomial growth function with respect to the action of G. In general this case is the exception rather than the rule. To overcome this difficulty we consider a sequence of gradient flows that converge to the one defined by the variational equations.

Theorem 1. *Let G be a finitely generated group acting on Λ with finitely many orbits, and with small stabilizers. Let σ be a cocycle on G and let H be an interaction potential which is C^2-bounded on $\mathcal{O}_\sigma$ and which is G-invariant and has periodic phase. Assume that either H satisfies the strong twist condition or that H satisfies the twist condition and is of finite range. Then there is a solution of the variational equations which lies in $\mathcal{B}_\sigma$.*

Suppose that H is of finite range. For any finite subset F in Λ consider the function H_F on configuration space defined by

$$H_F(u) = \sum_{B\cap F\neq\emptyset} H_B(u).$$

Let $-\nabla H_F$ be the gradient vector field defined by this function and let ϕ_F be the flow defined by the differential equation

$$\frac{d}{dt}\phi_F(u,t) = -\nabla H_F(\phi_F(u,t)).$$

Proposition 13. *The flow ϕ_F is defined on $\mathcal{O}_\sigma$ for all times.*

Indeed, the gradient ∇H_F is a Lipschitz vector field in $\mathcal{O}_\sigma$ with the Banach space topology.

Proposition 14. *The flow $\phi_F(t,u)$ defined by the differential equation*

$$\frac{d}{dt}\phi_F(t,u) = -\nabla H_F(\phi_F(t,u))$$

preserves the order structure on the space of configurations.

The proof of this proposition is along the lines of the one in [6]. Briefly, the idea is that the twist condition implies that the corresponding linearized equation is given by a matrix with positive entries off the diagonal. This is shown to imply the monotonicity of the gradient flow.

A consequence of this and the fact that the interaction potential has periodic phase is that ϕ_t induces a flow on $\mathcal{B}_\sigma/\mathcal{R}$.

Proposition 15. *If the interaction potential H is ferromagnetic, then H_F is non-increasing along the flow, that is*

$$\frac{d}{dt}H_F(u) \leq 0$$

We compute the derivative along the flow $\phi : \mathcal{O}\times\mathbb{R}\to\mathcal{O}$ defined by the differential equation. By the chain rule we have that:

$$\begin{aligned}
\frac{d}{dt}H_F(\phi_F(u,t)) &= -\nabla H_F(u)\frac{d}{dt}\phi_F(u,t) \\
&= -\left(\frac{\partial}{\partial p}\sum_B H_B(u)\right)_{p\in BF}\cdot\left(\frac{\partial}{\partial p}\sum_B H_B(u)\right)_{p\in BF} \\
&= \sum_{p\in BF}\left(\frac{\partial}{\partial p}H_B(u)\right)^2 \leq 0.
\end{aligned}$$

By the order preserving property, we can consider the flow on $\mathcal{B}_\sigma/\mathcal{R}$. The last detail we need is:

Proposition 16. *The space $\mathcal{B}_\sigma/\mathcal{R}$ is compact (for the pointwise convergence metric).*

Proof. Let F be a fundamental domain for the action of G on Λ. If u is a configuration, we can apply one of the operators $\mathcal{R}$ so that $u(p) \in [0,1]$ for $p \in F$. If furthermore u is a Birkhoff configuration, then

$$\sigma(\gamma) \leq u(\gamma p) - u(p) \leq \sigma(\gamma) + 1$$

Hence we can view $\mathcal{B}_\sigma/\mathcal{R}$ as a subspace of an infinite product of circles. It is also closed, hence compact.

Note that the potential H_F induces a map on this quotient space because of the invariance of the interactions H_B under the transformations $\mathcal{R}$.

Since H_F is bounded below in this compact space, and it is non-increasing along the orbits of the flows, there must be a sequence $t_n \to \infty$ for which

$$\frac{d}{dt}\phi_F(u, t_n) \to 0.$$

Therefore, if we start with a Birkhoff configuration u, we obtain a sequence of Birkhoff configurations $\phi(u, t_n)$ which is compact in the space $\mathcal{B}_\sigma/\mathcal{R}$. Thus there is a convergent subsequence whose limit is a critical point u_F.

Consider now an increasing sequence $F_0 \subset F_1 \subset \cdots$ of finite sets exhausting Λ. For each of then we have a critical point u_n in $\mathcal{B}_\sigma$ for the gradient flow ∇H_n. After modification by the operators $\mathcal{R}$, which preserves the property of being a critical point, we see that these sequence of points has a subsequence which converges point-wise to u, a configuration in $\mathcal{B}_\sigma$.

To finish the proof of the theorem we only need the following:

Proposition 17. *This configuration u satisfies the variational equations $V(u) = 0$.*

Proof. We have to check that $V(u)(p) = 0$ for any p in Λ. Fix p and choose n large enough so that if B contains p then $B \subset F_n$. The convergence $u_n \to u$ is uniform on F_n, and $\nabla H_n(w)(p) = V(w)(p)$. It follows that u satisfies the variational equations.

6. Properties of ground state configurations

For a configuration u and a point p in Λ define the energy of u due to p as

$$E_p(u) = \sum_{B \ni p} \frac{1}{|B|} H_B(u).$$

For X a finite subset of Λ let

$$h_X(u) = \frac{1}{|X|} \sum_{p \in X} E_p(u).$$

We would like to take the limit as X approaches Λ, but there is no canonical way to do so. We will regard it as a function on finite parts of Λ, $X \in \mathcal{F}(\Lambda) \mapsto h_X(u)$.

One possibility would be to take a fundamental domain X for G, list the elements of G as $\{e, \gamma_1, \gamma_2, \dots\}$ and set $X_n = X \cup \ldots \cup \gamma_n X$. Then, if u is a configuration which has integer periods with respect to a finite index subgroup G' of G, we see that the limit

$$\lim_{n \to \infty} h_{X_n}(u)$$

exists and is equal to the specific energy of u in a fundamental domain for G'.

We want to mention one property about the specific energy of configurations. In [2] this property is proved under the assumption that the configuration is minimal and satisfies the Birkhoff condition. It appears that considerably less is needed. We include a proof for completeness.

Proposition 18. *Let the interaction potential H be of finite range, G-invariant and satisfy the strong ferromagnetic condition. Let σ be a cocycle on G. If u is a configuration which belongs to $\mathcal{O}_\sigma$ then there is a constant $K(u, \sigma)$ such that*

$$H(u, X) \leq K(\sigma, u)\ \mathit{Card}(X),$$

for any finite subset X of Λ.

Proof. Let u be a configuration in $\mathcal{O}_\sigma$. Let u_σ be a configuration defined by the cocycle σ. Applying the ferromagnetic hypothesis in the development provided by Taylors theorem we have that for any B with $H_B \neq 0$ there are positive constants $\alpha(B)$, $\beta(B)$ such that

$$H_B(u) \leq \alpha(B) \sum_{p,q \in B} (u(p) - u(q))^2 + \beta(B).$$

This follows from Taylor's theorem. The constants α and β depend on σ, on the distance between u and u_σ, and on B. There are only finitely many B's up to translation, so the invariance of H allows us to choose α and β independently of B.

Next note that for any p, q and for u in $\mathcal{O}_\sigma$ we have previously shown that

$$|u(p) - u(q)| \leq C + |\sigma(\gamma_{pq})|,$$

where C is a constant and γ is any transformation that takes p to q. Therefore, if the interaction potential is of bounded range, there is a constant C such that

$$|u(p) - u(q)| \leq C$$

for any p, q belonging to the same set B with $H_B \neq 0$.

We now put all this information together. Let X be a finite set and $u \in \mathcal{O}_\sigma$. We have

$$\begin{aligned} H(u, X) &= \sum_{B \cap X \neq \emptyset} H_B(u) \\ &\leq \sum_B \sum_{p,q \in B} \alpha(u(p) - u(q))^2 + \beta \\ &\leq \sum_B n_B^2 (C^2 + \beta) \\ &\leq K(u, \sigma) \mathrm{Card}(X) \end{aligned}$$

Aubry's fundamental lemma for the Frenkel-Kontorova model shows the non-intersecting property of ground state configurations. The analysis of Aubry and Mather of configurations on the line which are minimal for the Lagrangian shows that they satisfy a non-intersecting property, namely, if $u : Z \to R$ is a ground state configuration, then it is Birkhoff.

In higher dimensions this need not be true, and Blank [2] provides the following example:

$$H(u_1, u_2, u_3) = (u_1 - u_2 + b_1)^2 + (u_1 - u_3 + b_2)^2$$

which admits $f(x_1, x_2) = x_1 x_2$ as a solution to the variational equations.

Blank showed that a similar property holds for minimum energy configurations for the model he studies in [2]. The following proposition exhibits a similar property for the models we study. From a geometric point of view, it is essentially the maximum principle for minimal surfaces.

The space Λ has no structure, but given an interaction potential $H = \{H_B\}$, those sets B for which $H_B \neq 0$ define a sort of topology on Λ. So if X is a finite subset of Λ, we will denote by $N(X)$ the union of those sets B which intersect X and with $H_B \neq 0$, and by X^0 the subset of those elements p of X for which there is some B which meets X in some other points different from p. Although these definitions may look strange, they are simply verified for models whose supporting set Λ is corresponds to the vertices of a graph and the interaction potential is defined in terms of the connecting edges.

First we need some new terminology. Say that a subset X of Λ is connected if for any pair of elements p, q of it, there is a sequence $p_0, \ldots, p_k$ in X with $p_0 = p$, $p_k = q$, and sets B_i, each containing a pair p_i, p_{i+1}, and for which $H_{B_i} \neq 0$.

The meaning of this condition is obvious. Each connected component of Λ behaves in its own independent way, and there is no restriction in assuming our models have connected state space.

Proposition 19. *Let u and v be ground state configurations for the ferromagnetic potential H on the configuration space of Λ. Suppose that there is a finite connected set X such that $u \geq v$ on $N(X)$. Then, either $u = v$ or $u > v$ in all of X.*

Proof. We use the standard technique of the Hilbert integral in calculus of variations. Any ground state configuration must satisfy the variational equations. Thus, if $p \in X$ we get

$$\sum_{B \ni p} \frac{\partial}{\partial p}[H_B(u) - H_B(v)] = 0$$

and replacing the terms in the sum by their integrals,

$$\sum_{B \ni p} \int_0^1 \frac{d}{dt}\frac{\partial}{\partial p} H_B(tu + (1-t)v)dt = 0.$$

This expression may be written as:

$$\sum_{B \ni p} \sum_{q \in B} [u(q) - v(q)] \int_0^1 \frac{\partial^2}{\partial q \partial p} H_B(tu + (1-t)v)dt = 0.$$

Let $p \in X^0$ be such that $u(p) = v(p)$. For $q \neq p$ in X^0, the term $u(q) - v(q)$ in the sum above is ≥ 0 by hypothesis, while the integral term is strictly negative due to the ferromagnetic condition. This forces $u = v$ through all of $N(p)$. Now the fact that X is connected allows us to extend the equality $u = v$ to all X.

Corollary 3. *Let u, v be two critical configurations on Λ, which is assumed to be connected. If $u \leq v$, then either $u = v$ or $u < v$.*

Corollary 4. *Let X be a connected finite subset of Λ and let u,v be to configurations which agree outside X and such that they both minimize $H(\cdot, X)$ with boundary conditions $u_{\Lambda - X}$. Then $u = v$ on X.*

7. Existence of ground state configurations

The result we just proved guarantees the existence of Birkhoff critical configurations with arbitrary rotation number. In this section we show that Birkhoff ground states exist with arbitrary rotation cocycle. Here we have to assume that the group is residually finite, and that the space Λ is connected with respect to the interaction potential to be considered.

Theorem 2. *Let G be a finitely generated residually finite group acting on the set Λ with finitely many orbits and with small stabilizers. Let H be a finite range interaction potential on Λ, making Λ connected, which is G-invariant, has periodic phase, and satisfies the strong ferromagnetic condition. If σ is a cocycle on G, then there are Birkhoff configurations on $\mathcal{B}_\sigma$ which are ground states.*

We start with a rational cocycle $\sigma : G \to \mathbb{Q}$, integer valued on the finite index subgroup G'. We let F be a connected fundamental domain for G'. Consider the Hamiltonian

$$H_F = \sum_{B \cap F \neq \emptyset} H_B$$

We also consider an exhaustion $F = F_0 \subset F_1 \cdots$ of Λ by fundamental domains with the property that the sets B which intersect F_n are contained in F_{n+1}, and they correspond to a tower of finite index subgroups of G'.

We denote by H_n the Hamiltonian corresponding to F_n. The problem is to minimize H_n with periodic boundary conditions. These boundary conditions give us the constrains for a Lagrange multiplier problem.

A configuration u which is G_n-periodic is completely determined by its values on F_n. If q is a point outside F_n then there is a unique p in F such that $u(q) = u(p) + \sigma(\gamma)$ for some γ in G_n. (There could be several γ's but the result is independent of that because the small stabilizers condition.)

Let N denote the union of all B's with $H_B \neq 0$ which intersect F. The problem is then to minimize H for configurations u on N subject to the constrains determined by the periodicity condition. We obtain one linear equation of the form

$$g_q(u) = u(q) - u(p) - \sigma(\gamma)$$

for each point q in $N \setminus F$. For p in F denote by $\eta(p)$ the number of points equivalent to it in N.

We start by discussing the problem of minimizing H_n over G_n-periodic configurations. First, we consider H_n as a function on R^{F_n}. This can be done because any element in this space extends in a unique way to a configuration on Λ, and then we apply H_n. Now the twist condition implies that this function attains its minimum over a compact set. It follows that we obtain a periodic configuration u_n which minimizes H_n among periodic configurations.

Proposition 20. *The function H_n has a minimum on the space of G_n periodic configurations.*

We want to show that critical configurations of this system are ordered. The method of Lagrange multipliers provides a system of equations which we now describe. First there are real numbers λ_q, one for each q in $N \setminus F$, and λ_0 which is either 0 or 1 (and so that not all λ's are zero). The equations are as follows: if $p \in F$ has no other point of its orbit in N,then

$$\frac{\partial}{\partial p} H(u, N) = \sum_{B \ni p} \frac{\partial}{\partial p} H_B(u) = 0,$$

if the orbit of $p \in F$ meets N in the points $q_1, \cdots, q_k$ (other than p), then

$$\lambda_0 \sum_{B \ni p} \frac{\partial}{\partial p} H_B(u) = \lambda_1 + \cdots + \lambda_k$$

for the scalars λ corresponding to the points in $N \backslash F$ equivalent to p, and if $q \in N \backslash F$ is in the orbit of $p \in F$ we get

$$\lambda_0 \sum_{A \ni q} \frac{\partial}{\partial q} H_A(u) = -\lambda_*$$

where the sum is over the sets A that meet F.

We then see that we can take $\lambda_0 = 1$ and the last two equations can be combined into one which is independent of the scalars λ and holds for any critical configuration:

$$\sum_{B \ni p} \frac{\partial}{\partial p} H_B(u) + \sum_{q \in O_p \cap N} \sum_{A \ni q} \frac{\partial}{\partial q} H_A(u) = 0$$

(here O_p denotes the orbit of p in Λ).

This last equation can be rewritten in the form

$$\sum_{B \ni p} n_{B,p} \frac{\partial}{\partial p} H_B(u) = 0$$

because of the periodicity of the configuration and of the interaction potential. The index $n_{B,p}$ is a positive integer which comes out of the rearrangement of the sum over the A's.

Now if the interaction potential satisfies the strong ferromagnetic condition, the same technique of the Hilbert integral allows us to obtain a sort of Aubry's fundamental lemma.

Proposition 21. *If the interaction potential satisfies the strong twist condition, then two G_n-periodic configurations u and v that minimize H_n and such that $u \geq v$ are either equal or $u > v$.*

Proof. The proof is essentially the same as the one in the previous section. First start with a point in F_n which is equivalent to no point of $N(F_n)$. If $u = v$ at this point, the proof in the previous section shows the $u = v$ in the whole neighborhood of the point.

For points of F_n that have some equivalent in $N(F_n)$, the last equation we have written allows us to use again the Hilbert integral technique.

In conclusion, if $u(p) = v(p)$ at some point of F_n, we obtain that $u = v$ through all of F_n, as we are assuming the fundamental domains to be connected.

Hence, if u, v are periodic and minimize H_n we can consider the configurations $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$ which are also periodic, and the twist condition implies

$$H_n(u) + H_n(v) \geq H_n(u \vee v) + H_n(u \wedge v)$$

Hence these configurations also minimize H_n. By applying the argument of the previous paragraph to the pairs u, $u \vee v$, etcetera, we conclude that either $u > v$ or $v > u$ or they are equal.

Before proceeding we note the following fact.

Lemma 1. *If u is a G_n-periodic configuration, so is $\mathcal{T}_\gamma u$ for any γ in G.*

Proof. If $\gamma_1 \in G_n$, then by normality of G_n in G, there is γ_2 in G_n such that $\gamma\gamma_1 = \gamma_2\gamma$. Hence $\sigma(\gamma_1) = \sigma(\gamma_2)$ and

$$\mathcal{T}_\gamma u(\gamma_1 p) = u(\gamma\gamma_1 p) = u(\gamma_2\gamma p) = u(\gamma p) + \sigma(\gamma_2).$$

¿From this the Birkhoff property follows as we now show.

Proposition 22. *Suppose that u is G_n-periodic and minimizes H_n. Then it is a Birkhoff configuration.*

Proof. We show that for any γ in G we have

$$|\mathcal{T}_\gamma u - u - \sigma(\gamma)| \leq 1.$$

If $[\sigma(\gamma)]$ denotes the integer part of $\sigma(\gamma)$, it follows from the previous lemma that the configurations $\overline{u} = T_\gamma u - [\sigma(\gamma)]$ and $\underline{u} = \overline{u} - 1$ are also G_n-periodic and minimize H_n. Hence the differences $\overline{u} - u$ and $\underline{u} - u$ have constant sign. Clearly these differences are invariant by G_n, and so we can determine their sign by adding over a fundamental domain for G_n.

To do this, observe that for a G_n-invariant configuration v we have

$$\sum_{F_n} \mathcal{T}_\gamma v - v = \sigma(\gamma)|F_n|$$

and in fact the sum can be taken over any fundamental domain for G_n. Hence

$$\sum_{F_n} \overline{u} - u = (\sigma(\gamma) - [\sigma(\gamma)])|F_n| \geq 0,$$

and similarly

$$\sum_{F_n} \underline{u} - u = (\sigma(\gamma) - [\sigma(\gamma)] - 1)|F_n| \leq 0.$$

It follows that $0 \leq \mathcal{T}_\gamma u - u - [\sigma(\gamma)] \leq 1$, which is equivalent to the Birkhoff property.

This discussion has the following consequence. We start with a tower of connected fundamental domains $F = F_0 \subset \cdots \subset F_n \subset \cdots$ corresponding to normal finite index subgroups of $G_0 \supset G_1 \supset \ldots$ of G, then we obtain a sequence of configurations $u_0, u_1, \ldots$, where u_n is periodic with respect to G_n and minimizes H_n. Furthermore, since these configurations satisfy the Birkhoff property and the cocycle is integer valued on G_0, they are all periodic with respect to G_0. By invoking the operators $\mathcal{R}$, we may assume that this sequence $\{u_n\}$ has a subsequence which converges pointwise to a configuration u. Hence u is a Birkhoff configuration. Furthermore,

Proposition 23. *The configuration u is a ground state for H.*

Indeed, let v be a configuration that agrees with u outside a finite subset X of Λ. Then $X \subset F_n$ for some large n. We can find a larger m such that every set B which meets F_n is inside F_m. Now restrict v to F_m and extend to Λ by periodicity with respect to G_m. This does not change the value of $H(v, F_n)$. Then, for $k \geq n$,

$$H(v, F_n) \geq H(u_k, F_n).$$

Since u_n converges uniformly to u on F_n, we take the limit with respect k and the ground state property of u is verified.

The case in which the cocycle σ is arbitrary is now immediate. We approximate σ by rational cocycles. For each one the argument just described produces a Birkhoff ground state. By using the operators $\mathcal{R}$, we may assume this sequence converges. The limit is then a Birkhoff ground state for σ.

8. Examples and problems

The typical example is the following. Let G be a group with a finite generating set $S = S^{-1}$. With it we construct a graph whose vertices are the elements of G and whose edges are labeled by the elements of S, that is, there is an edge from g to h if $h = gs$, and we denote it by (g, s, gs).

This graph is made into a metric space by taking each edge to be isometric to the unit interval in the real line. Furthermore, there is a natural left action of G on Γ, namely, $g(v, s, vs) = (gv, s, gvs)$, which is effective and transitive.

There is a natural potential H defined by

$$H_B(u) = (u(p) - u(q))^2$$

if p and q are connected by an edge, and trivial otherwise. The ground state configurations of this model are harmonic functions on the group G, where the discrete Laplacian is defined by

$$\Delta u(p) = \frac{1}{n(p)} \sum_{q \sim p} u(q) - u(p),$$

where $n(p)$ is the number of edges emanating from p, and $q \sim p$ means that q is connected to p by an edge.

It is then elementary that any cocycle on G defines a harmonic function. This corresponds to the integrable case.

The Frenkel-Kontorova model in the group G involves also the site potentials

$$H_p(u) = V(u(p))$$

where V is a periodic function satisfying $V'' < 0$. The variational equations are

$$n(g)u(g) - \sum_{s\in S} u(gs) + V'(u(g)) = 0.$$

Most of what was done for Z carries over to this general situation.

In physics there are two types of lattices which are frequently used. One is the Euclidean lattice Z^d, which is amenable, and the other goes by the name of Bethe lattice, non-amenable. This one is typically used to exemplify strange phenomena in comparison with the Euclidean lattices (and non-amenability is responsible for this), and also because computations for models on it are simplified by the fact that Bethe lattices are trees.

¿From our point of view, these lattices are the graphs of free products of groups of the form $G_1 * \cdots * G_n$, where each G_i is a copy of the infinite cyclic group Z or of the two element group $Z/2Z$. They are finitely generated groups and it is elementary how to compute their first cohomology group. Therefore our theory applies to them.

To conclude, we mention some problems that appear to have some interest. Of course, the theorem of existence of quasi-periodic solutions is only the first step in the very rich Aubry-Mather theory and it wold be interesting to see how much generalizes to the present setting. On the other hand, there are problems that seem to be suggested by the present formulation. The first concerns Gibbs states for the model Λ, H, G. It would be interesting to know what type of ground states are contained in the support of a Gibbs state. For instance, under what conditions is it true that the Dirac mass supported on a Birkhoff ground state is a Gibbs state? Examples related to this question are the Shlosman staircases (see [3]).

In a more geometrical vein, it would be interesting to develop a theory of minimal surfaces in Riemannian manifolds corresponding to the relation between the Frenkel-Kontorova model and geodesics on a torus. For manifolds of dimension three the theory of minimal surfaces admits a simplicial version (Jaco and Rubinstein). Given a compact three manifold M with a triangulation, a simplicial minimal surface is described by how is should intersect the simplices of the triangulation. Passing to the universal cover of M we obtain a space Λ, namely, the set of simplices, as well as an action of the fundamental group of M on Λ. An interaction potential can be written down in terms of the gluing data, so that solutions to the corresponding variational equations are simplicial minimal surfaces.

We also note that, given the interpretation of S as energy and of the rotations numbers as density, it is interesting physically to study the function that, to a rotation number associates the energy per particle of the minimal configuration. This has been studied when $\Lambda = Z^d$ numerically in [9] and rigorously in [11].

We also call attention to the models of quasi-crystals based on aperiodic tilings [10]. These have actions of semi-groups that correspond to dilations. It would be interesting to know whether some version of Aubry-Mather theory could be developed for them.

References

1. S. Aubry and P.Y. Le Daeron, *The discrete Frenkel-Kontorova model and its extensions; I. Exact results for the ground states*, Physica D **8** (1983), pp. 381–422.
2. M.L. Blank, *Metric properties of minimal solutions of discrete periodic variational problems*, Nonlinearity **2** (1989), pp. 1–22.
3. H.-O. Georgii, *Gibbs measures and phase transitions*, de Gruyter Studies in Mathematics 9, Walter de Gruyter, Berlin–New York (1988).
4. C. Golé, *A new proof of the Aubry-Mather's theorem*, Math. Z. **210** (1992), pp. 441–448.
5. M.Kléman and J.F. Sadoc, *A tentative description of the crystallography of amorphous solids*, J. Physique Lett. **40** (1979), pp. 569–574.
6. H. Koch, R. de la Llave and C. Radin, *Aubry-Mather theory for configurations on lattices.* Preprint
7. J. Mather, *Existence of quasiperiodic orbits for twists homeomorphisms of the annulus*, Topology **21** (1982), pp. 457–467.
8. V. Pokrovsky and A. Talapov, *Theory of incommensurate crystals*, Sov. Sci. Reviews, Supplement series 1 (1984).
9. F. Vallet, *Thermodynamique unidimensionelle, et structures bidimensionelles de quelques modeles pour des systemes incommensurables*, Thesis, Univ. Paris VI (1986).
10. M. Senechal, *Quasicrystals and Geometry* Cambrigde University Press, Cambrigde (1995)
11. W. M. Sen, *Phase Locking in Multi-dimensional Frenkel-Kontorova models*, Preprint (1994).

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