Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results.

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Abstract

We prove new results on stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators. In particular, our results imply that the absolutely continuous spectrum of free and periodic Schrödinger operators is preserved under all perturbations $V(x)$ satisfying $|V(x)| \leq C(1+x)^{-\alpha}$, $\alpha > \frac{1}{2}$. This result is known to be optimal on the power scale. We derive a new general criterion for stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under slowly decaying perturbations. We also consider more general classes of perturbing potentials which are not necessarily power decaying. In all cases we show that the main term of the asymptotic behavior of the generalized eigenfunctions has WKB form for almost all energies. The proof uses new maximal function and norm estimates and a.e. convergence results for certain multilinear integral operators.

1. Introduction and main results.

In this paper, we study the stability of the absolutely continuous spectrum of one-dimensional Schrödinger operators under perturbations by slowly decaying potentials. Suppose that H_U is a Schrödinger operator defined on $L^2(0,\infty)$ by the differential expression

$$H_U = -\frac{d^2}{dx^2} + U(x)$$

and some self-adjoint boundary condition at zero. We assume that U is some bounded function such that the operator H_U has absolutely continuous spectrum. The presence of the absolutely continuous spectrum has direct consequences for the physical properties of the quantum particle described by the operator H_U (see, e.g. [22], [2]). If we perturb this operator by some decaying potential $V(x)$, the Weyl criteria implies that the essential spectrum of the operators H_U and H_{U+V} coincide. We would like to find conditions on the rate of decay of $V(x)$ which ensure that the absolutely continuous spectrum of the unperturbed operator H_U is also preserved.

This problem has a long history as one of the most natural questions in quantum mechanics, and we briefly recall the main results. It has been known for a long time that if the perturbation $V(x)$ is absolutely integrable, then the absolutely continuous spectrum of the original operator is preserved. Up to a recent time, this result was basically all what was known on the preservation of the absolutely continuous spectrum of Schrödinger operators under decaying perturbations in general situation. Substantially more information is available in the case when $U(x) = 0$. There has been much work on proving the absolute continuity of the spectrum for the Schrödinger operators with potentials of slower decay, but satisfying some additional special assumptions. We mention a result going back to Weidmann [30] that if a potential V may be represented as a sum of a function of bounded variation and an absolutely integrable function, then the spectrum of the operator H_V on $R^+ = (0, \infty)$ is purely absolutely continuous. Many authors developed a scattering theory for long-range potentials satisfying certain conditions on its derivatives. See, for example, [1], [5], [11]. These results hold in any dimension and the proofs use a method of approximating the scattering trajectories by the solutions of the classical Hamilton-Jacobi equation. The weakest conditions on the long-range part of the potential under which the wave operators are known to exist, are given in [11]. For potentials satisfying $|V(x)| \leq C(1+|x|)^{-\frac{1}{2}-\epsilon}$, for instance, one can infer the existence of the wave operators if also $|D^\alpha V(x)| \leq C_1(1+|x|)^{-\frac{3}{2}-\epsilon}$ for every multiindex α with $|\alpha| = 1$.

Another class of results describe spectral behavior of spherically symmetric (i.e. essentially one-dimensional) specific oscillating potentials, the typical example being potentials of the type $V(x) = \frac{\sin x^\alpha}{x^\beta}$ with α, β positive. Such potentials in general do not satisfy conditions on the decay of the derivatives which are needed for the method of the above works to be applicable. We mention papers [3], [4], [10], [19] and [31] in which further references may be found. The spectrum of the operator H_V for such potentials turns out to be absolutely continuous with perhaps some isolated embedded eigenvalues when $\alpha = 1$. These potentials generalize the celebrated Wigner-von Neumann example [29]. Wigner and von Neumann were first to discover an example which shows that already for potentials decaying as a Coulomb potential at infinity, i.e. $V(x) = O(\frac{1}{1+|x|})$, the spectrum need not to be purely absolutely continuous and positive embedded eigenvalues may occur. Moreover, Naboko [21] and later Simon [25] found different constructions which show that for potentials decaying arbitrarily slower than at a Coulomb rate, one can already have very striking spectral phenomena. Namely, for every function $C(x)$ going monotonously to infinity as x goes to infinity at an arbitrarily slow rate one can find a potential $V(x)$ satisfying

$$|V(x)| \leq \frac{C(x)}{1+|x|},$$

such that the corresponding Schrödinger operator H_V has a dense set of eigenvalues in R^+.

A new general class of potentials preserving the absolutely continuous spectrum of the free Schrödinger operator was recently found by one of us in [12]. Namely, if the potential V satisfies $|V(x)| \leq C(1+|x|)^{-\frac{3}{4}-\epsilon}$ with some $\epsilon > 0$, with no additional assumptions, then the whole positive semi-axis $(0, \infty)$ is an essential support of the absolutely continuous part of the spectral measure. We recall that the set E is an essential support of the measure μ if $\mu(X) > 0$ for any $X \subset E$ of positive Lebesgue measure. Of course, as the examples of

Naboko and Simon show, rich embedded singular spectrum may occur; however it is indeed embedded in the sense that there is an underlying absolutely continuous spectrum. One can describe the set where the singular part of the spectral measure might be supported in R^+ rather explicitly in terms of the properties of the Fourier transform of $x^{\frac{1}{4}}V(x)$ [12].

The result of [12] was further improved in [13], where a general criterion was proven, which implies the stability of the absolutely continuous spectrum of the operator H_U under perturbations $V(x)$ satisfying $|V(x)| \le C(1+x)^{-\frac{2}{3}-\epsilon}$ given that certain operator, composed from the generalized eigenfunctions of H_U, satisfies $L^2 - L^2$ estimate. In particular, it was shown that the absolutely continuous spectrum of free and periodic one-dimensional Schrödinger operators was stable under all perturbations by potentials satisfying $|V(x)| \le C(1+x)^{-\frac{2}{3}-\epsilon}$. Later, this result for the case $U = 0$ was also proved by S. Molchanov by a different method [20].

On the other hand, there exists work on random potentials by Kotani and Ushiroya [17] which provides a bound for the best possible result that one can hope to prove. The results of [17] imply that (in the case $U = 0$) there exist potentials $V(x)$ satisfying $V(x) \le C(1+x)^{-\frac{1}{2}}$, which lead to the purely singular spectrum on R^+. We remark that combining methods used in recent works [15] and [16], one can show that the $(1+x)^{-\frac{1}{2}}$ rate of decay is also critical for perturbations of the periodic Schrödinger operators.

In this paper we establish some results on the preservation of absolutely continuous spectrum of optimal nature for power decaying potentials. In particular, we prove

Theorem 1.1. *Suppose that the potential $V(x)$ is such that there exist $\epsilon > 0$ and $p \le 2$ so that $V(x)x^\epsilon \in L^p(R^+)$. Then the whole positive semi-axis $R^+ = (0, \infty)$ is an essential support of the absolutely continuous part of the spectral measure of the operator H_V. Moreover, for a.e. $\lambda \in R^+$, there exist solutions $\phi_\lambda(x)$ and $\overline{\phi}_\lambda(x)$ of generalized eigenfunction equation*

$$-y'' + V(x)y = \lambda y$$

with the asymptotic behavior which is of pure WKB form in the main term:

$$\phi_\lambda(x) = \exp\left(i\sqrt{\lambda}x - \frac{i}{2\sqrt{\lambda}}\int\limits_0^x V(t)\,dt\right)(1 + o(1)).$$

Of course this result also implies the stability of the absolutely continuous spectrum for all $V(x)$ satisfying $|V(x)| \le C(1+x)^{-\frac{1}{2}-\epsilon}$ with some $\epsilon > 0$.

We have a similar result for perturbations of periodic Schrödinger operators. Namely if U is continuous periodic, $S = \cup_{n=0}^{\infty}(a_n, b_n)$ is the band spectrum of the unperturbed operator, and $\theta(x, \lambda)$ are Bloch functions, we have

Theorem 1.2. *If the potential $V(x)$ is as in Theorem 1.1, the set S is an essential support of the absolutely continuous part of the spectral measure of the operator H_{U+V}. For a.e. $\lambda \in S$, there exist solutions $\psi_\lambda(x)$, $\overline{\psi}_\lambda(x)$ of the equation*

$$-y'' + (U(x) + V(x))y = \lambda y \tag{1}$$

with the asymptotic behavior

$$\psi_\lambda(x) = \theta(x,\lambda)\exp\left(\frac{i}{2\Im(\theta\overline{\theta}')}\int\limits_0^x V(t)|\theta^2(t,\lambda)|\,dt\right)(1+o(1)). \tag{2}$$

Both theorems will follow from a certain general criterion. Suppose that H_U is an operator such that all solutions of the equation

$$-y'' + U(x)y = \lambda y \tag{3}$$

are bounded for a.e. $\lambda \in S$, where S is a certain set of positive Lebesgue measure. It is known (see, e.g., [25], [28]) that in this case the set S belongs to an essential support of the absolutely continuous part of the spectral measure. Let us pick a family of solutions $\theta(x,\lambda)$, $\lambda \in S$, of the equation (3), such that $\theta(x,\lambda)$ are uniformly bounded over S and $\theta(x,\lambda)$, $\overline{\theta}(x,\lambda)$ are linearly independent for every $\lambda \in S$. It is easy to see that we can always find such family. We have

Theorem 1.3. *Suppose that the potential $V(x)$ is such that there exist $\epsilon > 0$ and $p \le 2$ so that $x^\epsilon V(x) \in L^p$. Assume that we can choose $\theta(x,\lambda)$ so that an operator*

$$(Kf)(\lambda) = \chi(S)\int\limits_0^\infty \theta(x,\lambda)^2 \exp\left(\frac{i}{\Im(\theta\overline{\theta}')}\int\limits_0^x V(t)|\theta(t,\lambda)|^2\,dt\right) f(x)\,dx \tag{4}$$

satisfies $L_2(R^+,dx) - L^2(S,d\lambda)$ bound on functions f of compact support. Then the absolutely continuous spectrum of H_U supported on the set S is preserved under perturbation by V, i.e. the set S belongs to an essential support of the absolutely continuous part of the spectral measure of operator H_{U+V}. Moreover, for a.e. $\lambda \in S$, there exist solutions $\psi_\lambda(x)$, $\overline{\psi}_\lambda(x)$ of the equation (1) with the asymptotic behavior (2).

Remarks. 1. We notice that the assumption of the boundedness of all solutions at a.e. energies corresponding to the essential support of absolutely continuous spectrum is rather natural. Almost all known examples of one-dimensional Schrödinger operators with the absolutely continuous spectrum satisfy this assumption. Only recently there appeared rigorous counterexamples to the conjecture that this is true in general (see [18]), but the corresponding potentials are of rather special form and in particular are not bounded from below.

2. All three main theorems that we prove have natural analogs for the whole axis problems. We will not focus on this aspect. All proofs may be generalized to the whole axis case in a straightforward manner following [13].

3. While we seem to come quite close, we cannot yet treat potentials under L^p, $1 < p \le 2$ assumption only.

The main new technique we develop in this paper involves norm bounds and a.e. convergence results for multilinear integral operators of certain type, which may be of interest on its own. The plan of the paper is as follows. In Section 2 we discuss the basic scheme of asymptotic integration. In Section 3 we formulate first key results on the estimates for maximal functions of certain integral operators. In Section 4 we establish norm estimates for multilinear integral operators, and in Section 5 prove corresponding a.e. convergence results. Section 6 contains the conclusion of the proof of all main results. In the Appendix we discuss generalization of our results to the case of potentials which may have strong local singularities.

Independently, results related to ours were obtained by Christian Remling [24] by a very different method, based in part on the ideas from [13] and [20]. Some of the results we prove here (along with some of the results of [24]) were announced in [7]. The announcement [7] also contains a list of open problems which we find most interesting.

2. Asymptotic integration and bounded eigenfunctions.

To prove the stability of the absolutely continuous spectrum of the operator H_U, we will use the following Lemma:

Lemma 2.1. *Let the potential $W(x)$ be locally integrable and satisfy*

$$\sup_x \int\limits_{|x-t|\leq 1} |W_-(t)|\, dt < \infty$$

(W_- denotes the negative part of the potential W). Suppose that for every energy λ from the set E of positive Lebesgue measure, all solutions of the equation

$$-\frac{d^2}{dx^2}y(x) + W(x)y(x) = \lambda y(x)$$

are bounded. Then the set E belongs to an essential support of the absolutely continuous part of the spectral measure of the operator H_W.

The proof of this Lemma may be found in [28] (see also [26]). We notice that in different formulations, the fact that bounded solutions imply absolutely continuous spectrum was known for a long time (see, e.g., [6]). Lemma 2.1 is the most convenient statement for our purpose.

The general plan of our proof is similar to [13] and may be described as follows. By assumption, we know that for every $\lambda \in S$, all solutions of the generalized eigenfunction equation (3) for the unperturbed operator are bounded. Examples with imbedded eigenvalues [21], [25], [16] show that if $V(x)$ is not short-range, we cannot hope in general that for every $\lambda \in S$ we still have only bounded solutions for a perturbed equation; there may exist rather rich, dense in S set for which we will have decaying (L^2) and therefore also growing solutions. Our goal will be to show that nevertheless for a.e. $\lambda \in S$, we still have only bounded solutions. This will ensure that the absolutely continuous spectrum is preserved, although embedded singular spectrum may occur.

Thus, our goal is to study the solutions of the equation

$$-y'' + (U(x) + V(x))y = \lambda y.$$

We rewrite this equation as a system

$$y_1' = \begin{pmatrix} 0 & 1 \\ U+V-\lambda & 0 \end{pmatrix} y_1,$$

where y_1 is now a vector $\begin{pmatrix} y \\ y' \end{pmatrix}$. Let us apply a variation of the parameters transformation with solutions of the unperturbed equation

$$y_1 = \begin{pmatrix} \theta(x,\lambda) & \overline{\theta}(x,\lambda) \\ \theta'(x,\lambda) & \overline{\theta}'(x,\lambda) \end{pmatrix} y_2,$$

to bring the equation to a more symmetric form

$$y_2' = \frac{i}{2\Im(\theta\overline{\theta}')} \begin{pmatrix} V(x)|\theta(x,\lambda)|^2 & V(x)\overline{\theta}(x,\lambda)^2 \\ -V(x)\theta(x,\lambda)^2 & -V(x)|\theta(x,\lambda)|^2 \end{pmatrix} y_2. \tag{5}$$

Notice that $2i\Im(\theta,\overline{\theta'})$ is a Wronskian of two solutions θ and $\overline{\theta}$ and hence is independent of x. Let us introduce the notation

$$p(x,\lambda) = \frac{1}{2\Im(\theta\overline{\theta'})} \int_0^x V(y)|\theta(y,\lambda)|^2\, dy.$$

It will be convenient to apply to (5) an additional transformation

$$y_2 = \begin{pmatrix} \exp(ip(x,\lambda)) & 0 \\ 0 & \exp(-ip(x,\lambda)) \end{pmatrix} y_3.$$

We arrive at the following equation for y_3 :

$$y_3' = \frac{i}{2\Im(\theta\overline{\theta}')} \begin{pmatrix} 0 & V(x)\overline{\theta}(x,\lambda)^2 \exp(2ip(x,\lambda)) \\ -V(x)\theta(x,\lambda)^2 \exp(-2ip(x,\lambda)) & 0 \end{pmatrix} y_3. \tag{6}$$

We follow the idea of "$(I+Q)$" asymptotic integration originating in Harris-Lutz [9]: to find some invertible transformation of equation (6) which would make off-diagonal terms absolutely integrable and then apply Levinson's theorem (see, e.g. [8]) to find the asymptotic behavior of solutions of the resulting equation. If we succeed, we can go back and find the asymptotic behavior of the solutions of the original equation.

Let

$$y_3 = (1-|q|^2)^{-\frac{1}{2}} (I+Q)\, y_4, \tag{7}$$

where I is an identity matrix,

$$Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix},$$

and $q(x,\lambda)$ is some function to be defined. A computation gives for y_4

$$y_4' = (1-|q|^2)^{-1} \left(\frac{i}{2\Im(\theta\overline{\theta}')} \begin{pmatrix} V\Re(\theta^2 q \exp(-2ip)) & V\overline{\theta}^2 \exp(2ip) + V\theta^2 q^2 \exp(-2ip) \\ -V\theta^2 \exp(-2ip) - V\overline{\theta}^2 \overline{q}^2 \exp(2ip) & -V\Re(\theta^2 q \exp(-2ip)) \end{pmatrix} + \right.$$

$$+\begin{pmatrix} \frac{1}{2}(q\overline{q}'-q'\overline{q}) & -q' \\ -\overline{q}' & \frac{1}{2}(q'\overline{q}-q\overline{q}') \end{pmatrix}\Bigg)\Bigg) y_4. \tag{8}$$

We summarize the main result of this section in

Theorem 2.2. *Suppose that for some given λ there exists a function $q(x,\lambda) \in C^1$, $q(x,\lambda) \stackrel{x\to\infty}{\longrightarrow} 0$, such that*

$$q'(x,\lambda)+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\overline{\theta}^2(x,\lambda)\exp(2ip(x,\lambda))+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\theta(x,\lambda)^2\exp(-2ip(x,\lambda))q^2(x,\lambda)\in L^1. \tag{9}$$

Then all solutions of the generalized eigenfunction equation (1) are bounded. Moreover, there are two solutions $\psi(x,\lambda)$, $\overline{\psi}(x,\lambda)$ with the asymptotic behavior

$$\psi(x,\lambda)=\theta(x,\lambda)\exp(ip(x,\lambda))\exp\left(\frac{i}{2\Im(\theta\overline{\theta})}\int\limits_0^x(1-|q|^2)^{-1}V\Re(\theta^2 q\exp(-2ip))\,dt\right)(1+o(1)). \tag{10}$$

Remark. We note that in all applications that we will have the last cumbersome term in the product giving the asymptotic behavior will turn out to be integrable.

Proof. The proof follows immediately from Levinson's theorem (see, e.g. [8]) and equation (8). We may consider this system of equations only for x large enough, so that $|q(t,\lambda)|<1$ for all $t>x$ and the transformation (7) is non-singular. By the assumption of the theorem, the off-diagonal terms are absolutely integrable. The diagonal terms are purely imaginary and hence Levinson's theorem is applicable. Asymptotic behavior of the solutions (10) follows directly from the explicit solution of the equation (8) with diagonal terms omitted and application of transformations we applied to the original system of equations. □

To complete the proof of Theorem 1.3, we need to construct the function $q(x,\lambda)$ verifying (9) and conditions given in Theorem 2.2 for a.e. $\lambda\in S$. The main problem is that if we try to solve equation

$$q'(x,\lambda)+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\overline{\theta}^2(x,\lambda)\exp(2ip(x,\lambda))+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\theta^2(x,\lambda)\exp(-2ip(x,\lambda))q^2(x,\lambda)=0 \tag{11}$$

by iteration, we obtain expressions involving multilinear integral operators of certain type. We need to show that these expressions converge a.e. $\lambda\in S$ in order to ensure $q(x,\lambda)\stackrel{x\to\infty}{\longrightarrow}0$ for a. e. λ, and to make sure that (9) is satisfied after some number of iterations. The first approximation to the solution would be

$$q^{(0)}(x,\lambda)=\frac{i}{2\Im(\theta\overline{\theta})}\int\limits_x^\infty V(t)\overline{\theta}^2(t,\lambda)\exp(2ip(t,\lambda))\,dt.$$

Again, we have to justify this formula by proving that the conditional integral is well-defined a.e. λ. This is relatively simple and has been already done in [13]. In the next section, we formulate the main result from [13] that we will use and make a few comments.

3. A.e. convergence for integral operators.

Let the operator K be defined on the measurable bounded functions f of compact support by

$$(Kf)(\lambda) = \int_0^\infty k(\lambda, x) f(x)\, dx, \tag{12}$$

where $k(\lambda, x)$ is a measurable and bounded function on $I \times R^+$. To study the a.e. convergence of the integral defining $Kf(\lambda)$ on functions from L^p, we study the corresponding maximal function. Denote by $M_K f(\lambda)$ the maximal function

$$M_K f(\lambda) = \sup_N \left| \int_0^N k(\lambda, x) f(x)\, dx \right| . \tag{13}$$

In [13], the following was proved:

Theorem 3.1. *Suppose that an operator K, defined by* (12) *with bounded kernel $k(\lambda, x)$, satisfies the norm estimate $\|Kf\|_2 \le C_2 \|f\|_2$ for all bounded functions of compact support. Then for every $q > 2$ and p such that $q^{-1} + p^{-1} = 1$, we have the following estimate for the maximal function:*

$$\|M_K f\|_q \le C_q \|f\|_p \;\; for \;\; every \;\; f \in L_p \tag{14}$$

As a consequence, the integral

$$\int_0^N k(\lambda, x) f(x)\, dx$$

converges as $N \to \infty$ for almost every value of λ if $f \in L_p$.

For the proof of Theorem 3.1 we refer to [13]; we also sketch in the appendix a proof of a very similar result (Lemma A.3). The proof of Theorem 3.1 may be obtained from that proof by a simple modification.

Finally, we note that it is a general principle for singular integral operators of this type that a couple of L^p-type norm estimates, $L^{p_1} - L^{q_1}$ and $L^{p_2} - L^{q_2}$ valid on functions of compact support imply the estimates of type (14) (but may be in general involving Lorentz spaces) for the corresponding maximal function (and hence a.e. convergence) for all intermediate L^p, $p_1 < p < p_2$. See [14] for more details.

4. Norm estimates for multilinear transforms.

In this section, we study the questions related to the norm estimates for certain multilinear transforms. The results of this and next sections will enable us to fulfill the plan sketched in the end of Section 2 and find the function $q(x, \lambda)$ with the needed properties for a.e. λ by iteration of (11).

Suppose that the functions $k_i(\lambda, x)$, $i = 1, ...n$ are defined on $I \times R^+$, where I is some measurable set in R. We assume that the operators

$$(K_i f)(\lambda) = \int_{R^+} k_i(\lambda, x) f(x)\, dx$$

satisfy the bounds

$$\|K_i f\|_{L^q(I,d\lambda)} \le C_i \|f\|_{L^p(R^+,dx)} \tag{15}$$

on functions of compact support for some $2 > p \ge 1$ and $q > p$.

Let $n \ge 2$. Let A be any set of ordered pairs $\alpha = (i_\alpha, i_{\alpha'})$, with $1 \le i_\alpha, i_{\alpha'} \le n$. Let $|A|$ denote the cardinality of A. By $\chi_E(x)$ we denote a characteristic function which is equal to one when $x \in E$ and is zero otherwise.

Consider the multilinear operator T_n given by

$$T_n(f_1, ..f_n)(\lambda) = \int\limits_{R^n} \prod_{j=1}^{n} f_j(x_j) k_j(x_j, \lambda) \prod_{\alpha \in A} \chi_{R^+}(x_{i_\alpha} - x_{i_{\alpha'}})\, dx, \tag{16}$$

$x = (x_1, ...x_n)$. Notice that if there were no "diagonal" characteristic functions, the expression (16) would decompose into a product of one-dimensional integrals, and the analysis would become trivial.

Remark. We do not rule out the possibility that some of the characteristic functions in (16) are contradictory and the whole expression is zero.

Our goal in this section is to prove the following property:

Theorem 4.1. *Suppose that the multilinear operator T_n is given by (16) with kernels $k_j(\lambda, x_j)$ satisfying (15). Then for any functions $f_i \in L^p(R^+, dx)$, $i = 1, ...n$, such that the integral (16) converges absolutely for a.e. λ, we have*

$$\|T_n(f_1, ...f_n)\|_{s_n} \le C_n \prod_{i=1}^{n} \|f_i\|_p,$$

where $s_n^{-1} = nq^{-1}$. The constant C_n depends only on n and constants in the norm bounds (15) for operators K_i.

Remark. Notice that the conclusion of Theorem 4.1 holds in particular when $s_n < 1$.

By assumption, the value of $T_n(f_1, ...f_n)(\lambda) = g(\lambda)$ is well-defined for a.e. λ by the absolutely convergent integral. Our strategy will be to divide the domain of integration into disjoint pieces and represent the function g as a sum of terms coming from integration over these disjoint pieces, formally:

$$g(\lambda) = \sum_{i=1}^{\infty} g_i(\lambda).$$

Because of the absolute convergence, we have that the sum $\sum_{i=1}^{n} g_i(\lambda)$ converges to $g(\lambda)$ for a.e. λ as $n \to \infty$. We show, choosing the functions g_i in a convenient way, that the sum also converges absolutely in the appropriate space L_{s_n}, thus proving Theorem 4.1.

In the proof of Theorem 4.1, we will need a certain representation of the function

$$f(x_1) f(x_2) \chi_{R^+}(x_2 - x_1)$$

as a sum of products of two functions depending only on x_1 and x_2 respectively. Let us first introduce a decomposition of R^+ associated with the function f. Normalize the function f so that $\|f\|_p^p = 1$. By χ_E we will denote the characteristic function of the set E. Let $E(1,1)$ and $E(1,2)$ be disjoint intervals such that

$$\|f(x)\chi_{E(1,1)}\|_p^p = \|f(x)\chi_{E(1,2)}\|_p^p = 2^{-1},$$

$E(1,1) \cup E(1,2) = R^+$ and $E(1,1)$ lies entirely to the right of $E(1,2)$ (i.e. for any $x \in E(1,1)$, $y \in E(1,2)$ we have $x \le y$). We note that $E(1,2)$ is half-infinite and assume $E(1,1)$ contains its right end for the above decomposition to hold. We also remark that the decomposition is not necessarily unique (f might vanish on some set so that this decomposition will be non-unique), and we just take some decomposition. In future we will omit such inessential details. We continue to decompose each of the intervals $E(1,l)$ in a similar manner, obtaining on the m^{th} step 2^m intervals $\{E(m,l)\}_{l=1}^{2^m}$, such that $\cup_{l=1}^{2^m} E(m,l) = R^+$, $\|f(x)\chi(E(m,l)\|_p^p = 2^{-m}$ for $j = 1, ...2^m$, the intervals are disjoint and $E(m,l)$ lies entirely to the left from $E(m,i)$ if $l < i$. In notation $E(m,l)$, we refer to m as "generation" of this interval and to l as "index". Of importance, in particular, will be the following evident property of intervals $\{E(m,l)\}_{m=1,\, l=1}^{\infty,\, j=2^m}$: any two intervals are either disjoint or one is contained in another.

We proceed to decompose the "diagonal" characteristic functions in a convenient way.

Lemma 4.2. *The following identity holds:*

$$\chi_{R^+}(x_2 - x_1) f(x_1) f(x_2) = \left(\sum_{m=1}^{\infty} \sum_{l=1, l \text{odd}}^{2^m} \chi_{E(m,l)}(x_1) \chi_{E(m,l+1)}(x_2) \right) f(x_1) f(x_2). \tag{17}$$

Proof. Let us denote by H_{12} the set

$$H_{12} = \{x \in R^2, x = (x_1, x_2) | x_1 < x_2\},$$

and by supp$f$ the closure of the set of the points x such that for every interval I, such that $x \in I$, $|f(x)|$ is positive on the set of positive Lebesgue measure in I. The claim will follow if we show that

$$H_{12} \cap (\text{supp}_{x_1} f \times \text{supp}_{x_2} f) = \cup_{m=1}^{\infty} \cup_{l=1, l\text{odd}} (E(m,l) \times E(m,l+1))$$

and the sets under the union on the right hand side are disjoint. The latter fact is easy to see: if $E(m,l) \subset E(s,i)$, l odd, $s \neq m$, then necessarily $m > s$ and $E(m,l+1)$ also belongs to $E(s,i)$, not $E(s,i+1)$. On the other hand, we show that for every $y_2, y_1 \in \text{supp} f$, $y_1 < y_2$, there exist two sets $E(m,l)$, $E(m,l+1)$, with l odd, such that $y_1 \in E(m,l)$ and $y_2 \in E(m,l+1)$. Let $\|f\chi_{(y_1,y_2)}\|_p^p = a > 0$. Here we assume that f is normalized and use the condition that y_1, y_2 lie in supp$f$ to infer that $a > 0$. Choose s so that $2^{-s} \ge a \ge 2^{-s-1}$. If y_1, y_2 lie in one set of generation s, $E(s,l)$, then necessarily $y_1 \in E(s+1, 2l-1)$ and $y_2 \in E(s+1, 2l)$. If y_1 and y_2 lie in different sets of generation m, $E(s,l)$ and $E(s,l+1)$, then either l is odd or $y_1 \in E(s-1, l/2^r)$ and $y_2 \in E(s-1, l/2^r+1)$, where r is such that $l/2^r$ is odd. □

Remark. In particular, if supp$f = R^+$, we get a representation of diagonal characteristic function $\chi_{R^+}(x_2 - x_1)$ as a sum of products of characteristic functions of some intervals in x_1 and x_2 variables.

Proof of Theorem 4.1. Without loss of generality, we assume throughout the proof that $\|f_i\|_p = \frac{1}{n}$ for all $i = 1, ...n$. Let

$$f(x) = \left(\sum_{i=1}^{n} |f_j(x)|^p\right)^{\frac{1}{p}}.$$

Consider the family of the intervals $\{E(m,l)\}$ associated with the function f. An important property of this family is that

$$\|f_i(x)\chi(E(m,l))\|_p^p \leq 2^{-m} \tag{18}$$

for all i, l, m. Write

$$A = \{\alpha_1, ...\alpha_{|A|}\}.$$

We begin by substituting the result of Lemma 4.2 into formula (16):

$$T_n(\overline{f}) = \sum_{m_1}^{\infty} ... \sum_{m_{|A|}=1}^{\infty} \sum_{l_1=1}^{\prime} ... \sum_{l_{|A|}=1}^{\prime} \int_{R^n} dx \prod_{j=1}^{n} k_j(\lambda, x_j) f_j(x_j) \prod_{t=1}^{|A|} \chi_{E(m_t,l_t)}(x_{i_t}) \chi_{E(m_t,l_t+1)}(x_{i'_t}), \tag{19}$$

where $\sum\limits_{l_t}^{\prime}$ means the sum over all odd integers $l_t \in [1, 2^{m_t}]$, and $i_t = i_{\alpha_t}$, $i'_t = i'_{\alpha_t}$. Thus

$$T(\overline{f})(\lambda) \leq \sum_{m_1}^{\infty} ... \sum_{m_{|A|}=1}^{\infty} F^{\overline{m}}(\overline{f})(\lambda), \tag{20}$$

where $\overline{m} = (m_1, ...m_{|A|})$ and

$$F^{\overline{m}}(\overline{f})(\lambda) = \sum_{l_1=1}^{\prime} ... \sum_{l_{|A|}=1}^{\prime} \int_{R^n} dx \prod_{j=1}^{n} |K_j(f_j \chi_{G(j,\overline{l})})(\lambda)|, \tag{21}$$

where $\overline{l} = (l_1, ...l_{|A|})$ (all variables l_t take only odd values), the set $G(j, \overline{l})$ depends on $\overline{m}$ (and on $|A|$), and

$$G(j, \overline{l}) = \left[\bigcap_{t:j=i_t} E(m_t, l_t)\right] \bigcap \left[\bigcap_{t:j=i'_t} E(m_t, l_t + 1)\right].$$

For many values of $\overline{l}$, the set $G(j, \overline{l})$ is empty for some j. Such terms contribute zero to the sum, and this is an observation underlying the estimate that we are going to derive for $F^{\overline{m}}(\overline{f})$. We aim to prove that

$$\|F^{\overline{m}}(\overline{g})\|_{s_n} \leq C_n 2^{-\gamma_n |\overline{m}|} \prod_{j=1}^{n} \|g_j\|_p^{(1-\beta_n)} \tag{22}$$

for any $g_1, ...g_n$ which satisfy (18). Here γ_n is some positive constant which depends only on n, C_n depends on n and constants in norm bounds (15) for the operators K_j, and β_n satisfies

$$1 \geq 1 - \beta_n > \max(\frac{p}{q}, \frac{p}{2}).$$

We may proceed to prove (22) by induction on $|A|$. The case $|A| = 0$ is immediate from hypothesis on K_j by Hölder inequality (in this case $\overline{m} = 0$).

It will be convenient to consider the graph Υ with vertices $\{1, ...n\}$ and edges (i_t, i'_t) joining i_t to i'_t for any t (and no other edges). To each edge we associate the generation m_t which corresponds to a generation in the decomposition (17) of $\chi_{R^+}(x_{i_t} - x_{i'_t})$ that we fixed in the sum (21). It suffices to treat the case where Υ is connected; the general case then follows by Hölder's inequality.

Fix $\overline{m}$. Relabel the indices so that $m_1 \leq ... \leq m_{|A|}$. For simplicity of notation, we also relabel pairs (i_t, i'_t) so that m_t still denotes the generation in the decomposition of $\chi_{R^+}(x_{i_t} - x_{i'_t})$. Let N be the largest index for which $m_N = m_1$. Drop from the sum (21) all terms for which there exists j such that $G(j, \overline{l}) = \emptyset$; such terms contribute 0. We say that an index l *remains* if the corresponding term has not been dropped.

We have the following

Lemma 4.3. *For any $1 \leq j \leq n$, either*

$$G(j, \overline{l}) \subset E(m_1, l_1) \ \ \forall \overline{l} \ \text{ remaining} \tag{23}$$

or

$$G(j, \overline{l}) \subset E(m_1, l_1 + 1) \ \ \forall \overline{l} \ \text{ remaining.} \tag{24}$$

Let $B_1 = \{j$: (23) holds $\}$, $B_2 = \{j$: (24) holds $\}$. Moreover if $2 \leq t \leq N$ (that is, if $m_t = m_1$) then

$$l_t = l_1 \ \ \forall \overline{l} \ \text{ remaining.} \tag{25}$$

Finally for each $t > N$ (so $m_t > m_1$), either i_t, i'_t are both in B_1 for all remaining l or they are both in B_2 for all remaining l. We say that $t \in B_1$, $t \in B_2$ respectively.

Proof. For any $m \geq 1$, l odd, set

$$\tilde{E}(m, l) = E(m, l) \cup E(m, l + 1).$$

First we prove that if $\overline{l}$ remains, then

$$\tilde{E}(m_t, l_t) \subset \tilde{E}(m_1, l_1) \tag{26}$$

for all t. Notice that both sets in (26) also belong to the family $E(m, l)$ (they are $E(m_t - 1, \frac{l_t - 1}{2})$, $E(m_1 - 1, \frac{l_1 - 1}{2})$ respectively; we may assume $E(0, 0) = R^+$). Therefore, to prove (26) it is sufficient to show that the two sets in (26) intersect, since in this case by martingale-type property one is contained in another.

Recall that m_1 is the generation which is fixed in decomposition of the characteristic function $\chi_{R^+}(x_{i_1} - x_{i'_1})$ in the sum (21) for $F^{\overline{m}}$. Pick any other m_t which is fixed in the decomposition of the characteristic function $\chi_{R^+}(x_{i_t} - x_{i'_t})$. Since the graph Υ is connected, we can find a path in Υ which connects either i_t or i'_t with either i_1 or i'_1, and does not contain the edges (i_1, i'_1), (i_t, i'_t). Suppose that this path goes from i_1 to i_t and passes successively through the edges with the corresponding generations $m_{t_1}, ...m_{t_r}$. For $G(j, \bar{l})$ to be non-zero for all j, we must have

$$\tilde{E}(m_1, l_1) \cap \tilde{E}(m_{t_1}, l_{t_1}) \neq \emptyset, \quad \tilde{E}(m_t, l_t) \cap \tilde{E}(m_{t_r}, l_{t_r}) \neq \emptyset, \quad \text{and}$$
$$\tilde{E}(m_{t_i}, l_{t_i}) \cap \tilde{E}(m_{t_{i+1}}, l_{t_{i+1}}) \neq \emptyset \ \text{ for all } \ i = 1, ...r-1. \tag{27}$$

Hence by our assumption that $m_1 \leq m_t$ for all t we see that $\tilde{E}(m_{t_1}, l_{t_1}) \subset \tilde{E}(m_1, l_1)$. But then by (27) also $\tilde{E}(m_1, l_1) \cap \tilde{E}(m_{t_2}, l_{t_2}) \neq \emptyset$, hence $\tilde{E}(m_{t_2}, l_{t_2}) \subset \tilde{E}(m_1, l_1)$. We continue in the same way concluding that $\tilde{E}(m_t, l_t) \subset \tilde{E}(m_1, l_1)$ and hence (26) holds.

The statements (23), (24) and (25) of the lemma now follow immediately from the martingale-type property of the sets $E(m, l)$ and the definition of the set $G(j, \bar{l})$. To prove the final statement, suppose that we know in addition that $m_t > m_1$. We can find a path in Υ which goes from i_t or i'_t to a vertex adjacent to an edge (i_s, i'_s) with the corresponding generation m_s equal to m_1 (i.e. with $s \leq N$), and contains only edges with the corresponding generations strictly less than m_1. An argument analogous to the above shows that in this case $\tilde{E}(m_t, l_t)$ is contained either in $E(m_1, l_s)$ or in $E(m_1, l_s + 1)$ for all remaining $\bar{l}$, depending on whether the vertex to which the path leads coincides with i_s or i'_s respectively. By (25) the lemma is proven. $\square$

Remark. It may happen that there exist two (or more) different paths from i_t (or i'_t) one of which leads to vertex j_1 where $f_{j_1}(x_{j_1})$ is multiplied by $\chi_{E(m_1,l_1)}(x_{j_1})$ while the other leads to vertex j_2 where $f_{j_2}(x_{j_2})$ is multiplied by $\chi_{E(m_1,l_1+1)}(x_{j_2})$. In this case, $G(j, \bar{l})$ is zero and hence $\bar{l}$ is not remaining.

By Lemma 4.3

$$F^{\overline{m}}(\overline{f})(\lambda) = \sum_{l_1}^{\prime} \sum_{l_{N+1}}^{\prime} ... \sum_{l_{|A|}}^{\prime} \prod_{j \in B_1} |K_j(f_j \chi_{E(m_1,l_1)} \chi_{G(j,\bar{l})})(\lambda)| \prod_{j \in B_2} |K_j(f_j \chi_{E(m_1,l_1+1)} \chi_{G(j,\bar{l})})(\lambda)| \leq$$
$$\leq \sum_{l_1}^{\prime} F^{\overline{m}^{(1)}, l_1}_{A_1, B_1}(\overline{f})(\lambda) F^{\overline{m}^{(2)}, l_1}_{A_2, B_2}(\overline{f})(\lambda),$$

where

$$F^{\overline{m}^{(1)}, l_1}_{A_1, B_1}(\overline{f})(\lambda) = \sum_{l_t : t \in A_1}^{\prime} \prod_{j \in B_1} |K_j(f_j \chi_{E(m_1,l_1)} \chi_{G(j,l)})(\lambda)|$$

and $\sum\limits_{l_t : t \in A_1}^{\prime}$ denotes the sum over all l_t such that $1 \leq l_t \leq 2^{m_t}$, l_t is odd, $t \in A_1$, and we write $m^{(1)} = (m_t)_{t \in A_1}$. Note that $F^{\overline{m}^{(1)}, l_1}_{A_1, B_1}(\overline{f})(\lambda)$ depends only on those f_j for which $j \in B_1$. The factor $F^{\overline{m}^{(2)}, l_1}_{A_2, B_2}(\overline{f})(\lambda)$ is defined similarly, but $E(m_1, l_1)$ is replaced by $E(m_1, l_1 + 1)$.

We may rewrite for each $j \in B_1$

$$G(j, l) = E(m_1, l_1) \bigcap G^1(j, (l_t)_{t \in A_1}),$$

where

$$G^1(j,(l_t)_{t\in A_1}) = \left[\bigcap_{t\in A_1, j=i_t} E(m_t,l_t)\right]\bigcap\left[\bigcap_{t\in A_1, j=i'_t} E(m_t,l_t+1)\right].$$

Indeed, by Lemma 4.3 all other sets which enter in the definition of $G(j,l)$ belong to $E(m_1,l_1{+}1)$ and hence are absent for $\overline{l}$ which remain. Thus $F^{\overline{m}^{(1)},l_1}_{A_1,B_1}(\overline{f})(\lambda)$ (and $F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{f})(\lambda)$) are the expressions of the same form as the original $F^{\overline{m}}$. Since $0<|A_1|<|A|$, both $F^{\overline{m}^{(1)},l_1}_{A_1,B_1}(\overline{f})(\lambda)$ and $F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{f})(\lambda)$ may be estimated by induction on $|A|$. Therefore

$$\|F^{\overline{m}^{(1)},l_1}_{A_1,B_1}(\overline{f})(\lambda)\|_{s_{|B_1|}} \le C2^{-\gamma_{|B_1|}|m^{(1)}|}\prod_{j\in B_1}\|f_j\chi_{E(m_1,l_1)}\|_p^{1-\beta_{|B_1|}}; \tag{28}$$

the similar bound also holds for $F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{f})(\lambda)$. Using (28), we are ready to estimate $\|F^{\overline{m}}(\overline{f})\|_{s_n}$. We distinguish between two cases: $s_n<1$ and $s_n>1$. Suppose first that $s_n<1$. Then

$$\|F^{\overline{m}}(\overline{g})\|_{s_n}^{s_n} \le \sum_{l_1}{}' \|F^{\overline{m}^{(1)},l_1}_{A_1,B_1}(\overline{g})(\lambda)\|^{s_n}_{s_{|B_1|}}\|F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{g})(\lambda)\|^{s_n}_{s_{|B_2|}}. \tag{29}$$

We used the fact that $\|\sum h_i(x)\|_s^s \le \sum\|h_i(x)\|_s^s$ when $s<1$ and Hölder's inequality. Plugging the estimate (28) and a similar bound for $F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{g})(\lambda)$ into (29), we find

$$\|F^{\overline{m}}(\overline{g})\|_{s_n}^{s_n} \le C_n 2^{-(\gamma_{|B_1|}|m^{(1)}|+\gamma_{|B_2|}|m^{(2)}|)s_n}\sum_{l_1}{}'\left(\prod_{j\in B_1}\|g_j\chi_{E(m_1,l_1)}\|_p^{(1-\beta_{|B_1|})s_n}\prod_{j\in B_2}\|g_j\chi_{E(m_1,l_1+1)}\|_p^{(1-\beta_{|B_2|})s_n}\right). \tag{30}$$

Pick $0<a_1,a_2<1$ such that

$$a_1(1-\beta_{|B_1|}) = a_2(1-\beta_{|B_2|}) = \frac{p}{q}.$$

We can find such a_1, a_2 by induction assumption. The sum in (30) may be estimated by Hölder's inequality in the following way:

$$\sum_{l_1}{}'\left(\prod_{j\in B_1}\|g_j\chi_{E(m_1,l_1)}\|_p^{(1-\beta_{|B_1|})s_n}\prod_{j\in B_2}\|g_j\chi_{E(m_1,l_1+1)}\|_p^{(1-\beta_{|B_2|})s_n}\right)\le$$

$$\le \prod_{j\in B_1}\max\nolimits_{l_1}\|g_j\chi_{E(m_1,l_1)}\|_p^{(1-a_1)(1-\beta_{|B_1|})s_n}\prod_{j\in B_2}\max\nolimits_{l_1}\|g_j\chi_{E(m_1,l_1+1)}\|_p^{(1-a_2)(1-\beta_{|B_2|})s_n}\prod_{j=1}^n\|g_j\|_p^{\frac{p}{q}s_n}.$$

Thus

$$\|F^{\overline{m}}(\overline{g})\|_{s_n} \le C_n 2^{-\min(\gamma_{|B_1|},\gamma_{|B_2|})(|m^{(1)}|+|m^{(2)}|)}\prod_{j=1}^n \sup\nolimits_l\|g_j\chi_{E(m_1,l)}\|_p^{\min(1-\beta_{|B_1|},1-\beta_{|B_2|})-\frac{p}{q}}\prod_{j=1}^n\|g_j\|_p^{\frac{p}{q}}.$$

Obviously $\|g_j\|_p \ge \sup_l\|g_j\chi_{E(m_1,l)}\|_p$, and $\sup_l\|g_j\chi_{E(m_1,l)}\|_p \le 2^{-m_1}$ by (18). Pick β_n so that

$$\min(1-\beta_{|B_1|},1-\beta_{|B_2|}) > 1-\beta_n > \max(\frac{p}{q},\frac{p}{2}),$$

and

$$\gamma_n = \min((\min(1-\beta_{|B_1|}, 1-\beta_{|B_2|}) - (1-\beta_n)), \gamma_{|B_1|}, \gamma_{|B_2|}).$$

Then

$$\|F^{\overline{m}}(\overline{g})\|_{s_n} \le C_n 2^{-\gamma_n|m|} \prod_{j=1}^{n} \|g_j\|_p^{1-\beta_n}. \tag{31}$$

There is only finite number of pairs B_1, B_2 such that $|B_1|+|B_2| = n$, and hence the constants γ_n, β_n may be chosen universally.

The case $s_n > 1$ is similar. Using triangle inequality and Hölder's inequality, we get

$$\|F^{\overline{m}}(\overline{g})\|_{s_n} \le \sum_{l_1}' \|F^{\overline{m}^{(1)},l_1}_{A_1,B_1}(\overline{g})(\lambda)\|_{s_{|B_1|}} \|F^{\overline{m}^{(2)},l_1}_{A_2,B_2}(\overline{g})(\lambda)\|_{s_{|B_2|}} \le$$

$$\le C_n 2^{-\min(\gamma_{|B_1|},\gamma_{|B_2|})(|m^{(1)}|+|m^{(2)}|)} \sum_{l_1}' \left(\prod_{j\in B_1} \|g_j\chi_{E(m_1,l_1)}\|_p^{(1-\beta_{|B_1|})} \prod_{j\in B_2} \|g_j\chi_{E(m_1,l_1+1)}\|_p^{(1-\beta_{|B_2|})} \right).$$

Provided that

$$\frac{1}{p}(|B_1|(1-\beta_{|B_1|}) + |B_2|(1-\beta_{|B_2|})) > 1, \tag{32}$$

we can apply the same argument as in the case $s_n < 1$ to prove (31). But (32) holds for all $|B_1|$, $|B_2| \ge 1$ since by induction hypothesis $1-\beta_r > \frac{p}{2}$ for all $r < n$. This completes the proof of (22).

Because of inequality (20) and (22) we find

$$\|T_n(\overline{f})\|_{s_n} \le C_n$$

for all f_i such that $\|f_i\|_p \le \frac{1}{n}$ with C_n depending only on n. Hence by multilinearity Theorem 4.1 holds. □

5. A.e. convergence of multilinear transforms

In the proof of the a.e. convergence, an important role will be played by the following operators. Let $D_1(\lambda), ...D_n(\lambda)$ be measurable functions of λ mapping I to $R^+ \cup \{\infty\}$. Let us denote

$$T_n^{D_1(\lambda),...D_n(\lambda)}(f_1, ...f_n)(\lambda)$$

an operator obtained from T_n by replacing the kernels $k_i(\lambda, x_i)$, $i = 1, ...n$ with

$$\tilde{k}_i(\lambda, x_i) = k_i(\lambda, x_i)\chi_{R^+}(D_i(\lambda) - x_i),$$

$i = 2, ...n$. Throughout this section, we will assume that the kernels $k_i(\lambda, x)$ are bounded and the integral operators K_i corresponding to these kernels satisfy $L^2 - L^2$ estimates. The results we prove here directly extend to more general situations, however the above conditions are exactly the case that we will need in applications and it is convenient to restrict our attention to it. We give the following natural definition:

Definition 5.1. *We say that the operator T_n converges on functions $f_1, ...f_n$ for some λ if the expressions*

$$T_n^{D_1,...D_n}(f_1,...f_n)(\lambda)$$

converge to a finite limit as all D_i tend to infinity in any way. Namely, there exists a number, which we denote $T_n^{\infty,...\infty}(f_1,...f_n)(\lambda)$ such that for any $\delta > 0$ there exists N_δ such that for any $D_i \geq N_\delta$, $i = 1, ...n$ (possibly infinite) we have

$$\left| T_n^{\infty,...\infty}(f_1,...f_n)(\lambda) - T_n^{D_1,...D_n}(f_1,...f_n)(\lambda) \right| < \delta.$$

Our first result is the following maximal estimate:

Theorem 5.2. *Let*

$$M_T(f_1,...f_n)(\lambda) = \sup_{D_1,...D_n} |T_n^{D_1,...D_n}(f_1,...f_n)(\lambda)|.$$

Then for any $f_i \in L^p$, $1 \leq p < 2$, we have

$$\|M_T(f_1,...f_n)\|_{s_n} \leq C_n \prod_{i=1}^{n} \|f_i\|_p, \tag{33}$$

where $s_n^{-1} = nq^{-1}$ and $q^{-1} + p^{-1} = 1$.

Proof. The proof uses a well-known trick going back to Kolmogorov and Silvester. Namely, it is sufficient to show that for any measurable $D_i(\lambda)$, $i = 1, ...n$, we have

$$\|T_n^{D_1(\lambda),...D_n(\lambda)}(f_1,...f_n)\|_{s_n} \leq C_n \prod_{i=1}^{n} \|f_i\|_p$$

with the constant C_n independent of $D_i(\lambda)$. But the expression $T_n^{D_1(\lambda),...D_n(\lambda)}(f_1,...f_n)(\lambda)$ is obtained from $T_n(f_1,...f_n)(\lambda)$ by replacing the kernels $k_i(\lambda, x)$ with the kernels

$$\tilde{k}_i(\lambda, x) = k_i(\lambda, x)\chi_{R^+}(D_i(\lambda) - x_i).$$

By Theorem 3.1, these kernels satisfy the estimates (15) for all p, q such that $p < 2$ and $q^{-1} = 1 - p^{-1}$, with constants in the norm bounds that do not depend on $D_i(\lambda)$. Therefore Theorem 4.1 is applicable and directly leads to (33). $\Box$

As one may expect, the maximal estimate (33) implies a.e. convergence.

Theorem 5.3. *The operator $T_n(f_1,...f_n)$ converges a.e. λ on any functions $f_i \in L^p$, $1 \leq p < 2$.*

Proof. Suppose, on the contrary, that for some $\epsilon > 0$ we have a set S_ϵ of positive Lebesgue measure such that for any N there exist $D_{ij} > N$, $i = 1, ...n$, $j = 1, 2$ such that

$$|T_n^{D_{11},...D_{n1}}(f_1,...f_n)(\lambda) - T_n^{D_{12},...D_{n2}}(f_1,...f_n)(\lambda)| > \epsilon$$

for every $\lambda \in S_\epsilon$. Let us denote $f_{i,N}(x_i) = f_i(x_i)\chi_{R^+}(N - x_i)$. For any $D_i > N$, we have

$$|T_n^{D_1,...D_n}(f_1,...f_n)(\lambda) - T_n^{D_1,...D_n}(f_{1,N},...f_{n,N})(\lambda)| \leq \sum_{j=1}^{n} |T_n^{D_1,...D_n}(g_{1j},...g_{nj})(\lambda)|,$$

where $g_{ij} = f_{i,N}$ if $i < j$, $g_{ij} = f_i$ if $i > j$, and $g_{ij} = f_i - f_{i,N}$ if $i = j$. In other words, we expand the difference into a telescopic sum. To estimate each term in the sum, we can apply Theorem 5.2. We get

$$\sup_{D_1,...D_n} \|T_n^{D_1,...D_n}(f_1,...f_n)(\lambda) - T_n^{D_1,...D_n}(f_{1,N},...f_{n,N})(\lambda)\|_{s_n}^{s_n} \leq$$

$$\leq \sum_{j=1}^{n} \|T_n^{D_1,...D_n}(g_{1j},...g_{nj})(\lambda)\|_{s_n}^{s_n} \leq C_n \sum_{j=1}^{n} \left(\prod_{i=1}^{n} \|g_{ij}\|_p\right)^{s_n}.$$

Clearly, the right-hand side goes to zero as $N \to \infty$, since every product contains the norm of $f_i - f_{i,N}$ for some i. On the other hand, the left-hand side is by assumption bounded from below by $(\frac{\epsilon}{2})^{s_n}|S_\epsilon|$ for every N. This gives a contradiction.

A straightforward adjustment of the above argument allows (for a.e. λ) to pass to the infinite limit in any order (i.e. for example first in D_n, then in D_{n-1} and so on). $\Box$

In order to prove a final Lemma that we will need in the iteration process, we need to consider a smaller class of multilinear operators than we did before. These are exactly the operators which appear in the process of iterating equation (11).

Definition 5.4. *We say that the multilinear transform T_n belongs to the class M_n, if*

$$T_n(f_1,...f_n)(\lambda) = \int_0^\infty ... \int_0^\infty dx_1...dx_n \prod_{j=1}^{n} k_j(\lambda, x_j) f_j(x_j) \prod_{j=2}^{n} \chi_{R^+}(x_j - x_{\sigma(j)}), \qquad (34)$$

where $k_j(\lambda, x)$ satisfy (15), and function σ takes values in $1,...n$ and satisfies $\sigma(j) < j$.

The operator T_n depends on the kernels k_j and the choice of function σ. It will be convenient to visualize the structure of the integral (34) defining the operator T_n as follows. To each operator T_n defined by (34) with some σ we associate a graph Γ with n vertices $\{v_j\}_{j=1}^n$. We think of the vertex v_j as of function $f_j(x_j)$. Two vertices v_i, v_j are connected by the directed edge e_{ij} (from i to j) iff the product under integral in (34) contains the characteristic function $\chi_{R^+}(x_j - x_i)$. In this case we say that v_i precedes v_j, $v_i \prec v_j$. It is easy to see that under all conditions on T_n the graph Γ is a connected tree. The directions on the edges (i.e. relation "$\prec$") induce a partial order on the vertices of Γ in which v_1 is the minimal element. We note also that a graph Γ together with the mentioned above partial order and the correspondence between kernels $k_i(\lambda, x_i)$ and vertices determine the operator T_n in a unique way.

Lemma 5.5. *Suppose that functions $f_i(x)$, $i = 1,...n$ satisfy $(1 + x)^{-\epsilon} f_i(x) \in L_p$ for some $\epsilon > 0$ and $p \leq 2$. Then for a.e. λ*

$$|T_n(f_{1,N}, f_2...f_n)(\lambda)| \leq C(\lambda)(1 + N)^{-n\delta}$$

for any $\delta < \epsilon$.

Proof. The proof is an exercise in integration by parts. Fix $\delta < \epsilon$. Let us introduce notation $f_i^\delta(x)$ for a function

$$f_i^\delta(x) = f_i(x)(1+x)^\delta.$$

By assumption, $f^\delta(x) \in L^p$ with $p < 2$. The proof uses induction in n; the claim is that if m arguments in the operator $T_n(g_1, ...g_n)$ are functions f_i (without weight) and the rest are the functions $f_j^\delta(x_j)$, then

$$|T_n(g_{1,N}, g_2, ...g_n)| \leq C(\lambda)(1+N)^{-m\delta}$$

for a.e. λ. For $n = 1$, the induction hypothesis is easily justified by Theorem 3.1 and integration by parts. Suppose the induction statement is true for $l \leq n-1$. It also true for n and $m = 0$ by Theorem 5.3. We will now show that if the induction statement is true also for n and any $m < m_0 \leq n-1$ then it is true for n, m_0. That will conclude the proof. Since $m_0 \geq 1$, find some function g_l which is equal to function f_l without weight. Recall that we have a correspondence between T_n and a tree graph Γ (describing effectively how the integrals nest). Take a path on Γ from vertex v_1 to v_l. Assume that this path passes through the vertices (in this order) $v_1, v_{l(1)}, ...v_{l(s)}, v_l$. We integrate by parts $(s+1)$ times, on the first step in x_1, differentiating the part of the under-integral expression

$$\left(\int_{x_1}^{\infty} dx_{l(1)} ... \right)$$

and integrating the rest from $x_{l(1)}$ to infinity. Such integration by parts is well-defined for a.e. λ by Theorem 5.3; we may regularize all infinite integrals in the original expression by integrating to some number M instead of infinity in all of them and then after integration by parts pass to the limit $M \to \infty$, obtaining for a.e. λ a well-defined equality by Theorem 5.3. We obtain the off-diagonal terms which have the right decay $(C(\lambda)N^{-\delta m_0})$ by induction hypothesis in n (they are products of the operators of order less than n). The graph corresponding to the new integral term is isomorphic to Γ, but the minimal vertex is now $v_{l(1)}$ (new integral term includes $\chi_{R^+}(x_1 - x_{l(1)})$ rather than $\chi_{R^+}(x_{l(1)} - x_1)$). Integration by parts allows to "travel" along the graph Γ. We continue integration by parts for a.e. λ so that in s steps the vertex v_l becomes minimal. All off-diagonal terms that we obtain are well-behaved (a.e. λ) by induction hypothesis in n. The final integral term looks as follows:

$$\int_N^\infty dx_l k_l(\lambda, x_l)(f_l(x_l)(1+x_l)^\delta)(1+x_l)^{-\delta} \left(\int_{x_l}^\infty ... \right) ... \left(\int_{x_l}^\infty ... \right).$$

Integrating by parts one last time in x_l, differentiating $(1+x_l)^{-\delta}$ and integrating the rest, we obtain

$$(1+N)^{-\delta} \int_N^\infty dx_l k_l(\lambda, x_l) f_{l,\delta}(x_l) \left(\int_{x_l}^\infty ... \right) ... \left(\int_{x_l}^\infty ... \right) -$$

$$- \int_N^\infty dx_l (1+x_l)^{-1-\delta} \int dx_l' k_l(\lambda, x_l') f_{l,\delta}(x_l') \left(\int_{x_l'}^\infty ... \right) ... \left(\int_{x_l'}^\infty ... \right).$$

These expressions are estimated from above by $C(\lambda)N^{-\delta m_0}$ by our induction hypothesis in m for n fixed. This completes the proof of the lemma. $\square$

6. Conclusion of the proof of main results.

Given the general results of Sections 4 and 5, we complete the proof of main results.

Proof of Theorem 1.3. We recall that it suffices to show that under the assumptions of the theorem for a.e. $\lambda \in S$ we can find a function $q(x,\lambda)$ such that $q(x,\lambda) \in C^1$, $q(x,\lambda) \to 0$, and the condition (9) holds:

$$q'(x,\lambda)+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\overline{\theta}^2(x,\lambda)\exp(2ip(x,\lambda))+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\theta^2(x,\lambda)\exp(-2ip(x,\lambda))q^2(x,\lambda) \in L^1.$$

By assumption, kernels

$$k_1(\lambda,x) = \overline{\theta}^2(x,\lambda)\exp(2ip(x,\lambda))$$

and $k_2(\lambda,x) = \overline{k_1(\lambda,x)}$ satisfy $L^2 - L^2$ and trivial $L^1 - L^\infty$ estimates and therefore the theory developed in Sections 4 and 5 applies to the multilinear operators T_n from classes M_n composed from the kernels $k_1(\lambda,x)$, $k_2(\lambda,x)$. We will construct the function $q(x,\lambda)$ by iteration. Let

$$q^{(0)}(x,\lambda) = -\frac{i}{2\Im(\theta\overline{\theta})}\int_x^\infty \overline{\theta}^2(t,\lambda)\exp(2ip(t,\lambda))V(t)\,dt,$$

well-defined for a.e. $\lambda \in S$ by Theorem 3.1. Given $q^{(n-1)}(x,\lambda)$, we define

$$q^{(n)}(x,\lambda) = -\frac{i}{2\Im(\theta\overline{\theta})}\int_x^\infty \overline{\theta}^2\exp(2ip)V(t)\,dt - \frac{i}{2\Im(\theta\overline{\theta})}\int_x^\infty \theta^2\exp(-2ip)V(t)(q^{(n-1)})^2\,dt. \quad (35)$$

We need the following

Lemma 6.1. *$q^{(n)}(x,\lambda)$ is equal to a sum of multilinear transforms of classes M_j (composed from the kernels $k_1(\lambda,x)$ and $k_2(\lambda,x)$ and with all arguments equal to $V(x)$) and is defined a.e. $\lambda \in S$. Moreover,*

$$q^{(n)}(x,\lambda) - q^{(n-1)}(x,\lambda) = \sum_i T_{j_i,x}^{\infty,\dots\infty}(V,\dots V),$$

where sum is finite and the orders j_i satisfy $2n-1 \le j_i \le 2^{n+1}-1$.

Proof. We use induction. For $n=0$, all statements are obvious (assume $q^{(-1)}(x,\lambda)=0$). Suppose they are also true for $m \le n-1$. The fact that $q^{(n)}(x,\lambda)$ is a sum of multilinear transforms of classes M_l with some l follows immediately from the induction hypothesis and formula (35). The fact that $q^{(n)}$ is well-defined for a.e. $\lambda \in S$ is then a consequence of Theorem 5.3. Note that

$$q^{(n)}(x,\lambda) - q^{(n-1)}(x,\lambda) =$$

$$= -\frac{i}{2\Im(\theta\overline{\theta})}\int_0^\infty \theta^2\exp(-2ip)V(t)(q^{(n-1)}(t,\lambda)-q^{(n-2)}(t,\lambda))(q^{(n-1)}(t,\lambda)+q^{(n-2)}(t,\lambda))\,dt.$$

By induction assumption, every term on the right-hand side is a multilinear transform from class M_j of order j no less then $(2n-3)+2 = 2n-1$ and no higher than $(2^n-1)2+1 = 2^{n+1}-1$. $\square$

Now we can complete the proof of Theorem 1.3. If $x^\epsilon V(x) \in L^p$, $p \leq 2$, Lemma 5.5 implies that for any multilinear operator of order l_i

$$\left|T^{\infty,\ldots\infty}_{l_i,x}(V,...V)(\lambda)\right| \leq C(\lambda)(1+x)^{-\delta l_i}$$

for a.e. $\lambda \in S$ and any $\delta < \epsilon$. Notice that

$$(q^{(n)})'(x,\lambda)+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\overline{\theta}^2(x,\lambda)\exp(2ip(x,\lambda))+\frac{i}{2\Im(\theta\overline{\theta})}V(x)\theta^2(x,\lambda)\exp(-2ip(x,\lambda))(q^{(n)})^2(x,\lambda) =$$

$$= \frac{i}{2\Im(\theta\overline{\theta})}V(x)\theta^2(x,\lambda)\exp(-2ip(x,\lambda))\left((q^{(n)})^2(x,\lambda)-(q^{(n-1)})^2(x,\lambda)\right).$$

Pick n so that $(2n-1)\delta > \frac{1}{2}$. Then by Lemma 6.1 we find that the expression on the right hand side is a.e. λ absolutely integrable, since it is a.e. λ a product of some bounded function, L^p function $V(x)$ and L^q function $(1+x)^{-\frac{1}{2}-\eta}$, $\eta > 0$ (take $q^{-1} = 1-p^{-1}$).

Hence $q^{(n)}(x,\lambda)$ satisfies the condition (9) and therefore we can take $q(x,\lambda) = q^{(n)}(x,\lambda)$ for a.e. $\lambda \in S$. Then the first claim of Theorem 1.3 follows from Theorem 2.2 and Lemma 2.1. As a set of full measure in S it suffices to take the set where all multilinear transforms $T_l(V,...V)$ (composed from $k_1(\lambda,x)$, $k_2(\lambda,x)$) of order not lager than $2^{n+1}-1$ converge. It remains to prove the formula (2) for the asymptotic behavior of the eigenfunctions. Notice that the asymptotic behavior stated in Theorem 2.2 differs from the asymptotic behavior we need to show to prove Theorem 1.3 only by the presence of an additional multiplier

$$\exp\left(\frac{i}{2\Im(\theta\overline{\theta})}\int_0^x (1-|q|^2)^{-1}V\Re(\theta^2 q\exp(-2ip))\,dt\right) \tag{36}$$

in the asymptotic formula for the solutions in Theorem 2.2. But note that the limit of the integral

$$\int_0^N (1-|q(t,\lambda)|^2)^{-1}V(t)\Re(\theta^2(t,\lambda)q(t,\lambda)\exp(-2ip(t,\lambda)))\,dt$$

as $N \to \infty$ exists for a.e. λ. Indeed, we can expand $(1-|q|^2)^{-1}$ into absolutely convergent series in $|q^2|$. Then the whole expression becomes represented as a sum of multilinear transforms $T^{N,\infty,\ldots\infty}_{l,x}(V,...V)(\lambda)$. Starting from some l, the under-integral expression will become absolutely integrable over the whole axis for a.e. λ by Lemma 5.5; in the remaining finite sum every term is convergent by Theorem 5.3. Therefore, for a.e. λ, the expression (36) can be written as $C(\lambda)(1+o(1))$ and hence can be omitted in the asymptotic expression for the solutions. This completes the proof. $\square$

Proof of Theorems 1.1 and 1.2. It remains only to verify the L^2-L^2 bounds for the corresponding operator (4). It is convenient to choose $\theta(x,\lambda) = \exp(i\sqrt{\lambda}x)$ in the free case and $\theta(x,\lambda)$ Bloch functions in the periodic case. The corresponding $L^2 - L^2$ bounds were already shown in [13]. For the sake of completeness and since the argument is not very long, we provide here a sketch of the proof for the free case. For the periodic case, the proof is analogous given standard information on the properties of the Bloch functions, see [13] for details.

Without loss of generality, we can restrict our attention to some interval (a,b), $0 < a < b < \infty$. Consider $\phi(\lambda) \in C_0^\infty(0,\infty)$, $1 \geq \phi(\lambda) \geq 0$ such that $\phi(\lambda) = 1$ when $\lambda \in (a,b)$. Clearly it suffices to show $L^2 - L^2$ bound on functions of compact support for an operator

$$Kf(\lambda) = \phi(\lambda) \int\limits_0^\infty \exp\left(2i\sqrt{\lambda}x - \frac{i}{\sqrt{\lambda}} \int\limits_0^x V(t)\,dt\right) f(x)\,dx.$$

We have

$$\|Kf\|^2_{L^2(a,b)} = \int\limits_0^\infty \int\limits_0^\infty dxdyf(x)f(y) \int d\lambda\phi(\lambda) \exp\left(2i\sqrt{\lambda}(x-y) - \frac{i}{\sqrt{\lambda}} \int\limits_y^x V(t)\,dt\right).$$

Let us denote by $Z(x,y)$ the kernel

$$Z(x,y) = \int d\lambda\phi(\lambda) \exp\left(2i\sqrt{\lambda}(x-y) - \frac{i}{\sqrt{\lambda}} \int\limits_y^x V(t)\,dt\right).$$

Let us integrate by parts in λ in the expression for $Z(x,y)$ N times, integrating $\exp(2i\sqrt{\lambda}(x-y))$ and differentiating the rest. We obtain

$$|Z(x,y)| \leq C(\phi,a,b,N)\min(1,|x-y|^{-N}) \left| \int\limits_y^x V(t)\,dt \right|^N \leq C(\phi,a,b,N)\min(1,|x-y|^{-N(1-\frac{1}{q})})\|V\|_p. \tag{37}$$

Taking N large enough, for instance such that $N(1-\frac{1}{q}) > 1$, we see that an operator with the kernel $Z(x,y)$ maps L^2 to L^2 (by Schur's test, for example). Therefore also

$$\|Kf\|^2_{L^2(a,b)} \leq C(\phi,a,b)\|f\|_2^2. \qquad \square$$

We conclude the paper by formulating one simple generalization of Theorem 1.3 (which implies the corresponding generalizations of Theorems 1.1 and 1.2).

Theorem 6.2 *Fix a potential $V(x)$. Suppose that there exists a monotone differentiable function $d(x)$, $d(x) > 0$, $d(x) \xrightarrow{x\to\infty} 0$, $d'(x) \leq 0$, such that $V(x)d(x)^{-1} \in L^p$ with some $p < 2$ and $V(x)d(x)^N \in L^1$ for some integer N. Then under the assumptions of Theorem 1.3, all conclusions of Theorem 1.3 hold for perturbation of H_U by $V(x)$.*

Proof. Going through the proofs of Lemma 5.5 and Theorem 1.3, we substitute $x^{-\epsilon}$ with $d(x)$. Given that $d(x)^{-1}V(x) \in L^p$, $p < 2$, all proofs go unchanged, and in the final step instead of $V(x)(1+x)^{-\delta(2n-1)}$ being absolutely integrable we need to check that for some N,

$$V(x)d(x)^N \in L^1.$$

This is exactly what we assumed in the statement of Theorem 6.2. □

We note one additional particular class of potentials which we may treat using Theorem 6.2. Namely, take any $V(x)$ such as in Theorem 1.3, take a sequence $\{x_n\}$ and insert intervals of arbitrary size I_n at each point x_n. Let $\tilde{V}(x)$ be a potential obtained from V by adding such intervals I_n where $\tilde{V}$ is zero. Then it is easy to construct a function $d(x)$ with the properties as in Theorem 6.2. For the details of such construction we refer to [12] (where it was derived in a slightly different context).

Appendix: Singular potentials.

In Appendix, we discuss the preservation of the absolutely continuous spectrum for potentials with strong local singularities. The proof turns out to be almost entirely parallel to the non-singular case, so we mostly sketch the arguments with a few exceptions. We should note, however, that the result has some interest in it. In the explicit construction of power decaying potentials such that the corresponding Schrödinger operators have purely singular spectrum [23] one can try to use the possible singularity of the potential to get singular spectrum under stronger decay conditions. The results of the appendix show that such plan does not work out, at least the fundamental exponent $\frac{1}{2}$ and virtually all results we have shown before extend to the situation where strong local singularities are allowed.

We will consider the potentials from the spaces $l^p(L^1)$, $1 \leq p \leq 2$, with the norm given by

$$\|f\|_{l^p(L^1)} = \left(\sum_{n=0}^{\infty} \left(\int_n^{n+1} |f(x)|\, dx \right)^p \right)^{\frac{1}{p}}.$$

The main result we show is

Theorem A.1. *Suppose that U is continuous periodic (in particular, the free case $U = 0$ is of course included). Let the perturbation $V(x)$ be such that there exist $p \leq 2$ and $\epsilon > 0$ with $V(x)x^{-\epsilon} \in l^p(L^1)$. Then all conclusions of Theorems 1.2 hold, in particular the absolutely continuous spectrum S of the operator H_U is preserved under the perturbation by V and for a.e. $\lambda \in S$ there exist solutions of the equation (1) with the pure WKB asymptotic behavior in the main term (i.e. with asymptotic behavior given by (2)).*

As before, Theorem A.1 is a corollary of the following general criterion (an analog of Theorem 1.3):

Theorem A.2. *Suppose that the potential $V(x)$ is such that there exist $\epsilon > 0$ and $p \leq 2$ so that $x^{\epsilon}V(x) \in l^p(L^1)$. Assume that an operator*

$$(Kf)(\lambda) = \chi(S) \int_0^{\infty} \theta(x,\lambda)^2 \exp\left(\frac{i}{\Im(\theta\overline{\theta}')} \int_0^x V(t)|\theta(t,\lambda)|^2\, dt \right) f(x)\, dx$$

satisfies the bound

$$\|Kf(\lambda)\|_{L^2(S)} \le C\|f\|_{l^2(L^1)}$$

on functions f of compact support. Then the absolutely continuous spectrum of H_U supported on the set S is preserved under perturbation by V, i.e. the set S belongs to the essential support of the absolutely continuous part of the spectral measure of operator H_{U+V}. Moreover, for a.e. $\lambda \in S$, there exist solutions $\psi_\lambda(x)$, $\overline{\psi}_\lambda(x)$ of the equation (1) with the asymptotic behavior (2).

We will indicate the changes in the proof of Theorem 1.3 which are necessary to prove Theorem A.2. First, we need the following substitute of Theorem 3.1.

Lemma A.3. *Suppose that an integral operator K with uniformly bounded kernel $k(\lambda, x)$ satisfies $l^2(L^1) - L^2$ estimate. Then the maximal function given by*

$$(M_K f)(\lambda) = \sup_N \left| \int_0^N k(\lambda, x) f(x)\, dx \right|$$

satisfies

$$\|M_K f\|_q \le C_q \|f\|_{l^p(L^1)}$$

for any $p < 2$ and $q^{-1} = 1 - p^{-1}$. Therefore, the integral defining $(M_K f)(\lambda)$ converges a.e. λ for any $f \in l^p(L^1)$.

Proof. Given the function f with $\|f\|_{l^p(L^1)} = 1$, we consider the family of intervals $E(m,l)$ similar to the L^p case. Namely, first we consider intervals $E(1,1)$ and $E(1,2)$, such that their union is the whole half-axis, the first interval lies to the left of the second and

$$\|f\chi_{E(1,1)}\|^p_{l^p(L^1)} = \|f\chi_{E(1,2)}\|^p_{l^p(L^1)}.$$

We note that in contrast to the L^p case, we cannot in general say that $\|f\chi_{E(1,i)}\|^p_{l^p(L^1)} = 2^{-1}$. However, it is easy to see that

$$\|f\chi_{E(1,i)}\|^p_{l^p(L^1)} \le 2^{-1}.$$

If the sets $E(1,1)$, $E(1,2)$ are unions of the integer intervals $(l, l+1)$, we have equality. Otherwise each set $E(1,i)$ is a union of integer intervals and a set A_i, such that $A_1 \cup A_2 = (m, m+1)$ for some m. In this case

$$\left(\int |f\chi_{A_1}|\, dx\right)^p + \left(\int |f\chi_{A_2}|\, dx\right)^p \le \left(\int_m^{m+1} |f(x)|\, dx\right)^p$$

(since $p > 1$), and hence

$$\|f\chi_{E(1,2)}\|^p_{l^p(L^1)} + \|f\chi_{E(1,1)}\|^p_{l^p(L^1)} \le 1.$$

We decompose each of the intervals $E(1,i)$ further as in L^p case. On the step m, we obtain a family of intervals $E(m,l)$, such that $\cup_l E(m,l) = R^+$, $E(m,l_1)$ lies to the left from $E(m,l_2)$

if $l_1 \le l_2$. We choose these intervals so that the total number of the intervals of generation m is 2^m and $\|f\chi_{E(m,l)}\|^p_{l^p(L^1)} \le 2^{-m}$ for every l. Such family is obtained by splitting every interval of given power into two equal pieces, with the same arguments as on the first step.

Let

$$(M_{K,m}f)(\lambda) = \sup_l |M_K(f\chi_{E(m,l)})(\lambda)|.$$

Then

$$(M_K f)(\lambda) \le \sum_{m=1}^{\infty} (M_{K,m}f)(\lambda).$$

This follows from the construction of the family $E(m,l)$. Hence

$$\|M_K f\|_q \le \sum_{n=1}^{\infty} \|M_{K,m}f\|_q.$$

On the other hand,

$$\|M_{K,m}f\|_q^q \le \sum_{l=1}^{2^m} \|K(f\chi_{E(m,l)})\|^q_{L^q} \le \sum_{l=1}^{2^m} \|f\chi_{E(m,l)}\|^q_{l^p(L^1)} \le 2^{m(1-\frac{q}{p})}.$$

Therefore,

$$\|M_K f\|_q \le \sum_{m=1}^{\infty} 2^{m\left(\frac{1}{q}-\frac{1}{p}\right)} \le C_q \|f\|_{l^p(L^1)}$$

(we assumed $\|f\|_{l^p(L^1)} = 1$, but the bound extends to all f by sublinearity of M_K). A.e. convergence follows from the maximal estimate in a standard way. □

Proof of Theorem A.2. The proofs of multilinear transform properties and a.e. convergence estimates go exactly the same way as before. The family $E(m,l)$ has the same properties as in L^p case, in particular martingale-type property (two sets either disjoint or one is contained in another). The Lemma 4.3 clearly remains valid. The function

$$f(x) = \left(\sum_{i=1}^{n} |f_i(x)|^p\right)^{\frac{1}{p}},$$

used to construct the family $E(m,l)$ in the proof may be replaced by

$$f(x) = \sum_{i=1}^{n} |f_i(x)|.$$

The only other changes we need to make in the proof is to change throughout $\|\cdot\|_p$ to $\|\cdot\|_{l^p(L^1)}$. □

To prove Theorem A.1, we need to show the $l^2(L^1) - L^2(I)$ bounds.

Proof of Theorem A.1. We will show the proof of the needed norm bound only for the case $U = 0$. The general periodic case follows from the properties of the Bloch functions in a way parallel to the free case. We refer to [13] for necessary information and a similar argument.

Clearly we can restrict our attention to some compact interval $I=(a,b)$, $b>a>0$. It is sufficient to show that the operator

$$(Kf)(\lambda)=\phi(\lambda)^2\int\limits_0^\infty \exp\left(2i\sqrt{\lambda}x-\frac{i}{\sqrt{\lambda}}\int\limits_0^x V(t)\,dt\right)f(x)\,dx$$

satisifes $l^2(L^1)-L^2(I)$ bound on functions of compact support, where $\phi\in C_0^\infty(R^+)$ and $\phi(\lambda)=1$ for $\lambda\in I$. Note that

$$(Kf)(\lambda)=\sum_{l=0}^\infty \phi(\lambda)\exp\left(2i\sqrt{\lambda}l-\frac{i}{\sqrt{\lambda}}\int\limits_0^l V(t)\,dt\right)\phi(\lambda)\int\limits_0^1 \exp\left(2i\sqrt{\lambda}y-\frac{i}{\sqrt{\lambda}}\int\limits_l^{y+l} V(t)\,dt\right)f(y+l)\,dy. \tag{38}$$

We can write $Kf(\lambda)$ as follows:

$$Kf(\lambda)=\sum_{l=0}^\infty \phi(\lambda)\exp\left(2i\sqrt{\lambda}l-\frac{i}{2\sqrt{\lambda}}\int\limits_0^l V(t)\,dt\right)f(\lambda,l),$$

where the expression $f(\lambda,l)$ has the following property: for every $m\geq 0$,

$$|\partial_\lambda^m f(\lambda,l)|\leq C_m(I,\phi)\mathrm{sup}_n\left(\int\limits_n^{n+1}|V(x)|\,dx\right)^m f_l$$

where $f_l=\int_l^{l+1}|f(x)|\,dx$. This property of $f(\lambda,l)$ is evident from (38).

We compute

$$\|Kf\|^2_{L^2(I)}\leq\sum_{l=0}^\infty\sum_{m=0}^\infty\int d\lambda\phi^2(\lambda)\exp\left(2i\sqrt{\lambda}(l-m)-\frac{i}{2\sqrt{\lambda}}\int\limits_m^l V(t)\,dt\right)f(\lambda,l)\overline{f}(\lambda,m).$$

Let us integrate by parts, differentiating $f(\lambda,l)\overline{f}(\lambda,m)$ and integrating the rest. By virtually the same computation as one which led us to (37) we obtain

$$\left|\int d\lambda\phi(\lambda)\exp\left(2i\sqrt{\lambda}(l-m)-\frac{1}{2\sqrt{\lambda}}\int\limits_m^l V(t)\,dt\right)\right|\leq C_r\mathrm{min}(1,|l-m|)^{-\frac{r}{p}}$$

for every positive integer r. Therefore, taking into account the properties of $f(\lambda,n)$, we obtain

$$\|Kf\|^2_{L^2(I)}\leq C_r\sum_{l=0}^\infty\sum_{m=0}^\infty \min(1,|m-l|)^{-\frac{r}{p}}f_m f_l.$$

Taking r large enough ($\frac{r}{p}>1$ will do) we see that the operator K satisfies the required bound. □

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