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The Riemann–Hilbert problem associated with the quantum Nonlinear Schrödinger equation.

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Abstract

We consider the dynamical correlation functions of the quantum Nonlinear Schrödinger equation. In a previous paper [3] we found that the dynamical correlation functions can be described by the vacuum expectation value of an operator-valued Fredholm determinant. In this paper we show that a Riemann–Hilbert problem can be associated with this Fredholm determinant. This Riemann–Hilbert problem formulation permits us to write down completely integrable equations for the Fredholm determinant and to perform an asymptotic analysis for the correlation function.

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1 Introduction

We consider exactly solvable models of statistical mechanics in one space and one time dimension. The Quantum Inverse Scattering Method and the Algebraic Bethe Ansatz are effective methods for a description of the spectrum of these models. Our aim is the evaluation of correlation functions of exactly solvable models. Our approach is based on the determinant representation for correlation functions. It consists of a few steps: first the correlation function is represented as a vacuum expectation-value of an operator-valued Fredholm determinant, second the Fredholm determinant is described by a classical completely integrable equation, third the classical completely integrable equation is solved by means of the Riemann–Hilbert problem, fourth, the vacuum expectation-value of the asymptotics for the operator-valued Fredholm determinant is calculated. This permits us to evaluate the long distance and large time asymptotics of the correlation function. Our method is described in [5].

The quantum Nonlinear Schrödinger equation can be described in terms of the canonical Bose fields $\psi(x,t),\ \psi^\dagger(x,t),\ (x,t\in \mathbf{R})$ with the standard commutation relations

$$[\psi(x,t),\psi^\dagger(y,t)] = \delta(x-y),\ [\psi(x,t),\psi(y,t)] = [\psi^\dagger(x,t),\psi^\dagger(y,t)] = 0. \tag{1.1}$$

The Hamiltonian of the model is

$$H = \int dx\, \left(\partial_x\psi^\dagger(x)\partial_x\psi(x) + c\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x) - h\psi^\dagger(x)\psi(x)\right). \tag{1.2}$$

Here $0 < c < \infty$ is the coupling constant and $h > 0$ is the chemical potential. The Hamiltonian H acts in the Fock space with the vacuum vector $|0\rangle$. The vacuum vector $|0\rangle$ is characterized by the relation:

$$\psi(x,t)|0\rangle = 0. \tag{1.3}$$

The vacuum vector $\langle 0|$ is characterized by the relation:

$$\langle 0|\psi^\dagger(x,t) = 0,\ \langle 0|0\rangle = 1. \tag{1.4}$$

The spectrum of the model was first described by E. H. Lieb and W. Liniger [7], [6]. The Lax representation for the corresponding classical equation of motion

$$i\frac{\partial}{\partial t}\psi = [\psi, H] = -\frac{\partial^2}{\partial x^2}\psi + 2c\psi^\dagger\psi\psi - h\psi, \tag{1.5}$$

was found by V. E. Zakharov and A. B. Shabat [9]. The Quantum Inverse Scattering Method for the model was formulated by L. D. Faddeev and E. K. Sklyanin [1].

In this paper we shall consider the thermodynamics of the model. The partition function $\mathcal{Z}$ and free energy F are defined by

$$\mathcal{Z} = \operatorname{tr}\left(e^{-\frac{H}{T}}\right) = e^{-\frac{F}{T}}. \tag{1.6}$$

The free energy F can be expressed in terms of the Yang–Yang equation [8]

$$\begin{aligned} \varepsilon(\lambda) &= \lambda^2 - h - \frac{T}{2\pi}\int_{-\infty}^{\infty} \frac{2c}{c^2+(\lambda-\mu)^2} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu, & (1.7)\\ F &= -\frac{T}{2\pi}\int_{-\infty}^{\infty} \ln\left(1+e^{-\frac{\varepsilon(\mu)}{T}}\right) d\mu. & (1.8) \end{aligned}$$

The correlation function, which we shall study in this paper, is defined by

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = \frac{\operatorname{tr}\left(e^{-\frac{H}{T}}\psi(0,0)\psi^\dagger(x,t)\right)}{\operatorname{tr} e^{-\frac{H}{T}}}. \tag{1.9}$$

In a previous paper [3] we obtained the determinant representation for this correlation function. It was shown in the paper [4] that the Fredholm determinant thus obtained can be expressed in terms of solutions of an non-Abelian nonlinear Schrödinger equation. More precisely, the second logarithmic derivatives of the Fredholm determinant with respect to distance and time are densities of the conservation laws of this equation. However, such a description of the determinant is not complete, since the non-Abelian Nonlinear Schrödinger equation has an infinite set of solutions, and it is not clear a priori, which of them describes the correlation function.

On the contrary, the Riemann–Hilbert problem associated with the Fredholm determinant is uniquely solvable. This is the main advantage of the approach based the application of the Riemann–Hilbert problem.

The plan of this paper is the following. In section 2 we review the determinant representation and recall definitions and notations used in [3], [4]. In section 3 we formulate the Riemann–Hilbert problem. We prove the equivalence of the integral equations considered in [4] and the Riemann–Hilbert problem. The Lax representation of the non-Abelian Nonlinear Schrödinger equation is considered in section 4. Section 5 is devoted to the modified formulation of the Riemann–Hilbert problem, which is especially useful for asymptotic analysis.

2 Determinant representation for the correlation function

In this section we summarize the formulations and results obtained in the previous papers [3], [4] for the reader's convenience. Our starting point is the determinant representation for the dynamical correlation function of the local fields obtained in [3]:

$$\langle\psi(0,0)\psi^\dagger(x,t)\rangle_T = -\frac{e^{-iht}}{2\pi}(0|\frac{\det\left(\tilde{I}+\widetilde{V}\right)}{\det\left(\tilde{I}-\frac{1}{2\pi}\widetilde{K}_T\right)}\int_{-\infty}^{\infty}\hat{b}_{12}(u,v)dudv|0). \tag{2.1}$$

In order to describe explicitly the r.h.s. of (2.1) we present here the system of definitions and notations, used in [4].

2.1 Vectors and operators of the Hilbert space

Consider ket-vector $|E^R(\lambda)\rangle$

$$|E^R(\lambda)\rangle = \begin{pmatrix} E_1^R(\lambda|u) \\ E_2^R(\lambda|u) \end{pmatrix}, \tag{2.2}$$

belonging to a Hilbert space $\mathcal{H}$. Here $E_j^R(\lambda|u)$ are two-variable functions. It is convenient to present the space $\mathcal{H}$ as the tensor product of two spaces $\mathcal{H} = \widetilde{\mathcal{H}}\otimes\widehat{\mathcal{H}}$. Each of the discrete components $E_j^R(\lambda|u)$, being a function of the variable λ, belongs to the space $\widetilde{\mathcal{H}}$. In turn, the space $\widehat{\mathcal{H}}$ consists of two-component functions of the variable u.

In order to define scalar products in the space $\widehat{\mathcal{H}}$ we introduce the bra-vector $\langle E^L(\lambda)|$

$$\langle E^L(\lambda)| = \left(E_1^L(\lambda|u), E_2^L(\lambda|u)\right), \tag{2.3}$$

which also can be treated as vector of $\mathcal{H}$. Then the scalar product in $\widehat{\mathcal{H}}$ is given by standard formula

$$\langle E^L(\lambda)|E^R(\mu)\rangle = \int\limits_{-\infty}^{\infty}\left(E_1^L(\lambda|u)E_1^R(\mu|u) + E_2^L(\lambda|u)E_2^R(\mu|u)\right)\,du. \tag{2.4}$$

Below we shall consider two types of operators acting in $\mathcal{H}$. The first type of operators of which $\tilde{I}+\widetilde{V}$ in (2.1) is an example, acts in the space $\widetilde{\mathcal{H}}$ only. By definition

$$\left(\tilde{I}+\widetilde{V}\right)\circ|E^R(\mu)\rangle = \begin{pmatrix} E_1^R(\lambda|u) + \int\limits_{-\infty}^{\infty}\widetilde{V}(\lambda,\mu)E_1^R(\mu|u)d\mu \\ \\ E_2^R(\lambda|u) + \int\limits_{-\infty}^{\infty}\widetilde{V}(\lambda,\mu)E_2^R(\mu|u)d\mu \end{pmatrix}, \tag{2.5}$$

$$|E^R(\lambda)\rangle\circ\left(\tilde{I}+\widetilde{V}\right) = \begin{pmatrix} E_1^R(\mu|u) + \int\limits_{-\infty}^{\infty}E_1^R(\lambda|u)\widetilde{V}(\lambda,\mu)d\lambda \\ \\ E_2^R(\mu|u) + \int\limits_{-\infty}^{\infty}E_2^R(\lambda|u)\widetilde{V}(\lambda,\mu)d\lambda \end{pmatrix}. \tag{2.6}$$

Here $\tilde{I}$ is the identity operator in $\widetilde{\mathcal{H}}$. The action on the bra-vectors $\langle E^L(\lambda)|$ is quite similar. Below we shall mark the operators of this type by "tilde" and denote their action by the sign "$\circ$".

The second type of operator acts in the space $\widehat{\mathcal{H}}$. It is convenient to treat these operators as 2×2 matrices with operator-valued entries:

$$\hat{A} = \begin{pmatrix} \hat{A}_{11}(u,v) & \hat{A}_{12}(u,v) \\ \hat{A}_{21}(u,v) & \hat{A}_{22}(u,v) \end{pmatrix}. \tag{2.7}$$

The action of the operators (2.7) is given by

$$\hat{A}\cdot|E^R(\lambda)\rangle = \begin{pmatrix} \displaystyle\sum_{k=1}^{2}\int\limits_{-\infty}^{\infty} \hat{A}_{1k}(u,v)E_k^R(\lambda|v)\,dv \\ \displaystyle\sum_{k=1}^{2}\int\limits_{-\infty}^{\infty} \hat{A}_{2k}(u,v)E_k^R(\lambda|v)\,dv \end{pmatrix}. \tag{2.8}$$

$$\langle E^L(\lambda)|\cdot\hat{A} = \left(\sum_{j=1}^{2}\int\limits_{-\infty}^{\infty} E_j^L(\lambda|u)\hat{A}_{j1}(u,v)\,du; \sum_{j=1}^{2}\int\limits_{-\infty}^{\infty} E_j^L(\lambda|u)\hat{A}_{j2}(u,v)\,du\right) \tag{2.9}$$

This type of operator will be marked by "hat".

We would like to draw to the attention of the reader that we use "hat" not only for the operator-valued matrices of type (2.7), but for their matrix elements also. We hope, that this system of notation will not cause misunderstanding.

Finally, we define the trace of "hat"-operators as

$$\mathrm{Tr}\,\hat{A} = \mathrm{tr}\,\hat{A}_{11} + \mathrm{tr}\,\hat{A}_{22} = \int\limits_{-\infty}^{\infty}\left(\hat{A}_{11}(u,u) + \hat{A}_{22}(u,u)\right)\,du, \tag{2.10}$$

where

$$\mathrm{tr}\,\hat{A}_{jk} = \int\limits_{-\infty}^{\infty}\hat{A}_{jk}(u,u)\,du. \tag{2.11}$$

2.2 Dual fields

The important objects, on which the r.h.s. of (2.1) depends, are auxiliary quantum operators — dual fields, acting in an auxiliary Fock space. They are introduced in order to remove two-body scattering and to reduce the model to free fermionic. One can find the detailed definition and properties of dual fields in Section 5 and Appendix C of [3]. Here we repeat them in brief.

Consider an auxiliary Fock space having vacuum vector $|0)$ and dual vector $(0|$. The three dual fields $\psi(\lambda)$, $\phi_{D_1}(\lambda)$ and $\phi_{A_2}(\lambda)$ acting in this space are defined as

$$\begin{aligned}
\phi_{A_2}(\lambda) &= q_{A_2}(\lambda) + p_{D_2}(\lambda),\\
\phi_{D_1}(\lambda) &= q_{D_1}(\lambda) + p_{A_1}(\lambda),\\
\psi(\lambda) &= q_{\psi}(\lambda) + p_{\psi}(\lambda).
\end{aligned} \tag{2.12}$$

Here $p(\lambda)$ are annihilation parts of dual fields: $p(\lambda)|0) = 0$; $q(\lambda)$ are creation parts of dual fields: $(0|q(\lambda) = 0$. Thus, any dual field is the sum of annihilation and creation parts (the dual field $\psi(\lambda)$ should not be confused with the field $\psi(x,t)$, which appears in the expression for the Hamiltonian of the model).

The only nonzero commutation relations are

$$\begin{aligned}
&[p_{A_1}(\lambda), q_{\psi}(\mu)] = [p_{\psi}(\lambda), q_{A_2}(\mu)] = \ln h(\mu,\lambda),\\
&[p_{D_2}(\lambda), q_{\psi}(\mu)] = [p_{\psi}(\lambda), q_{D_1}(\mu)] = \ln h(\lambda,\mu),\\
&[p_{\psi}(\lambda), q_{\psi}(\mu)] = \ln[h(\lambda,\mu)h(\mu,\lambda)], \quad \text{where} \quad h(\lambda,\mu) = \frac{\lambda-\mu+ic}{ic}.
\end{aligned} \tag{2.13}$$

Recall that c is the coupling constant in (1.2). It follows immediately from (2.13) that the dual fields belong to an Abelian sub-algebra

$$[\psi(\lambda), \psi(\mu)] = [\psi(\lambda), \phi_a(\mu)] = [\phi_b(\lambda), \phi_a(\mu)] = 0, \tag{2.14}$$

where $a, b = A_2, D_1$. The properties (2.13), (2.14), in fact, permit us to treat the dual fields as complex functions, which are holomorphic in some neighborhood of the real axis.

2.3 Fredholm determinant

Now we are ready to describe the determinant representation (2.1).

The important factor entering the r.h.s. of (2.1) is the Fredholm determinant of the integral operator $\tilde{I} + \widetilde{V}$, acting on the real axis as

$$\left(\tilde{I} + \widetilde{V}\right) \circ f(\mu) = f(\lambda) + \int_{-\infty}^{\infty} \widetilde{V}(\lambda,\mu) f(\mu)\, d\mu, \tag{2.15}$$

where $f(\lambda)$ is some trial function. Recall that $\tilde{I}$ is the identity operator in the space $\widetilde{\mathcal{H}}$.

The kernel $\widetilde{V}(\lambda,\mu)$ can be presented in the form

$$\widetilde{V}(\lambda,\mu) = \frac{\langle E^L(\lambda)|E^R(\mu)\rangle}{\lambda-\mu}, \tag{2.16}$$

where the components of vectors $\langle E^L(\lambda)|$ and $|E^R(\mu)\rangle$ are equal to

$$E_1^R(\lambda|u) = E_2^L(\lambda|u) = E_+(\lambda|u), \tag{2.17}$$

$$E_2^R(\lambda|u) = -E_1^L(\lambda|u) = E_-(\lambda|u). \tag{2.18}$$

Due to this specification we have

$$\langle E^L(\lambda)|E^R(\lambda)\rangle = 0, \tag{2.19}$$

hence the kernel $\widetilde{V}(\lambda,\mu)$ possesses no singularity at $\lambda = \mu$. Later on, the orthogonality property (2.19) will play an important role.

The functions $E_\pm$ introduced in [4] are equal to:

$$E_+(\lambda|u) = \frac{1}{2\pi}\frac{Z(u,\lambda)}{Z(u,u)}\left(\frac{e^{-\phi_{A_2}(u)}}{u-\lambda+i0} + \frac{e^{-\phi_{D_1}(u)}}{u-\lambda-i0}\right)\sqrt{\vartheta(\lambda)}$$
$$\times e^{\psi(u)+\tau(u)+\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda)-\psi(\lambda)-\tau(\lambda))}, \tag{2.20}$$

$$E_-(\lambda|u) = \frac{1}{2\pi}Z(u,\lambda)e^{\frac{1}{2}(\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda)-\psi(\lambda)-\tau(\lambda))}\sqrt{\vartheta(\lambda)}, \tag{2.21}$$

where the function $Z(\lambda,\mu)$ is defined by

$$Z(\lambda,\mu) = \frac{e^{-\phi_{D_1}(\lambda)}}{h(\mu,\lambda)} + \frac{e^{-\phi_{A_2}(\lambda)}}{h(\lambda,\mu)}. \tag{2.22}$$

Here $\psi(\lambda)$, $\phi_{D_1}(\lambda)$ and $\phi_{A_2}(\lambda)$ are just the dual fields (2.12).

Recall also that the function $\vartheta(\lambda)$ is the Fermi weight

$$\vartheta(\lambda) = \left(1+\exp\left[\frac{\varepsilon(\lambda)}{T}\right]\right)^{-1}. \tag{2.23}$$

This function defines the dependence of the correlation function on temperature and chemical potential. The function $\tau(\lambda)$ is the only function depending on time and distance:

$$\tau(\lambda) = it\lambda^2 - ix\lambda. \tag{2.24}$$

Thus, functions $E_\pm(\lambda|u)$ depend on time t, distance x, temperature T and chemical potential h, but this dependence, as a rule, suppressed in our notations.

In order to define the function $\hat{b}_{12}(u,v)$, we introduce vectors $\langle F^L(\lambda)|$ and $|F^R(\mu)\rangle$, belonging to the space $\mathcal{H}$

$$\langle F^L(\lambda)| = \Big(F_1^L(\lambda|u), F_2^L(\lambda|u)\Big); \qquad |F^R(\mu)\rangle = \begin{pmatrix} F_1^R(\mu|u) \\ \\ F_2^R(\mu|u)\end{pmatrix}, \tag{2.25}$$

as solutions of the integral equations (see [4])

$$\left(\tilde{I}+\widetilde{V}\right)\circ\langle F^L(\mu)|\equiv\langle F^L(\lambda)|+\int\limits_{-\infty}^{\infty}\widetilde{V}(\lambda,\mu)\langle F^L(\mu)|\,d\mu=\langle E^L(\lambda)|, \tag{2.26}$$

$$|F^R(\lambda)\rangle\circ\left(\tilde{I}+\widetilde{V}\right)\equiv|F^R(\mu)\rangle+\int\limits_{-\infty}^{\infty}|F^R(\lambda)\rangle\widetilde{V}(\lambda,\mu)\,d\lambda=|E^R(\mu)\rangle. \tag{2.27}$$

Let us define operator $\widehat{B}$ as

$$\widehat{B}=\int\limits_{-\infty}^{\infty}|F^R(\lambda)\rangle\langle E^L(\lambda)|\,d\lambda=\int\limits_{-\infty}^{\infty}|E^R(\mu)\rangle\langle F^L(\mu)|\,d\mu. \tag{2.28}$$

The components of this operator are

$$B_{jk}(u,v)=\int\limits_{-\infty}^{\infty}F_j^R(\lambda|u)E_k^L(\lambda|v)\,d\lambda,\qquad j,k=1,2. \tag{2.29}$$

The function $\hat{b}_{12}(u,v)$ is the matrix element of the operator $\hat{b}$ defined by

$$\hat{b}=\widehat{B}+\hat{g}. \tag{2.30}$$

Here $\hat{g}$ is

$$\hat{g}=\begin{pmatrix}0&-1\\0&0\end{pmatrix}\hat{g}_{12}(u,v),\qquad \hat{g}_{12}(u,v)=\delta(u-v)e^{\psi(v)+\tau(v)}. \tag{2.31}$$

Due to the above definition we have

$$\hat{b}_{ab}(u,v)=\widehat{B}_{ab}(u,v)\qquad\text{for all } a,b \text{ except } a=1,b=2, \tag{2.32}$$

$$\hat{b}_{12}(u,v)=\widehat{B}_{12}(u,v)-\hat{g}_{12}(u,v) \tag{2.33}$$

The importance of operators $\hat{b}$ and $\hat{g}$ will be clear later, after the formulation of the Riemann–Hilbert problem.

The last factor in the representation of the correlation function, which is not explained yet, is the determinant of the operator $\tilde{I}-\frac{1}{2\pi}\widetilde{K}_T$. This operator also acts on the whole real axis. Its kernel is given by

$$\widetilde{K}_T(\lambda,\mu)=\left(\frac{2c}{c^2+(\lambda-\mu)^2}\right)\sqrt{\vartheta(\lambda)}\sqrt{\vartheta(\mu)}. \tag{2.34}$$

This determinant does not depend on time and distance, therefore it can be considered as a constant factor.

Thus, we have described the r.h.s. of (2.1). The temperature correlation function of local fields is proportional to the vacuum expectation value in the auxiliary Fock space of the Fredholm determinant of the integral operator. The auxiliary quantum operators—dual fields—enter the kernels $\widetilde{V}$ and $\hat{b}_{12}(u,v)$. However, due to the property (2.14) the Fredholm determinant is well defined.

3 Riemann–Hilbert problem

Consider the Riemann–Hilbert problem for the operator $\hat{\chi}(\lambda)$ acting in the space $\widehat{\mathcal{H}}$ and depending on the complex parameter λ:

$$\hat{\chi}(\lambda|u,v) = \begin{pmatrix} \hat{\chi}_{11}(\lambda|u,v) & \hat{\chi}_{12}(\lambda|u,v) \\ \hat{\chi}_{21}(\lambda|u,v) & \hat{\chi}_{22}(\lambda|u,v) \end{pmatrix}. \tag{3.1}$$

The matrix $\hat{\chi}(\lambda|u,v)$ is holomorphic for $\lambda \in \mathbf{C} \setminus \mathbf{R}$, and satisfies the normalization condition at $\lambda = \infty$

$$\hat{\chi}(\infty|u,v) = \hat{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(u-v). \tag{3.2}$$

The boundary values on the real axis are related by

$$\hat{\chi}_-(\lambda) = \hat{\chi}_+(\lambda)\hat{G}(\lambda), \quad \lambda \in \mathbf{R}, \tag{3.3}$$

where we set $\hat{\chi}_\pm(\lambda) = \lim_{\epsilon\to+0} \hat{\chi}(\lambda \pm i\epsilon)$. The jump matrix $\hat{G}(\lambda)$ here is given by

$$\hat{G}(\lambda|u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(u-v) + 2\pi i \hat{H}(\lambda|u,v), \tag{3.4}$$

where

$$\hat{H}(\lambda|u,v) = |E^R(\lambda)\rangle\langle E^L(\lambda)| = \begin{pmatrix} E_1^R(\lambda|u)E_1^L(\lambda|v) & E_1^R(\lambda|u)E_2^L(\lambda|v) \\ E_2^R(\lambda|u)E_1^L(\lambda|v) & E_2^R(\lambda|u)E_2^L(\lambda|v) \end{pmatrix}. \tag{3.5}$$

Recall that due to (2.17), (2.18)

$$\begin{aligned} E_1^R(\lambda|u) &= E_+(\lambda|u), \quad E_2^R(\lambda|u) = E_-(\lambda|u), \\ E_1^L(\lambda|v) &= -E_-(\lambda|v), \quad E_2^L(\lambda|v) = E_+(\lambda|v), \end{aligned} \tag{3.6}$$

where the functions $E_\pm(\lambda|u)$ are defined in (2.20) and (2.21). However, we do not use the explicit expressions for $E_\pm$ in this section.

One should understand the r.h.s of the jump condition (3.3) as an operator product in the space $\widehat{\mathcal{H}}$. Another words, more detailed the equality (3.3) means

$$\left(\hat{\chi}_-(\lambda|u,v)\right)_{jk} = \sum_{l=1}^{2} \int_{-\infty}^{\infty} \left(\hat{\chi}_+(\lambda|u,w))\right)_{jl} \left(\hat{G}(\lambda|w,v)\right)_{lk} dw. \tag{3.7}$$

Thus, we are dealing with an infinite dimensional Riemann–Hilbert problem.

The orthogonality property (2.19)

$$\langle E^L(\lambda)|E^R(\lambda)\rangle = 0 \tag{3.8}$$

implies $\mathrm{Tr}\,\hat{H}^n(\lambda) = 0$ for $\quad n \geq 1$. Hence,

$$\det \hat{G}(\lambda) = 1. \tag{3.9}$$

We assume that the Riemann–Hilbert problem is solvable. We note also that in the class of integral operators which we are dealing with, the analyticity of the operator with respect to the parameter λ implies the analyticity of its determinant. Therefore, due to the Liouville theorem and equations (3.2), (3.9), we conclude that $\det \hat{\chi}(\lambda) = 1$. Hence, as in to the usual matrix case the solvability of our Riemann–Hilbert problem implies the uniqueness of the solution. It means also that the inverse matrix $\hat{\chi}^{-1}(\lambda)$ exists and is holomorphic for $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

The integral equations (2.26), (2.27)

$$\left(\tilde{I} + \widetilde{V}\right) \circ \langle F^L(\mu)| = \langle E^L(\lambda)|, \tag{3.10}$$

$$|F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) = |E^R(\mu)\rangle, \tag{3.11}$$

were used in [4] for derivation of the differential equation, describing the Fredholm determinant. Now we prove the equivalence of these integral equations and the Riemann–Hilbert problem (3.2), (3.3). Namely, we establish that the solution of the Riemann–Hilbert problem can be presented in terms of the solutions of integral equations and vice versa.

The following integral representation is the basis of this paper.

Theorem 3.1 *A solution of the Riemann–Hilbert problem (3.3) has the following integral representation*

$$\hat{\chi}(\lambda) = \hat{I} - \int_{-\infty}^{\infty} \frac{|F^R(\mu)\rangle\langle E^L(\mu)|}{\mu - \lambda} d\mu, \tag{3.12}$$

where $|F^R(\mu)\rangle$ *is the solution of the integral equation (3.11):*

$$|F^R(\lambda)\rangle \circ \left(\tilde{I} + \widetilde{V}\right) = |E^R(\mu)\rangle.$$

The inverse of this solution has the following integral representation:

$$\hat{\chi}^{-1}(\lambda) = \hat{I} + \int\limits_{-\infty}^{\infty} \frac{|E^R(\mu)\rangle\langle F^L(\mu)|}{\mu - \lambda}\, d\mu, \tag{3.13}$$

where $\langle F^L(\lambda)|$ *is the solution of integral equation (3.10):*

$$\left(\tilde{I} + \widetilde{V}\right) \circ \langle F^L(\mu)| = \langle E^L(\lambda)|.$$

Proof. The operator $\hat{\chi}(\lambda)$ defined in (3.12) is a holomorphic function everywhere except the real axis and possesses the correct asymptotic behavior when $\lambda \to \infty$. On the other hand, on the real axis we have

$$\begin{aligned}
\hat{\chi}_+(\lambda)\widehat{G}(\lambda) &= \hat{\chi}_+(\lambda) + 2\pi i \Bigg[|E^R(\lambda)\rangle\langle E^L(\lambda)| \\
&- \int\limits_{-\infty}^{\infty} \frac{|F^R(\mu)\rangle\langle E^L(\mu)|E^R(\lambda)\rangle\langle E^L(\lambda)|}{\mu - \lambda - i0}\, d\mu \Bigg] \\
&= \hat{\chi}_+(\lambda) + 2\pi i \left[|E^R(\lambda)\rangle - \int\limits_{-\infty}^{\infty} |F^R(\mu)\rangle \widetilde{V}(\mu, \lambda)\, d\mu \right] \langle E^L(\lambda)| \\
&= \hat{I} - \int\limits_{-\infty}^{\infty} \frac{|F^R(\mu)\rangle\langle E^L(\mu)|}{\mu - \lambda - i0}\, d\mu + 2\pi i |F^R(\lambda)\rangle\langle E^L(\lambda)| \\
&= \hat{I} - \int\limits_{-\infty}^{\infty} \frac{|F^R(\mu)\rangle\langle E^L(\mu)|}{\mu - \lambda + i0}\, d\mu = \hat{\chi}_-(\lambda).
\end{aligned} \tag{3.14}$$

Thus, the operator-valued matrix $\hat{\chi}(\lambda)$ satisfies all the conditions of the Riemann–Hilbert problem (3.2), (3.3).

Just the same method allows one to check representation (3.13). The theorem is proved.

A direct corollary of the representation (3.12) is the following asymptotic expansion of $\hat{\chi}_\pm(\lambda)$ at $\lambda \to \infty$

$$\hat{\chi}_\pm(\lambda) = \hat{I} + \frac{\widehat{B}}{\lambda} + \frac{\widehat{C}}{\lambda^2} + \ldots, \tag{3.15}$$

where

$$\widehat{B} = \int\limits_{-\infty}^{\infty} |F^R(\lambda)\rangle\langle E^L(\lambda)|\, d\lambda, \quad \widehat{C} = \int\limits_{-\infty}^{\infty} \lambda |F^R(\lambda)\rangle\langle E^L(\lambda)|\, d\lambda, \qquad \ldots \tag{3.16}$$

We see that the operator $\widehat{B}$ introduced in the previous section (see (2.28)) appears as the first coefficient in the asymptotic expansion (3.15).

Theorem 3.2 *For arbitrary $\lambda \in \mathbf{C}$*

$$|F^R(\lambda)\rangle = \hat{\chi}(\lambda)|E^R(\lambda)\rangle, \tag{3.17}$$

$$\langle F^L(\lambda)| = \langle E^L(\lambda)|\hat{\chi}^{-1}(\lambda). \tag{3.18}$$

In particular, for $\lambda \in \mathbf{R}$

$$|F^R(\lambda)\rangle = \hat{\chi}_+(\lambda)|E^R(\lambda)\rangle = \hat{\chi}_-(\lambda)|E^R(\lambda)\rangle, \tag{3.19}$$

$$\langle F^L(\lambda)| = \langle E^L(\lambda)|\hat{\chi}_+^{-1}(\lambda) = \langle E^L(\lambda)|\hat{\chi}_-^{-1}(\lambda). \tag{3.20}$$

Proof. Using the definition of vectors $|F^R(\lambda)\rangle$ and $\langle F^L(\lambda)|$ and integral representations for $\hat{\chi}(\lambda)$ and $\hat{\chi}^{-1}(\lambda)$ we have

$$\begin{aligned}\hat{\chi}(\lambda)|E^R(\lambda)\rangle &= |E^R(\lambda)\rangle - \int_{-\infty}^{\infty} \frac{|F^R(\mu)\rangle\langle E^L(\mu)|E^R(\lambda)\rangle}{\mu-\lambda}\, d\mu \\ &= |E^R(\lambda)\rangle - \int_{-\infty}^{\infty} |F^R(\mu)\rangle\widetilde{V}(\mu,\lambda)\, d\mu = |F^R(\lambda)\rangle,\end{aligned} \tag{3.21}$$

$$\begin{aligned}\langle E^L(\lambda)|\hat{\chi}^{-1}(\lambda) &= \langle E^L(\lambda)| + \int_{-\infty}^{\infty} \frac{\langle E^L(\lambda)|E^R(\mu)\rangle\langle F^L(\mu)|}{\mu-\lambda}\, d\mu \\ &= \langle E^L(\lambda)| - \int_{-\infty}^{\infty} \widetilde{V}(\lambda,\mu)\langle F^L(\mu)|\, d\mu = \langle F^L(\lambda)|.\end{aligned} \tag{3.22}$$

Due to the property $\langle E^L(\lambda)|E^R(\lambda)\rangle = 0$, we find for $\lambda \in \mathbf{R}$

$$\begin{aligned}\hat{\chi}_-(\lambda)|E^R(\lambda)\rangle &= \hat{\chi}_+(\lambda)\Big(\hat{I} + 2\pi i|E^R(\lambda)\rangle\langle E^L(\lambda)|\Big)|E^R(\lambda)\rangle \\ &= \hat{\chi}_+(\lambda)|E^R(\lambda)\rangle,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\langle E^L(\lambda)|\hat{\chi}_-^{-1}(\lambda) &= \langle E^L(\lambda)|\Big(\hat{I} - 2\pi i|E^R(\lambda)\rangle\langle E^L(\lambda)|\Big)\hat{\chi}_+^{-1}(\lambda) \\ &= \langle E^L(\lambda)|\hat{\chi}_+^{-1}(\lambda).\end{aligned} \tag{3.24}$$

Here we have used

$$\hat{G}^{-1}(\lambda) = \hat{I} - 2\pi i|E^R(\lambda)\rangle\langle E^L(\lambda)|.$$

The theorem is proved.

The latest theorem allows one to consider the transformation

$$|E^R(\lambda)\rangle \to |F^R(\lambda)\rangle = \hat{\chi}(\lambda)|E^R(\lambda)\rangle \tag{3.25}$$

as a gauge transformation, which is analytic in the complex plane $\lambda \in \mathbf{C}$.

Thus, we have proved the equivalence of the integral equations (3.10), (3.11) and the Riemann–Hilbert problem (3.2), (3.3). It is interesting to mention that integral equations allows us to consider the transformation $|F^R(\lambda)\rangle \to |E^R(\lambda)\rangle$ as transformation in the space $\widetilde{\mathcal{H}}$. At the same time, the solution $\hat{\chi}(\lambda)$ of the Riemann–Hilbert problem defines the same transformation in the space $\widehat{\mathcal{H}}$.

4 Lax representation

In the previous paper [4] we had obtained a generalization of the Nonlinear Schrödinger equation, which describes the Fredholm determinant $\det\left(\tilde{I}+\widetilde{V}\right)$

$$\begin{aligned} -i\partial_t\hat{b}_{12}(u,v) &= -\partial_x^2\hat{b}_{12}(u,v) \\ &+2\int\limits_{-\infty}^{\infty} dw_1dw_2\hat{b}_{12}(u,w_1)\hat{b}_{21}(w_1,w_2)\hat{b}_{12}(w_2,v), \end{aligned} \tag{4.1}$$

$$\begin{aligned} i\partial_t\hat{b}_{21}(u,v) &= -\partial_x^2\hat{b}_{21}(u,v) \\ &+2\int\limits_{-\infty}^{\infty} dw_1dw_2\hat{b}_{21}(u,w_1)\hat{b}_{12}(w_1,w_2)\hat{b}_{21}(w_2,v). \end{aligned} \tag{4.2}$$

In this section we derive the equations (4.1), (4.2) and their Lax representation in terms of the Riemann–Hilbert problem (3.3). Namely, we prove that Riemann–Hilbert problem (3.3) implies equations (4.1), (4.2).

In the paper [4] we have obtained the system of linear differential equations for vectors $|E^R(\mu)\rangle$ and $\langle E^L(\lambda)|$. Let us recall this system

$$\partial_x|E^R(\mu)\rangle = \widehat{L}(\mu)|E^R(\mu)\rangle, \qquad \partial_x\langle E^L(\lambda)| = -\langle E^L(\lambda)|\widehat{L}(\lambda), \tag{4.3}$$

$$\partial_t|E^R(\mu)\rangle = \widehat{M}(\mu)|E^R(\mu)\rangle, \qquad \partial_t\langle E^L(\lambda)| = -\langle E^L(\lambda)|\widehat{M}(\lambda), \tag{4.4}$$

where we have set

$$\widehat{L}(\lambda) = \lambda\hat{\sigma} + [\hat{g},\hat{\sigma}], \quad \widehat{M}(\lambda) = -\lambda\widehat{L}(\lambda) + \partial_x\hat{g}. \tag{4.5}$$

Here the operator $\hat{g}$ is defined in (2.31), the operator $\hat{\sigma}$ is given by

$$\hat{\sigma} = \frac{1}{2i}\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)\delta(u-v). \tag{4.6}$$

The proof of the system (4.3), (4.4) is based on the obvious relations

$$\partial_x e^{\tau(\lambda)} = -i\lambda e^{\tau(\lambda)}, \qquad \partial_t e^{\tau(\lambda)} = i\lambda^2 e^{\tau(\lambda)}. \tag{4.7}$$

It is easy to check that the system (4.3), (4.4) is compatible. Indeed, the compatibility condition

$$\partial_t \widehat{L} - \partial_x \widehat{M} + [\widehat{L}, \widehat{M}] = 0$$

implies

$$[\partial_t \hat{g}, \hat{\sigma}] - \partial_x^2 \hat{g} + \Big[[\hat{g}, \hat{\sigma}] \cdot \partial_x \hat{g}\Big] = 0, \tag{4.8}$$

which in turn is equivalent to

$$i\partial_t \hat{g}_{12}(u,v) - \partial_x^2 \hat{g}_{12}(u,v) = 0.$$

The last equality is obviously valid due to (2.31).

Our aim now is to find the system describing the derivatives of vectors $|F^R(\lambda)\rangle$ and $\langle F^L(\lambda)|$ with respect to x and t. The solution of this problem is given by the following theorem.

Theorem 4.1 *The vectors $|F^R(\lambda)\rangle$ and $\langle F^L(\lambda)|$ satisfy the following system of differential equations*

$$\partial_x |F^R(\lambda)\rangle = \widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle, \qquad \partial_x \langle F^L(\lambda)| = -\langle F^L(\lambda)|\widehat{\mathcal{L}}(\lambda), \tag{4.9}$$

$$\partial_t |F^R(\lambda)\rangle = \widehat{\mathcal{M}}(\lambda)|F^R(\lambda)\rangle, \qquad \partial_t \langle F^L(\lambda)| = -\langle F^L(\lambda)|\widehat{\mathcal{M}}(\lambda), \tag{4.10}$$

where we have set

$$\widehat{\mathcal{L}}(\lambda) = \lambda\hat{\sigma} + [\hat{b}, \hat{\sigma}], \tag{4.11}$$

$$\widehat{\mathcal{M}}(\lambda) = -\lambda\widehat{\mathcal{L}}(\lambda) + \partial_x \hat{b}. \tag{4.12}$$

Here the operators $\hat{b}$ and $\hat{\sigma}$ are defined in (2.30) and (4.6).

Proof. It follows immediately from (3.17), (3.18) and (4.3), (4.4) that the derivatives of vector $|F^R(\lambda)\rangle$ with respect to x and t can be presented in the form

$$\partial_x |F^R(\lambda)\rangle = \widehat{\mathcal{L}}(\lambda)|F^R(\lambda)\rangle, \tag{4.13}$$

$$\partial_t |F^R(\lambda)\rangle = \widehat{\mathcal{M}}(\lambda)|F^R(\lambda)\rangle, \tag{4.14}$$

where

$$\widehat{\mathcal{L}} = \partial_x \hat{\chi}(\lambda) \cdot \hat{\chi}^{-1}(\lambda) + \hat{\chi}(\lambda) \cdot \widehat{L}(\lambda) \cdot \hat{\chi}^{-1}(\lambda),$$

$$\widehat{\mathcal{M}} = \partial_t \hat{\chi}(\lambda) \cdot \hat{\chi}^{-1}(\lambda) + \hat{\chi}(\lambda) \cdot \widehat{M}(\lambda) \cdot \hat{\chi}^{-1}(\lambda). \tag{4.15}$$

It was shown in the previous section that the gauge transformation (3.25) is continuous across the real axis. The direct corollary of this property is that matrices $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{M}}$ possess no cuts on the real axis. Let us prove this directly.

Let $\widehat{\mathcal{L}}_\pm(\lambda) = \lim_{\epsilon\to+0} \widehat{\mathcal{L}}(\lambda \pm i\epsilon)$. Then we have for $\mathrm{Im}\,\lambda = 0$

$$\widehat{\mathcal{L}}_\pm(\lambda) = \partial_x\hat{\chi}_\pm(\lambda)\cdot\hat{\chi}_\pm^{-1}(\lambda) + \hat{\chi}_\pm(\lambda)\cdot\widehat{L}(\lambda)\cdot\hat{\chi}_\pm^{-1}(\lambda). \tag{4.16}$$

Using the jump condition (3.3) we have

$$\begin{aligned}
\widehat{\mathcal{L}}_-(\lambda) &= \partial_x\hat{\chi}_-(\lambda)\cdot\hat{\chi}_-^{-1}(\lambda) + \hat{\chi}_-(\lambda)\cdot\widehat{L}(\lambda)\cdot\hat{\chi}_-^{-1}(\lambda)\\
&= \partial_x\hat{\chi}_+(\lambda)\cdot\hat{\chi}_+^{-1}(\lambda) + \hat{\chi}_+(\lambda)\cdot\partial_x\widehat{G}(\lambda)\cdot\widehat{G}^{-1}(\lambda)\cdot\hat{\chi}_+^{-1}(\lambda)\\
&+ \hat{\chi}_+(\lambda)\cdot\widehat{G}(\lambda)\cdot\widehat{L}(\lambda)\cdot\widehat{G}^{-1}(\lambda)\cdot\hat{\chi}_+^{-1}(\lambda).
\end{aligned} \tag{4.17}$$

Via (4.3), (4.4) and due to (3.4), (3.5) we find

$$\partial_x\widehat{G}(\lambda) = [\widehat{L}(\lambda), \widehat{G}(\lambda)].$$

After the substitution of this expression into (4.17) we arrive at

$$\widehat{\mathcal{L}}_-(\lambda) = \partial_x\hat{\chi}_+(\lambda)\cdot\hat{\chi}_+^{-1}(\lambda) + \hat{\chi}_+(\lambda)\cdot\widehat{L}(\lambda)\cdot\hat{\chi}_+^{-1}(\lambda) = \widehat{\mathcal{L}}_+(\lambda). \tag{4.18}$$

Thus, $\widehat{\mathcal{L}}$ is continuous across the real axis. One can perform just the same consideration for matrix $\widehat{\mathcal{M}}$ also.

Due to the analyticity properties of $\hat{\chi}(\lambda)$, $\hat{\chi}^{-1}(\lambda)$ and their normalization conditions, we conclude that matrices $\widehat{\mathcal{L}}(\lambda)$ and $\widehat{\mathcal{M}}(\lambda)$ are holomorphic for $\lambda \in \mathbf{C}$ and possess the asymptotics

$$\widehat{\mathcal{L}}(\lambda) \to \lambda\hat{\sigma} + O(1), \qquad \widehat{\mathcal{M}}(\lambda) \to -\lambda^2\hat{\sigma} + O(\lambda). \tag{4.19}$$

Due to the Liouville theorem we conclude that $\widehat{\mathcal{L}}(\lambda)$ is a linear function of λ, while $\widehat{\mathcal{M}}(\lambda)$ is a quadratic function of λ. Using the asymptotic expansion (3.15) we find

$$\widehat{\mathcal{L}}(\lambda) = \lambda\hat{\sigma} + [\hat{b}, \hat{\sigma}], \tag{4.20}$$

$$\widehat{\mathcal{M}}(\lambda) = -\lambda^2\hat{\sigma} - \lambda[\hat{b}, \hat{\sigma}] + \partial_x\hat{g} + [\hat{\sigma}, \widehat{C}] + [\hat{b}, \hat{\sigma}]\cdot\widehat{B} - \widehat{B}\cdot[\hat{g}, \hat{\sigma}]. \tag{4.21}$$

The formula (4.20) exactly reproduces the expression (4.11). In order to reduce the expression for $\widehat{\mathcal{M}}(\lambda)$ to the formula (4.12), one should take into account that relations (4.15) provide us an infinite set of identities between decomposition coefficients $\widehat{B},\ \widehat{C},\ldots$ and their derivatives with respect to x and t. For example, the first of the relations (4.15) implies

$$\partial_x \hat{\chi}(\lambda) = \widehat{\mathcal{L}}(\lambda)\hat{\chi}(\lambda) - \hat{\chi}(\lambda)\widehat{L}(\lambda). \tag{4.22}$$

One can substitute the asymptotic expansion for $\hat{\chi}(\lambda)$ and explicit expressions for $\widehat{\mathcal{L}}(\lambda)$ and $\widehat{L}(\lambda)$ into the last formula. After this, comparing coefficients at negative powers of λ, we obtain the mentioned above set of identities. In particular, for λ^{-1} we have

$$\partial_x \widehat{B} = [\hat{\sigma}, \widehat{C}] + [\hat{b}, \hat{\sigma}] \cdot \widehat{B} - \widehat{B} \cdot [\hat{g}, \hat{\sigma}], \tag{4.23}$$

Therefore, we arrive at

$$\widehat{\mathcal{M}}(\lambda) = -\lambda^2 \hat{\sigma} - \lambda[\hat{b}, \hat{\sigma}] + \partial_x \hat{b}. \tag{4.24}$$

Thus, we have proved differential equations (4.9), (4.10) for the vector $|F^R(\lambda)\rangle$. The equations for vector $\langle F^L(\lambda)|$ can be found in a quite similar way.

Since a gauge transformation does not disturb the compatibility condition, we obtain for the pair (4.9) and (4.10)

$$\partial_t \widehat{\mathcal{L}} - \partial_x \widehat{\mathcal{M}} + [\widehat{\mathcal{L}}, \widehat{\mathcal{M}}] = 0,$$

what implies in turn

$$[\partial_t \hat{b}, \hat{\sigma}] - \partial_x^2 \hat{b} + \Big[[\hat{b}, \hat{\sigma}] \cdot \partial_x \hat{b}\Big] = 0. \tag{4.25}$$

The matrix operator-valued equation (4.25) is equivalent to the four scalar operator-valued partial differential equations. It is easy to check that the equations for the diagonal part of $\hat{b}$ are valid automatically due to identity (4.23):

$$\begin{aligned} \partial_x \hat{b}_{11}(u,v) &= i \int\limits_{-\infty}^{\infty} \hat{b}_{12}(u,w)\hat{b}_{21}(w,v)\, dw, \\ \partial_x \hat{b}_{22}(u,v) &= -i \int\limits_{-\infty}^{\infty} \hat{b}_{21}(u,w)\hat{b}_{12}(w,v)\, dw \end{aligned} \tag{4.26}$$

These equations, being substituted into the antidiagonal part of (4.25), give the non-Abelian Nonlinear Schrödinger equation (4.1), (4.2). More compactly they can be written in the following form

$$\begin{aligned} -i\partial_t \hat{b}_{12} &= -\partial_x^2 \hat{b}_{12} + 2\hat{b}_{12}\hat{b}_{21}\hat{b}_{12}, \\ i\partial_t \hat{b}_{21} &= -\partial_x^2 \hat{b}_{21} + 2\hat{b}_{21}\hat{b}_{12}\hat{b}_{21}. \end{aligned} \tag{4.27}$$

One should understand here the nonlinear terms in the sense of integral operator products.

Remark. The method of derivation of the Nonlinear Schrödinger equation described in this section is closely connected with the "dressing procedure" proposed in [10].

5 Modification of the Riemann–Hilbert problem

In the previous sections we have demonstrated that integral operator $\tilde{I}+\widetilde{V}$ generates in a natural way the Riemann–Hilbert problem. The latter, in turn, defines the dressing gauge transformation, which permits us to obtain the exactly solvable classical differential equations. As was shown in the paper [4], the Fredholm determinant of the operator $\tilde{I}+\widetilde{V}$ can be described in terms of the solutions of the Riemann–Hilbert problem and differential equations mentioned. Namely, the logarithmic derivatives of the determinant with respect to distance and time are expressed in terms of the operator $\hat{\chi}$ asymptotic expansion coefficients $\widehat{B}$ and $\widehat{C}$. Let us present here the list of logarithmic derivatives (see [4]):

$$\begin{aligned}
&\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) = i \operatorname{tr} \widehat{B}_{11},\\
&\partial_t \log\det\left(\tilde{I}+\widetilde{V}\right) = i \operatorname{tr}(\hat{C}_{22}-\hat{C}_{11}-\widehat{B}_{21}\hat{g}_{12}),\\
&\partial_x\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) = -\operatorname{tr}(\hat{b}_{12}\widehat{B}_{21}),\\
&\partial_t\partial_x \log\det\left(\tilde{I}+\widetilde{V}\right) = i \operatorname{tr}(\partial_x\hat{b}_{12}\cdot\widehat{B}_{21}-\partial_x\widehat{B}_{21}\cdot\hat{b}_{12}).
\end{aligned} \tag{5.1}$$

Thus, the solution $\hat{\chi}$ of the Riemann–Hilbert problem allows to reconstruct the Fredholm determinant up to a constant factor, which does not depend on x and t. Hence, the calculation of the correlation function of local fields is reduced to the solving of the operator-valued Riemann-Hilbert problem.

In the present section we formulate a new Riemann–Hilbert problem, which appears to be more convenient from the point of vie of an asymptotic analysis. The detailed asymptotic investigation of this problem will be given in our forthcoming publication. Here we restrict our selves only to basic formulations.

Consider the following substitution

$$\hat{\chi}(\lambda) = \hat{\chi}^{(m)}(\lambda)\hat{\chi}^{0}(\lambda), \tag{5.2}$$

where the triangular matrix $\chi^0(\lambda|u,v)$ is defined by

$$\hat{\chi}^0(\lambda|u,v) = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\delta(u-v) + \begin{pmatrix} 0 & -1\\ 0 & 0\end{pmatrix}\frac{\hat{g}_{12}(u,v)}{u-\lambda}. \tag{5.3}$$

The expansion of $\hat{\chi}^0$ at $\lambda \to \infty$ is given by

$$\hat{\chi}^0(\lambda) = \hat{I} - \frac{\hat{g}}{\lambda} - \frac{i\partial_x \hat{g}}{\lambda^2} + \cdots. \tag{5.4}$$

Due to (3.15) we find for the asymptotic expansion of $\hat{\chi}^{(m)}$

$$\hat{\chi}^{(m)}(\lambda) = \hat{I} + \frac{\hat{b}}{\lambda} + \frac{\hat{c}}{\lambda^2} + \ldots, \tag{5.5}$$

where the operator-valued matrix $\hat{b}$ was defined in (2.30): $\hat{b} = \widehat{B} + \hat{g}$; and

$$\hat{c} = \hat{C} + \hat{B}\hat{g} + i\partial_x \hat{g}. \tag{5.6}$$

The new operator-valued matrix $\hat{\chi}^{(m)}(\lambda)$ satisfies the modified Riemann–Hilbert problem:

$$\begin{aligned} \hat{\chi}^{(m)}(\lambda) &\to \hat{I}, \qquad \lambda \to \infty, \\ \hat{\chi}^{(m)}_{-}(\lambda) &= \hat{\chi}^{(m)}_{+}(\lambda)\widehat{G}^{(m)}(\lambda), \qquad \lambda \in \mathbf{R}. \end{aligned} \tag{5.7}$$

The modified jump matrix $\widehat{G}^{(m)}$ is equal to

$$\widehat{G}^{(m)}(\lambda) = \hat{\chi}^0_{+}(\lambda)\widehat{G}(\lambda)(\hat{\chi}^0_{-})^{-1}(\lambda), \tag{5.8}$$

and possesses the following entries

$$\widehat{G}^{(m)}_{11}(\lambda|u,v) = \delta(u-v) - \delta(u-\lambda)Z(v,\lambda)\vartheta(\lambda)e^{\phi_{D_1}(\lambda)}, \tag{5.9}$$

$$\widehat{G}^{(m)}_{12}(\lambda|u,v) = -2\pi i(1-\vartheta(\lambda))\delta(u-\lambda)\delta(v-\lambda)e^{\psi(\lambda)+\tau(\lambda)} \tag{5.10}$$

$$\widehat{G}^{(m)}_{21}(\lambda|u,v) = -\frac{i}{2\pi}\vartheta(\lambda)Z(u,\lambda)Z(v,\lambda)e^{\phi_{D_1}(\lambda)+\phi_{A_2}(\lambda)-\psi(\lambda)-\tau(\lambda)}, \tag{5.11}$$

$$\widehat{G}^{(m)}_{22}(\lambda|u,v) = \delta(u-v) - Z(u,\lambda)\delta(v-\lambda)\vartheta(\lambda)e^{\phi_{A_2}(\lambda)}. \tag{5.12}$$

To derive the above matrix elements, we have used the relation

$$Z(u,u) = e^{-\phi_{D_1}(u)} + e^{-\phi_{A_2}(u)}.$$

Remark. One should understand the equations (5.9)–(5.12) in the weak topology sense. Namely,

$$\begin{aligned} H_1(u_1,u_2) = H_2(u_1,u_2), &\qquad (Weak\ topology) \iff \\ \int_{-\infty}^{\infty} f_1(u_1)H_1(u_1,u_2)f_2(u_2)du_1du_2 &= \int_{-\infty}^{\infty} f_1(u_1)H_2(u_1,u_2)f_2(u_2)du_1du_2. \end{aligned}$$

where $f_1(u_1), f_2(u_2)$ are test functions.

The transformation, considered above, is the direct generalization of the approach proposed in [2] for the free fermionic limit of the quantum Nonlinear Schrödinger equation. The main advantage of the modified Riemann–Hilbert problem (5.7) is the simple explicit dependency of the jump matrix $\widehat{G}^{(m)}(\lambda)$ on the variables x and t. This allows us, in particular, to use another method for deriving the differential equations (4.27). Indeed, the jump condition for the matrix

$$\hat{\chi}^{(e)}(\lambda|u,v) = \hat{\chi}^{(m)}(\lambda|u,v)\exp\left(-\frac{1}{2}\tau(\lambda)\sigma_3\right) \tag{5.13}$$

can be written as

$$\hat{\chi}_-^{(e)}(\lambda) = \hat{\chi}_+^{(e)}(\lambda)\widehat{G}^{(e)}(\lambda), \qquad \lambda\in\mathbf{R}, \tag{5.14}$$

where the jump matrix

$$\widehat{G}^{(e)}(\lambda) = \exp\left(-\frac{1}{2}\tau(\lambda)\sigma_3\right)\widehat{G}^{(m)}(\lambda)\exp\left(\frac{1}{2}\tau(\lambda)\sigma_3\right), \tag{5.15}$$

does not depend on x and t. Hence, the logarithmic derivatives

$$\mathcal{F}_x(\lambda) = \left(\partial_x\hat{\chi}^{(e)}(\lambda)\right)(\hat{\chi}^{(e)})^{-1}(\lambda), \tag{5.16}$$

$$\mathcal{F}_t(\lambda) = \left(\partial_t\hat{\chi}^{(e)}(\lambda)\right)(\hat{\chi}^{(e)})^{-1}(\lambda), \tag{5.17}$$

possess no cut on the real axis. Therefore they are holomorphic for $\lambda\in\mathbf{C}$ and have the following asymptotics:

$$\mathcal{F}_x(\lambda)\to\lambda\hat{\sigma}, \qquad \mathcal{F}_t(\lambda)\to-\lambda^2\hat{\sigma}. \tag{5.18}$$

Due to the Liouville theorem we conclude that $\mathcal{F}_x(\lambda)$ and $\mathcal{F}_t(\lambda)$ are linear and quadratic functions of λ respectively. Using the asymptotic expansion (5.5) we arrive at

$$\mathcal{F}_x(\lambda) = \lambda\hat{\sigma}+[\hat{b},\hat{\sigma}], \tag{5.19}$$

$$\mathcal{F}_t(\lambda) = -\lambda^2\hat{\sigma}-\lambda[\hat{b},\hat{\sigma}]+\partial_x\hat{b}. \tag{5.20}$$

Thus, we again have obtained the Lax representation (4.11), (4.12).

Finally, let us rewrite the logarithmic derivatives of the Fredholm determinant in terms on new decomposition coefficients $\hat{b}$ and $\hat{c}$. We have

$$\begin{aligned}
&\partial_x\log\det\left(\tilde{I}+\widetilde{V}\right) = i\,\mathrm{tr}\,\hat{b}_{11},\\
&\partial_x\partial_x\log\det\left(\tilde{I}+\widetilde{V}\right) = -\,\mathrm{tr}(\hat{b}_{12}\hat{b}_{21}),\\
&\partial_t\log\det\left(\tilde{I}+\widetilde{V}\right) = i\,\mathrm{tr}(\hat{c}_{22}-\hat{c}_{11}),\\
&\partial_t\partial_x\log\det\left(\tilde{I}+\widetilde{V}\right) = i\,\mathrm{tr}(\partial_x\hat{b}_{12}\cdot\hat{b}_{21}-\partial_x\hat{b}_{21}\cdot\hat{b}_{12}).
\end{aligned} \tag{5.21}$$

Summary

The main purpose of this paper was the formulation of the Riemann–Hilbert problem associated with the correlation function of the quantum Nonlinear Schrödinger equation. We used the Riemann–Hilbert problem in order to derive the non-Abelian classical Nonlinear Schrödinger equation, which describes the Fredholm determinant. As we have seen, the solution of this equation is completely described by the solution of the Riemann–Hilbert problem. This permits us to reduce the calculation of the Fredholm determinant to the investigation of the Riemann–Hilbert problem. The detailed asymptotic analysis of the latter will be performed in our forthcoming publication.

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References

[1] L. D. Faddeev and E. K. Sklyanin, Quantum-mechanical approach to completely integrable field theory models, *Sov. Phys. Dokl.* **23**, 902-904 (1978)

[2] A. R. Its, A. G. Izergin, V .E. Korepin and G. G. Varzugin, Large time and distance asymptotics of the correlator of the impenetrable bosons at finite temperature, Physica **D54**, 351, (1991).

[3] T. Kojima, V. E. Korepin and N. A. Slavnov, Determinant Representation for Dynamical Correlation Functions of the Quantum Nonlinear Schrödinger Equation, Preprint ITP-SB-96-71 (1996) (accepted to Commun. Math. Phys.)

[4] T. Kojima, V. E. Korepin and N. A. Slavnov, Completely Integrable Equation for the Quantum Correlation Function of Nonlinear Schrödinger Equation, Preprint ITP-SB-96-75 (1996) (accepted to Commun. Math. Phys.)

[5] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge University Press, 1993

[6] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas I, *Phys. Rev.* **130**, 1605-1616 (1963)

[7] E. H. Lieb, Exact analysis of an interacting Bose gas II, *Phys. Rev.* **130**, 1616-1632 (1963)

[8] C. N. Yang and C. P. Yang, Thermodynamics of one dimensional system of bosons with repulsive delta-interaction, *J. Math. Phys.* **10**, 1115-1122 (1969)

[9] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP.* **34**, 62-69 (1978)

[10] V. E. Zakharov and A. B. Shabat, The integration of nonlinear equations of mathematical physics by means of inverse scattering method. *Functs. Analiz i jego Proloj.* **13**, 13-22 (1979)