Existence of Exponentially Small Separatrix Splitting and Homoclinic Connections Between Whiskered Tori in Weakly Hyperbolic Near-Integrable Hamiltonian Systems

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Abstract

In this paper we study a generalization of Arnold's original example in which he discussed the existence of a mechanism for instability caused by the splitting of the homoclinic manifolds of the weakly hyperbolic tori, that has subsequently been referred to as *"Arnold diffusion"* in case when the number of degrees of freedom $n \geq 3$. Namely, we consider a widely studied model of a pendulum weakly coupled with $n-1$ rotors with the degeneracies in the unperturbed Hamiltonian, corresponding to different time-scales, existing in the problem.

Using an alloy of the iterative and direct methods developed within the last years we give exponentially small upper bounds for the splitting measure of transversality for the case of an even, analytic perturbation, thus improving the estimate of Gallavotti [1994], which he calls quasiflat, and generalizing the analogous recent estimate of Delshams et al. [1996] for the rapidly quasiperiodically forced pendulum to a much larger class of Hamiltonian systems.

In particular, the exponentially small upper bound for the transversality measure of the splitting applies when the Hamiltonian has extra degeneracies, namely when the frequencies on a torus become near-resonant. In fact, we show that in such a case the quantity in question becomes smaller, which is the incarnation of the general fact that resonant regions in the action space are in fact more stable in the sense that they have larger Nekhoroshev exponent. Nevertheless, we emphasize that getting uniform estimates for an arbitrary $n \geq 3$ is very hard unless one makes some additional assumptions on the approximation properties of the frequency vector.

Although recent developments show that the first order of canonical perturbation theory, given by Melnikov integrals, generally cannot be accepted as the leading order answer for the splitting distance for the case of more than two degrees of freedom, including the rapidly quasiperiodically forced pendulum problem, we suggest an analytic perturbation, the majority of whose Fourier components are strictly non-zero, for which Melnikov integrals can be vindicated as the leading order approximation for the components of the splitting distance in different directions if the frequencies on the invariant tori satisfy certain arithmetic conditions. This allows us to bound the splitting distance from below.

Furthermore, having such a perturbation, for the case of three degrees of freedom, we use a simple number-theoretical argument to find the asymptotics of the Fourier series with exponentially small coefficients involved. This enables us to compute the numerous homoclinic orbits for the whiskered tori

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of asymptotically full measure, and by proving the domineering contribution of the first order of perturbation theory for the transversality measure, to suggest a leading order answer for this quantity, thus proving the existence of an infinite number of heteroclinic connections between tori with close diophantine frequencies.

We elucidate the numerous arithmetic issues that obstruct getting a compact leading-order approximation for the splitting size, most of which can be overcome in the case of three degrees of freedom, as our example shows. These obstacles can be also possibly avoided in the same fashion for an arbitrary $n \geq 3$ if one treats the case when the frequencies of the rotors are near a resonance of multiplicity $n-3$ or $n-2$.

Contents

1 Introduction

A central question in the study of near integrable analytic Hamiltonian systems is the stability of the action variables. In particular, consider a Hamiltonian of the form

$$H(\vec{I}, \vec{\varphi}) = H_0(\vec{I}) + \varepsilon H_1(\vec{I}, \vec{\varphi}, \varepsilon), \quad (\vec{I}, \vec{\varphi}) \in R^n \times T^n, \tag{1}$$

with associated Hamiltonian vector field

$$\begin{aligned} \dot{\vec{I}} &= -\varepsilon \frac{\partial H_1}{\partial \vec{\varphi}}(\vec{I}, \vec{\varphi}, \varepsilon), \\ \dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial \vec{I}}(\vec{I}) + \varepsilon \frac{\partial H_1}{\partial \vec{I}}(\vec{I}, \vec{\varphi}, \varepsilon). \end{aligned}$$

We are interested in the behavior of the $\vec{I}$ values for ε small. The KAM theorem gives a partial answer to this question, insuring that generically, the $\vec{I}$ variables are stable for most initial conditions. For those initial conditions not covered by the KAM theorem, Nekhoroshev's theorem provides an upper bound on the evolution of the $\vec{I}$ variables for *finite* times that are exponentially long with respect to ε.

In 1964 Arnold published an example of a three degrees of freedom Hamiltonian system containing a mechanism for a global instability of the action variables which has come to be known as *"Arnold diffusion"*. We begin by describing this example.

Arnold [1964] considered a Hamiltonian of the form

$$H = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) + \mu\varepsilon\left(\cos x - 1\right)\left(\sin\varphi_1 + \cos\varphi_2\right), \tag{2}$$

where $x \in T^1$, $y \in R$, $(I_1, I_2) \in R^2$, $(\varphi_1, \varphi_2) \in T^2$, and ε and μ are parameters. The Hamiltonian obtained by setting $\mu = 0$ is referred to as the *unperturbed Hamiltonian*, which one recognizes as the Hamiltonian for three uncoupled systems: a pendulum and two *rotors* described by two action-angle pairs of variables. Hamilton's canonical equations for this system are given by

$$\begin{aligned}
\dot{x} &= \tfrac{\partial H}{\partial y} = y, \\
\dot{y} &= -\tfrac{\partial H}{\partial x} = \varepsilon \sin x + \mu\varepsilon \sin x\left(\sin\varphi_1 + \cos\varphi_2\right), \\
\dot{\varphi}_1 &= \tfrac{\partial H}{\partial I_1} = I_1, \\
\dot{I}_1 &= -\tfrac{\partial H}{\partial \varphi_1} = -\mu\varepsilon\cos\varphi_1\left(\cos x - 1\right), \\
\dot{\varphi}_2 &= \tfrac{\partial H}{\partial I_2} = 1, \\
\dot{I}_2 &= -\tfrac{\partial H}{\partial \varphi_2} = \mu\varepsilon\sin\varphi_2\left(\cos x - 1\right).
\end{aligned} \tag{3}$$

We want to describe the geometry of the phase space for $\mu = 0$. In this case the system has a four dimensional normally hyperbolic invariant manifold given by

$$\mathcal{M}_0 = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, x = y = 0\right\}.$$

whose five dimensional stable and unstable manifolds coincide along

$$W^s\left(\mathcal{M}_0\right) = W^u\left(\mathcal{M}_0\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0\right\}.$$

$\mathcal{M}_0$ is foliated by a family of two-tori, and the trajectories on the two-tori are given by

$$\begin{aligned}
\varphi_1(t) &= I_1 t + \varphi_{10}, \\
\varphi_2(t) &= t + \varphi_{20}.
\end{aligned}$$

We denote these two-tori by $\mathcal{T}(I_1)$.

Recall that the dynamics is restricted to lie in the five dimensional energy surface given by

$$h = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right),$$

and the four dimensional $\mathcal{M}_0$ intersects the five dimensional energy surface in a three dimensional set. This three dimensional set, denoted by $\mathcal{M}_0^h$, is filled out by a family of two tori, and each two-torus has three-dimensional stable and unstable manifolds that coincide and are given by

$$W^s\left(\mathcal{T}(I_1)\right) = W^u\left(\mathcal{T}(I_1)\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0,\ I_1 \text{ fixed}\right\}.$$

The stable and unstable manifolds of a given torus are often referred to as the *whiskers* of the torus. Moreover, $W^s(\mathcal{T}(I_1))$ intersects $W^u(\mathcal{T}(I_1))$ non-transversely, which is a situation that we do not expect to persist for μ small, but nonzero, and it is to this situation that we now turn.

The particular form of the perturbation in (2) is such that it does not affect $\mathcal{M}_0$ or any of the tori on it. Nevertheless, for any given torus the perturbation may cause the stable and unstable manifolds to intersect transversely. Moreover, because the tori are arbitrarily close on $\mathcal{M}_0^h$ the stable and unstable manifolds of nearby tori may intersect and, consequently, give rise to a mechanism for a global drift in the trajectories. Following Lochak [1995], we refer to this as the *Arnold mechanism* and we now describe it in a bit more detail. The tori can be denoted by the parameter I_1. Arnold showed that one could find a sequence of tori, $\{I_1^i\}$, $i = 1, \ldots, N$, with neighboring tori sufficiently close, i.e. $|I_1^i - I_1^{i+1}| = O(|\mu|^2)$, such that the unstable manifold of $\mathcal{T}_\mu(I_1^i)$ transversely intersects the stable manifold of $\mathcal{T}_\mu(I_1^{i+1})$, $i = 1, \ldots, N-1$. This sequence of whiskered tori is said to form a *transition chain*, and Arnold uses it to prove the following theorem.

Theorem 1.1 (Arnold) *For every ε, $r > 0$ there exists a $\mu_0 > 0$ such that for all $0 < \mu \leq \mu_0$ there are invariant tori $\mathcal{T}_\mu(I_1)$ and $\mathcal{T}_\mu(I_1')$, with $|I_1 - I_1'| > r$, which are connected by a transition chain.*

He then shows that the transition chain is shadowed by a true orbit of the system and in this way one obtains an $O(1)$ drift in the I_1 variable over some finite time interval.

In carrying out this program in general there are two main difficulties that must be overcome. However, before discussing these we first explain how Arnold's example is related to the general problem of stability of the actions in near integrable systems.

For the unperturbed Hamiltonian consider a *multiplicity one* or *simple* resonance, i.e. there exists *one* independent integer vector, $\vec{k}$, such that

$$\vec{k} \cdot \partial_{\vec{I}} H_0(\vec{I}) = 0. \tag{4}$$

For nondegenerate $H_0(\vec{I})$ (4) defines a codimension one surface in the action space or, equivalently, a hyperplane in the frequency space. Near this *resonance surface* the Hamiltonian can be transformed to a normal form of the following form

$$H(y, x, \vec{I}, \vec{\varphi}, \sqrt{\varepsilon}) = h_0(\vec{I}, \varepsilon) + P(y, x, \vec{I}, \varepsilon) + \mu F(y, x, \vec{I}, \vec{\varphi}, \sqrt{\varepsilon}), \tag{5}$$

where $(y, x) \in R^2$, $(\vec{I}, \vec{\varphi}) \in R^{n-1} \times T^{n-1}$, and $|\mu| \leq \mu_0 = O(\varepsilon^d)$, the latter being such that $d \geq \frac{3}{2}$. Precise statements on the size of domains and analyticity conditions can be found in Rudnev and Wiggins [1996]. We regard μ as the perturbation parameter in (5). Then the important point in relation to Arnold's example is that under certain non-degeneracy conditions, for fixed $\vec{I}$ and ε, $P(y, x, \vec{I}, \varepsilon)$ is the Hamiltonian of a one degree-of-freedom Hamiltonian system whose phase portrait is qualitatively that of a simple pendulum. In particular, it has hyperbolic equilibria connected by homoclinic orbits. Thus, we see that (5) has the structure of the Hamiltonian of a pendulum coupled to $n-1$ action-angle pairs, or *rigid rotors*. Therefore, one can view Arnold's example as a model for the global dynamics near a multiplicity one resonance surface, which raises the question "is Arnold's mechanism of transition chains a generic mechanism causing the drift of trajectories along a multiplicity one resonance surface"? In attempting to answer this question (which is *not* answered in this paper) we must face the two main difficulties described above, and to which we now return.

Exponentially Small Splitting of Separatrices: The first step is to establish transverse homoclinic and heteroclinic intersections between tori that survive the perturbation. In this regard, Arnold's example was quite clever in that it avoided major technical difficulties, yet it illustrated the essential phenomena, i.e., a mechanism giving rise to an $O(1)$ drift in the action variables. His perturbation was such that it did not affect any of the tori. But most important was the role played by the two independent parameters ε and μ. Arnold used Poincaré-Melnikov (further Melnikov) integrals to measure the splitting. The integral turned out to be exponentially small with respect to ε, yet if μ and ε are regarded as independent, it gives a valid approximation to the distance. However, if μ is regarded as a function of ε, Melnikov integrals are only a valid measure of the distance provided $|\mu| \leq \exp\left(-\frac{1}{\varepsilon^a}\right)$, $a \geq \frac{1}{2}$.

As one can see from our description of the general problem of the dynamics near a multiplicity one resonance, it is important to be able to measure the splitting in cases where $|\mu| \leq \mu_0 = O(\varepsilon^d)$ for some positive $d \geq \frac{3}{2}$. In this paper we develop techniques for doing this. We establish a uniform exponentially small upper bound for the measure of transversality of the splitting and study how it changes when the frequency vector becomes near-resonant. These aspects were avoided in Arnold's example by his choice of μ exponentially small with respect to ε. Such a choice allowed him totally to ignore all the difficulties caused by the small divisors, which become a major obstruction for getting uniform bounds and estimates.

In principle, we show that there is quite a large class of perturbations for which the Melnikov function can be justified as the leading order answer for the components of the splitting distance due to the fact that the splitting is exponentially small in all the orders of perturbation theory. Nevertheless, it becomes much harder to deal with the transversality measure, for the latter is represented by a determinant of a matrix which, as we show, can be extremely ill-conditioned, therefore, neglecting the higher order terms can lead to a disastrous error.

Note, that the original perturbation in Arnold [1964] contains only two harmonics, and one of the angle variables is actually time, so it has a constant frequency. Yet in the perturbation for the general problem of the dynamics near a multiplicity one resonance the perturbation is a general Fourier series containing an infinite number of harmonics. This results in the appearance of small divisors in the exponents in all orders of perturbation theory, starting from the Melnikov integrals, which further complicate matters. We consider a generalization of Arnold's example where the perturbation contains an infinite number of harmonics, and for this example we give an asymptotically exact answer for the splitting distance for the number of degrees of freedom $n \geq 3$, moreover, when $n = 3$ we provide a compact leading order answer not only for the splitting distance, but also for the measure of transversality, provided that $|\mu| \leq \mu_0 = O(\varepsilon^d)$, with d to be specified.

Construction of Transition Chains and Shadowing Orbits: Once the existence of transverse homoclinic and heteroclinic orbits is established then one can attempt to construct transition chains. There are severe difficulties here that have not been overcome so far.

If one assumes the existence of such a transition chain, another problem is to "shadow" it with a real orbit and estimate the "diffusion speed". Recently, there's been a lot of progress in this matter due to Bessi [1996](A,B), where the variational methods were applied to yield a close to optimal value for the "diffusion speed" in Arnold's original example, and the geometric constructions in Marco [1996], where the basic "windowing" techniques for constructing the "shadowing orbits", originally due to Easton [1981], were clarified and applied to heteroclinic intersections. However, this is not the subject of this paper. Rather, we are concerned solely with the problem of exponentially small splitting of separatrices.

Before describing the content and results obtained in this paper, we give a brief survey of known results on the splitting of separatrices. For the bigger picture of "Arnold diffusion" we would refer a reader to Lochak [1995], who gives a superb review of the state-of-the-art.

It is well known that the Nekhoroshev theorem gives an upper bound on the "speed of diffusion", stipulating that the latter cannot be faster than $\sim \exp(-C\varepsilon^{-a})$, where $C > 0$ is a constant, whose optimal value is most likely not uniform for different models. Clearly, the most important quantity in this estimate is a, or the "Nekhoroshev exponent", which nowadays is believed to have an optimal value of $\frac{1}{2n}$, where n is the number of "non-resonant degrees of freedom". So, in a way, "Arnold diffusion" is beyond the resolution of canonical perturbation theory. Moreover, after Nekhoroshev [1975] found his now celebrated theorem (in fact, a brief note announcing this theorem appeared in 1971), for some 15 years or so (until the first "Western" paper by Benettin et al. [1985] came out), this theorem had not been given the attention it deserves, at least "in the West".

Today "Arnold diffusion" is one of the major outstanding conjectures in the theoretical study of near-integrable Hamiltonian systems. Recently, there has been a lot of attention on this subject in the mathematical community. Nevertheless, it isn't an exaggeration to say that until several years ago it had been virtually ignored by mathematicians, at least outside of Russia.

In the meantime physicists had a certain interest in the problem and came up with a number of hypotheses, most of which are still far from being rigorously justified or rejected. Chirikov [1979] published a report of great insight and importance, which is undoubtedly a major contribution to the foundations whereupon the "Arnold diffusion" rests. He systemized the problem into a number of connected issues and suggested answers to certain important questions, in particular emphasizing the kinship existing between the size of the splitting and the speed of the diffusion, the latter being exponentially slow with respect to the Nekhoroshev exponent, for which he had predicted the value $\frac{1}{2n}$. Yet somehow until recently Chirikov's work had fallen out of sight of mathematicians, so apparently there had been very little research done on the subject for quite a few years.

Benettin et al. [1985] found a $\frac{1}{n^2}$ estimate for the Nekhoroshev exponent, the same estimate then appeared in Benettin and Gallavotti [1986], and the latter paper suggested that it would be difficult to do any better. But eventually Chirikov's remarkable insight as a physicist was confirmed by Lochak [1992] who used simultaneous approximation to invent a conceptually new proof of Nekhoroshev's theorem, without the traditional hassle of dealing with small divisors and Fourier series, which almost immediately and naturally justified the $\frac{1}{2n}$ for the exponent (in fact, the first version of the paper had $\frac{1}{2n+2}$, which then quickly became $\frac{1}{2n}$ after a trick due to Neishtadt was incorporated into the algebraic part of the proof). Around the same time Pöschel [1993] published an improved traditional proof, where he also succeeded in obtaining the exponent $\frac{1}{2n}$.

As we mentioned earlier, the skeleton in the phase space near which the diffusion occurred in Arnold's original paper were the "transition chains", due to the splitting of separatrices of the hyperbolic tori, caused by the perturbation. In Chirikov's work there is an argument concerning the relation between the speed of diffusion and the splitting transversality measure (rewritten and discussed in Paragraph V.2 of Lochak [1992], although the latter author suggested that rigorous justification of this hypothesis would be very hard). Nevertheless, the most recent works of Bessi, of which we mention Bessi [1996](A,B), show that things may change very quickly, provided that new ideas in the field eventually arrive and get implemented. Bessi's works consider different variations of original Arnold's example (2), always requiring that μ be sufficiently (exponentially) small with respect to ε, so that the transition chains can be constructed, and the Melnikov function can be justified as the leading order approximation for the splitting measurement. Bessi [1996](A) deals exactly with the Hamiltonian (2) and estimates the speed of diffusion as $\exp(-C\varepsilon^{-\frac{1}{2}})$. Bessi [1996](B) withal considers the case of four degrees of freedom, when the number of the harmonics in the perturbation is infinite, quite similar to the one we will be dealing with, and shows the existence of an $O(1)$ drift in the action variables during a time interval whose length does not exceed $\exp(C\varepsilon^{-\frac{1}{4}})$. In particular, he notes that the value of the exponents (respectively 1/2 and 1/4) is the same as the value of the exponent in the splitting distance function (provided that the latter is given by the Melnikov function with μ being exponentially small with respect to ε).

It was not until 1992 when the first preprint of Chierchia and Gallavotti [1994] came out, which along with Lochak [1992] gave a second wind of interest to the Arnold diffusion problem. The former authors applied Arnold's original framework to study the instability of the unperturbed integrals of motion in what they had defined as "a-priori unstable" systems, namely those whose unperturbed part involves a hyperbolic pair of variables with an $O(1)$ Lyapunov exponent. Unlike (1), which they referred to as an "a-priori stable" system, and we also assume this terminology, the a-priori unstable, or *strongly hyperbolic* systems possess homoclinic orbits in their unperturbed parts. In fact, these systems do not really belong to the domain of the Arnold diffusion problem, for the degeneracy, or the *singular* perturbation essence of it (which in (2) was reflected by the fact that the pendulum and hyperbolicity disappears when ε is zero) is the centerpiece thereof.

In our opinion, the most significant contribution of Chierchia and Gallavotti [1994] for the Arnold diffusion problem was that the authors had developed a recursion calculus for the Lindstedt series, representing the perturbed stable and unstable manifolds of the surviving invariant tori up to all orders of perturbation theory, which can be applied whether the problem is degenerate, or not. As a model problem, where this algorithm was applied, the authors considered a pendulum weakly coupled with two or more rotors. If the Lyapunov exponent of the pendulum is of the same order as the frequencies of the rotors, then this is an example of an a-priori unstable system. For such a system, if the rotors are away from resonances, the first order calculation will suffice to evaluate the splitting distance.

Studying model problems for the a-priori stable systems near a simple resonance, represented by the Hamiltonian (5), one always encounters the fact that the problem possesses two or more time-scales, which is an effect of the resonance. Formally it is reflected through the dependence of the Lyapunov exponent of the pendulum upon ε in such a way that it goes to zero when the latter goes to zero, as it does in Arnold's example. Thus the problem becomes singular. The hyperbolicity disappears when ε is zero, or the problem is *weakly hyperbolic*. The whisker calculus can still be applied to this model, and the convergence issues turn out to be fairly robust with respect to the above degeneracy: one can construct the Normal form for the whiskered tori, using a version of the KAM theorem (see Rudnev and Wiggins [1996]) or even prove their existence directly (see Gallavotti [1994] for the motivation and Gentile [1995] for the proof).

If one computes the first order of perturbation theory for the splitting distance, or the Melnikov function (Wiggins [1988]), for a weakly hyperbolic model, it turns out to be seemingly useless (and totally useless if the perturbation has a finite number of harmonics, when the possibility of having a compact answer no longer exists), for the Jacobian of the Melnikov (vector-) function is at most exponentially small in ε. The order by order analysis of the Lindstedt series, fulfilled in Gallavotti [1994] resulted in what he called a *quasiflat* upper bound for the splitting size (for a finite number of harmonics in the perturbation).

We will further characterize the difference between quasiflatness and exponential smallness, professed in the present paper in the case of three and more degrees of freedom (in the case of a pendulum coupled with only one rotor these notions are essentially the same).

Consider Arnold's example itself, with $\mu \sim \varepsilon^d$ with d large enough to insure all the necessary convergences, or its generalization (7), studied in this paper (see below), similar to what Gallavotti [1994] is devoted to, where it is called the "Thirring model". Suppose, the splitting distance $\vec{\Delta}$, defined as the difference of the unperturbed integrals of motion, or the action variables $\vec{I}^{u,s}(t,\vec{\alpha})$ on the unstable and the stable manifolds of the invariant torus with the non-resonant frequency $\vec{\omega}$ is given by a Fourier series in some angular parameter $\vec{\alpha} \in T^{n-1}$, where n is the number of degrees of freedom (the number of rotors + one for a pendulum) on some Poincaré section, say at $t=0$:

$$\vec{\Delta}(\vec{\alpha}) = \vec{I}^u(0,\vec{\alpha}) - \vec{I}^s(0,\vec{\alpha}) = \sum_{\vec{k}} \vec{\Delta}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}).$$

Then the Fourier coefficients $\Delta_{i\vec{k}}$, where $i=1,..,n-1$ and $\vec{k} \in Z^{n-1}$ are bounded from above as follows:

$$\begin{aligned} |\Delta_{i\vec{k}}| &\preceq \quad e^{-|\vec{k}|\sigma_0} \sup_{|\vec{l}|\le K} \exp\left(-\frac{\pi}{2}\frac{\vec{\omega}\cdot\vec{l}}{\sqrt{\varepsilon}}\right) \quad \text{quasiflat,} \\ |\Delta_{i\vec{k}}| &\preceq \quad e^{-|\vec{k}|\sigma_0} \exp\left(-\frac{\pi}{2}\frac{\vec{\omega}\cdot\vec{k}}{\sqrt{\varepsilon}}\right) \qquad\qquad \text{exp. small.} \end{aligned} \tag{6}$$

The sign $\preceq$ we use in the sense that the right-hand side can be in fact multiplied by some finite power of ε and a constant, and then it will be an upper bound. Besides, $K \ge |\vec{k}|$ is some kind of a cutoff parameter, depending on a particular perturbation. Also $|\vec{k}| = \sum_{j=1}^{n-1} |k_j|$, and the factor $e^{-|\vec{k}|\sigma_0}$ controls the convergence of this Fourier series in some complex neighborhood of a torus T^{n-1} in the $\vec{\alpha}$-variables of the analyticity width $\sigma_0 > 0$.

The key difference is that not only is the exponentially small estimate more handsome, easier to deal with, and seemingly "smaller", but what's more important is that in the latter estimate the harmonics with some $k_i = 0$ solely do not contribute to the transversality, which is typically measured by the quantity

$$\Upsilon = |\det \partial_{\vec{\alpha}} \vec{\Delta}(\vec{\alpha})|,$$

because $\sin(\vec{k}\cdot\vec{\alpha})$ will not depend on the ith component of $\vec{\alpha}$. This eventually accounts for the fact that when the rotors frequencies approach a resonance, the transversality measure of the splitting becomes *smaller*. The former estimate does not allow to see this and will possibly predict a much *larger* value for the quantity in question (no longer beyond all powers of the perturbation parameter), for the *supremum* can be achieved on such a term. Nevertheless, we prove that Υ is *always* exponentially small with respect to ε, *as long as there is at least one frequency which is fast.*

Finally, Chierchia and Gallavotti [1994] construct transition chains *in the a-priori unstable case*, avoiding the resonances between the rotors by taking a finite number of harmonics in the perturbation, and give a superexponentially slow estimate for the speed of diffusion. The latter estimate was dramatically improved (to the extent of showing that the a-priori unstable systems are indeed beyond the Arnold diffusion problem in its original sense due to Arnold, Nekhoroshev and Chirikov) recently by Bernard [1996] up to the perturbation parameter squared by basically following Bessi [1996](A) in a variational construction of the shadowing orbits.

The calculus for the invariant tori and their whiskers was part of a number of achievements in developing the Lindstedt series method, inspired by the work of Eliasson [1988], where the existence of the KAM tori was established through the proof of convergence of the Lindstedt series. This approach was furthered by the Italian scientists who succeeded in coming up with a number of new results in recent years. For more details see Chierchia and Falcolini [1993], Gallavotti [1994], Gentile [1995], and the references therein. We will further refer to these methods as *direct methods.*

Outside Italy, the degenerate splitting for the systems with more than two degrees of freedom has not been studied so extensively, but there has been much success with understanding the splitting of separatrices for the rapidly periodically forced pendulum dynamics in many different ways, and, in particular, the vindication of the Melnikov function for this problem (in most of the cases), which has been so far the only simple quantitative method for evaluation of the splitting distance. There has been a variety of methods, developed in a series of papers treating the rapidly periodically forced pendulum (allowing different extent of generalization), such as Holmes et al. [1988], Delshams and Seara [1992], Ellison et al. [1993], and Gelfreich [1993], where the exponential smallness of the splitting was established, and thus the first-order answer for the splitting distance was justified (although even there one can come up with "bad" perturbations, which do not have the first Fourier mode, and they won't be amenable to Melnikov's method). Regarding the sequence of papers that we have just mentioned, our opinion is that the work of Ellison et al. [1993] is the most universal (although more difficult than the others), and it can be extended to the case of more than two degrees of freedom to result in a *quasiflat* upper bound, similar to the one obtained by Gallavotti [1994].

As for the universal leading order *answer* being given by the Melnikov function for more than two degrees of freedom, the works of Gallavotti [1994] and Simó [1994] show that this is fairly hopeless. Simó [1994] elucidates the apparent difficulties for the case of three degrees of freedom, and numerically observes a significant dependence of the intersection pattern upon the actual value of ε (Chirikov had also predicted that!), although the computations therein relied on the Melnikov function, which wasn't justified.

In the recent preprint from Barcelona, Delshams et al. [1996] studied a *linear case* of a "slow" pendulum, rapidly and quasiperiodically forced with the fixed frequency vector $\vec{\omega} = (1, \frac{\sqrt{5}+1}{2})$, the latter being a number whose continued fraction contains ones only. By these authors the use of the Melnikov function for a specifically chosen meromorphic perturbation has been justified within the framework that stemmed from a series of papers by Lazutkin and collaborators (see Lazutkin et al. [1989] and self-quotations referenced therein), and further developed by Delshams and Seara [1992], Gelfreich [1993]. Delshams et al. [1996] suggested that in more than two degrees of freedom the splitting must also be exponentially small, which is much stronger than the quasiflatness of Gallavotti [1994]. Unfortunately, the proofs in the former paper can not be extended to the *nonlinear case* where the frequencies of the rotors are not fixed. The obstacle is that the Normal form stipulates integrability on a Cantor set only, whereas in the linear case the problem is integrable in a full neighborhood of the single invariant torus. In the linear case Lazutkin's ideas of defining the splitting distance as the difference in the values of a certain first integral on the unstable and the stable manifolds and the ensuing proof of exponential smallness accomplished in Delshams et al. [1996] can hardly be utilized, for a single integral of motion is not enough to measure the splitting of n-dimensional whiskers embedded in the $(2n-1)$-dimensional energy surface, whereas there is a well-known generic reality of the non-existence of any integrals of motion but the energy, in any full neighborhood in the phase space.

Nevertheless, in a certain sense, Delshams et al. [1996] can be regarded as the first announced and strictly proved result on exponentially small splitting in more than two degrees of freedom, for it does not follow from Gallavotti [1994] that splitting is exponentially small, if one restricts oneself to the linear case of the quasiperiodically forced pendulum.

Indeed, the direct methods are usually not so transparent because of combinatorics. Moreover, the quasiflat estimate probably cannot be improved within this approach even for the linear problem, for one simply cannot see the exponential smallness behind it. In our paper we will end up showing that by a near-identity change of variables a quasiflat estimate can be made to look exponentially small, and we will come back to this topic not once further on. In addition, when the perturbation has an infinite number of harmonics, directly analyzing successive orders of perturbation theory, and drawing the induction proofs, becomes increasingly difficult due to not only combinatorial growth of the number of summands contributing in each order, which makes the series look Gevrey-1, but also due to the phenomenon that we refer to as *mode mixing*, which is expressed by the presence of the *supremum* in the first estimate of (6), to which we will also devote some attention further.

In this paper we are concerned with several main issues. Before we compile them, we want to fix the terminology once and for all.

- By the *splitting distance*, or the *splitting distance function*, we mean a vector, which is the difference in the actions on the unstable and the stable manifolds at time $t = 0$.

- By the *splitting size*, or the *transversality measure*, we mean the determinant Υ of the matrix, which is the Jacobian of the splitting distance function.

First, we will establish the exponentially small *upper bound* for the transversality measure of splitting of the homoclinic manifolds of the invariant tori surviving the perturbation for more than two degrees of freedom.

Second, we will discuss how the splitting exponent changes when the frequencies of the rotors become near-resonant, and this will be in compliance with the new phenomenon of *stability of resonances*, pointed out by Lochak [1992]. Under certain assumptions on the approximation properties of the frequency vector we will illustrate that the higher the multiplicity of a resonance, the larger the exponent in the upper bound for the transversality measure, so the smaller the splitting size. This situation is very similar to what happens to the optimal value of the Nekhoroshev exponent in the neighborhood of a resonance of a specified multiplicity, reaching $\frac{1}{2}$ for a full resonance.

Third, by considering a special example of a perturbation with an infinite number of harmonics we will validate the use of the Melnikov function to give the leading order behavior of the splitting distance in the case of non-resonant frequencies. This will enable us to obtain a *lower bound* for the splitting distance in this particular example by using a simple number-theoretical argument, and obtain the Nekhoroshev exponent in this lower bound.

Finally, we will focus on a particular problem of three degrees of freedom, for which using Continued fractions theory for evaluating the small divisors we will get the exact asymptotics for the splitting distance and its measure of transversality, for an asymptotically full measure set of frequencies and *for most of the values of the perturbation parameter.* Apropos of the statement in italics, we will demonstrate how changing the perturbation parameter can dramatically affect the intersection pattern and the size of the splitting itself, depending on the approximation properties of a particular frequency vector. This phenomenon, (once proposed by Chirikov) seems to be rather similar to what Lochak [1990] suggests, and most likely has the same nature as the intermittency phenomenon described in Lochak [1993] as an illustration why the spatial Nekhoroshev exponent is unlikely to be $\frac{1}{2}$ uniformly in the whole phase space.

We believe that the example in question is important and rather general. First of all, it illustrates the optimality of our upper bound for the splitting size at least for the models above. Also, it elucidates that generically, when the initial Fourier modes of the perturbation generate an $(n-1)$-dimensional lattice in Z^{n-1} for the case of n degrees of freedom, one can still consider Melnikov's method as a way to compute the answer to the problem, or at least as an estimate, if certain high (depending on ε and the approximation properties of the frequency vector) harmonics are present in the perturbation; otherwise they will appear in the higher orders of the Lindstedt series or can possibly be created by applying a finite number of near-identity symplectic changes of coordinates. The latter approach is well-developed and allows some flexibility if one pursues engineering certain harmonics in the perturbation for a given frequency vector, trying to maximize the corresponding Fourier coefficients. Note, that in Delshams and Seara [1992] or Gallavotti [1994] in the case of two degrees of freedom, if the perturbation didn't originally have the first Fourier mode, one would have to undertake the same procedure of putting *the perturbation* into a suitable Normal form by a sequence of near-identity transformations, before applying the Melnikov method, or to compute the higher order terms in the Lindstedt series until encountering the first Fourier mode. In principle, the situation with three degrees of freedom is rather similar, but made much more subtle by the appearance of small divisors in the exponents, which makes a "good" (the one that stands for the asymptotics of the splitting) Fourier mode not to be simply the first one, as it is in the case of two degrees of freedom, but to depend on the value of ε and the approximation properties of $\vec{\omega}$.

These aspects of the matter are studied in details in the last section of this paper, which involves no more than just simple arithmetics and the application of Continued fraction theory. The latter allows one to go much further and not only to prove the domineering role of the Melnikov function for a class of analytic perturbations, but also to come up with the constructive procedure, yielding two harmonics that determine the leading-order behavior of the splitting distance and the splitting size with exponential precision. This will give us an opportunity to visualize the whole pattern of the homoclinic connections plus to establish their generic non-degeneracy. It will also demonstrate how sensitive with respect to $\varepsilon \to 0$ the intersection pattern appears to be, illustrating certain non-uniformity in ε numerically observed in Simó [1994]. This non-uniformity seems to be inherent for the phenomena whose magnitude goes beyond all orders of perturbation theory, and may be related to what Lochak [1993] calls the intermittency phenomenon.

We consider a generalization of Arnold's example, similar to one, suggested by Lochak [1990], and studied by Chierchia and Gallavotti [1994], Gallavotti [1994], etc. representing an n-degrees of freedom Hamiltonian system, consisting of a pendulum weakly coupled with $n-1$ fast rotors via a small momenta-independent potential.

Generalized Arnold Model

$$H(y,x,\vec{I},\vec{\varphi}) = \sum_{j=1}^{n-1} \frac{I_j^2}{2} + \frac{y^2}{2} + \varepsilon(\cos x - 1) + \mu F(x,\vec{\varphi}), \tag{7}$$

where

$$(y,x) \in R\times T, \quad (\vec{I},\vec{\varphi}) \in R^{n-1}\times T^{n-1}, \quad T = R/Z.$$

We require that F be a trigonometric polynomial in the x-variable of finite degree ν_0 and that the average of F in the $\vec{\varphi}$-variables be zero for all x. Moreover, we require that F be *even* in the $(x,\vec{\varphi})$-variables, what will allow us to develop the symmetry argument that will be extremely useful in our analysis.

The multiplier ε in front of the pendulum potential energy term will be considered "small enough", real and positive, it's due to this factor that we call this Hamiltonian weakly hyperbolic; μ will be a small complex independent of ε parameter, which will lie in a circle of a small radius $\mu_0(\varepsilon) \sim \varepsilon^d$ to be defined. Withal, we will not pursue finding the optimal value for d.

As we have already mentioned several times earlier, in a certain way, this Hamiltonian can be viewed as a very simple representation of a Normal form for the n-degrees of freedom a-priori stable system, representing the global structure near a multiplicity-one resonant surface. More about the above Normal forms can be found in Rudnev and Wiggins [1996].

We have already not once mentioned the linear (and simpler) problem, or the rapidly quasiperiodically forced pendulum, which can be described by the Hamiltonian

$$H(y,x,\vec{I},\vec{\varphi}) = \vec{\omega}\cdot\vec{I} + \frac{y^2}{2} + \varepsilon(\cos x - 1) + \mu F(x,\vec{\varphi}), \tag{8}$$

with a diophantine constant frequency $\vec{\omega}$. This system can be useful to illustrate the ideas.

As a framework for our analysis, we use the standard approach, whose main steps are constructing the Normal form for the surviving whiskered tori, thus obtaining the local time-angle parameterizations for them, then analytically continuing in time their local stable and unstable manifolds for a wide complex range of parameters, to insure that the domains of parameterization for the global stable and the unstable manifold overlap, by a standard extension result that we prove closely to how it was performed by Delshams and Seara [1992]. So, we parameterize the manifolds in the *intrinsic variables*, namely those, pertaining to the Normal form. The use of these variables makes the time-angle representation of the trajectories on the whiskers look quite simple, in particular it prevents the mixing of the Fourier modes in the second and higher orders of perturbation theory.

Then we incorporate some direct methods and proceed with studying the trajectories in the configuration space (which is an n-torus). We actually show how the intrinsic time-angle parameterization and the manifolds themselves can be constructed directly explicitly order by order using the corresponding second-order system of ODE's. The convergence of this procedure will follow from the Normal form theorem and the extension result. This somewhat invalidates the purposefulness of the previous two steps in the spirit of Gentile [1995], who proved the convergence of the formal Lindstedt series for the whiskers.

Along with the parameterization of the manifolds we construct the initial conditions for the (configuration) trajectories (depending on the angle parameter). For negative time these trajectories are on the unstable manifold, whereas for positive time they are on the stable one. The time-derivatives for the trajectories generally turn out to be discontinuous at time-zero. This is the difference between the first derivatives of the trajectories at time equal to minus and plus zero, or the difference in the momenta (actions), that gives the measurement of the splitting distance. From the parity of the system, the splitting distance becomes an odd function of the angle parameter. This quantity, or more exactly the related Jacobian which measures the transversality, turns out to be exponentially small.

The splitting thus measured can be moved by $O(\mu)$ with the flow to the fixed Poincaré section $x = \pi$, where it is traditionally measured, and so on this section it will certainly also be exponentially small. Nevertheless, the corresponding $O(\mu)$ time-shift will depend on the angle parameter, and as a result, the harmonics mixing phenomenon will occur in the second and higher orders of standard perturbation theory, for the computation will involve multiplication of the Fourier series. That's why one cannot see the exponential smallness right on the fixed Poincaré section, it is "disturbed", and a seemingly optimal quasiflat (see Gallavotti [1994]) estimate ensues.

We emphasize that in this respect the case of three or more degrees of freedom is much more subtle in the following sense. If we have a two degrees of freedom case with one rotor only, the splitting distance will be a periodic function of one parameter. Therefore, if it looks exponentially small in one coordinate system, then a near-identity transformation, periodic in this parameter, will not make the situation obscure. Nevertheless, when there are two or more rotors, then a near-identity change of coordinates, quasiperiodic in the two or more parameters, will make the expression for the splitting distance cease looking exponentially small, donning a quasiflat mien instated. Thus, if one has the second "disturbed" coordinate system right from the start, there is no way to figure out the first (unique) one (in a certain sense it's like getting rid of the "noise"), and see the exponential smallness behind the quasiflat estimate. The latter, though, can be very misleading in the sense that we've already mentioned, and thought of as an optimal estimate can lead to the consequences that will contradict the Nekhoroshev theorem, such as the Arnold diffusion with the speed, algebraic in ε.

Another severe difficulty that we further emphasize is that the most significant quantity, characterizing the splitting, namely its measure of transversality, or the splitting size, is given as a determinant if $n \geq 3$, therefore, one has to be extremely careful with the error analysis, for the matrix in question will oftentimes turn out to be extremely ill-conditioned, especially when the frequencies are near-resonant.

For the Normal form, we use the analogue of the KAM theorem for the model (7), or the Birkhoff Normal form theorem in the linear case (8). The "hyperbolic" version of the KAM theorem was first proved by Graff [1974], the more recent references also embracing the singular case are Treshchev [1991], Eliasson [1994], Chierchia and Gallavotti [1994], and Rudnev and Wiggins [1996]; the latter contains the estimates and norm relations that we will often use. The proof of the Birkhoff Normal form theorem for the quasiperiodically forced pendulum can be found in Delshams et al. [1996]. It goes along the familiar lines of a KAM-like iterative scheme and being simpler, allows a larger maximum perturbation size then the KAM theorem applied to the model (7).

2 Problem Set-Up. Discussion of the Unperturbed System. Statement of the Main Results.

For the Hamiltonian (7) we assume that $\varepsilon > 0$ and $\mu \in C$ are two independent parameters. We refer to the Hamiltonian (7) with $\mu = 0$ as the unperturbed Hamiltonian.

The perturbation F in this model will be considered analytic in the $\vec{\varphi}$-variables in a complex strip of the width $\sigma_0 > 0$ with its sup-norm being $O(1)$ for all real x. If F is a finite trigonometric polynomial in $\vec{\varphi}$, it is certainly analytic in any strip, but the above condition will impose a constraint upon the coefficients of the polynomial.

We consider a *time-angle even* perturbation, namely the one satisfying $F(x,\vec{\varphi}) = F(-x,-\vec{\varphi})$, and one way to write it down is

$$F(x,\vec{\varphi}) = \sum_{\vec{k}\in Z^{n-1}} F_{\vec{k}}(x) e^{i(\vec{k}\cdot\vec{\varphi})}, \tag{9}$$

where the Fourier coefficients $F_{\vec{k}}(x)$ such that $F_{\vec{k}}(x) = F_{-\vec{k}}(-x)$ are trigonometric polynomials in $x \in T$ satisfying for all $x \in T$:

$$|F_{\vec{k}}(x)| \leq \exp(-|\vec{k}|\sigma_0), \tag{10}$$

where in the standard notation

$$|\vec{k}| = \sum_{i=1}^{n-1} |k_i|.$$

In addition, as we have already mentioned in the Introduction, we assume that

$$F_{\vec{0}}(x) =< F >= 0, \ \forall x \in T,$$

where $< \cdot >$ stands for the angle-average.

If $\mu = 0$, the Hamiltonian (7) describes the uncoupled motion of a pendulum in the (x, y) plane (these variables we will further call *hyperbolic*) and $n-1$ rotors, rapidly (compared to the time-scale of the pendulum) gyrating with constant frequencies.

The Hamiltonian vector field generated by (7) is given by:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \varepsilon \sin x - \mu g(x, \vec{\varphi}), \\ \dot{\vec{\varphi}} &= \vec{I}, \\ \dot{\vec{I}} &= -\mu \vec{f}(x, \vec{\varphi}). \end{aligned} \tag{11}$$

Here $g(x, \vec{\varphi}) = \partial_x F(x, \vec{\varphi}),\ \vec{f}(x, \vec{\varphi}) = \partial_{\vec{\varphi}} F(x, \vec{\varphi})$. This system was originally considered by Lochak [1990] and is a generalization of (2).

For $\mu = 0$ the motion, described by the Hamiltonian (7), is very simple. There is a $2n-2$ dimensional invariant manifold

$$\mathcal{M}_0 \equiv \{(x, y, \vec{\varphi}, \vec{I}) : (x, y) = (0, 0)\},$$

which is foliated by $(n-1)$-tori with frequencies $\vec{\omega}(\vec{I}) = \vec{I}$. The manifold $\mathcal{M}_0$ is normally hyperbolic with the Lyapunov exponent $\lambda_0 = \sqrt{\varepsilon}$. Its $2n-1$ dimensional stable and unstable manifolds coincide and form an invariant manifold $\mathcal{M}_0^h$ which is foliated by the coinciding stable and unstable manifolds (whiskers) of each individual torus. The separatrix of the pendulum is defined by the relation

$$\frac{y^2}{2} + \varepsilon(\cos x - 1) = 0. \tag{12}$$

The unperturbed trajectories in $\mathcal{M}_0^h$ are given by

$$\begin{aligned} x_0(\tau) &= 4 \arctan\left(e^{-\sqrt{\varepsilon}\tau}\right), \\ y_0(\tau) &= \frac{2\sqrt{\varepsilon}}{\cosh(\sqrt{\varepsilon}\tau)}, \\ \vec{\varphi}_0(\vec{\omega}, \tau, \vec{\alpha}, t_0) &= \vec{\alpha} + \vec{\omega}(\tau + t_0), \\ \vec{I}_0(\vec{\omega}) &= \vec{\omega}. \end{aligned} \tag{13}$$

where t is real, $\tau = t - t_0$ being "pendulum time", $\vec{\omega} \in D \subseteq R^n$, and $\vec{\alpha}$ and t_0 correspond to the initial condition at $t = 0$, and will be further interpreted as complex parameters. Further, representing the dependencies in terms of τ and t_0, rather than t will turn out to be fairly convenient for dealing with the perturbed problem.

For a $2n$-solution vector of (13) we will use the notation $\tilde{\Gamma}_0(\vec{\omega}, \tau, \vec{\alpha}, t_0)$, namely

$$\tilde{\Gamma}_0(\vec{\omega}, \tau, \vec{\alpha}, t_0) \equiv (x_0(\tau), y_0(\tau), \vec{\varphi}_0(\vec{\omega}, \tau, \vec{\alpha}, t_0), \vec{I}_0(\vec{\omega})),$$

often omitting the parameter $\vec{\omega}$ from this notation, when it's clear that we are dealing with a certain whiskered torus of a certain frequency.

As one can see $\tilde{\Gamma}_0(\tau,\vec{\alpha},t_0)$ is in fact analytic for $\tau \neq \frac{\iota}{\sqrt{\varepsilon}}\left(\frac{\pi}{2}+m\pi\right)$ for $m\in Z$ since $y_0(\tau)$ has simple poles at the latter points.

We will also use other notation for the unperturbed (and further perturbed) trajectories. Namely from (13) it's clear that we can write

$$\tilde{\Gamma}_0(\tau,\vec{\alpha},t_0)=\Gamma_0(\tau,\vec{\psi}),$$

where $\vec{\psi}=\vec{\alpha}+\vec{\omega}t=\vec{\alpha}+\vec{\omega}(\tau+t_0)$.

Our analysis will simultaneously apply to the tori of the normally hyperbolic manifold $\mathcal{M}_0$, whose frequencies $\vec{\omega}\equiv\partial_{\vec{I}}H|_{\mu=0}$ enjoy the classical diophantine condition, formulated in terms of two parameters γ,ϖ, $0<\gamma\leq 1,\ \varpi\geq n-2$, the latter will be considered fixed. In other words, we require that $\vec{\omega}\in\Omega_\gamma$ for

$$\Omega_\gamma\equiv\{\vec{\omega}:|\vec{\omega}\cdot\vec{k}|\geq\gamma|\vec{k}|^{-\varpi},\ \forall\vec{k}\neq\vec{0}\}. \tag{14}$$

Apropos of the parameter γ, we can let it depend on ε as follows:

$$\gamma=\gamma_0\varepsilon^b,\ \text{with}\ 0\leq b\leq\frac{1}{2}, \tag{15}$$

where γ_0 is $O(1)$ (and can be assumed to be exactly 1 without loss of generality). Note that if $b>\frac{1}{2}$, then (7) is no longer a model for the near-resonant behavior of an a-priori stable system (see Rudnev and Wiggins [1996] for details).

Once again, we will be dealing with a time-angle even analytic perturbation (9), so the latter can be represented by a convergent Fourier series as follows:

$$F(x,\vec{\varphi})=\sum_{k:\ |k_0|\leq\nu_0,\vec{k}\neq\vec{0}}F_k\exp(-|\vec{k}|\sigma_0)\cos(k_0x+\vec{k}\cdot\vec{\varphi}) \tag{16}$$

with the notation $k=(k_0,\vec{k}),\ k_0\in Z,\ \vec{k}\in Z^{n-1}$, where all the F_k are uniformly bounded from above, and $\nu_0>0$ being an integer constant. We also require (10) to be satisfied. It is to this class of perturbations that our exponentially small upper bound will apply.

As an example we will compute the leading order answer for the case when the frequency is away from the resonances for the following particular case of (16):

$$F(x,\vec{\varphi})=\sum_{\vec{k}\neq\vec{0}}P_{\vec{k}}(x)\exp(-|\vec{k}|\sigma_0)\cos(\vec{k}\cdot\vec{\varphi}), \tag{17}$$

namely when all the Fourier coefficients $F_{\vec{k}}(x)$ of F in the representation (9) have a maximum exponent in terms of $|\vec{k}|$:

$$F_{\vec{k}}(x)\sim\exp(-|\vec{k}|\sigma_0).$$

The functions $P_{\vec{k}}(x)=P_{-\vec{k}}(x)$ will be even trigonometric polynomials of finite degrees $\nu_{\vec{k}}$, not exceeding some integer constant $\nu_0\geq 1$ (in case when $\nu_0=0$ the pendulum and the rotors are uncoupled), for all $\vec{k}\neq\vec{0}$ let $P_{\vec{k}}(x)$ be bounded for all real $x\in T$.

For simplicity, we can assume that the Fourier coefficients of (16) and (17) are bounded from above to insure that the sup-norm

$$|F|\leq 1\ \ \forall x\in T,\ |\Im\vec{\varphi}|\leq\sigma_0.$$

Since we are dealing with the time-angle even perturbations, apropos of (17) we have to assume that all $P_{\vec{k}}(x)$ are even functions of the variable x; then they can be expressed as follows:

$$P_{\vec{k}}(x)=\sum_{j=0}^{\nu_{\vec{k}}}2^{-j}(2j-1)!A_{j\vec{k}}(1-\cos x)^j, \tag{18}$$

where the integers $\nu_{\vec{k}}$ are such that $0 \le \nu_{\vec{k}} \le \nu_0$, and $0 \le A_{j,-\vec{k}} = A_{j\vec{k}}$ by parity. The factors $2^{-j}(2j-1)!$ will further play a normalizing role.

So far, the restriction of (16) to (17) has solely pursued the purpose of warding off some very special cases and easing the algebra.

The next requirement is absolutely necessary to insure that the Melnikov function is at all useful. We will also assume that

$$\forall \vec{k} \neq \vec{0}, \forall j \in \{1,..,\nu_{\vec{k}}\}\ A_{j\vec{k}} \ge 0 \text{ and } \exists j_{\vec{k}} \in \{1,..,\nu_{\vec{k}}\}:\ A_{j_{\vec{k}}\vec{k}} \ge C_1 > 0 \tag{19}$$

for some positive constant C_1. Apparently, this is one of the ways to insure that all the Fourier modes in the Melnikov function (91) are nonzero.

It is a well known fact, claimed by the KAM theorem (and the Birkhoff Normal form theorem for the linear model (8)), that if the perturbation is small enough, it does not destroy, but only slightly deforms the tori, whose frequencies are in Ω_γ, as well as their local whiskers. In particular, the upper bound for the size of the perturbation is proportional to γ^2. The phase trajectories on such tori can be represented as quasiperiodic functions of $\vec{\psi} = \vec{\alpha} + \vec{\omega}t$, where $\vec{\alpha}$ stands for an initial condition. Moreover, for $\tau = t - t_0$ the Normal form enables one to represent the whiskers as graphs over $(\tau, \vec{\psi})$. The local unstable and stable whiskers can be analytically continued using the vector field itself to justify the choice of the "Poincaré section" $t = t_0 = 0$, where the splitting is going to be measured. This fact is stated further in more details in the Extension lemma. Furthermore, since in the unperturbed problem the actions of the rotors are the integrals of motion on the manifolds homoclinic to the invariant tori, their difference on the perturbed stable and unstable manifolds on the above "Poincaré section" $t = t_0 = 0$ can be used to measure how these manifolds break up under the perturbation, and it will be a quasiperiodic function of the parameter $\vec{\alpha}$ only. Moreover, for the nonlinear problem (7) there is an opportunity for heteroclinic intersections.

Regarding this, for a whiskered torus with the diophantine frequency $\vec{\omega}$ we will define the homoclinic splitting distance function as follows:

$$\begin{aligned}
\Delta(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= (\Lambda,\vec{\Delta})(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\
&\text{where} \\
\Lambda(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= y^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon) - y^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\
\vec{\Delta}(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= \vec{I}^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon) - \vec{I}^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon).
\end{aligned} \tag{20}$$

We will sometimes use the notation $\Lambda = \Delta_0$ and $\vec{\Delta} = (\Delta_1,..,\Delta_{n-1})$. We will often omit the $\vec{\omega}$-dependence of the homoclinic splitting, thinking of a fixed torus. Note, that for the rapidly quasiperiodically forced pendulum (8) the frequency is fixed, and the Λ measurement suffices. We will usually omit $\vec{\omega}$ in the above notation for the homoclinic splitting. We will also often drop μ and ε from the above dependencies, keeping in mind that there is no splitting for $\mu = 0$.

The measure of transversality of the splitting will be the quantity

$$\Upsilon = |\det \partial_{\vec{\alpha}} \vec{\Delta}|, \tag{21}$$

quite sufficient if one recalls that Λ and $\vec{\Delta}$ are dependent via energy conservation.

This is the measure of transversality, or the size of the splitting which is meant to be exponentially small when we say that splitting is exponentially small and in which one is interested in the Arnold diffusion problem.

There will be few last pieces of notation that we want to introduce. We will often encounter the quantities

$$\begin{aligned}
p_* &= \tfrac{\pi}{2\sqrt{\varepsilon}} - \sqrt{\varepsilon}, \\
\sigma_* &= \sigma_0 - \sqrt{\varepsilon}.
\end{aligned} \tag{22}$$

The first one is related to the pole of the unperturbed pendulum separatrix (see (13), whereas the second one indicates that after all the construction we will end up losing very little analyticity in the angle variables.

Apropos of our strategy of making estimates, we do not fancy finding any realistic smallness conditions for ε, always following the convention that ε is small enough to insure for any constants $K, d = O(1)$ and a small positive δ that

$$K \geq \varepsilon^{\delta},$$

$$1 \geq K\varepsilon^{-d} e^{-\varepsilon^{-\delta}}.$$

This ruse will save us the effort of evaluating many non-significant constants, and we will be utilizing it all the time. Also, once δ is introduced, we endow it with a role of taking care of the constants in the future as well, wherever it's in its power.

We need some extra assumptions about the diophantine properties of the frequencies that we will be dealing with.

Assumption 2.1 *Along with (14) assume that for a fixed number $s \in \{1, .., n-1\}$, the frequency $\vec{\omega}$ can be decomposed in the following way:*

$$\vec{\omega} = (\sqrt{\varepsilon}\vec{\omega}_0, \vec{\omega}_{\vec{s}}),$$

where $\vec{\omega}_{\vec{s}} = (\omega_{n-s}, .., \omega_{n-1})$ is such that any l-subvector of $\vec{\omega}_{\vec{s}}$ for $l = 2, .., s$ (so it includes $\vec{\omega}_{\vec{s}}$ itself) satisfies the diophantine condition (14) *with $\gamma = gamma_0$ (see (15)) and $\varpi = l - 2\delta$ for some small positive δ such that $0 < \delta \ll \frac{1}{2}$.*

Finally, there will be a little more notation. For a frequency-vector $\vec{\omega}$ we denote

$$\omega_+ = \max_{i=1,..,n-1} |\omega_i|.$$

Without loss of generality we can think that $\omega_+ = |\omega_1|$. We will also need "the second maximum", defined as

$$\omega_- = \max_{i=2,..,n-1} |\omega_i|,$$

and again without loss of generality we can think that $\omega_- = |\omega_2|$.

We formulate our main results, to the proof and analysis of which the rest of the paper is dedicated.

Theorem 2.1 (Exponential smallness of the splitting size) *Suppose, $n \geq 3$. Given a diophantine frequency vector $\vec{\omega} \in \Omega_\gamma$, satisfying Assumption 2.1 with $1 \leq s \leq n-1$, suppose that in the Hamiltonian (7) the perturbation is given by (16).*

For any small positive number δ, such that $0 < \delta \ll \frac{1}{2}$, there exists a small number $\varepsilon_0 > 0$, also depending upon all the other parameters of the problem, excluding μ and ε only, such that if $0 < \varepsilon < \varepsilon_0$ and for

$$\mu_0 = \varepsilon^{\max\left(\frac{19}{4} + \frac{n}{2} + \varpi + \delta,\ 2\nu_0 + 1 + \delta\right)}$$

for all $\mu \in C$ such that $|\mu| \leq \mu_0$, the homoclinic splitting is exponentially small.

Namely, the splitting distance Δ, defined in (20) is given by a convergent Fourier series in $\alpha \in T^{n-1}$ whose coefficients obey the following upper bound:

$$|\Delta_{\vec{k}}| \leq \frac{|\mu|}{\mu_0} \frac{\sqrt{\varepsilon}}{\vec{k} \cdot \vec{\omega}} e^{-|\vec{k}\cdot\vec{\omega}| p_* - |k| \sigma_*},$$

with the quantities p_* and σ_* coming from (22).

Moreover, the measure of transversality of the splitting respects the upper bound

$$\Upsilon_{\vec{k}} \leq \left(\frac{|\mu|}{\mu_0}\right)^{n-1} \exp\left\{-\sigma_0 \left(\frac{\pi\gamma_0}{\sqrt{\varepsilon}}\right)^{\frac{1}{s-2\delta}}\right\}.$$

Besides,

$$\vec{\alpha} :\ \alpha_i = [0 \, or \, \pi \quad for \ i = 1, .., n-1$$

yield homoclinic connections, i.e., the splitting distance at these points is zero.

Remark: One can clearly see that the smaller s, or, equivalently, the higher the multiplicity of a resonance to which $\vec{\omega}$ is close, the smaller the splitting size becomes. This phenomenon is reminiscent of Lochak's stabilizing properties of resonances obtained in his new proof of the Nekhoroshev theorem (Lochak [1992]). In particular, if $s = n-1$ and the frequency is diophantine with ϖ approaching $n-2$, then the above upper bound approaches the lower bound for the splitting distance, provided by the next theorem. Besides, the case $s=1$ is much simpler than the others, and it can be shown that the exponent in this case is exactly $\frac{1}{2}$. The case $s=2$ can possibly be treated deeper in terms of Continued fractions following the template that we further develop for the nonresonant case of $n=3$.

Theorem 2.2 (Asymptotics of the splitting distance) *Suppose, $n \geq 3$. Given a diophantine frequency vector $\vec{\omega} \in \Omega_\gamma$, satisfying Assumption 2.1 with $s = n-1$, suppose in the Hamiltonian (7) the perturbation is given by (17) and it satisfies (18), (19).*

For any small positive number δ, such that $0 < \delta \ll \frac{1}{2}$, there exists a small number $\varepsilon_0 > 0$, also depending upon all the other parameters of the problem, excluding μ and ε only, such that if $0 < \varepsilon < \varepsilon_0$ and for

$$\mu_0 = \varepsilon^{\max\left(\frac{15}{4}+\frac{n}{2}+\varpi+\delta,\ 2\nu_0+1+\delta\right)}$$

for all $\mu \in C$ such that $|\mu| \leq \mu_0$, the homoclinic splitting distance $\vec{\Delta}$ can be asymptotically found as

$$\vec{\Delta}(\vec{\omega},\vec{\alpha},\mu) = \mu\vec{M}\left\{1 + O\left(\frac{\mu}{\mu_0}\right) + O\left(e^{-\varepsilon^{-\frac{1}{2(n-1)}}}\right)\right\}$$

where

$$\vec{M}(\vec{\omega},\vec{\alpha}) = \sum_{\vec{k}\in Z^{n-1},\ |k|<K(\varepsilon)} \vec{k}\frac{\pi|\vec{k}\cdot\vec{\omega}|}{\varepsilon}\exp\left(-|\vec{k}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right)\sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}}\prod_{l=1}^{j-1}\left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon} + 4l^2\right)\sin(\vec{k}\cdot\vec{\alpha})$$

is the Melnikov function with $K(\varepsilon) = \left[\varepsilon^{-\frac{1}{2}+\delta}\right]$ and $\vec{\alpha} \in W_{\sigma_*}T^{n-1}$.

Besides, for $\vec{\alpha} \in W_{\sqrt{\varepsilon}}T^{n-1}$ the homoclinic splitting distance function $\vec{\Delta}$ satisfies the following lower bound in the sup-norm: for $i = 1,..,n-1$:

$$|\Delta_i|^{\infty}_{\sqrt{\varepsilon}} \geq \frac{|\mu|}{\mu_0}\exp\left(-C\varepsilon^{-\frac{1}{2(n-1)}}\right).$$

with

$$C = \omega_+(n-1)\left(\frac{2\sigma_0(n-1)}{\omega_-(n-2)}\right)^{\frac{n-2}{n-1}}\left(\frac{\pi}{2\omega_-}\right)^{\frac{1}{n-1}}.$$

Remark: In fact, computing the leading order for the splitting distance function, one ends up with (91), which coincides with the above expression for the Melnikov function up to an exponentially small error. Withal, to obtain the lower bound for the splitting distance we have implemented a trivial fact from the approximation theory, analogous to the well-known Dirichlet theorem, which immediately gave us the Nekhoroshev exponent, as it was predicted by Chirikov [1979], just as simultaneous approximation gave it to Lochak [1992] in his proof of the Nekhoroshev theorem. More about how the size of the splitting relates to the speed of the "Arnold diffusion" can be found in the latter two references, to a large extent it has been lately made rigorous by Bessi [1996](A,B).

In the case of three degrees of freedom one can move much farther. Namely, the bounds in the two preceding Theorems dealt with the powers of ε in the exponents only, not really worrying much about the constants by which these powers of ε were multiplied. Namely, the philosophy of the estimates was such that for some positive constants C_1 and C_2 we would always think that $\varepsilon^d \gg \exp(-C_1\varepsilon^{-a}) \approx \exp(-C_2\varepsilon^{-a}) \gg \exp(-\varepsilon^{-a-\delta})$. But if $n=3$ one can impinge upon a more ambitious task of computing the exact values of these constants and making the estimates much more accurate thinking that if $C_2 > C_1$, then $\exp(-C_1\varepsilon^{-a}) \gg \exp(-C_2\varepsilon^{-a})$. In this sense, using a number of trivial facts from the theory of Continued fractions, we will be able to find two terms in the Melnikov function, which exceed all the others and account for its leading-order behavior.

To be able to formulate the theorem we rewrite for a 2-vector $\vec{\omega}$:

$$\vec{\omega} = \omega_+(\pm 1, \beta),$$

where we have assumed that the absolute value of the first component of the frequency vector $\vec{\omega}$ is larger than the absolute value of the second component, and that $\omega_+ = |\omega_1|$ is independent of ε.

Apropos of the dot products $\vec{k}\cdot\vec{\omega}$ we can see that from parity properties of the Melnikov function $\vec{M}$ from Theorem 2.2 it's enough to consider $k_2 \geq 0$. We can rewrite

$$|\vec{k}\cdot\vec{\omega}| = \omega_+|\pm k_1 + k_2\beta| \equiv \omega_+|-p + q\beta|,$$

where $\beta = \frac{\omega_2}{\omega_+}$, $q = k_2 \geq 0$, and $p = \mp k_1$.

Surely $0 < |\beta| < 1$. Suppose, β can be represented by a Continued fraction expansion

$$\beta = [a_0, a_1, ...],$$

where a_0 equals the integer part of β, so it's either 0 or -1, whereas all the following entries are positive integers. A brief recourse into the theory of Continued fractions is taken in Section 6.4.1. To formulate the theorem we need the fact that the continued fraction uniquely encodes the sequence $\{\frac{p_n}{q_n}\}$ of the best approximations to the irrational β, a fraction $\frac{p}{q}$ being called the best approximation if no other fraction with the same or smaller denominator approximates the given irrational with the same precision or better.

As for the other terms $a_i,\ i \geq 1$ in the Continued fraction for β, we assume the following:

Assumption 2.2 *Given the ultraviolet cutoff parameter $q_\varepsilon = [\varepsilon^{-\frac{1}{2}+\delta_0}]$, where $[\cdot]$ stands for an integer part, and $0 < \delta_0 \ll \frac{1}{2}$, suppose that for $i = 1, .., n_\varepsilon$, the latter being defined as*

$$n_\varepsilon = \max\{i \geq 1 :\ q_i \leq q_\varepsilon\},$$

where q_i is the denominator in the ith best approximation to β (an ith convergent), the entries a_i in the Continued fraction for β are bounded as follows:

$$a_i \leq \bar{a} = \bar{a}_0\varepsilon^{-\delta_2}$$

for some rather small non-negative δ_2.

Note, that Section 6.4, which is entirely devoted to establishing the Theorem to be formulated, does not use the diophantine condition (14), dealing with Assumption 2.2 instead; in particular the above defined quantity $\bar{a}$ must not be confused with γ from (14).

The upshot of the case of three degrees of freedom is that first, one can easily establish that in the series for the splitting distance function under the ultraviolet cutoff, those terms that correspond to the best approximations to β significantly exceed all the rest. Thus, the series can be indexed by one integer rather than two. Analyzing the exponents in the members of this series, we establish that the sequence of these exponents has either one single minimum (V-shape) for some $\vec{k}^* = (-p_{n_*}, q_{n_*})$, where p_{n_*} and q_{n_*} are the numerator and the denominator of the n_*th best approximation to β, or two neighboring local minima (W-shape) for some for $\vec{k}^* = (-p_{n_*}, q_{n_*})$, and $\vec{k}^{**} = (-p_{n_*+2}, q_{n_*+2})$. In the former case we will be able to choose the second minimum exponent, which will be indexed by either $n_* - 1$ or $n_* + 1$, in the latter case we will prove that n_* and $n_* + 2$ also yield the two minimum exponents! . Thus, we will find the two maximum terms in the Fourier series for the splitting distance, which will account for its leading-order behavior, as well as the measure of transversality. Besides, since the exponents explicitly depend on ε, we will have to avoid its certain values to insure that the above minima of the exponents are sharp enough.

For the reader, not well familiar with the theory of Continued fractions, the ensuing statement of the Theorem may seem confusing. Indeed, this Theorem is lead to through a fairly lengthy argument of Section 6.4 and is by no means merely the existence statement, although it may seem to be replete with various possibilities. Conversely, given $\vec{\omega}$ and ε, we suggest a simple algorithm which will allow to find the aforesaid maximum terms with no ambiguity.

Theorem 2.3 (Splitting distance and size for $n=3$) *Let the number of degrees of freedom $n=3$. Suppose that the smallness conditions of Theorem 2.1 are satisfied. Suppose, β satisfies Assumption 2.2. Suppose, the positive numbers δ_j for $j=0,1,2,3$, satisfy the following inequalities:*

$$\begin{array}{lllllll} \frac{1}{2} & & & -3\delta_2 & -\delta_3 & \geq 0, \\ \frac{1}{2} & & -\delta_1 & -2\delta_2 & -\delta_3 & \geq 0, \\ \frac{1}{4} & -\delta_0 & & -\frac{1}{2}\delta_2 & -3\delta_3 & \geq 0, \\ \frac{1}{4} & & -\delta_1 & -\frac{3}{2}\delta_2 & -2\delta_3 & \geq 0, \\ & \delta_0 & & -2\delta_2 & -2\delta_3 & \geq 0, \\ & & \delta_1 & -2\delta_2 & -\delta_3 & \geq 0. \end{array} \tag{23}$$

For the asymptotically full-measure set of the values of β and the asymptotically full-measure set of the values of $\varepsilon < \varepsilon_0$, there exist two indices $n_* > 0$ and $n_{**} > 0$ such that either $n_{**} = n_* + 1$, or $n_{**} = n_* + 2$, depending on $\vec{\omega}, \varepsilon, \sigma_0$, such that for $\vec{k}^* = (\mp p_{n_*}, q_{n_*})$ and $\vec{k}^{**} = (\mp p_{n_{**}}, q_{n_{**}})$, where $\frac{p_{n_*}}{q_{n_*}}$ and $\frac{p_{n_{**}}}{q_{n_{**}}}$ are the accordingly indexed best approximations to β, the splitting distance function can be asymptotically found as:

$$\vec{\Delta}(\vec{\omega}, \vec{\alpha}) = 2\mu\frac{\pi}{\varepsilon} \sum_{\vec{k}\in\{\vec{k}^*,\vec{k}^{**}\}} \vec{k}|\vec{k}\cdot\vec{\omega}| \sum_{j=1}^{\nu_{\vec{k}^*}} A_{j\vec{k}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon} + 4l^2\right) \mathcal{E}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha})$$

$$\times \left[1 + O\left(e^{-\varepsilon^{-\delta_3}}\right)\right],$$

where

$$\mathcal{E}_{\vec{k}} = \exp\left(-|k|\sigma_0 - \frac{\pi|\vec{k}\cdot\vec{\omega}|}{2\sqrt{\varepsilon}}\right) \equiv \exp(-E_{\vec{k}}),$$

with the relative error not exceeding

$$O\left(\frac{|\mu|}{\mu_0} + e^{-\varepsilon^{-\delta_3}}\right).$$

One of the members of the set $\{E_{\vec{k}^*}, E_{\vec{k}^{**}}\}$ (the second minimum exponent denoted as E_2 with the according notation q_2 for the denominator of the corresponding Continued fraction) exceeds another (the absolute minimum exponent, denoted as E_1 with the according notation q_1 for the denominator of the corresponding Continued fraction) by at least a quantity $O\left(e^{\varepsilon^{-\delta_3}}\right)$.

The absolute minimum exponent lies within the following bounds:

$$\sqrt{\frac{2\pi\omega_+\sigma_0}{(\bar{a}_0+2)}}\varepsilon^{-\frac{1}{4}+\frac{1}{2}\delta_2} \leq E_1 \leq 8\sqrt{\pi\omega_+\sigma_0}\varepsilon^{-\frac{1}{4}}.$$

The second minimum exponent has the upper bound:

$$E_2 \leq 8\sqrt{\pi\omega_+\sigma_0(\bar{a}_0+2)}\ \varepsilon^{-\frac{1}{4}-\frac{1}{2}\delta_2}.$$

The denominators of the corresponding Continued fractions lie within the following intervals

$$\begin{array}{lcccl} \mathcal{K}(\varepsilon)\frac{1}{2(\bar{a}+1)} & < & q_1 & < & 2\mathcal{K}(\varepsilon), \\ \mathcal{K}(\varepsilon)(\bar{a}+2)^{-\frac{3}{2}} & < & q_2 & < & \mathcal{K}(\varepsilon)(\bar{a}+2)^{\frac{1}{2}}, \end{array}$$

where $\bar{a} = \bar{a}_0\varepsilon^{-\delta_2}$.

Moreover, for the zeroes of the splitting distance function which can be approximately computed from its above asymptotics and whose number is at least $4|\beta|\varepsilon^{-\frac{1}{2}+2\delta_2+\delta_3}$*, the quantity* $\Upsilon \neq 0$ *will be asymptotically given as*

$$\Upsilon = 4\mu^2 \left(\frac{\pi}{\varepsilon}\right)^2 \mathcal{E}_{\vec{k}^*}\mathcal{E}_{\vec{k}^{**}} \prod_{\vec{k}\in\{\vec{k}^*,\vec{k}^{**}\}} |\vec{k}\cdot\vec{\omega}| \sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}} \prod_{l=1}^{j-1}\left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon} + 4l^2\right).$$

with the same maximum relative error as for the splitting distance function above.

These results hold true for $|\beta| \in \mho \subset (0,1)$*, such that the measure of its complement* $\bar{\mho} = (0,1)\setminus\mho$ *is bounded as*

$$mes\ \bar{\mho} \leq \varepsilon^{\frac{\delta_2}{2}},$$

and for all $0<\varepsilon<\varepsilon_0$*, except for an open set, which is the union of a countable number of open intervals centered at the sequence of the so-called critical values, which accumulates at zero and is nowhere dense. The union of all these intervals has the relative measure of the order* $O(\varepsilon^{\delta_1-2\delta_2})$*.*

Remark 0: Clearly, (23) has a solution in positive real numbers. One can assign e.g. $\delta_0=\delta_1=\frac{1}{12}$, $\delta_2=\delta_3=\frac{1}{64}$.

3 The Normal Form. Local Dynamics on the Whiskers

3.1 The KAM theorem

Now we are at the point where we need to formulate our version of the KAM theorem that would quantify what has been discussed in the preceding passages to accoutre us with the local Normal form for the surviving weakly hyperbolic tori and their local whiskers for $\varepsilon>0$. A proof may be found in Rudnev and Wiggins [1996]. Although this is a purely technical matter, but the important thing for establishing the lower bounds and the leading order behavior of the splitting distance function would be to lose very little analyticity in the angle variables in the process of transforming to the Normal form. It is well known that the analyticity loss is caused through the use of the Cauchy inequalities in the standard estimates for the KAM-like iterative schemes.

Our theorem, in its most general formulation, applies to the a-priori unstable systems. However, the degenerate nature of the system (7) as expressed by the dependence of the Lyapunov exponent on ε, does not cause extra convergence difficulties if the smallness conditions for the size of the perturbation are satisfied (as is shown in Treshchev [1991] and in a different setting in Rudnev and Wiggins [1996]). In the utmost generality an a-priori unstable Hamiltonian is real-analytic in some complex extension of the domain, specified below, and may be represented as follows:

$$H(y,x,\vec{I},\vec{\varphi},\varepsilon,\mu) = h_0(\vec{I},\varepsilon) + P(y,x,\vec{I},\varepsilon) + \mu F(y,x,\vec{I},\vec{\varphi},\varepsilon,\mu), \tag{24}$$

where

$$(y,x)\in R\times R,\ \vec{I}\in D\subseteq R^{n-1}, \vec{\varphi}\in T^{n-1}.$$

The dependence of the above Hamiltonian upon a real positive fixed parameter ε is optional, whereas μ is a small complex perturbation parameter. The perturbation F must be uniformly bounded in some norm in some complex extension of the domain of $H(\cdot)$. Note that the quantities standing behind the notation for the functions, variables, and parameters in this section generally are not the same as in the main body of the paper.

More specifically, the projection of the domain of the Hamiltonian (24) on the real space is the direct product $\bar{B}^2\times D\times T^{n-1}$, where $\bar{B}^2$ stands for an open disk in the (y,x)-plane and $D\subseteq R^{n-1}$ is an open set in the action space. By its complex extension we understand the direct product of B^2_κ - a complex C^2-ball of radius κ for some positive κ, and the complex neighborhoods of D, and T^{n-1}, such that for some positive r,σ one has $|\Im\vec{I}|\leq r$, $|\Im\vec{\varphi}|\leq\sigma$ with the standard maximum-component notation for the vector quantities. We shall use the notation $D_{r,\sigma,\kappa}$ for the above complex extension of our domain.

Generally we will write $W_\sigma T^l$ for the above complex extension of an l-torus, omitting the superscript l for $l = 1$, and B^l_κ for an l- dimensional complex ball, also using $\bar{B}^l_\kappa$ for its projection onto the real space and sometimes omitting the superscript l for $l = 1$.

For $\mu = 0$ the Hamiltonian (24) must satisfy some basic non-degeneracy conditions, namely the Hessian matrix $Q = \partial^2_{\vec{I}} H|_{\mu=0}$ and its inverse must have non-vanishing determinants for all $\vec{I}, y, x$, both being uniformly bounded from below by some positive constant $R_0(\varepsilon)$. Note, that ε is considered fixed and strictly positive. Also for all $\vec{I}$ the origin $(y, x) = (0, 0)$ must be a hyperbolic fixed point with the real part of the Lyapunov exponent $\lambda(\vec{I}, \varepsilon)$

$$\lambda(\vec{I}, \varepsilon) \equiv \sqrt{\left|(\partial^2_{xy} P)^2 - \partial^2_{yy} P \partial^2_{xx} P\right|_{(y,x)=(0,0)}}$$

being uniformly bounded away from zero by some positive quantity $\lambda_0(\varepsilon)$. We say that such a Hamiltonian is *non-degenerate with the parameters* R_0, λ_0.

In the standard notation the frequency is defined as $\vec{\omega} = \partial_{\vec{I}} H|_{\mu=0}$; when we speak of the frequency map Ψ, we assume that it acts as the identity upon the variables $(x, y, \vec{\varphi})$, namely

$$\Psi: \ (x, y, \vec{\varphi}, \vec{I}) \to (x, y, \vec{\varphi}, \vec{\omega}(y, x, \vec{I}, \varepsilon)).$$

Naturally, for the Hamiltonian (7) the frequency map is the identity, so $R_0 = 1$, besides $\lambda_0 = \sqrt{\varepsilon}$.

Further we will be usually omitting the dependencies on ε, for it will be assumed to be incorporated in the parameters describing the "unperturbed dynamics" such as λ, etc.

The first fact to mention is that for $\mu = 0$ there is a canonical transformation $\mathcal{R}: \ (x^*, y^*, \vec{\varphi}^*, \vec{I}) \to (x, y, \vec{\varphi}, \vec{I})$, that does not affect the action variables, such that the unperturbed Hamiltonian in the new variables depends only on $\vec{I}$ and the product $x^* y^*$. This transformation is defined in the direct product of some small neighborhood of the origin in the hyperbolic variables, the size thereof depending on the bound λ_0 for the Lyapunov exponent, some open subset of the complex extension of D in the action variables, and a complex neighborhood of an $(n-1)$-torus of possibly smaller analyticity width than the original value of σ in the angle variables, This is a long since well-known fact, a particular case of which is Birkhoff Normal form near a hyperbolic equilibrium, see Moser [1956]. A version of the (iterative) proof of this fact can be found in Chierchia and Gallavotti [1994], or it can be straightforwardly synthesized from the more general hyperbolic KAM-type proof. We borrow the statement from the latter mentioned reference with some slight casual modifications appropriate for our use.

Lemma 3.1 *For every $\vec{I}^* \in D$, $\varepsilon > 0$ there exists a complex ball $B^{n-1}_{r^*}$ centered at $\vec{I}^*$ in the action space with the radius $r^*(\vec{I}^*) \leq r$, such that for every $\vec{I} \in B^{n-1}_{r^*}$ there exists a real analytic canonical transformation $\mathcal{R}: \ (x^*, y^*, \vec{\varphi}^*, \vec{I}) \to (x, y, \vec{\varphi}, \vec{I})$, acting upon the action variables as an identity and not changing the value of the Lyapunov exponent. The domain of the new variables $(x^*, y^*, \vec{\varphi}^*, \vec{I})$ is $B^2_{\kappa^*} \times W_{\sigma_1} T^{n-1} \times B^{n-1}_{r^*}$ with some smaller analyticity parameters $0 < \kappa^* < \kappa$, $0 < \sigma_1 < \sigma$, where κ^* can be made proportional to the bound λ_0 for the real part of the Lyapunov exponent.*

This transformation casts the unperturbed Hamiltonian into the normal form:

$$H|_{\mu=0} \circ \mathcal{R} = h_0(\vec{I}, \varepsilon) + P^*(J, \vec{I}, \varepsilon). \tag{25}$$

where $J = x^* y^*$, the Lyapunov exponent $\lambda(\vec{I}, \varepsilon)$ being equal to $\partial_J P^*(J, \vec{I}, \varepsilon)_{J=0}$.

With our particular choice of Hamiltonian (7) or (8) the pendulum and the rotors are uncoupled for $\mu = 0$, and this implies the following (technically) important facts:

- The transformation $\mathcal{R}$ exists globally for all $\vec{I} \in D$ and does not depend on it and on the angles $\vec{\varphi}$ either.
- Putting the Hamiltonian (7) into the Normal form as stated in Lemma 3.1 does not cause any analyticity loss in the $\vec{\varphi}$-variables.

- The transformation $\mathcal{R}$ will be defined in the hyperbolic variables in a complex ball of radius $\kappa^* = R\sqrt{\varepsilon}$ for some constant $R > 0$, independent of ε.

To avoid too much notation we shall further omit the star indices.

The theorem that we formulate next pertains to the a-priori unstable Hamiltonians, already put into the Normal form of Lemma 3.1, namely:

$$\begin{aligned} H &= h_0(\vec{I},\varepsilon) + P(J,\vec{I},\varepsilon) + \mu F(y,x,\vec{I},\vec{\varphi},\varepsilon,\mu) \\ &\equiv h(J,\vec{I},\varepsilon) + \mu F(y,x,\vec{I},\vec{\varphi},\varepsilon,\mu), \end{aligned} \tag{26}$$

where $J = xy$. As we have already mentioned, the above Hamiltonian will be analytic with some positive analyticity parameters r, σ in the variables $\vec{I}, \vec{\varphi}$, some κ in the hyperbolic variables y, x, and e.g. $|\mu| \leq 1$.

We will briefly describe the properties of the norm that was used in Rudnev and Wiggins [1996] to prove the KAM theorem for the a-priori unstable systems, which generalizes the Fourier norm, used by Pöschel [1993], for the case when hyperbolicity is present, with a little notation and one simple technical fact that we will need.

Given the function $u(y,x,\cdot)$, analytic in $(y,x) \in B^2_\kappa$ we define as $\overline{u(y,x,\cdot)}$ the absolute sum of its Taylor series in y, x coordinates near $(y,x) = (0,0)$. Namely if

$$u(y,x,\cdot) = \sum_{i,j=0}^{\infty} u(\cdot)_{ij} y^i x^j,$$

then

$$\overline{u(y,x,\cdot)} \equiv \sum_{i,j=0}^{\infty} |u(\cdot)_{ij} y^i x^j|.$$

Also, if $u(y,x,\cdot,\vec{\varphi})$ is analytic in in the angles $\vec{\varphi}$ in a complex neighborhood of an l-dimensional torus $\vec{\varphi} \in W_\sigma T^l$ and given as a Fourier series

$$u(y,x,\cdot,\vec{\varphi}) = \sum_{\vec{k} \in Z^l} u_{\vec{k}}(y,x,\cdot) e^{\iota(\vec{k}\cdot\vec{\varphi})},$$

we define its Absolute norm as

$$|u|_{\sigma,\kappa} \equiv \sup \sum_{\vec{k} \in Z^l} \overline{u_{\vec{k}}(y,x,\cdot)} e^{|\vec{k}|\sigma},$$

where the *supremum* is taken throughout all the values of $(y,x) \in B^2_\kappa$ and the whole domain of all the other variables and parameters, denoted by $(\cdot)$.

In a particular case, when the function u depends on the variables $(y,x,\vec{I},\vec{\varphi})$ and $l = n-1$ in the domain described above, instead of $|u|_{\sigma,\kappa}$ we will write $|u|_{r,\sigma,\kappa}$ to specify exactly the domain of analyticity.

If a function depends solely upon the angle variables, the Absolute norm becomes the Fourier norm used by Pöschel [1993]. If this is the case, then the index κ certainly gets dropped in the notation. In particular, for $\sigma = 0$ we call it the Fourier norm "with zero width", equal to the sum of the absolute values of the Fourier coefficients.

How the Absolute norm is related to the common *sup*-norm (for the latter we will often use the notation $|u|^\infty_{r,\sigma,\kappa}$ to specify the analyticity parameters), is stated in the following Proposition, the simple proof of which can be found in Rudnev and Wiggins [1996].

Proposition 3.1 *For the sup–norm* $|\cdot|^\infty_{r,\sigma,\kappa}$ *on* $D_{r,\sigma,\kappa}$ *for all* $\chi > 0,\ \eta > 0$ *one has:*

$$|u|^\infty_{r,\sigma,\kappa} \leq |u|_{r,\sigma,\kappa} \leq \frac{(\kappa+\eta)^2}{\eta^2} \coth^{n-1}\left(\frac{\chi}{2}\right) |u|^\infty_{r,s+\chi,\kappa+\eta}. \tag{27}$$

By rescaling μ we can achieve $|F|_{r,\sigma,\kappa} = 1$ in (26).

The hyperbolic KAM theorem, whose statement is coming up, yields the Normal form for the Hamiltonian (26) on a Cantor set of whiskered tori with frequencies in Ω_γ, if the perturbation is small enough. In particular, in the smallness condition the loss of analyticity in the angle variables is taken into account explicitly, with the analyticity and non-degeneracy parameters of the system having optional dependence on ε.

Theorem 3.1 *Suppose the Hamiltonian (26) is analytic in $D_{r,\sigma,\kappa}$, and non-degenerate with the parameters R_0, λ_0, and $|\mu| \leq 1$.*

If in the whole domain for the variables and parameters in the Absolute norm

$$|F|_{r,s,\kappa} \leq 1, \tag{28}$$

then given $\sigma_1 < \sigma$, one can find some small constant $r_0 = r_0(r, \sigma - \sigma_1, \varpi, R_0, \kappa, \lambda_0)$, so that there exists a small positive $\mu_0 = \mu_0(R_0, \lambda_0, r_0, \sigma - \sigma_1, \kappa, \varpi)$, such that for all $\mu :\ |\mu| \leq \mu_0$ there exists a symplectic near identity map

$$\Xi :\ (q, p, \vec{\psi}, \vec{I'}) \to (x, y, \vec{\varphi}, \vec{I}),$$

casting the Hamiltonian (26) into the form

$$H \circ \Xi = H_+(J', \vec{I'}, \varepsilon, \mu),$$

where $J' = pq$ with the projection of the domain thereof on the action space being a Cantor set $\Psi_+^{-1}\Omega_\gamma$, with $\Psi_+^{-1} : (\cdot, \vec{\omega}) \to (\cdot, \vec{I'}(\vec{\omega}, J', \varepsilon, \mu))$ being the inverse of the new frequency map Ψ_+ given as $\Psi_+ : (\cdot, \vec{I'}) \to (\cdot, \partial_{\vec{I'}} H_+)$.

The map Ξ is real analytic in $q, p, \vec{\psi}, \mu$ for $(q, p) \in B^2_{\frac{\kappa}{2}}$, $\vec{\psi} \in W_{\sigma_1} T^{n-1}$, $\mu \in B^1_{\mu_0}$; for real J', μ it is C^∞ in the sense of Whitney on the set $\Psi_+^{-1}\Omega_1$. If the initial Hamiltonian is C^∞ in ε, then the maps Ξ, Ψ_+, Ψ_+^{-1} are C^∞ in ε as well.

The transformation Ξ induces a transformation $\Phi = \Psi \circ \Xi \circ \Psi_+^{-1}$, such that the projection of its domain on the frequency space of the rotors is the set Ω_γ, and it is real analytic in the variables $(q, p, \vec{\psi})$ in the same domains as the above transformation Ξ and C^∞ - in the sense of Whitney - in the frequency variables on the set Ω_γ.

In particular, one can take

$$r_0 = \min\left(\frac{1}{2}(4\varpi + 20)^{-(\varpi+1)} \left(\frac{\sigma - \sigma_1}{3}\right)^{\varpi+1}, \frac{1}{2}\lambda_0^{\frac{\varpi+1}{\varpi}}, r\right), \tag{29}$$

and

$$\mu_0 \leq \gamma^2 \min\left(\frac{2^{-2\varpi-13} r_0^{\frac{2\varpi+1}{\varpi+1}} (\sigma - \sigma_1)}{27 R_0^2}, \frac{2^{-2\varpi-14} r_0^{\frac{\varpi}{\varpi+1}} \kappa^2}{27}, \frac{2^{-11} r_0^2}{9 R_0^5}, \frac{\lambda_0 \kappa^2}{300}\right). \tag{30}$$

In addition, provided the above smallness conditions are satisfied, the new Hamiltonian H_+ is non-degenerate with the parameters R_0', λ_0' which differ from R_0, λ_0 by no more than a factor $\left(1 + \frac{|\mu|}{\mu_0}\right)$.

Remark: The constant μ_0 depends on the analyticity and non-degeneracy parameters of the problem, thus in particular, on ε. Moreover, it turns out to be proportional to the power of the analyticity loss in the angle variables, which we shall also choose depending on ε. The latter will be further rather important, so we'll reserve for it the notation:

$$\chi(\varepsilon) = \sigma - \sigma_1.$$

This parameter $\chi(\varepsilon)$ we'll choose the same as χ in the equation (27) for the norm relations.

3.2 The Normal form. Parameterization of the local whiskers

Now we would like to apply the results of the previous section to our specific choice of the Hamiltonian (7) with the perturbation (16) for the case of n degrees of freedom.

The first step will be to apply Lemma 3.1 to put the unperturbed Hamiltonian into the Normal form near the origin in the hyperbolic variables by some symplectic real-analytic transformation $\mathcal{R}$, which, as we mentioned, will act as the identity upon the $(\vec{I},\vec{\varphi})$-variables. The projection of the domain of this transformation on the "new" hyperbolic variables will be a complex ball B^2_κ, where $\kappa = R\sqrt{\varepsilon}$ with some choice of a positive constant $R<1$, independent of ε.

In other words, we can express this as follows:

$$\begin{aligned} x &= \Theta_0^1(y^*,x^*,\sqrt{\varepsilon}), \\ y &= \Theta_0^2(y^*,x^*,\sqrt{\varepsilon}), \end{aligned} \tag{31}$$

where the functions $\Theta_0^{1,2}$ will be analytic in $(y^*,x^*)\in B^2_\kappa$ and form a symplectic pair.

The Hamiltonian (7) will be remolded into

$$H\circ\mathcal{R} = \frac{\vec{I}^2}{2} + \sqrt{\varepsilon}P^*(J,\sqrt{\varepsilon}) + \mu F^*(y^*,x^*,\vec{\varphi}), \tag{32}$$

with $P^*(J,\sqrt{\varepsilon})$ analytic in $J=x^*y^*$ and C^∞ in $\sqrt{\varepsilon}$. Moreover, $\partial_J P^*|_{J=0}=1$. In addition, in the whole domain the *sup*-norm $|F^*|\le 1$.

Furthermore, the parameter r indicating the analyticity width in the action variables can be chosen equal to 1. Besides, given σ_0, according to (27) the Absolute norm of the perturbation F in (17) in the domain $D_{r,\sigma_0-\chi(\varepsilon),\kappa}$ won't exceed $O(\chi^{-n+1})$. That's why we will satisfy this requirement if for some $\delta>0$ we first choose μ_0 such that

$$\mu_0 = \varepsilon^{\frac{n-1}{2}+2b+\delta}, \tag{33}$$

where b comes from (15).

Besides, as we have already mentioned $R_0=1$, $\lambda_0=\sqrt{\varepsilon}$.

We can now apply Theorem 3.1, with $\sigma=\sigma_0-\chi(\varepsilon)$, $\sigma_1=\sigma_0-2\chi(\varepsilon)$ if we satisfy the smallness conditions (29), (30).

For the future we'll assume

$$\chi(\varepsilon) = \frac{1}{4}\sqrt{\varepsilon}, \tag{34}$$

which will save us some algebra, because we will have to lose analyticity in the angle variables four times: evaluating the Absolute Norm of the perturbation (17), applying the KAM theorem, proving further the Extension lemma, and differentiating the splitting distance function to get the splitting size.

Then for any $\delta>0$ if ε small enough, we can choose μ_0 as follows:

$$\mu_0 = \varepsilon^{\frac{n+1}{2}+\varpi+\delta}. \tag{35}$$

For the linear Hamiltonian (8) the analogous result will be the Birkhoff Normal form theorem, see Delshams et al. [1996]. The proof of the latter can actually be extracted from the hyperbolic KAM proof if one looks at it carefully, and hence the smallness condition (35) will embrace this particular case. We repeat, that we are not concerned with optimal values for μ_0, so the latter statement can be interpreted as "there is a finite $d>0$ such that for $|\mu|\le\mu_0\le\varepsilon^d$ the Normal form can be constructed".

We can easily derive a more dynamical corollary of Lemma 3.1 and Theorem 3.1 (see Chierchia and Gallavotti [1994] or Rudnev and Wiggins [1996] for details), which will also embrace the linear model.

Corollary 3.1 *Suppose, for the Hamiltonian (7) or (8) the condition (35) is satisfied. Consider a set of variables* $(p,q,\vec{\omega},\vec{\psi})$ *such that* $(p,q)\in B^2_{\kappa/2}$, $\vec{\omega}\in\Omega_\gamma$, $\vec{\psi}\in W_{\sigma_1}T^{n-1}$.

There exist uniquely defined functions $\Theta_0^{1,2}(p,q,\sqrt{\varepsilon})$, *which are the same as in (31), and for* $i=1,..,4$ *the functions* $\Theta_1^i(p,q,\vec{\omega},\vec{\psi},\sqrt{\varepsilon},\mu)$. *These functions are analytic in the variables* $p,q,\vec{\psi}$ *in the above domains,* *also analytic in the parameter μ for $|\mu|\leq\mu_0$, the latter satisfying (35), and C^∞ in $\sqrt{\varepsilon}$, and the sup-norm of the latter group of functions is uniformly bounded by 1 in the direct product of the domains for their variables and parameters.*

There exist n-dimensional invariant surfaces which in terms of these functions can be described as follows:

$$\begin{aligned} x &= \Theta_0^1(p,q,\sqrt{\varepsilon})+\tfrac{\mu}{\mu_0}\Theta_1^1(p,q,\vec{\omega},\vec{\psi},\sqrt{\varepsilon},\mu),\\ y &= \Theta_0^2(p,q,\sqrt{\varepsilon})+\tfrac{\mu}{\mu_0}\Theta_1^2(p,q,\vec{\omega},\vec{\psi},\sqrt{\varepsilon},\mu),\\ \vec{\varphi} &= \vec{\psi}+\tfrac{\mu}{\mu_0}\Theta_1^3(p,q,\vec{\omega},\vec{\psi},\sqrt{\varepsilon},\mu),\\ I &= \vec{\omega}+\tfrac{\mu}{\mu_0}\Theta_1^4(p,q,\vec{\omega},\vec{\psi},\sqrt{\varepsilon},\mu) \end{aligned} \tag{36}$$

with $\Theta_1^3=0$ for the linear model.

For $(p,q)=(0,0)$ the equations (36) describe the invariant $(n-1)$-whiskered tori whereupon the motion is quasiperiodic with frequencies $\vec{\omega}\in\Omega_\gamma$. Their local unstable whiskers correspond to $q\neq 0, p=0$, and their local stable whiskers to $q=0, p\neq 0$.

Henceforth in this section we'll assume that $\vec{\omega}$ is diophantine and fixed, so the following discussion will pertain to a particular torus, and $\vec{\omega}$ will be treated as a parameter.

By Theorem 3.1 and Corollary 3.1, if one satisfies the estimate (35), then for all $|\mu|\leq\mu_0$ the surviving perturbed invariant weakly hyperbolic tori and their whiskers will be $\frac{\mu}{\mu_0}$-close to their unperturbed counterparts, the Lyapunov exponent on the whiskers being

$$\lambda'=\sqrt{\varepsilon}\left(1+O\left(\frac{\mu}{\mu_0}\right)\right).$$

Apparently, the presence of δ in (35) will insure that the $O\left(\frac{\mu}{\mu_0}\right)$ term will be "as small as needed". The trajectories on the tori will be given by the relations (36) with

$$(p,q)=(0,0),\ \vec{\psi}=\vec{\alpha}+\vec{\omega}t \tag{37}$$

for real t and $\vec{\alpha}\in W_{\sigma_1}T^{n-1}$. For simplicity we rescale $\frac{\kappa}{2}\to\kappa$.

The trajectories on the local unstable and the local stable manifolds $W^{u,s}$ of the torus with the frequency $\vec{\omega}$ described by (36) are given as follows. For a complex number t_0 on the local unstable manifold we will have:

$$\vec{\psi}=\vec{\alpha}+\vec{\omega}t,\ p=0,\ q=\exp(\lambda'\tau),\ \text{with } \lambda'\Re\tau\leq\log\kappa, \tag{38}$$

where $\tau=t-t_0$ with $t\in R,\ \vec{\alpha}\in W_{\sigma_1}T^{n-1}$, and no restriction on the imaginary part of the parameter t_0.

In the same fashion on the local stable manifold we have:

$$\vec{\psi}=\vec{\alpha}+\vec{\omega}t,\ p=\exp(-\lambda'\tau),\ q=0,\ \text{with } \lambda'\Re\tau\geq-\log\kappa. \tag{39}$$

In other words, the local unstable and stable manifolds can be parameterized as

$$W^{u,s}_{loc}=W^{u,s}_{loc}(\vec{\omega},\tau,\vec{\alpha}+\vec{\omega}t),$$

if the parameters (t_0,t) satisfy the inequalities in (38), (39) for all $\vec{\alpha}\in W_{\sigma_1}T^{n-1}$. Most of the time we will be dealing with a selected torus with a fixed diophantine frequency, so we will omit the $\vec{\omega}$-dependencies whenever it is not misleading.

One can see from (38) and (39) that a perturbed trajectory on the local unstable (stable) whisker is a function of τ and $\vec{\psi}=\vec{\alpha}+\vec{\omega}t$, denoted

$$\tilde{\Gamma}_{loc}(\tau,\vec{\alpha},t_0)=\Gamma_{loc}(\tau,\vec{\psi}). \tag{40}$$

Now we introduce the quantities $T^*(\varepsilon), T(\varepsilon)$ as

$$T^* \equiv -\frac{3}{2}\sqrt{\varepsilon}\log\kappa = -\frac{3}{2}\sqrt{\varepsilon}(\log\sqrt{\varepsilon} + \log R) \equiv -\frac{3}{4}\sqrt{\varepsilon}\log\varepsilon + T,$$

where T is proportional to $\sqrt{\varepsilon}$. Clearly, by choosing the perturbation small enough (it's sufficient to have a small ε and any $|\mu| \leq \mu_0$) one can achieve that for $\Re\tau \geq T^*$ both the projections of the perturbed and the unperturbed stable whiskers onto the (y, x)-coordinate plane are in B_κ^2, the same can be done for the unstable whiskers for $\Re\tau \leq -T^*$. By Theorem 3.1, in these neighborhoods the perturbed whiskers are $\frac{\mu}{\mu_0}$-close to the unperturbed ones.

We want to extend the above parameterization of the whiskers by $(\tau, \vec{\alpha}, t_0)$ over the time intervals (negative or positive) big enough so that the τ-projections of the domains of parameterization for the stable and the unstable manifolds overlap. This will allow us to strictly define the splitting distance as a function of the above parameterizations. This part of the standard argument is given by the Extension lemma, which is very similar to the one proved in Delshams and Seara [1992]. Although its proof is rather lengthy and cumbersome, the Extension lemma is no more than the statement that the trajectories on the perturbed global whiskers stay close to the unperturbed ones for a sufficiently long time; in particular, they pass $x = \pi$ still being close. Clearly, this can be achieved by making the perturbation small enough, even if for some values of the parameters the unperturbed trajectories pass close to poles of finite order. There is no need for the Extension lemma if one pursues the direct approach to the whiskers calculus.

Before we expound this very technical part, we want to make several observations about symmetries. Although the parity properties are trivial to establish, they reveal a lot about how the parameterizations of the stable and the unstable whiskers are related to each other. Besides, the parity properties in the "old", or the original variables and the intrinsic, or the Normal form variables, turn out to be certainly the same. This will give us different ways to measure the homoclinic splitting and elucidate how these different ways are related to each other.

3.3 Parity Properties

The fact that the Hamiltonian (7) is even in the variables $(x, \vec{\varphi})$ tells us that for the vector field (11), generated by this Hamiltonian, if $(x(t), y(t), \vec{\varphi}(t), \vec{I}(t))$ is a solution of (11), so is $(-x(-t), y(-t), -\vec{\varphi}(-t), \vec{I}(-t))$. Also, due to our choice of the perturbation as an even function of $(x, \vec{\varphi})$, the variables $(-x, y, -\vec{\varphi}, \vec{I})$ will be canonical with respect to the Hamiltonian $-H$ (or time $-t$).

Furthermore, if

$$\tilde{\Gamma}^u \equiv (x(t), y(t), \vec{\varphi}(t), \vec{I}(t))$$

is a trajectory on the unstable manifold of an invariant torus with the frequency $\vec{\omega}$ then

$$\tilde{\Gamma}^s \equiv (-x(-t), y(-t), -\vec{\varphi}(-t), \vec{I}(-t))$$

will be a trajectory on the stable manifold of this torus.

It follows from KAM theory, namely from Corollary 3.1, that the local unstable and stable whisker of this torus are parameterized by the new Normal form coordinates $(q, \vec{\psi})$ for the local unstable and $(p, \vec{\psi})$ for the local stable manifolds.

Now if one compares the Normal forms for $H(y, x, \vec{I}, \vec{\varphi})$ and $-H(y, -x, \vec{I}, -\vec{\varphi})$ (the "$-$" sign before the latter corresponds to the inversion of time), it's easy to conclude that they are the same except a change of t to $-t$ (or equivalently a change of H_+ to $-H_+$ in the Normal form), which for the linear Normal form equations of motion on the local unstable whiskers (38) and the local stable whiskers (39) becomes tantamount to permuting p and q and changing $\vec{\psi}$ to $-\vec{\psi}$. In other words, the Normal form transformation Ψ, whose existence is guaranteed by the KAM theorem, will have the following property: if

$$(x, y, \vec{\varphi}, \vec{I}) = \Xi(q, p, \vec{\psi}, \vec{I}'),$$

then

$$(-x, y, -\vec{\varphi}, \vec{I}) = \Xi(p, q, -\vec{\psi}, \vec{I}').$$

In particular, in terms of the parameterization (40) of the local whiskers, it means that if $\tilde{\Gamma}^u_{loc} = \Gamma^u_{loc}(\tau, \vec{\psi})$, is a trajectory on the unstable whisker for $\Re\tau \leq -T^*$, then $\tilde{\Gamma}^s_{loc} = \Gamma^s_{loc}(-\tau, -\vec{\psi})$ is the corresponding trajectory on the stable whisker. This implies, for example, that if $\Re\tau \leq -T^*$, then

$$\vec{I}^u_{loc}(\tau, \vec{\psi}) = \vec{I}^s_{loc}(-\tau, -\vec{\psi}),$$

and the same parity relation locally holds for the pendulum momentum $y^{u,s}_{loc}$, whereas for the functions $\vec{\varphi}^{u,s}_{loc}$ and $x^{u,s}_{loc}$ the above symmetry takes place with the *minus* sign. Further, we will also refer to $(x, \vec{\varphi})$ as the *coordinate* and $(y, \vec{I})$ as the *momentum* components of $\tilde{\Gamma}$.

Besides, on the invariant torus itself, namely when $p = q = 0$, the coordinate components of $\tilde{\Gamma}$ will clearly be represented by the odd functions of $\vec{\psi}$, whereas the momentum components, or their time derivatives, will be even in $\vec{\psi}$.

If one thinks in terms of locally defined functions (for further they will be extended so their domains overlap, and their coordinate components will be proved continuous at $\tau = 0$, whereas the difference of the action components will measure the splitting distance), defined in the following way:

$$\tilde{\Gamma}_{loc} = \Gamma_{loc}(\tau, \vec{\psi}) = \begin{cases} \Gamma^u_{loc}(\tau, \vec{\psi}) & \text{for } \Re\tau \leq -T^* \\ \Gamma^s_{loc}(\tau, \vec{\psi}) & \text{for } \Re\tau \geq T^*, \end{cases} \tag{41}$$

similar to how it was done in Gallavotti [1994], then for $|\tau| \geq T^*$, the following time-angle properties will hold:

$$\begin{array}{rcl} x_{loc}(\tau, \vec{\psi}) & = & -x_{loc}(-\tau, -\vec{\psi}), \\ \vec{\varphi}_{loc}(\tau, \vec{\psi}) & = & -\vec{\varphi}_{loc}(-\tau, -\vec{\psi}), \\ y_{loc}(\tau, \vec{\psi}) & = & y_{loc}(-\tau, -\vec{\psi}), \\ \vec{I}_{loc}(\tau, \vec{\psi}) & = & \vec{I}_{loc}(-\tau, -\vec{\psi}). \end{array} \tag{42}$$

This implies that the coordinate components $(x_{loc}, \vec{\varphi}_{loc})$ of $\tilde{\Gamma}_{loc}$ are *time-angle odd* (namely with respect to the simultaneous change of τ to $-\tau$ and $\vec{\psi}$ to $-\vec{\psi}$, keeping in mind that for $\Re\tau \leq -T^*$ one deals with the unstable and for $\Re\tau \geq T^*$ with the stable local whisker), whereas their time-derivatives, or the momentum components $(y_{loc}, \vec{I}_{loc})$, are *time-angle even*.

4 The Extension Lemma

Conceptually the Extension lemma contains a very straightforward argument, which in terms of real parameterization of the trajectories would boil down to an application of Gronwall's inequality. When the manifolds are parameterized in the complex region, we just have to make sure that away from the real axis (since this mostly concerns a parameter t_0 with a large imaginary part), the perturbation F does not grow too wild, as well as the Green's function of the linearized system, this will eventually lead to the new smallness condition for $\mu_0(\varepsilon)$, more stringent than (35). The other thing is to make sure that the imaginary part of the (perturbed) angles $\vec{\varphi}$ does not get larger than the analyticity width σ_1, where we can obtain an estimate on the *sup*-norm of the angle-gradient of F.

In other words, the Normal form and the Extension lemma together are nothing but the affirmation of Existence and Uniqueness of the whiskers for some semiinfinite time intervals including zero. To establish Existence and Uniqueness one can also use direct argument as Ellison et al. [1993] (for $n = 2$) or Gentile [1995] (for $n \geq 2$).

The proof we give is at some places schematic. For more details the reader is referred to Delshams and Seara [1992], where the same type of proof is delivered for the forced pendulum problem ($n = 2$, $\nu_0 = 1$).

We shall consider a domain Σ, such that

$$\Sigma = C \setminus \left\{\tau \in C : \ |\Re\tau| < \sqrt{\varepsilon}, \ |\Im\tau| > p_*\right\}$$

along with its subset

$$\Sigma^u = \left\{\tau \in C : \ \Re\tau \leq -\sqrt{\varepsilon}\right\} \bigcup \left\{\tau \in C : \ -\infty < \Re\tau \leq T^*, \ |\Im\tau| \leq p_*\right\},$$

with $p_* = \frac{\pi}{2\sqrt{\varepsilon}} - \sqrt{\varepsilon}$.

We also define

$$\sigma_2 = \sigma_0 - \frac{3}{4}\sqrt{\varepsilon}.$$

Lemma 4.1 (The Extension Lemma) *Let $\tilde{\Gamma}_0(\tau, \vec{\alpha}, t_0) \equiv (x_0(\tau), y_0(\tau), \vec{\varphi}_0(\tau, \vec{\alpha}, t_0), \vec{\omega})$ as in (13) be the parameterization of the unperturbed manifold homoclinic to the invariant torus with the frequency $\vec{\omega} \in \Omega_\gamma$, defined for $t \in R$, all complex $\vec{\alpha}$ and $\tau \in \Sigma$.*

Let

$$\tilde{\Gamma}^u(\tau, \vec{\alpha}, t_0) \equiv (x^u(\tau, \vec{\alpha}, t_0), y^u(\tau, \vec{\alpha}, t_0), \vec{\varphi}^u(\tau, \vec{\alpha}, t_0), \vec{I}^u(\tau, \vec{\alpha}, t_0))$$

be a trajectory of the perturbed problem (11) on the unstable manifold of the invariant torus with the frequency $\vec{\omega}$ for $\tau \in \Sigma$ and $\vec{\alpha} \in W_{\sigma_2} T^{n-1}$, with the initial conditions at τ_0 such that $\Re \tau_0 = -T^*$ that differ from $\tilde{\Gamma}_0(\tau_0, \vec{\alpha}, t_0)$ by the quantity $O\left(\frac{\mu}{\mu_0}\right)$, where μ_0 is given by (35), namely:

$$\sup |\tilde{\Gamma}^u(\tau, \vec{\alpha}, t_0) - \tilde{\Gamma}_0(\tau, \vec{\alpha}, t_0)| = O\left(\frac{\mu}{\mu_0}\right),$$

where the supremum is taken over the components of $\tilde{\Gamma}^u$ and $\tilde{\Gamma}_0$.

Also suppose that μ_0 satisfies (35) and for any small $\delta > 0$ and $\varepsilon > 0$ small enough

$$|\mu| \le \max\left(\mu_0 \varepsilon^{\frac{13}{4}}, \varepsilon^{2\nu_0 + 1 + \delta}\right).$$

Then the solution $\tilde{\Gamma}^u(\tau, \vec{\alpha}, t_0)$ exists and is unique for $\tau \in \Sigma^u$, and $\vec{\alpha} \in W_{\sigma_2} T^{n-1}$; besides it obeys the following uniform estimate:

$$\sup |\tilde{\Gamma}^u(\tau, \vec{\alpha}, t_0) - \tilde{\Gamma}_0(\tau, \vec{\alpha}, t_0)| = O\left(\frac{\mu}{\mu_0}\sqrt{\varepsilon}\right), \tag{43}$$

where the supremum is taken over the components of $\tilde{\Gamma}^u$ and $\tilde{\Gamma}_0$, and μ_0 has been redefined as

$$\mu_0 = \max\left(\mu_0 \varepsilon^{\frac{13}{4}}, \varepsilon^{2\nu_0 + 1 + \delta}\right).$$

Remark: Obviously, the same (up to an obvious sign change) result will hold for the trajectories on the stable manifold (from parity). Besides, we prove here actually more than we need, namely that the perturbed manifolds not only remain close to the unperturbed ones, but also return to a small neighborhood of the equilibrium point in the hyperbolic variables.

Recalling (35), one can rewrite the smallness condition of the Extension lemma as follows:

$$\mu_0 = \varepsilon^{\max\left(\frac{15}{4} + \frac{n}{2} + \varpi + 2b + \delta,\ 2\nu_0 + 1 + \delta\right)}. \tag{44}$$

Furthermore, following Gallavotti [1994], in the same fashion as we have done in Section 3.3, we define the discontinuous function

$$\tilde{\Gamma}(\tau, \vec{\alpha}, t_0) = \begin{cases} \tilde{\Gamma}^u(\tau, \vec{\alpha}, t_0) & \text{for } \Re\tau \le 0_- \\ \tilde{\Gamma}^s(\tau, \vec{\alpha}, t_0) & \text{for } \Re\tau \ge 0_+, \end{cases} \tag{45}$$

for $\tau = t - t_0$. Then the Extension lemma, for it yields the analytic continuation of $\tilde{\Gamma}_{loc}$, will insure that the following time-angle properties will take place:

$$\begin{aligned} x(\tau, \vec{\alpha}, t_0) &= -x(-\tau, -\vec{\alpha}, -t_0), \\ \vec{\varphi}(\tau, \vec{\alpha}, t_0) &= -\vec{\varphi}(-\tau, -\vec{\alpha}, -t_0), \\ y(\tau, \vec{\alpha}, t_0) &= y(-\tau, -\vec{\alpha}, -t_0), \\ \vec{I}(\tau, \vec{\alpha}, t_0) &= \vec{I}(-\tau, -\vec{\alpha}, -t_0). \end{aligned} \tag{46}$$

In terms of the notation (45) we can formulate a Corollary which is the "analytic continuation" of the local Corollary 3.1. The dependencies asserted by this Corollary will be explicitly constructed in the following section.

Corollary 4.1 *Suppose, the smallness condition (44) holds and ε is small enough. For $|\mu| \leq \mu_0$, $\tau \in \Sigma$, $\vec{\alpha} \in W_{\sigma_2}T^{n-1}$ the solution $\tilde{\Gamma}(\tau, \vec{\alpha}, t_0)$ of the system of equations (11) on the whiskers of the invariant $(n-1)$-torus with the frequency $\vec{\omega} \in \Omega_\gamma$ is analytic in the parameters $t_0, \vec{\alpha}, \mu$ in the above domains.*

Moreover, it is formally quasiperiodic in the variable $\vec{\psi} = \vec{\alpha} + \vec{\omega}t$, namely it can be represented as a series

$$\tilde{\Gamma}(\tau, \vec{\alpha}, t_0) = \Gamma(\tau, \vec{\psi}) = \sum_{\vec{k} \in Z^{n-1}} \Gamma_{\vec{k}}(\tau) e^{\imath \vec{k} \cdot \vec{\psi}}, \tag{47}$$

whose coefficients asymptote to constants as $\tau \to \infty$.

Remark: In fact one can derive from the proof of the Extension lemma that $\tilde{\Gamma}(\tau, \vec{\alpha}, t_0)$ can be expressed as a convergent sum of monomials, each of which has the form

$$C_{j_1, j_2, \vec{k}} \tau^{j_1} e^{-j_2 \operatorname{sign}(\Re\tau)\sqrt{\varepsilon}\tau + \imath \vec{k} \cdot (\vec{\alpha} + \vec{\omega}t)},$$

with $j_1, j_2 \geq 0$ and some coefficients $C_{j_1, j_2, \vec{k}} = C_{j_1, j_2, \vec{k}}(\vec{\omega}, \varepsilon)$. Nevertheless, the next section gives a direct proof of this statement.

5 Exponential Smallness: Analytic Considerations

In this section we will develop the formalism for the rigorous definition and estimation of the separatrix splitting distance function, which follows a template similar to that followed in the work of Ellison et al. [1993] and Gallavotti [1994]. An advantage of our approach is that through the use of the Normal form coordinates in Corollary 4.1 and the symmetries of the original Hamiltonian, already pointed out in Section 3.3, we will considerably simplify the dependencies of the functions in question. This will lead to the fundamental difference between our approach and the abovementioned works which we now briefly describe. In these earlier papers the authors fix the initial conditions $x(0) = \pi$, $\vec{\varphi}(0) = \vec{\alpha}$ at time-zero and seek solutions of (11) which are asymptotically quasiperiodic at infinity (minus or plus for either the unstable or the stable manifold). Clearly, the coordinate components $(x, \vec{\varphi})$ of $\tilde{\Gamma}$ for such a parameterization are continuous at time-zero. Thus the splitting distance can be defined as the difference of the momenta $(y, \vec{I})$ on this Poincaré section. Such an approach leads to the following representation for the trajectories on the whiskers

$$\tilde{\Gamma}(\tau, \vec{\alpha}, t_0) = \sum_{\vec{k} \in Z^{n-1}} \tilde{\Gamma}_{\vec{k}}(\tau, \vec{\alpha}) e^{\imath (\vec{k} \cdot \vec{\omega})t}. \tag{48}$$

This representation coincides with our representation given in (47) at the first order of perturbation theory but differs at higher orders. At higher orders a phenomenon that we refer to as *mode mixing* occurs, namely $\tilde{\Gamma}$ no longer depends only on the composite variable $\vec{\psi} = \vec{\alpha} + \vec{\omega}t$, but separately on $(\vec{\alpha}, \vec{\omega}t)$. Further, the abovementioned authors establish that in computing the splitting distance one does not have to worry about $\vec{k} = \vec{0}$, and for $\vec{k} \neq \vec{0}$ the contour of integration used to evaluate the splitting distance (see below) can be shifted into the complex plane, insuring the quasiflat estimate (see the Introduction and below).

We pursue a different approach for the construction of the trajectories, also requiring that they become asymptotically quasiperiodic at infinity, but with the additional requirement that they possess the functional dependencies of the representation (47) of Corollary 4.1. So, we have an additional task of building the "initial conditions" at $\tau = 0$ to insure that the representation (47) holds, and proving that the coordinate components in their (45) representation are continuous functions of τ. Once this is done, we can define the splitting distance as the difference of the momentum components at $\tau = 0$. The advantage of this approach is that due to the simplicity of the dependencies in the representation (47) the exponentially small estimate will follow right away. We also hope to persuade the reader that the splitting distance thus defined differs from the one on the fixed Poincaré section by a factor $(1 + O(\mu))$, where the $O(\mu)$ term is a quasiperiodic function of $\vec{\alpha}$, which is "responsible" for the mode mixing in the standard approach and for not allowing one to see the exponential smallness underlying the quasiflat estimate.

Before proceeding we want to emphasize that the main results of this section are as follows:

1. Using an iterative solution technique, we will show that the trajectories in the unstable and stable whiskers have the functional dependencies described above. One might wonder why we required the Normal form and the Extension lemma developed earlier. The Normal form gave us the desired solutions *locally* near the tori having the desired analyticity properties. The Extension lemma enabled us to continue those solutions outside a neighborhood of the tori and, most importantly, to control the analyticity properties of the continued solutions. In the course of showing the desired functional dependencies of the solutions in this section we will use the analyticity properties given by the Extension lemma. We can view the results of this section as showing that the simple local parameterization of solutions provided by the Normal form extend outside the local neighborhood of the tori.

   We remark that Gallavotti [1994] suggested, and Gentile [1995] rigorized, that one can establish the existence of the whiskered tori directly according to the asymptotics of the trajectories of the system (11) at infinity, using the Lindstedt series approach. Possibly this can also be done in our setting using an analogue of the iterative scheme that we develop below, however we have not chosen to follow this approach.

2. With the functional dependencies of the trajectories in hand, we then easily prove two results that are crucial for proving exponential smallness of the splitting distance and splitting size. Proposition 5.1 proves a parity property on the coordinate components of the variations of the solutions at the point where the splitting is measured. Lemma 5.1 gives a regularity result on the coordinate components of the trajectories with respect to the initial conditions where the splitting is measured.

3. Using the abovementioned results we are finally able to obtain exponentially small upper bounds on the splitting distance and splitting size.

We will fix $t_0 = 0$, then $\tau = t$ and real. Analyticity in τ, asserted by the Extension lemma, will be further used by us to argue that one can righteously lift the contour of integration to compute the splitting distance.

We let $t = 0_\pm$ in (45) and start by considering the functions

$$\begin{aligned}
&x(0_-,\vec{\alpha}) - x(0_+,\vec{\alpha}),\\
&\vec{\varphi}(0_-,\vec{\alpha}) - \vec{\varphi}(0_+,\vec{\alpha}),\\
&y(0_-,\vec{\alpha}) - y(0_+,\vec{\alpha}),\\
&\vec{I}(0_-,\vec{\alpha}) - \vec{I}(0_+,\vec{\alpha}).
\end{aligned}$$

According to (46) these functions stand for the difference of the phase space coordinates on the unstable and the stable manifolds on the "Poincaré section" $t = 0$. We will use the parity properties (46) to establish that the $(x,\vec{\varphi})$-coordinates are *continuous* on such a "Poincaré section", then the differences in the action variables will account for the splitting distance. This would be obvious if we were considering the whiskers as graphs over the $(x,\vec{\varphi})$-variables, as was done in Gallavotti [1994], but this requires a proof for the intrinsic (Normal form) parameterization.

We will prove the following Lemma:

Lemma 5.1 *For $t = t_0 = 0$*

$$x(0_-,\vec{\alpha}) = x(0_+,\vec{\alpha}) = \pi + \xi(\vec{\alpha},\mu)$$

$$\vec{\varphi}(0_-,\vec{\alpha}) = \vec{\varphi}(0_+,\vec{\alpha}) = \vec{\alpha} + \vec{\varsigma}(\vec{\alpha},\mu),$$

where the functions $\xi(\vec{\alpha},\mu)$, $\vec{\varsigma}(\vec{\alpha},\mu)$ are $O\left(\frac{\mu}{\mu_0}\right)$, being quasiperiodic and analytic in $\vec{\alpha}$, in $W_{\sigma_2}T^{n-1}$.

The proof of Lemma 5.1 will be part of the discussion which is coming up.

We shall rewrite the equations for the vector field (11), generated by the Hamiltonian (7) as a second-order system of ODE's

$$\begin{aligned}
&\ddot{x} - \varepsilon \sin x + \mu g(x,\vec{\varphi}) = 0,\\
&\ddot{\vec{\varphi}} + \mu \vec{f}(x,\vec{\varphi}) = 0.
\end{aligned} \tag{49}$$

The unperturbed ($\mu = 0$) homoclinic solution of (49) will be given by (13), where the initial condition at $t = 0$ is chosen to insure

$$x_0(0) = \pi, \quad \vec{\varphi}_0(0, \vec{\alpha}) = \vec{\alpha}.$$

The latter will become a parameter describing different trajectories. We will show that this is the same parameter as the Normal form parameter.

We are seeking a family of solutions $(x(t, \vec{\alpha}), \vec{\varphi}(t, \vec{\alpha}))$ of (49) that become asymptotically quasiperiodic at infinity with exponential speed (for $|t|$ large these will correspond to the solutions given by the version of the KAM theorem in Theorem 3.1). These solutions will be defined for $t \le 0_-$ on the unstable and for $t \ge 0_+$ on the stable manifold with the following initial condition:

$$x(0, \vec{\alpha}) = \pi + O(\mu), \quad \vec{\varphi}(0, \vec{\alpha}) = \vec{\alpha} + O(\mu). \tag{50}$$

The $O(\mu)$ terms in (50), depending quasiperiodically on $\vec{\alpha}$, will be further determined in a special way to insure that the dependencies of the functions $x, \vec{\varphi}$ are the same as in (47). Then, if we can do that, from analyticity it will follow that $\vec{\alpha}$ coincides with the Normal form parameter.

Rather than solve (49) directly, we will consider the following *variations*

$$\begin{aligned}
\xi(\tau, \vec{\alpha}, t_0) &= x(\tau, \vec{\alpha}, t_0) - x_0(\tau), \\
\vec{\varsigma}(\tau, \vec{\alpha}, t_0) &= \vec{\varphi}(\tau, \vec{\alpha}, t_0) - \vec{\varphi}_0(\tau, \vec{\alpha}, t_0), \\
\eta(\tau, \vec{\alpha}, t_0) &= \dot{\xi}(\tau, \vec{\alpha}, t_0) = y(\tau, \vec{\alpha}, t_0) - y_0(\tau), \\
\vec{\zeta}(\tau, \vec{\alpha}, t_0) &= \dot{\vec{\varsigma}}(\tau, \vec{\alpha}, t_0) = \vec{I}(\tau, \vec{\alpha}, t_0) - \vec{\omega}.
\end{aligned} \tag{51}$$

The equations satisfied by the variations have the advantage of being amenable to iterative solution techniques.

The variations ξ and $\vec{\varsigma}$ satisfy the following second order system of ODE's:

$$\begin{aligned}
&\ddot{\xi} + \varepsilon(\cos x_0 \cdot \xi - h(\tau, \xi)) + \mu g(x_0 + \xi, \vec{\varphi}_0 + \vec{\varsigma}) = 0, \\
&\ddot{\vec{\varsigma}} + \mu \vec{f}(x_0 + \xi, \vec{\varphi} + \vec{\varsigma}) = 0.
\end{aligned} \tag{52}$$

where we've denoted

$$h(\tau, \xi) = (\sin(x_0(\tau) + \xi) - \sin x_0(\tau) + \cos x_0(\tau) \cdot \xi). \tag{53}$$

So far, we are dealing with the case $t_0 = 0$, but we will nevertheless sometimes retain it, along with $\tau = t - t_0$, as a complex variable instead of t in certain expressions for further cross-reference and whenever we want to emphasize analyticity. Our notation has not assumed the functional dependencies of (47) so far, for we want to construct them explicitly.

We next develop the set-up for the solution of (52) by reformulating it in an integrable equation form that can be solved by iterative methods. We begin by linearizing (52) near $(x_0, \vec{\varphi}_0)$ to obtain

$$\begin{aligned}
&\ddot{\xi} + \varepsilon \cos x_0(\tau) \cdot \xi = 0, \\
&\ddot{\vec{\varsigma}} = 0.
\end{aligned} \tag{54}$$

First, we solve the linearized pendulum equation. As is well known, a possible choice for the solutions of (54), such that at $\tau = t = 0$ one of them equals one and the other equals zero,[1] will be as follows:

[1] In Delshams and Seara [1992] $\xi_2(\tau)$ was chosen in such a way that it had a *double* zero at either of $\tau = t - t_0 = \pm\frac{\pi}{2\sqrt{\varepsilon}}$, which allowed the authors to improve the possible dependence $\mu_0(\varepsilon)$ versus e.g. Ellison et al [1993]. We will also resort to this subterfuge during the proof of the Extension lemma. Nevertheless, exercising this trick in this section would encumber the following argument for we would no longer have the condition $\xi_2(0) = 0$.

$$\xi_1(t) = \frac{1}{2\sqrt{\varepsilon}}\dot{x}_0(t) = \frac{1}{\cosh\sqrt{\varepsilon}t}, \qquad \xi_2(t) = \frac{1}{2\sqrt{\varepsilon}}\sinh\sqrt{\varepsilon}t + \frac{1}{2}\frac{t}{\cosh\sqrt{\varepsilon}t} \tag{55}$$

For further use we will write down the fundamental matrix X of the (54):

$$X = \begin{pmatrix} \frac{1}{\cosh\sqrt{\varepsilon}t} & \frac{1}{2\sqrt{\varepsilon}}\sinh\sqrt{\varepsilon}t + \frac{1}{2}\frac{t}{\cosh\sqrt{\varepsilon}t} \\ -\sqrt{\varepsilon}\frac{\sinh\sqrt{\varepsilon}t}{\cosh^2\sqrt{\varepsilon}t} & \frac{1}{2}\cosh\sqrt{\varepsilon}t + \left(\frac{1}{2}\frac{1}{\cosh\sqrt{\varepsilon}t} - \sqrt{\varepsilon}\frac{t\sinh\sqrt{\varepsilon}t}{\cosh^2\sqrt{\varepsilon}t}\right) \end{pmatrix}. \tag{56}$$

The second linearized equation, (or actually set of equations) will obviously have the following basic solutions:

$$\vec{\varsigma}_1(t) = 1, \quad \vec{\varsigma}_2(t) = t,$$

where the slightly abusive notation that we've used means that the equalities in question occur for all the components of the vectors involved.

The solution of (52) for the variations ξ and $\vec{\varsigma}$ and their time-derivatives $\eta = \dot{\xi}$ and $\vec{\zeta} = \dot{\vec{\varsigma}}$ is the same as the solution to the following system of integral equations:

$$\begin{aligned}
&\xi(t) = \xi_2(t)\left(\eta(0) - \mu\int_0^t \xi_1(s)\tilde{f}(s,\xi,x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds\right) + \xi_1(t)\left(\xi(0) + \mu\int_0^t \xi_2(s)\tilde{f}(s,\xi,x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds\right), \\
&\eta(t) = \dot{\xi}_2(t)\left(\eta(0) - \mu\int_0^t \xi_1(s)\tilde{f}(s,\xi,x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds\right) + \dot{\xi}_1(t)\left(\xi(0) + \mu\int_0^t \xi_2(0)\tilde{f}(s,\xi,x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds\right), \\
&\vec{\varsigma}(t) = \vec{\varsigma}(0) + \vec{\zeta}(0)t - \mu\int_0^t (t-s)\vec{f}(x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds. \\
&\vec{\zeta}(t) = \vec{\zeta}(0) - \mu\int_0^t \vec{f}(x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds.
\end{aligned} \tag{57}$$

For brevity we've omitted $\vec{\alpha}$ here and used the notation

$$\tilde{f}(s,\xi,x_0+\xi,\vec{\varphi}_0+\vec{\varsigma}) = -\frac{\varepsilon}{\mu}h(s,\xi) + g(x_0+\xi,\vec{\varphi}_0+\vec{\varsigma}),$$

where h comes from (53). Note, that in the above expressions $\xi(0,\vec{\alpha})$ and $\vec{\varsigma}(0,\vec{\alpha})$ are not set equal to zero. We reserve the freedom to choose the initial conditions appropriately in order to obtain the desired dependencies for the functions of interest.

At this point we want to further emphasize the notational conventions that we will adhere to.

- We take for granted the fact that all the functions involved are defined on the unstable manifold for $\rho = \operatorname{sign} t = -1$ and on the stable manifold for $\rho = 1$.
- We will often omit the parameter $\vec{\alpha}$ from our notation, keeping in mind that it's always there.
- Apropos of the functions $\vec{f}$ and $\tilde{f}$, we will often write their dependencies as $\vec{f}(t,\vec{\alpha})$, $\tilde{f}(t,\vec{\alpha})$ or even $\vec{f}(t)$ and $\tilde{f}(t)$ under the integration sign, etc.

¿From the last equation of (57) one can see that $\vec{\zeta}$ as a function of t is a primitive of $-\mu\vec{f}(t)$, which allows us to rewrite the integral in the third equation in (57) as follows:

$$-\mu\int_0^t s\vec{f}(x_0+\xi,\vec{\varphi}_0+\vec{\varsigma})ds = t\vec{\zeta}(t,\vec{\alpha}) - \int_0^t \vec{\zeta}(s,\vec{\alpha})ds, \tag{58}$$

Clearly, if the formal Fourier series in $\vec{\omega}t$ for $\vec{\zeta}$ (by formal we mean that its coefficients may generally depend on (t,α)) has a zeroth component such that the latter does not vanish in ∞, the former integral cannot be bounded as $\tau \to \infty$. Moreover, this also leads to the conclusion that the zeroth Fourier component of $\vec{f}$ is integrable at infinity.

The fact that the solutions of (52) on the unstable and the stable manifolds that we are looking for must tend to quasiperiodic functions as $\tau \to \infty$ naturally leads to the technical necessity of introducing some kind of improper integration, or a way to obtain time-primitives of the functions involved, as it was done in different ways by many authors, in particular by Gelfreich [1993], Ellison et al. [1993], and Gallavotti [1994]. We take an alloy of the second and the third ones as a template, for it does not require a separate section to be described.

We can view the improper integration operation (not necessarily for $t_0 = 0$) as acting on a function space $\mathcal{B}$, whose elements will be represented as sums of monomials,

$$C_{j_1,j_2,\vec{k}} \tau^{j_1} e^{-j_2 \rho \sqrt{\varepsilon}\tau + \iota \vec{k}\cdot(\vec{\alpha}+\vec{\omega}t)} \tag{59}$$

with some coefficients $C_{j_1,j_2,\vec{k}}$ for integer $j_1 \geq 0,\ j_2 \geq -1$, where $\tau = t - t_0,\ \tau \in \Sigma$. These sums, given τ, t_0, will converge uniformly in $\vec{\alpha}:\ |\Im\vec{\alpha}| \leq \sigma_2 = \sigma_0 - \frac{3}{4}\sqrt{\varepsilon}$, and given $\vec{\alpha}$ in τ in any finite subset of Σ. Each of these monomials possesses an indefinite integral expressed in terms of a sum of the monomials of the same kind.

We define $\mathcal{B}_0$ as the subset of $\mathcal{B}$ whose elements are bounded at infinity.

¿From the KAM theorem and the Extension lemma we know that the functions that we will be dealing with all become quasiperiodic (hence, bounded) at infinity, admitting the representation (48), the latter being more general so far than (47), with the Fourier coefficients analytic in τ for $\tau \in \Sigma$. We will argue that in fact they are in $\mathcal{B}_0$.

Hence, as for the Fourier components for $\vec{k} \neq \vec{0}$, the worst possible case for behavior at infinity that we will have to deal with will be the second integral in the first two expressions of (57), and for large, but fixed $|t|$ the integrand will behave as a convergent series

$$\sum_{\vec{k}} u_{\vec{k}}(t) e^{\rho\sqrt{\varepsilon}t + \iota(\vec{k}\cdot\vec{\omega})t}$$

with uniformly bounded in Σ coefficients $u_{\vec{k}}$. So, given $\tau \in \Sigma$, for $\rho = \mathrm{sign}\,(\Re\tau)$ we can define an improper integral

$$\mathop{\int\!\!\!\!\!\bowtie}_{\rho\infty}^{\tau} = \int_{\sqrt{\varepsilon}+\iota\Im\tau}^{\tau} + \mathop{\int\!\!\!\!\!\bowtie}_{\rho\infty}^{\sqrt{\varepsilon}+\iota\Im\tau},$$

where the latter is an integral in the complex plane along the straight line $\mathcal{L}_{\vec{k}}$, which can be parameterized as

$$\mathcal{L}_{\vec{k}} = \left\{ (\sqrt{\varepsilon}, \Im\tau) + \left(\rho, \mathrm{sign}(\vec{k}\cdot\vec{\omega})\frac{2\sqrt{\varepsilon}}{k\cdot\vec{\omega}}\right) s,\ \text{for } s \geq 0 \right\}.$$

Note that the presence of $\sqrt{\varepsilon}$ guarantees that $\mathcal{L}_{\vec{k}} \in \Sigma,\ \forall \vec{k} \neq \vec{0}$

As for the case $\vec{k} = \vec{0}$ (although we will not use it at all), we will not ever bother with lifting the contour into the complex plane, always integrating functions of a real variable t. Nevertheless, to be consistent, for $\tau \in \Sigma$ and a function $u_{\vec{0}}(\tau) \in \mathcal{B}$ we will always be able to find a positive constant R large enough to insure the convergence of

$$\int_{\rho\infty}^{\tau} u_{\vec{0}}(s) e^{-\rho R s} ds,$$

taken along the horizontal line in the complex plane. Following Gallavotti [1994], we define the improper integral as the zero coefficient of the Laurent series for the above quantity in R:

$$\mathop{\int\!\!\!\!\!\bowtie}_{\rho\infty}^{\tau} u_{\vec{0}}(s) ds = \frac{1}{2\pi\iota} \oint \frac{1}{R} \int_{\rho\infty}^{\tau} u_{\vec{0}}(s) e^{-\rho R s} ds.$$

By Cauchy's theorem we can always write $\int_0^\tau = \mathop{\int\!\!\!\!\!\bowtie}_{\rho\infty}^{\tau} - \mathop{\int\!\!\!\!\!\bowtie}_{\rho\infty}^{0}$.

Apparently, the improper integration operator maps the elements of $\mathcal{B}$ into the elements of $\mathcal{B}$, in particular, mapping $\mathcal{B}_0$ into itself. Equipped with a *sup*-norm, $\mathcal{B}_0$ is a Banach space.

The main purpose why one has to introduce some kind of an improper integration is because it gives a way to compute the time-primitives of the functions - elements of $\mathcal{B}_0$, so we will use the notation

$$\mathcal{J}[u](\tau) = \int_{\rho\infty}^{\tau} u(s)ds. \tag{60}$$

Furthermore, considering the above mentioned fact that at infinity $\vec{\zeta}$ and $\vec{f}$ become quasiperiodic functions with zero averages, we readily establish that

$$\begin{aligned} \eta(0_\rho) &= -\mu \int_{\rho\infty}^{0} \xi_1(s)\tilde{f}(s)ds, \\ \vec{\zeta}(0_\rho) &= -\mu \int_{\rho\infty}^{0} \vec{f}(s)ds, \end{aligned} \tag{61}$$

where $\rho = \pm$, provided that $\tilde{f}$ and $\vec{f}$ are in $\mathcal{B}_0$, what we assume so far and will shortly prove. Note, that to derive the first relation we have incorporated the fact that in the first equation of (57) the first term will otherwise grow exponentially with t.

Using the notation (60) we can take into account (61) to rewrite the first two equations of (57) as follows:

$$\begin{aligned} \xi(t) &= \xi_1(t)\left(\xi(0) + \mu\mathcal{J}[\xi_2\tilde{f}]\,|_0^t\right) - \mu\xi_2(t)\mathcal{J}[\xi_1\tilde{f}](t). \\ \vec{\varsigma}(t) &= \vec{\varsigma}(0) - \mu\mathcal{J}\left[\mathcal{J}[\vec{f}]\right]||_0^t \end{aligned} \tag{62}$$

This tells us how to choose the initial conditions. Namely we will require that

$$\begin{aligned} \xi(0) &= \mu\mathcal{J}[\xi_2\tilde{f}](0), \\ \vec{\varsigma}(0) &= -\mu\mathcal{J}\left[\mathcal{J}[\vec{f}]\right](0). \end{aligned} \tag{63}$$

Such a choice will prevent the mode mixing described earlier, and the next step will be to show that it is actually possible to find the solution of (62) with the desired properties. We do this by using the standard Picard iterations. Namely, we choose

$$\xi^0 \equiv 0,\ \vec{\varsigma}^0 \equiv \vec{0}$$

and proceed in an obvious fashion for $l = 1, 2, ...$

$$\begin{aligned} \xi^l(t) &= \xi_1(t)\left(\xi^l(0) + \mu\mathcal{J}[\xi_2\tilde{f}^{l-1}]\,|_0^t\right) - \mu\xi_2(t)\mathcal{J}[\xi_1\tilde{f}^{l-1}](t). \\ \vec{\varsigma}^l(t) &= \vec{\varsigma}(0) - \mu\mathcal{J}\left[\mathcal{J}[\vec{f}^{l-1}]\right]|_0^t\ , \end{aligned} \tag{64}$$

with the choice of the initial conditions

$$\begin{aligned} \xi^l(0) &= \mu\mathcal{J}[\xi_2\tilde{f}^{l-1}](0), \\ \vec{\varsigma}^l(0) &= -\mu\mathcal{J}\left[\mathcal{J}[\vec{f}^{l-1}]\right](0), \end{aligned}$$

where naturally

$$\begin{aligned} \tilde{f}^{l-1} &= -\tfrac{\varepsilon}{\mu}h(s, \xi^{l-1}) + \mu g(x_0 + \xi^{l-1}, \vec{\varphi}_0 + \vec{\varsigma}^{l-1}), \\ \vec{f}^{l-1} &= +\mu\vec{f}(x_0 + \xi^{l-1}, \vec{\varphi}_0 + \vec{\varsigma}^{l-1}). \end{aligned}$$

These iterations will converge due to the KAM theorem and the argument given in the proof of the Extension lemma, provided that μ is in compliance with the smallness condition (44). But since $\mathcal{B}_0$ is a Banach space, they will converge in $\mathcal{B}_0$.

Furthermore, the following Proposition is quite easy to establish:

Proposition 5.1 *For $l = 1, 2, ..$ the variations ξ^l and $\vec{\varsigma}^l$ are in $\mathcal{B}_0$, besides, the initial conditions $\xi^l(0, \vec{\alpha})$, $\vec{\varsigma}^l(0, \vec{\alpha})$ are represented by the odd functions of the parameter $\vec{\alpha}$.*

Proof: By induction in l.

For $l=1$ we have (see(16)):

$$\vec{f}(x_0,\vec{\varphi}_0) = \quad -\sum_{k:\ |k_0|\le\nu_0,\vec{k}\neq\vec{0}} (\vec{k}\cdot\vec{\omega})F_k \exp(-|\vec{k}|\sigma_0)\sin(k_0x_0(t)+\vec{k}\cdot(\vec{\alpha}+\vec{\omega}t)),$$

$$\tilde{f}(x_0,\vec{\varphi}_0) = \quad -\sum_{k:\ |k_0|\le\nu_0,\vec{k}\neq\vec{0}} k_0F_k \exp(-|\vec{k}|\sigma_0)\sin(k_0x_0(t)+\vec{k}\cdot(\vec{\alpha}+\vec{\omega}t)),$$

each term in the sums being time-angle odd, or equivalently $(t,\vec{\alpha})$-odd; what's more, one can always decompose it as follows:

$$\sin(k_0x_0(t)+\vec{k}\cdot(\vec{\alpha}+\vec{\omega}t)) = t\text{-odd}\cdot\vec{\alpha}\text{-even} + t\text{-even}\cdot\vec{\alpha}\text{-odd}.$$

Evidently, a time-primitive of a time-even function is a time-odd function, thus recalling that $\xi_2(t)$ is time-odd, it's easy to convince oneself from (64) that for $l=1$ this Proposition holds.

Furthermore, we clearly have

$$\xi^1(t,\vec{\alpha}) = \xi^1(t,\vec{\alpha}+\vec{\omega}t),\quad \vec{\varsigma}^1(t,\vec{\alpha}) = \vec{\varsigma}^1(t,\vec{\alpha}+\vec{\omega}t),$$

in other words $\xi^1,\vec{\varsigma}^1\in\mathcal{B}_0$.

Now we assume that $(\xi,\vec{\varsigma})^{l-1}$ are $(t,\vec{\alpha})$-odd and admit the representation (47), then clearly $\tilde{f}(x_0+\xi^{l-1},\vec{\varphi}_0+\vec{\varsigma}^{l-1})$ and $\vec{f}(x_0+\xi^{l-1},\vec{\varphi}_0+\vec{\varsigma}^{l-1})$ have the same parities, admit the representation (47), and can be expressed as a sum

$$t\text{-odd}\cdot\vec{\alpha}\text{-even} + t\text{-even}\cdot\vec{\alpha}\text{-odd},$$

so we only have to repeat the rest of the argument for $l=1$ to argue that ξ^l and $\vec{\varsigma}^l$ are in $\mathcal{B}_0$, and the initial conditions $\xi^l(0),\vec{\varsigma}^l(0)$ are the odd functions of $\vec{\alpha}$.

Finally, as we mentioned, provided that (44) is satisfied, the above iteration process converges (at least for real $\vec{\alpha}$) in $\mathcal{B}_0$, for the latter is a Banach space, by the Extension lemma and real analyticity of the flow. This concludes the proof of this Proposition. □

Proof of Lemma 5.1: Suppose, we are on the unstable manifold, so $t\le 0_-$. When t gets small enough, we will eventually find ourselves in the domain where the Normal Form is defined. In this domain the trajectories of (11) admit the same representation (47) in terms of the Normal form parameter $\vec{\alpha}$. Which means that the two representations must coincide, for at $t=-\infty$ the trajectories asymptote to an invariant torus.

This implies that for the "combined trajectories" (45), the relations (46) hold. Applying Proposition 5.1 at $t=t_0=0$, we claim that Lemma 5.1 is proved. □

We finally notice that in the preceding discussion we have restricted ourselves to $t_0=0$. Nevertheless, for real $t_0\neq 0$, if we consider the splitting at $\tau=0$ the same result will hold if we change $\vec{\alpha}\to\vec{\alpha}+\vec{\omega}t_0$.

At last we return to the splitting distance, which will be specified in terms of (61). We consider an n-vector $\Delta=(\Lambda,\vec{\Delta})$, where

$$\begin{aligned}\Lambda(\vec{\alpha}) &= \eta(0_-,\vec{\alpha})-\eta(0_+,\vec{\alpha}) = -\mu\int_{-\infty}^{+\infty}\xi_1(s)\tilde{f}(s,\vec{\alpha})ds\\ \vec{\Delta}(\vec{\alpha}) &= \vec{\zeta}(0_-,\vec{\alpha})-\vec{\zeta}(0_+,\vec{\alpha}) = -\mu\int_{-\infty}^{+\infty}\vec{f}(s,\vec{\alpha})ds.\end{aligned}\tag{65}$$

Furthermore, Δ can be represented as a Fourier series in $\vec{\alpha}$, and we will have

$$\begin{aligned}\Lambda_{\vec{k}} &= -\mu\int_{-\infty}^{+\infty}\xi_1(s)\tilde{f}_{\vec{k}}(s)e^{\iota(\vec{k}\cdot\vec{\omega})s}ds,\\ \vec{\Delta}_{\vec{k}} &= -\mu\int_{-\infty}^{+\infty}\vec{f}_{\vec{k}}(s)e^{\iota(\vec{k}\cdot\vec{\omega})s}ds\end{aligned}\tag{66}$$

¿From the Extension lemma we know that $\tilde{f}_{\vec{k}}(\tau)$ and $\vec{f}_{\vec{k}}(\tau)$ are analytic in τ for $\tau\in\Sigma$, therefore this definition certainly makes sense.

By the parity properties both $\vec{f}$ and $\tilde{f}$ are time-angle odd, so the splitting distance Δ will be an odd function of $\vec{\alpha}$, and $\Delta_{\vec{k}} = -\Delta_{-\vec{k}}$. [2]

The odd parity of Δ follows from the fact that

$$\eta(0_+, \vec{\alpha}) = \eta(0_-, -\vec{\alpha}),$$
$$\vec{\zeta}(0_+, \vec{\alpha}) = \vec{\varsigma}(0_-, -\vec{\alpha}).$$

see also (65). Therefore, according to Corollary 4.1 and the Remark that follows it, we do not have to deal with $\vec{k} = \vec{0}$ at all. Also,

$$\vec{\alpha}: \ \alpha_i = [0 \,\mathrm{or}\, \pi \quad \text{for } i = 1, .., n-1$$

yields homoclinic connections, i.e., at these points the splitting distance is zero.

For $\vec{k} \neq \vec{0}$ we have to estimate the integrals in (66). Assume that $\vec{k} \cdot \vec{\omega} > 0$.

Also, since $\tilde{f}$ and $\vec{f}$ are time-angle odd, they can be represented as the sum t-odd $\cdot\, \vec{\alpha}$-even $+$ t-even $\cdot\, \vec{\alpha}$-odd, and obviously the first summand does not contribute into the integrals (65). This apropos of (66) means that we can assume $\tilde{f}_{\vec{k}} = -\tilde{f}_{-\vec{k}}$ and $\vec{f}_{\vec{k}} = -\vec{f}_{-\vec{k}}$. Furthermore, from parity we have to compute only an integral from $-\infty$ to zero, and then multiply it by 2.

By the analyticity stipulated by the Extension lemma we can lift the contour of integration and compute for $\vec{\Delta}$:

$$\int_{\rho\infty}^{0} \vec{f}_{\vec{k}}(s) e^{\iota \vec{k}\cdot\vec{\omega} s} ds = \mathcal{J}[\vec{f}_{\vec{k}} e^{\iota \vec{k}\cdot\vec{\omega} s}](\iota p_*) - \iota \int_0^{p_*} \vec{f}_{\vec{k}}(\iota s) e^{-\vec{k}\cdot\vec{\omega} s} ds,$$

where $p_* = \frac{\pi}{2\sqrt{\varepsilon}} - \sqrt{\varepsilon}$.

Furthermore, for the $-\vec{k}$th Fourier component we can shift the contour of integration symmetrically with respect to the x-axis, therefore the second integrals will cancel out, for it's only a time-even component of $\vec{f}$ that contributes into the splitting.

To estimate the first integral (recall the definition of the operation $\mathcal{J}$), we notice that it easily follows from the KAM theorem and the proof of the Extension lemma that for both coordinate components x and $\vec{\varphi}$, the quantities $\xi_1(\tau)\tilde{f}_{\vec{k}}(\tau)$ and $\vec{f}_{\vec{k}}(\tau)$ in the integrands can be bounded from above by $O\left(\frac{\mu}{\mu_0}\sqrt{\varepsilon}\right) \exp(-|\vec{k}|\sigma_2)$, where μ_0 comes from (44) and $\sigma_2 = \sigma_0 - \frac{3}{4}\sqrt{\varepsilon}$. Therefore, we arrive at the estimate

$$|\Delta_{\vec{k}}| \leq \frac{|\mu|}{\mu_0} \frac{\sqrt{\varepsilon}}{\vec{k}\cdot\vec{\omega}} e^{-|\vec{k}\cdot\vec{\omega}|p_* - |k|\sigma_*}, \tag{67}$$

if one takes μ_0 from (44). In addition, the splitting distance function will be analytic in μ for $|\mu| \leq \mu_0$.

Proof of exponential smallness. Naturally, the quantity that one is interested in studying that is relevant to the splitting of separatrices when viewed as the key component of the Arnold diffusion problem is its transversality measure, or its size $\Upsilon = |\det \partial_{\vec{\alpha}} \vec{\Delta}|$.

¿From (67), Corollary 4.1 and the parity discussion it follows that the following representation is justified:

$$\vec{\Delta} = \frac{|\mu|}{\mu_0} \sqrt{\varepsilon} \sum_{\vec{k} \neq \vec{0}} A_{\vec{k}}(\mu, \varepsilon) \frac{1}{\vec{k}\cdot\vec{\omega}} e^{-|\vec{k}\cdot\vec{\omega}|p_* - |k|\sigma_*} \sin(\vec{k}\cdot\vec{\alpha}), \tag{68}$$

where all $A_{\vec{k}} = A_{-\vec{k}}$ do not exceed 1 in their absolute values.

If we fix $K(\varepsilon) = [\varepsilon^{-\frac{1}{2}+\delta}]$, for some small positive δ, where $[\cdot]$ stands for the integer part, and make a standard ultraviolet cutoff, then using (67), the sum of the absolute values of the coefficients $\Delta_{\vec{k}}$ (we will often

[2] In the first passage of this section we already mentioned that Ellison et al. [1993] and Gallavotti [1994] studied the splitting on the fixed Poincaré section $x = \pi$ at $t = 0$ (which implies that $\xi(0) = 0$, $\vec{\varsigma}(0) = \vec{0}$ and invalidates our representation (47), yielding (48) instead). In both papers it required quite complex ways to argue that the $\vec{k} = \vec{0}$ component in these representations does not contribute to the splitting distance. By an $O(\mu)$ time-shift, quasiperiodic in $\vec{\alpha}$, which can be constructed directly order by order, one can achieve in our representation that $x(0) = \pi$, this will imply that $\vec{\varphi}(0) = \alpha + O(\mu)$, the latter being an odd quasiperiodic function of $\vec{\alpha}$. The splitting distance will be also modified by an $O(\mu)$ quantity, quasiperiodic in $\vec{\alpha}$.

call the latter quantity the Fourier norm "with zero width"; it's known to exceed the *sup*-norm), restricted to the ultraviolet region $|\vec{k}| \geq K(\varepsilon)$ can be easily estimated as follows:

$$\sum_{|\vec{k}|\geq K} |\Delta_{\vec{k}}| \leq \frac{|\mu|}{\mu_0} \exp\left(-\varepsilon^{-\frac{1}{2}+2\delta}\right), \tag{69}$$

whereas for $|\vec{k}| \leq K(\varepsilon)$ using (14) we will have

$$\begin{aligned} |\Delta_{\vec{k}}| &\leq \tfrac{|\mu|}{\mu_0}\varepsilon^{-\frac{\varpi}{2}+\delta} \exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{\omega}\cdot\vec{k}|}{\sqrt{\varepsilon}}\right) \exp\left(\varepsilon^{\delta(|\vec{\omega}|+1)}\right) \\ &\leq \tfrac{|\mu|}{\mu_0}\varepsilon^{-\frac{\varpi}{2}} \exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{\omega}\cdot\vec{k}|}{\sqrt{\varepsilon}}\right). \end{aligned}$$

Therefore, we can estimate the determinant only considering the terms under the cutoff; if the estimate turns out to be much larger than the above estimate for the ultraviolet part, then neglecting it will be justified.

Furthermore, under the cutoff we can find the upper bound for the $\vec{k}$th component of the determinant (for the latter will also be given by the Fourier series in $\vec{\alpha}$, which will converge for $|\Im\vec{\alpha}| \leq \sigma_* = \sigma_0 - \sqrt{\varepsilon})$ if we do it for a finite sum of the following terms (recall the rule of computing determinants and the representation (68)):

$$\Upsilon_{\vec{k}} \leq \sum \left(\frac{|\mu|}{\mu_0}\sqrt{\varepsilon}\right)^{n-1} \left|\prod_{i=1}^{n-1} k^i_{j_i} \frac{1}{\vec{k}^i\cdot\vec{\omega}} \exp\left(-|\vec{k}^i|\sigma_0 - \frac{\pi}{2}\frac{|\vec{\omega}\cdot\vec{k}^i|}{\sqrt{\varepsilon}}\right)\right|. \tag{70}$$

In the above sum $\vec{k} = \vec{k}^1 + \vec{k}^2 + .. + \vec{k}^{n-1}$, the array $[j_1, j_2, .., j_{n-1}]$ is a permutation of the set $\{1, 2, .., n-1\}$, the sum is taken over all the permutations and all the ways to represent $\vec{k}$ as a sum of $\vec{k}^i \neq \vec{0}$ for $i = 1, 2, .., n-1$.

¿From the diophantine condition (14) and the cutoff rule, we can get rid of $\frac{1}{\vec{k}^i\cdot\vec{\omega}}$ if we rescale μ_0, having multiplied it by $\varepsilon^{\frac{\varpi}{2}}$ (this would also cause $\sqrt{\varepsilon}$ to go away, for γ in the diophantine condition (14) can be $O(\sqrt{\varepsilon})$).

Furthermore, as we have allowed γ so far go to zero when ε goes to zero, we can find ourselves in a near-resonant situation. To model this situation, we assume that some $r < n-1$ first components of $\vec{\omega}$ go to zero when ε goes to zero as $\sqrt{\varepsilon}$ the fastest. We use the notation $\vec{r} = (1, .., r)$ and $\vec{s} = (r+1, .., n-1)$ with $s = n-1-r$ to write

$$\vec{\omega} = (\vec{\omega}_{\vec{r}}, \vec{\omega}_{\vec{s}}) \text{ and accordingly } \vec{k} = (\vec{k}_{\vec{r}}, \vec{k}_{\vec{s}}).$$

One can probably play a lot with arithmetics and obtain many interesting parallels with the Nekhoroshev exponents, both time and spatial, if one considers different dependencies $\vec{\omega}_{\vec{r}}(\varepsilon)$, getting closer and farther away from the resonances. We content ourselves with several simple conclusions not to make this paper infinitely long.

Suppose $\vec{\omega}_{\vec{r}} = \sqrt{\varepsilon}\vec{\omega}_0$, where $\vec{\omega}_0$ does not depend on ε and has rationally independent components. Then (70) tells us that if $\vec{k}^i_s = \vec{0}$, $\forall\, i = 1, 2, .., n-1$, the corresponding term in the sum is zero. We emphasize this fact in our notation by formally introducing the Boolean variable

$$B_* = \{\exists i^* \in \{1, .., n-1\} : \ \vec{k}^{i^*}_{\vec{s}} \neq \vec{0}\}$$

and rewriting (70) in the following way:

$$\Upsilon_{\vec{k}} \leq \sum B_* \left(\frac{|\mu|}{\mu_0}\right)^{n-1} \varepsilon^{-\frac{n-1}{2}(\varpi+1)} \exp\left(-|\vec{k}^{i^*}_{\vec{s}}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{\omega}_{\vec{k}_s}\cdot\vec{k}^{i^*}_{\vec{s}}|}{\sqrt{\varepsilon}}\right). \tag{71}$$

This expression clearly shows that Υ is exponentially small. Moreover, there is a well-known fact, in particular reflected by the Dirichlet theorem (see below), that given a vector $\vec{\omega}$ whose components are independent over the rationals the problem of its linear approximation becomes easier as the dimension grows. This is the general motivation behind why the splitting size gets smaller in a resonant situation. Besides, it's already clear that if $r = n-2$, then the exponent in the estimate (70) will be nothing but $\frac{1}{2} - \delta$, where the latter (possibly rescaled) can be arbitrarily small.

The advantage of choosing $\vec{\omega}_{\vec{r}} = \sqrt{\varepsilon}\vec{\omega}_0$ is that under the cutoff one will have $|\vec{\omega}_{\vec{r}} \cdot \vec{k}| \leq |\vec{\omega}_0|\varepsilon^{\delta}$, therefore the estimate (71) will be valid. Otherwise, e.g. if one chooses $\vec{\omega}_{\vec{r}} = \varepsilon^{\frac{1}{4}}\vec{\omega}_0$, then the cutoff parameter should be chosen as $K(\varepsilon) = \varepsilon^{-\frac{1}{4}+\delta}$ and the exponent in the upper bound for the splitting size in the fully resonant situation will be $\frac{1}{4} - \delta$. In fact, the following Proposition is easy to establish:

Proposition 5.2 *If $\vec{\omega}$ is fully resonant, namely $r = n-2$, and $\vec{\omega}_{\vec{r}} = \varepsilon^b\vec{\omega}_0$, where $\frac{1}{2(n-1)} + 2\delta \leq b \leq \frac{1}{2}$ for any small positive δ and ε small enough, then*

$$|\Upsilon_k| \leq \left(\frac{|\mu|}{\mu_0}\right)^{n-1} e^{-K\varepsilon^{-b+\delta}},$$

where $\mu_0 = \varepsilon^d$ for a finite d depending upon the parameters of the problem and K being $O(1)$.

Choosing b smaller than $\frac{1}{2(n-1)}$ does not make any sense, for if this is the case the resonant upper bound will become comparable with the non-resonant case (see below).

So already at this point we are done with the analytic part of this paper, and the rest will be just arithmetics. We will devote a separate section to some motivation when the number of degrees of freedom is any $n \geq 3$, and then focus our attention to the case $n = 3$, for there one can use the Continued fractions theory to scrutinize the small divisors in the exponents. In particular, the case when $r = n-3$ can be most likely treated within this framework to yield the exponent $\frac{1}{4}$.

6 Exponential Smallness: Arithmetic Considerations

One can see that the arithmetics becomes a major obstruction to making any more or less general conclusions (especially, when $n > 3$), for the small divisors sit in the exponents, and the diophantine condition is not much help at all, for once being introduced to insure convergence of the now well-known KAM iterative scheme and further given a nice measure-theoretical interpretation, it is extremely awkward as far as any approximation business is concerned.

6.1 Conclusion of the proof of Theorem 2.1

We will use Assumption 2.1 on the frequency vector that will enable us to move an inch further with estimating (71) and to suggest, in particular, a simple upper bound for it.

Returning to (71), recall that we assume that

$$\vec{\omega} = (\sqrt{\varepsilon}\vec{\omega}_0, \vec{\omega}_{\vec{s}}),$$

where the vectors $\vec{\omega}_0$ and $\vec{\omega}_{\vec{s}}$ have ε-independent rationally independent components, and the latter satisfies Assumption 2.1 with $\gamma = \gamma_0$ and $\varpi \leq s - 2\delta$.

We fix s such that $s \in \{1, .., n-1\}$, being the dimension of a "non-resonant subvector" of $\vec{\omega}$. Then it becomes possible to estimate the exponent in it as follows:

$$|\vec{k}_{\vec{s}}^{i^*}|\sigma_0 + \frac{\pi}{2}\frac{|\vec{\omega} \cdot \vec{k}_{\vec{s}}^{i^*}|}{\sqrt{\varepsilon}} \geq \sigma_0 \left(\frac{\pi\gamma_0}{\sqrt{\varepsilon}}\right) \frac{1}{s - 2\delta},$$

just by finding its minimum as a function of $|\vec{k}_{\vec{s}}^{i^*}|$, which occurs when the latter is equal to $\frac{\pi\gamma_0(s-2\delta)}{2\sigma_0\sqrt{\varepsilon}}$. When we were computing the above minimum, we've taken the constant in the exponent actually smaller than it should be to get rid of the inverse powers of ε in the preexponential factor of (71) and to get rid of the summation sign, for the number of summands is clearly proportional to a certain finite (depending on the number of degrees of freedom) power of ε.

This concludes the proof of Theorem 2.1. □

6.2 Asymptotic Formula for the Splitting Distance for $n \geq 3$

In this section we'll be dealing with the case when the perturbation is given specifically by (17). The number of degrees of freedom is any finite $n \geq 3$. The purpose of the ensuing analysis will be to pay some tribute to the Melnikov function and show that in a certain way one can still rely on it as the leading order answer for the splitting distance.

We write

$$\vec{\Delta}(\vec{\omega}, \vec{\alpha}, \mu) = \mu \vec{M}(\vec{\omega}, \vec{\alpha}) + \left(\frac{\mu}{\mu_0}\right)^2 \vec{N}(\vec{\omega}, \vec{\alpha}, \mu), \tag{72}$$

where

$$\vec{M}(\vec{\omega}, \vec{\alpha}) = -\int_{-\infty}^{+\infty} \vec{f}(x_0(t), \vec{\psi}) dt \tag{73}$$

is the Melnikov integral with $\vec{\psi} = \vec{\alpha} + \vec{\omega} t$.

The Extension lemma along with the expression (68) imply that for all the absolute values of the Fourier coefficients of $\vec{N}$ will respect the bound:

$$|\vec{N}_{\vec{k}}(\vec{\omega}, \mu)| \leq \frac{\sqrt{\varepsilon}}{|\vec{k} \cdot \vec{\omega}|} \exp\left(-|\vec{k}|(\sigma_0 - \sqrt{\varepsilon}) - \frac{\pi}{2} \frac{|\vec{\omega} \cdot \vec{k}|}{\sqrt{\varepsilon}}(1 - \varepsilon)\right). \tag{74}$$

We recall that the perturbation in (17) is given as

$$F(x, \vec{\varphi}) = \sum_{\vec{k} \in Z^{n-1} \setminus \{0\}} P_{\vec{k}}(x) \exp(-|\vec{k}|\sigma_0) \cos(\vec{k} \cdot \vec{\varphi}),$$

where $P_{\vec{k}}(x)$ are even trigonometric polynomials in x, whose degrees do not exceed ν_0, having for all $\vec{k} \neq \vec{0}$ the following form:

$$P_{\vec{k}}(x) = \sum_{j=0}^{\nu_{\vec{k}}} A_{j\vec{k}} \frac{(2j-1)!}{2^j} (1 - \cos x)^j$$

with all $\nu_{\vec{k}} \leq \nu_0$ and the non-negative coefficients $A_{j\vec{k}}$ satisfying (19).

Then clearly from (181)

$$P_{\vec{k}}(x_0(t)) = \sum_{j=0}^{\nu_{\vec{k}}} (2j-1)! A_{j\vec{k}} \frac{1}{\cosh^{2j} \sqrt{\varepsilon}\tau}.$$

So, computing the Melnikov function will boil down to using the following formula for $\vec{k} \neq \vec{0}$, $j = 0, 1, 2, .., \nu_0$:

$$\int_{-\infty}^{+\infty} \frac{\cos[(\vec{k} \cdot \vec{\omega})t]}{\cosh^{2j}(\sqrt{\varepsilon}t)} dt = \frac{\pi(\vec{k} \cdot \vec{\omega})}{(2j-1)! \varepsilon \sinh\left(\frac{\pi}{2} \frac{\vec{k} \cdot \vec{\omega}}{\sqrt{\varepsilon}}\right)} \prod_{l=1}^{j-1} \left(\frac{(\vec{k} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2\right),$$

where for $j = 1$ there being no product term, and for $j = 0$ the integral turning into zero.

This implies that the Melnikov function can be expressed as follows:

$$\vec{M}(\vec{\omega}, \vec{\alpha}) = \sum_{\vec{k} \neq \vec{0}} \vec{k} \sin(\vec{k} \cdot \vec{\alpha}) \exp(-|\vec{k}|\sigma_0) \frac{\pi(\vec{k} \cdot \vec{\omega})}{\varepsilon \sinh\left(\frac{\pi}{2} \frac{\vec{k} \cdot \vec{\omega}}{\sqrt{\varepsilon}}\right)}$$
$$\times \sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2\right). \tag{75}$$

Our purpose will be to compare the Melnikov function that we have obtained with the bounds for the remainder $\vec{N}$ in (88), that we can easily derive from (90) and Assumption 2.1 that will be further used

Henceforward we'll be interested only in real values of $\vec{\alpha} \in T^{n-1}$, unless specified.

We fix the ultraviolet cutoff parameter $K(\varepsilon) = [\varepsilon^{-\frac{1}{2}+\delta}]$. Then repeating the argument in the end of Section 5 we will have the estimate under and above the cutoff as follows:

$$
\begin{aligned}
|\vec{N}_{\vec{k}}|\varepsilon^{\frac{\varpi}{2}} &\leq \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right), \quad |\vec{k}| < K(\varepsilon), \\
\sum_{|\vec{k}|\geq K(\varepsilon)} |\vec{N}_{\vec{k}}| &\leq \exp\left(-\varepsilon^{-\frac{1}{2}+2\delta}\right).
\end{aligned}
\tag{76}
$$

It's easy to see from (88) by comparing (91) with (92), that if $|\mu| \leq \mu_0$, the latter satisfying (44), then for $|\vec{k}| \leq |K|$ the contribution of the terms of the second and higher orders in μ in the splitting distance would be negligible, could we somehow handle the Fourier harmonics, where some component of $\vec{k}$ is zero, for the latter terms do not contribute to the same component of $\vec{M}$. The easiest way to do this is to introduce some additional assumptions on the frequency $\vec{\omega}$, such as e.g. Assumption 2.1.

The motivation comes from Diophantine approximation theory, and its keystone, the Dirichlet theorem, quoted below (see Schmidt [1980], p. 27), which suggests that given a vector with rationally independent components, its linear approximation by integer vectors, whose norm does not exceed a fixed value will be in most cases much more precise than it would be for its any subvector.

Theorem 6.1 (Dirichlet) *Suppose $\vec{\beta}$ is a real vector with components $\beta_1, .., \beta_l$, and $M > 1$ is an integer. Then there exists an integer l-vector $\vec{q}$ and an integer p such that*

$$1 \leq \max(|q_1|, .., |q_l|) \leq M^{\frac{1}{l}} \text{ and } |\vec{\beta}\cdot\vec{q} - p| \leq \frac{1}{M}. \tag{77}$$

Note, that by this time we've started running out of notation, for (p, q) have already been used in the Normal form section. Nevertheless, this notation is standard in Diophantine approximation theory and hopefully will not be confusing.

We recall that according to Assumption 2.1 with $s = n-1$, any l-subvector of $\vec{\omega}$ for $l = 2, .., n-1$ satisfies the diophantine condition (14) with $\gamma = \gamma_0$ and $\varpi = l - 2\delta$. Obviously, this set of vectors has full measure, which with more scrutiny could be made asymptotically full (if γ were taken equal to ε^{δ_2} for a very small positive δ_2, as we shall do in the following section).

This assumption will be sufficient to insure the dominating role of the Melnikov function, and therefore suggesting the lower bound for the splitting distance, which will all boil down to exersizing a couple of simple tricks.

Clearly, for $|\vec{k}| \leq K(\varepsilon)$ we can write in (91) with an exponentially small error

$$
\begin{gathered}
|\vec{k}\cdot\vec{\omega}| \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) \\
\text{instead of} \\
\exp(-|\vec{k}|\sigma_0)\frac{\vec{k}\cdot\vec{\omega}}{\sinh\left(\frac{\pi}{2}\frac{\vec{k}\cdot\vec{\omega}}{\sqrt{\varepsilon}}\right)},
\end{gathered}
\tag{78}
$$

if ε is small enough.

The Dirichlet theorem suggests how one can find a lower bound for the splitting distance function, because it allows one to estimate the dot products $\vec{k}\cdot\vec{\omega}$ in the exponents from above. So one can argue that there exists an integer vector $\vec{k}^*$ with its norm bounded as $O(M^{\frac{1}{n-2}})$, such that $|\vec{k}^*\cdot\vec{\omega}| \leq M^{-1}$, and then use a simple fact that the Fourier norm "with a zero width", which is the sum of the absolute values of all the Fourier components, will be greater than the absolute value of the particular Fourier coefficient in the Fourier series for the splitting distance function, corresponding to $\vec{k}^*$.

Unfortunately, for an arbitrary $\vec{\omega}$ it's hard to argue that all the $q's$ and p in the Dirichlet Theorem, or equivalently all the components of the aforementioned vector $\vec{k}^*$, are going to be nonzero, which one must necessarily have to be able to perform the estimate from below following this scheme because of the presence of $\vec{k}$ in the formula (91) for the Melnikov function.

One cannot totally avoid this difficulty, and given $\vec{\omega}$, the strategy for getting the lower bound may vary. The ensuing analysis can be applied to all the frequency vectors for which the ratio of the absolute value of the maximum modulus component and the absolute value of the second maximum modulus component is independent of ε.

We denote

$$\omega_+ = \max_{j=1,..,n-1} |\omega_j| = |\vec{\omega}|,$$

and without loss of generality we can assume that the maximum is achieved on the first component: $\omega_+ = |\omega_1|$. Then for the dot products $\vec{k}\cdot\vec{\omega}$ we can write

$$|\vec{k}\cdot\vec{\omega}| = \omega_+|p+\vec{q}\cdot\vec{\beta}|, \tag{79}$$

where $p = \pm k_1$ - the first component of the integer $(n-1)$-vector $\vec{k}$, which is multiplied by -1 if ω_1 is negative (the latter cannot be zero by the diophantine condition), $\vec{q} = (\vec{k}_2,..,k_{n-1}) \in Z^{n-2}$, and $\vec{\beta} = (\frac{\omega_2}{\omega_+},..,\frac{\omega_{n-1}}{\omega_+}) \in R^{n-2}$. We will also denote

$$\omega_- = \max_{j=2,..,n-1} |\omega_j|,$$

namely the "second maximum", then again, without loss of generality we can assume that $\omega_- = |\omega_2|$.

In the common notation $[\cdot]$ will stand for an integer part and $\{\cdot\}$ for a fractional part of a real number.

Then it's easy to establish the following Proposition, which is proved using the standard box argument, due to Dirichlet.

Proposition 6.1 *Given an $(n-1)$-vector $(\pm 1,\vec{\beta})$, where $\vec{\beta}\in R^{n-2}$ has rationally independent components, with $|\vec{\beta}| = \max_{j=1,..,n-2}|\beta_j|$, and a real number $M>2$ such that also*

$$M > \left(\frac{(n-3)(n-2)}{2}\right)^{n-2}, \tag{80}$$

there exists an $(n-1)$-integer vector $\vec{k}^*$, whose all components, except for possibly the first one, are nonzero, and such that

$$|\vec{k}^*| < 2(n-1)M^{\frac{1}{n-2}}\max(1,|\vec{\beta}|), \quad |\vec{k}^*\cdot\vec{\omega}| < \frac{1}{M}. \tag{81}$$

Proof: We consider the set $\mathcal{Q}$ of all the integer $(n-2)$-vectors $\vec{q}$ such that if $\vec{q}\in\mathcal{Q}$, then

$$\mathrm{sign}(q_j) = \mathrm{sign}(\beta_j), \;\; 1 \le |q_j| \le M^{\frac{1}{n-2}} + j, \;\;\; j = 1,..,n-2,$$

and if $\vec{q}^1,\vec{q}^2\in\mathcal{Q}$, then for all $j = 1,..,n-2$ one has $q_j^1 \neq q_j^2$. The number of such vectors is clearly not smaller than M.

Now we simply consider the set, consisting of all the fractional parts $\{\vec{q}\cdot\vec{\beta}\}$ (different, because $\vec{\beta}$ is linearly independent over the rationals), where $\vec{q}\in\mathcal{Q}$; all these points lie inside the interval $(0,1)$, so there is a pair $\vec{q}^1,\vec{q}^2\in\mathcal{Q}$ such that $\{\vec{q}^1\cdot\vec{\beta}\}$ and $\{\vec{q}^2\cdot\vec{\beta}\}$ differ by a quantity, which is less that than $\frac{1}{M}$.

If we take a vector $\vec{q}^*$ with components

$$q_j^* = q_j^1 - q_j^2, \;\;\; j = 1,..,n-2,$$

then by construction the components q_j^* of the vector $\vec{q}^*$ are all going to be nonzero.

To evaluate the norm (the sum of the absolute values of the components) of $\vec{q}^*$, we notice that

$$|q_j^*| \le M^{\frac{1}{n-2}} + j - 1, \;\;\; j = 1,..,n-2,$$

so,

$$|\vec{q}^*| \leq (n-2)M^{\frac{1}{n-2}} + \frac{(n-2)(n-3)}{2} < (n-1)M^{\frac{1}{n-2}}.$$

if (96) is abided with.

The integer part $[\vec{q}^* \cdot \vec{\beta}]$ is clearly bounded from above by $|\vec{q}^*| \max(1, |\vec{\beta}|)$, although it can not be guaranteed to be nonzero.

Thus we can choose $\vec{k}^* = (\mp[\vec{q}^* \cdot \vec{\beta}], \vec{q}^*)$ to satisfy the conditions of this Proposition. □

Using this Proposition we can argue that given $M > 2$ and satisfying (96) there exists an integer vector $\vec{k}^*$ with no component, except for possibly the first one, equal to zero, such that the corresponding exponential term in (94) will be bounded from below as follows:

$$\begin{aligned} \exp\left(-|\vec{k}^*|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}^*\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0 - \frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right) \\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-} - \frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right). \end{aligned} \tag{82}$$

Unfortunately, we don't know whether k_1^* is nonzero. If it is, then we repeat what we have just done, representing in the same notation $\vec{k}\cdot\vec{\omega} = \omega_-\vec{k}\cdot(\pm1, \vec{\beta})$, where now $\vec{\beta} = (\frac{\omega_1}{\omega_-}, \frac{\omega_3}{\omega_-}, .., \frac{\omega_{n-1}}{\omega_-})$, and again apply Proposition 6.2 to establish the existence of an integer vector $\vec{k}'$ which satisfies (97) and has all the nonzero components except for possibly the second one, yielding the following estimate for the corresponding exponential term:

$$\begin{aligned} \exp\left(-|\vec{k}'|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}'\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-} - \frac{\pi}{2}\frac{\omega_-}{M\sqrt{\varepsilon}}\right) \\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-} - \frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right). \end{aligned} \tag{83}$$

The absolute value of the exponent in the coinciding rightmost sides of (98) and (99) as a function of M has a minimum when

$$M = M^* = \left(\frac{\pi\omega_-(n-2)}{4(n-1)\sigma_0\sqrt{\varepsilon}}\right)^{\frac{n-2}{n-1}}, \tag{84}$$

which for ε small enough is certainly bigger than 2 and satisfies (96), provided that the quantity ω_- is ε-independent.

Substituting (100) into the lower bound at (98) or (99), we can rewrite it as:

$$\exp\left(-\omega_+(n-1)\left(\frac{2\sigma_0(n-1)}{\omega_-(n-2)}\right)^{\frac{n-2}{n-1}}\left(\frac{\pi}{2\sqrt{\varepsilon}}\right)^{\frac{1}{n-1}}\right). \tag{85}$$

So in the Fourier series for (88) there are at least two terms below the ultraviolet cutoff (labeled by $\vec{k}^*$ and $\vec{k}'$), whose exponentially small part is estimated from below by (101). That's why we can give an estimate *from below* for the Fourier norm "with zero width" of the splitting distance function, and consequently, using (27) its *sup*$-$ norm in a narrow complex neighborhood of T^{n-1} as well.

Recalling the definition of the Absolute norm, given in Section 3 (in this particular case it will be identical with the Fourier norm used in Pöschel [1993]), for *real only* values of $\vec{\alpha}$, specified by a zero subscript, the Fourier norm for the ith component of the homoclinic splitting-distance function ($i = 1, .., n-1$) will be bounded from below by the following expression (we neglect the exponentially small ultraviolet part of the remainder from (92), thinking that ε is small enough, and the exponentially small error has also been taken care of in the quite rough estimates for the norm of $\vec{k}^*$ or $\vec{k}'$. given by Proposition 6.2.

We use the notation

$$\vec{k}^{\delta_{i1}} = \left[\begin{array}{l} \vec{k}' \text{ for } i = 1 \\ \vec{k}^* \text{ for } i \neq 1, \end{array}\right.$$

which means that for $i = 1$ we take the mode, indexed by $\vec{k}'$, since we know for sure that $k'_1 \neq 0$, and for the rest of the components of the splitting distance function we take the mode indexed by $\vec{k}^*$, because we also know that $k^*_2, .., k^*_{n-1} \neq 0$. Thus, using (67) and (14), recalling that the ε-independent factor in the exponent is in fact larger than its true value, we arrive at the following estimate:

$$
\begin{aligned}
|\Delta_i|_0 \quad &\geq \quad \left(|\mu| \varepsilon^{-1+\frac{\varpi}{2}} \sum_{j=1}^{\nu_{\vec{k}^{\delta_{i1}}}} A_{j\vec{k}^{\delta_{i1}}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k}^{\delta_{i1}} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2 \right) - \left| \frac{\mu}{\mu_0} \right|^2 \varepsilon^{1-\frac{\varpi}{2}} \right) \\
&\times \quad \exp\left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right) \qquad (86) \\
&\geq \quad \frac{|\mu|}{\mu_0} \exp\left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right),
\end{aligned}
$$

and the same estimate will hold for the *sup*-norm $|\Delta_i|^{\infty}_{\sqrt{\varepsilon}}$ in a complex set $W_{\sqrt{\varepsilon}} T^{n-1}$, which is connected with the Fourier (Absolute) norm through (27), if μ_0 comes from (44).

The last thing to show is that the Melnikov function really dominates over the remainder. This will basically follow from (92) if we manage to compare the lower bound for the norm of the Melnikov function with the upper bound for the sum of all the terms in N_i such that $k_i = 0$ for $|\vec{k}| < K(\varepsilon)$ and $i = 1, .., n-1$.

But by our assumption on the frequency vector if at least one component of $\vec{k}$ is zero, then $|\vec{k} \cdot \vec{\omega}| \geq \gamma_0 |k|^{n-2-2\delta}$, therefore

$$
\min \left(|\vec{k}|\sigma_0 + \frac{\pi}{2} \frac{|\vec{k} \cdot \vec{\omega}|}{\sqrt{\varepsilon}} \right) \geq \sigma_0 \left(\frac{\pi \gamma_0}{\sigma_0 \sqrt{\varepsilon}} \right)^{\frac{1}{n-1-2\delta}} .
$$

To obtain this estimate we have simply used the diophantine condition with $\gamma = \gamma_0$ and $\varpi = n-2-2\delta$, where δ can be chosen much smaller than $\frac{1}{2}$, to find the minimum of the exponent. Therefore, if ε is small enough one ends up with the following estimate (by summing up (91), (92), (102), and the smallness condition (44)):

$$
\vec{\Delta} = \mu \vec{M} \left\{ 1 + O\left(\frac{\mu}{\mu_0} \right) + O\left(e^{-\varepsilon^{-\frac{1}{2(n-1)}}} \right) \right\}. \qquad (87)
$$

One can see now that by this time the proof of Theorem 2.2 has been completed. □

As one can see now that under certain fairly generic assumptions, it's not difficult at all to vindicate the Melnikov function as the leading-order answer for the splitting distance. In fact, all we need is that the perturbation possess an infinite number of harmonics.

It is much harder to follow through a similar argument for the transversality measure Υ, unless one is dealing with some fixed frequencies whose approximation properties are nice and uniform, and well known. Otherwise, it's very hard to deal with this quantity, for being a determinant, it becomes much more sensitive to any errors in its entries, at least if the problem is ill-conditioned, which seems to be the situation.

It is definitely extremely ill-conditioned in a near-resonant case, for from the previous section we know that the quantity Υ will be exponentially small, whereas the matrix $\partial_{\vec{\alpha}} \vec{\Delta}$ itself will have some entries which are proportional to ε to some power, which means that the condition number of such a matrix (the ratio of the largest and the smallest eigenvalues) will be *exponentially large*.

Even in a non-resonant situation, if we roughly rewrite (102) as

$$
|\Delta_i| \geq \mu e^{-K\varepsilon^{-\frac{1}{2(n-1)}}} (1 + O(\mu)),
$$

for some $O(1)$ constant K, we can estimate $\Upsilon \sim \mu^{n-1} e^{-K(n-1)\varepsilon^{-\frac{1}{2(n-1)}}}$. Thus, this quantity becomes smaller than the error in the preceding estimate, therefore, in general the leading order cannot be proved to dominate for the determinant unless there is an estimate on the condition number of the leading order matrix.

In other words, if we want to deal with the determinant and suggest some uniform estimates for exponentially small quantities, we have to worry not only about the power of ε in the exponents, but also about the values of certain constants, which is hardly manageable in a more or less general setting.

Fortunately, if the number of degrees of freedom $n = 3$, then one can actually do this. This is the situation that we are going to consider next.

6.3 Asymptotic Formula for the Splitting Distance for $n \geq 3$

In this section we'll be dealing with the case when the perturbation is given specifically by (17). The number of degrees of freedom is any finite $n \geq 3$. The purpose of the ensuing analysis will be to pay some tribute to the Melnikov function and show that in a certain way one can still rely on it as the leading order answer for the splitting distance.

We write

$$\vec{\Delta}(\vec{\omega}, \vec{\alpha}, \mu) = \mu \vec{M}(\vec{\omega}, \vec{\alpha}) + \left(\frac{\mu}{\mu_0}\right)^2 \vec{N}(\vec{\omega}, \vec{\alpha}, \mu), \tag{88}$$

where

$$\vec{M}(\vec{\omega}, \vec{\alpha}) = -\int_{-\infty}^{+\infty} \vec{f}(x_0(t), \vec{\psi}) dt \tag{89}$$

is the Melnikov integral with $\vec{\psi} = \vec{\alpha} + \vec{\omega} t$.

The Extension lemma along with the expression (68) imply that for all the absolute values of the Fourier coefficients of $\vec{N}$ will respect the bound:

$$|\vec{N}_{\vec{k}}(\vec{\omega}, \mu)| \leq \frac{\sqrt{\varepsilon}}{|\vec{k} \cdot \vec{\omega}|} \exp\left(-|\vec{k}|(\sigma_0 - \sqrt{\varepsilon}) - \frac{\pi}{2} \frac{|\vec{\omega} \cdot \vec{k}|}{\sqrt{\varepsilon}}(1 - \varepsilon)\right). \tag{90}$$

We recall that the perturbation in (17) is given as

$$F(x, \vec{\varphi}) = \sum_{\vec{k} \in Z^{n-1} \setminus \{0\}} P_{\vec{k}}(x) \exp(-|\vec{k}|\sigma_0) \cos(\vec{k} \cdot \vec{\varphi}),$$

where $P_{\vec{k}}(x)$ are even trigonometric polynomials in x, whose degrees do not exceed ν_0, having for all $\vec{k} \neq \vec{0}$ the following form:

$$P_{\vec{k}}(x) = \sum_{j=0}^{\nu_{\vec{k}}} A_{j\vec{k}} \frac{(2j-1)!}{2^j} (1 - \cos x)^j$$

with all $\nu_{\vec{k}} \leq \nu_0$ and the non-negative coefficients $A_{j\vec{k}}$ satisfying (19).

Then clearly from (181)

$$P_{\vec{k}}(x_0(t)) = \sum_{j=0}^{\nu_{\vec{k}}} (2j-1)! A_{j\vec{k}} \frac{1}{\cosh^{2j} \sqrt{\varepsilon}\tau}.$$

So, computing the Melnikov function will boil down to using the following formula for $\vec{k} \neq \vec{0}$, $j = 0, 1, 2, .., \nu_0$:

$$\int_{-\infty}^{+\infty} \frac{\cos[(\vec{k} \cdot \vec{\omega})t]}{\cosh^{2j}(\sqrt{\varepsilon}t)} dt = \frac{\pi(\vec{k} \cdot \vec{\omega})}{(2j-1)! \varepsilon \sinh\left(\frac{\pi}{2} \frac{\vec{k} \cdot \vec{\omega}}{\sqrt{\varepsilon}}\right)} \prod_{l=1}^{j-1} \left(\frac{(\vec{k} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2\right),$$

where for $j = 1$ there being no product term, and for $j = 0$ the integral turning into zero.

This implies that the Melnikov function can be expressed as follows:

$$\vec{M}(\vec{\omega}, \vec{\alpha}) = \sum_{\vec{k} \neq \vec{0}} \vec{k} \sin(\vec{k} \cdot \vec{\alpha}) \exp(-|\vec{k}|\sigma_0) \frac{\pi(\vec{k} \cdot \vec{\omega})}{\varepsilon \sinh\left(\frac{\pi}{2} \frac{\vec{k} \cdot \vec{\omega}}{\sqrt{\varepsilon}}\right)}$$
$$\times \sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2\right). \tag{91}$$

Our purpose will be to compare the Melnikov function that we have obtained with the bounds for the remainder $\vec{N}$ in (88), that we can easily derive from (90) and Assumption 2.1 that will be further used

Henceforward we'll be interested only in real values of $\vec{\alpha} \in T^{n-1}$, unless specified.

We fix the ultraviolet cutoff parameter $K(\varepsilon) = [\varepsilon^{-\frac{1}{2}+\delta}]$. Then repeating the argument in the end of Section 5 we will have the estimate under and above the cutoff as follows:

$$\begin{array}{lll} |\vec{N}_{\vec{k}}|\varepsilon^{\frac{\varpi}{2}} & \leq & \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right), \quad |\vec{k}| < K(\varepsilon), \\ \sum_{|\vec{k}|\geq K(\varepsilon)} |\vec{N}_{\vec{k}}| & \leq & \exp\left(-\varepsilon^{-\frac{1}{2}+2\delta}\right). \end{array} \tag{92}$$

It's easy to see from (88) by comparing (91) with (92), that if $|\mu| \leq \mu_0$, the latter satisfying (44), then for $|\vec{k}| \leq |K|$ the contribution of the terms of the second and higher orders in μ in the splitting distance would be negligible, could we somehow handle the Fourier harmonics, where some component of $\vec{k}$ is zero, for the latter terms do not contribute to the same component of $\vec{M}$. The easiest way to do this is to introduce some additional assumptions on the frequency $\vec{\omega}$, such as e.g. Assumption 2.1.

The motivation comes from Diophantine approximation theory, and its keystone, the Dirichlet theorem, quoted below (see Schmidt [1980], p. 27), which suggests that given a vector with rationally independent components, its linear approximation by integer vectors, whose norm does not exceed a fixed value will be in most cases much more precise than it would be for its any subvector.

Theorem 6.2 (Dirichlet) *Suppose $\vec{\beta}$ is a real vector with components $\beta_1, .., \beta_l$, and $M > 1$ is an integer. Then there exists an integer l-vector $\vec{q}$ and an integer p such that*

$$1 \leq \max(|q_1|, .., |q_l|) \leq M^{\frac{1}{l}} \text{ and } |\vec{\beta}\cdot\vec{q} - p| \leq \frac{1}{M}. \tag{93}$$

Note, that by this time we've started running out of notation, for (p, q) have already been used in the Normal form section. Nevertheless, this notation is standard in Diophantine approximation theory and hopefully will not be confusing.

We recall that according to Assumption 2.1 with $s = n-1$, any l-subvector of $\vec{\omega}$ for $l = 2, .., n-1$ satisfies the diophantine condition (14) with $\gamma = \gamma_0$ and $\varpi = l - 2\delta$. Obviously, this set of vectors has full measure, which with more scrutiny could be made asymptotically full (if γ were taken equal to ε^{δ_2} for a very small positive δ_2, as we shall do in the following section).

This assumption will be sufficient to insure the dominating role of the Melnikov function, and therefore suggesting the lower bound for the splitting distance, which will all boil down to exersizing a couple of simple tricks.

Clearly, for $|\vec{k}| \leq K(\varepsilon)$ we can write in (91) with an exponentially small error

$$\begin{array}{c} |\vec{k}\cdot\vec{\omega}| \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) \\ \text{instead of} \\ \exp(-|\vec{k}|\sigma_0)\frac{\vec{k}\cdot\vec{\omega}}{\sinh\left(\frac{\pi}{2}\frac{\vec{k}\cdot\vec{\omega}}{\sqrt{\varepsilon}}\right)}, \end{array} \tag{94}$$

if ε is small enough.

The Dirichlet theorem suggests how one can find a lower bound for the splitting distance function, because it allows one to estimate the dot products $\vec{k}\cdot\vec{\omega}$ in the exponents from above. So one can argue that there exists an integer vector $\vec{k}^*$ with its norm bounded as $O(M^{\frac{1}{n-2}})$, such that $|\vec{k}^*\cdot\vec{\omega}| \leq M^{-1}$, and then use a simple fact that the Fourier norm "with a zero width", which is the sum of the absolute values of all the Fourier components, will be greater than the absolute value of the particular Fourier coefficient in the Fourier series for the splitting distance function, corresponding to $\vec{k}^*$.

Unfortunately, for an arbitrary $\vec{\omega}$ it's hard to argue that all the $q's$ and p in the Dirichlet Theorem, or equivalently all the components of the aforementioned vector $\vec{k}^*$, are going to be nonzero, which one must necessarily have to be able to perform the estimate from below following this scheme because of the presence of $\vec{k}$ in the formula (91) for the Melnikov function.

One cannot totally avoid this difficulty, and given $\vec{\omega}$, the strategy for getting the lower bound may vary. The ensuing analysis can be applied to all the frequency vectors for which the ratio of the absolute value of the maximum modulus component and the absolute value of the second maximum modulus component is independent of ε.

We denote

$$\omega_+ = \max_{j=1,..,n-1} |\omega_j| = |\vec{\omega}|,$$

and without loss of generality we can assume that the maximum is achieved on the first component: $\omega_+ = |\omega_1|$. Then for the dot products $\vec{k}\cdot\vec{\omega}$ we can write

$$|\vec{k}\cdot\vec{\omega}| = \omega_+ |p + \vec{q}\cdot\vec{\beta}|, \tag{95}$$

where $p = \pm k_1$ - the first component of the integer $(n-1)$-vector $\vec{k}$, which is multiplied by -1 if ω_1 is negative (the latter cannot be zero by the diophantine condition), $\vec{q} = (k_2, .., k_{n-1}) \in Z^{n-2}$, and $\vec{\beta} = (\frac{\omega_2}{\omega_+}, .., \frac{\omega_{n-1}}{\omega_+}) \in R^{n-2}$. We will also denote

$$\omega_- = \max_{j=2,..,n-1} |\omega_j|,$$

namely the "second maximum", then again, without loss of generality we can assume that $\omega_- = |\omega_2|$.

In the common notation $[\cdot]$ will stand for an integer part and $\{\cdot\}$ for a fractional part of a real number.

Then it's easy to establish the following Proposition, which is proved using the standard box argument, due to Dirichlet.

Proposition 6.2 *Given an $(n-1)$-vector $(\pm 1, \vec{\beta})$, where $\vec{\beta} \in R^{n-2}$ has rationally independent components, with $|\vec{\beta}| = \max_{j=1,..,n-2} |\beta_j|$, and a real number $M > 2$ such that also*

$$M > \left(\frac{(n-3)(n-2)}{2}\right)^{n-2}, \tag{96}$$

there exists an $(n-1)$-integer vector $\vec{k}^*$, whose all components, except for possibly the first one, are nonzero, and such that

$$|\vec{k}^*| < 2(n-1)M^{\frac{1}{n-2}} \max(1, |\vec{\beta}|), \quad |\vec{k}^*\cdot\vec{\omega}| < \frac{1}{M}. \tag{97}$$

Proof: We consider the set $\mathcal{Q}$ of all the integer $(n-2)$-vectors $\vec{q}$ such that if $\vec{q} \in \mathcal{Q}$, then

$$\mathrm{sign}(q_j) = \mathrm{sign}(\beta_j), \;\; 1 \le |q_j| \le M^{\frac{1}{n-2}} + j, \;\; j = 1, .., n-2,$$

and if $\vec{q}^1, \vec{q}^2 \in \mathcal{Q}$, then for all $j = 1, .., n-2$ one has $q_j^1 \neq q_j^2$. The number of such vectors is clearly not smaller than M.

Now we simply consider the set, consisting of all the fractional parts $\{\vec{q}\cdot\vec{\beta}\}$ (different, because $\vec{\beta}$ is linearly independent over the rationals), where $\vec{q} \in \mathcal{Q}$; all these points lie inside the interval $(0,1)$, so there is a pair $\vec{q}^1, \vec{q}^2 \in \mathcal{Q}$ such that $\{\vec{q}^1\cdot\vec{\beta}\}$ and $\{\vec{q}^2\cdot\vec{\beta}\}$ differ by a quantity, which is less that than $\frac{1}{M}$.

If we take a vector $\vec{q}^*$ with components

$$q_j^* = q_j^1 - q_j^2, \;\; j = 1, .., n-2,$$

then by construction the components q_j^* of the vector $\vec{q}^*$ are all going to be nonzero.

To evaluate the norm (the sum of the absolute values of the components) of $\vec{q}^*$, we notice that

$$|q_j^*| \le M^{\frac{1}{n-2}} + j - 1, \;\; j = 1, .., n-2,$$

so,

$$|\vec{q}^*| \leq (n-2)M^{\frac{1}{n-2}} + \frac{(n-2)(n-3)}{2} < (n-1)M^{\frac{1}{n-2}}.$$

if (96) is abided with.

The integer part $[\vec{q}^* \cdot \vec{\beta}]$ is clearly bounded from above by $|\vec{q}^*| \max(1, |\vec{\beta}|)$, although it can not be guaranteed to be nonzero.

Thus we can choose $\vec{k}^* = (\mp[\vec{q}^* \cdot \vec{\beta}], \vec{q}^*)$ to satisfy the conditions of this Proposition. □

Using this Proposition we can argue that given $M > 2$ and satisfying (96) there exists an integer vector $\vec{k}^*$ with no component, except for possibly the first one, equal to zero, such that the corresponding exponential term in (94) will be bounded from below as follows:

$$\begin{aligned} \exp\left(-|\vec{k}^*|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{k}^* \cdot \vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0 - \tfrac{\pi}{2}\tfrac{\omega_+}{M\sqrt{\varepsilon}}\right) \\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\tfrac{\omega_+}{\omega_-} - \tfrac{\pi}{2}\tfrac{\omega_+}{M\sqrt{\varepsilon}}\right). \end{aligned} \tag{98}$$

Unfortunately, we don't know whether k_1^* is nonzero. If it is, then we repeat what we have just done, representing in the same notation $\vec{k} \cdot \vec{\omega} = \omega_- \vec{k} \cdot (\pm 1, \vec{\beta})$, where now $\vec{\beta} = (\frac{\omega_1}{\omega_-}, \frac{\omega_3}{\omega_-}, .., \frac{\omega_{n-1}}{\omega_-})$, and again apply Proposition 6.2 to establish the existence of an integer vector $\vec{k}'$ which satisfies (97) and has all the nonzero components except for possibly the second one, yielding the following estimate for the corresponding exponential term:

$$\begin{aligned} \exp\left(-|\vec{k}'|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{k}' \cdot \vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\tfrac{\omega_+}{\omega_-} - \tfrac{\pi}{2}\tfrac{\omega_-}{M\sqrt{\varepsilon}}\right) \\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\tfrac{\omega_+}{\omega_-} - \tfrac{\pi}{2}\tfrac{\omega_+}{M\sqrt{\varepsilon}}\right). \end{aligned} \tag{99}$$

The absolute value of the exponent in the coinciding rightmost sides of (98) and (99) as a function of M has a minimum when

$$M = M^* = \left(\frac{\pi\omega_-(n-2)}{4(n-1)\sigma_0\sqrt{\varepsilon}}\right)^{\frac{n-2}{n-1}}, \tag{100}$$

which for ε small enough is certainly bigger than 2 and satisfies (96), provided that the quantity ω_- is ε-independent.

Substituting (100) into the lower bound at (98) or (99), we can rewrite it as:

$$\exp\left(-\omega_+(n-1)\left(\frac{2\sigma_0(n-1)}{\omega_-(n-2)}\right)^{\frac{n-2}{n-1}}\left(\frac{\pi}{2\sqrt{\varepsilon}}\right)^{\frac{1}{n-1}}\right). \tag{101}$$

So in the Fourier series for (88) there are at least two terms below the ultraviolet cutoff (labeled by $\vec{k}^*$ and $\vec{k}'$), whose exponentially small part is estimated from below by (101). That's why we can give an estimate *from below* for the Fourier norm "with zero width" of the splitting distance function, and consequently, using (27) its *sup*$-$ norm in a narrow complex neighborhood of T^{n-1} as well.

Recalling the definition of the Absolute norm, given in Section 3 (in this particular case it will be identical with the Fourier norm used in Pöschel [1993]), for *real only* values of $\vec{\alpha}$, specified by a zero subscript, the Fourier norm for the ith component of the homoclinic splitting-distance function ($i = 1, .., n-1$) will be bounded from below by the following expression (we neglect the exponentially small ultraviolet part of the remainder from (92), thinking that ε is small enough, and the exponentially small error has also been taken care of in the quite rough estimates for the norm of $\vec{k}^*$ or $\vec{k}'$. given by Proposition 6.2.

We use the notation

$$\vec{k}^{\delta_{i1}} = \left[\begin{array}{l} \vec{k}' \text{ for } i = 1 \\ \vec{k}^* \text{ for } i \neq 1, \end{array}\right.$$

which means that for $i=1$ we take the mode, indexed by $\vec{k}'$, since we know for sure that $k_1' \neq 0$, and for the rest of the components of the splitting distance function we take the mode indexed by $\vec{k}^*$, because we also know that $k_2^*, .., k_{n-1}^* \neq 0$. Thus, using (67) and (14), recalling that the ε-independent factor in the exponent is in fact larger than its true value, we arrive at the following estimate:

$$\begin{aligned} |\Delta_i|_0 \quad &\geq \quad \left(|\mu| \varepsilon^{-1+\frac{\varpi}{2}} \textstyle\sum_{j=1}^{\nu_{\vec{k}^{\delta_{i1}}}} A_{j\vec{k}^{\delta_{i1}}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k}^{\delta_{i1}} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2 \right) - \left| \frac{\mu}{\mu_0} \right|^2 \varepsilon^{1-\frac{\varpi}{2}} \right) \\ &\times \quad \exp \left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right) \qquad (102) \\ &\geq \quad \frac{|\mu|}{\mu_0} \exp \left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right), \end{aligned}$$

and the same estimate will hold for the *sup*-norm $|\Delta_i|_{\sqrt{\varepsilon}}^{\infty}$ in a complex set $W_{\sqrt{\varepsilon}} T^{n-1}$, which is connected with the Fourier (Absolute) norm through (27), if μ_0 comes from (44).

The last thing to show is that the Melnikov function really dominates over the remainder. This will basically follow from (92) if we manage to compare the lower bound for the norm of the Melnikov function with the upper bound for the sum of all the terms in N_i such that $k_i = 0$ for $|\vec{k}| < K(\varepsilon)$ and $i = 1, .., n-1$.

But by our assumption on the frequency vector if at least one component of $\vec{k}$ is zero, then $|\vec{k} \cdot \vec{\omega}| \geq \gamma_0 |k|^{n-2-2\delta}$, therefore

$$\min \left(|\vec{k}| \sigma_0 + \frac{\pi}{2} \frac{|\vec{k} \cdot \vec{\omega}|}{\sqrt{\varepsilon}} \right) \geq \sigma_0 \left(\frac{\pi \gamma_0}{\sigma_0 \sqrt{\varepsilon}} \right)^{\frac{1}{n-1-2\delta}} .$$

To obtain this estimate we have simply used the diophantine condition with $\gamma = \gamma_0$ and $\varpi = n-2-2\delta$, where δ can be chosen much smaller than $\frac{1}{2}$, to find the minimum of the exponent. Therefore, if ε is small enough one ends up with the following estimate (by summing up (91), (92), (102), and the smallness condition (44)):

$$\vec{\Delta} = \mu \vec{M} \left\{ 1 + O\left(\frac{\mu}{\mu_0} \right) + O\left(e^{-\varepsilon^{-\frac{1}{2(n-1)}}} \right) \right\}. \qquad (103)$$

One can see now that by this time the proof of Theorem 2.2 has been completed. □

As one can see now that under certain fairly generic assumptions, it's not difficult at all to vindicate the Melnikov function as the leading-order answer for the splitting distance. In fact, all we need is that the perturbation possess an infinite number of harmonics.

It is much harder to follow through a similar argument for the transversality measure Υ, unless one is dealing with some fixed frequencies whose approximation properties are nice and uniform, and well known. Otherwise, it's very hard to deal with this quantity, for being a determinant, it becomes much more sensitive to any errors in its entries, at least if the problem is ill-conditioned, which seems to be the situation.

It is definitely extremely ill-conditioned in a near-resonant case, for from the previous section we know that the quantity Υ will be exponentially small, whereas the matrix $\partial_{\vec{\alpha}} \vec{\Delta}$ itself will have some entries which are proportional to ε to some power, which means that the condition number of such a matrix (the ratio of the largest and the smallest eigenvalues) will be *exponentially large*.

Even in a non-resonant situation, if we roughly rewrite (102) as

$$|\Delta_i| \geq \mu e^{-K\varepsilon^{-\frac{1}{2(n-1)}}} (1 + O(\mu)),$$

for some $O(1)$ constant K, we can estimate $\Upsilon \sim \mu^{n-1} e^{-K(n-1)\varepsilon^{-\frac{1}{2(n-1)}}}$. Thus, this quantity becomes smaller than the error in the preceding estimate, therefore, in general the leading order cannot be proved to dominate for the determinant unless there is an estimate on the condition number of the leading order matrix.

In other words, if we want to deal with the determinant and suggest some uniform estimates for exponentially small quantities, we have to worry not only about the power of ε in the exponents, but also about the values of certain constants, which is hardly manageable in a more or less general setting.

Fortunately, if the number of degrees of freedom $n = 3$, then one can actually do this. This is the situation that we are going to consider next.

6.4 Asymptotic Formula for the splitting distance and the splitting size size for the case of three degrees of freedom

When the number of degrees of freedom equals three, the arithmetic results of the previous section can be significantly furthered, for a wide class of non-resonant frequencies. The main reason for it is that in this case $\vec{\omega}$ is a 2-vector, and to evaluate the dot products $\vec{k} \cdot \vec{\omega}$ one can use well-developed theory of Continued fractions. This gives us an opportunity to obtain a compact answer for the leading order asymptotic behavior of the splitting distance function, which will in turn vindicate the leading-order answer for the measure of transversality Υ.

Notation in this section will be somewhat independent and standard for Continued fractions theory; in particular, since the number of degrees of freedom $n = 3$ is fixed, the symbol n will further play the role of an index variable.

The main character will be a vector-function $\vec{\mathcal{M}}$ on a 2-torus, given by a Fourier series

$$\vec{\mathcal{M}}(\vec{\alpha}) = \sum_{\vec{k} \in Z^2} \vec{A}_{\vec{k}} \mathcal{E}_{\vec{k}} \sin(\vec{k} \cdot \vec{\alpha}) \tag{104}$$

where $\vec{\alpha} \in T^2$, and the terms $\mathcal{E}_{\vec{k}}$ are represented as follows:

$$\mathcal{E}_{\vec{k}} = \exp\left(-|\vec{k}|\sigma_0 - \frac{\pi|\vec{k} \cdot \vec{\omega}|}{2\sqrt{\varepsilon}}\right) \equiv \exp(-E_{\vec{k}}). \tag{105}$$

The coefficients $\vec{A}_{\vec{k}} = -A_{-\vec{k}}$ will be all strictly nonzero, with their $\vec{k}$ dependence being such that they never grow or vanish faster than some finite power of $|\vec{k}|$ as $|\vec{k}| \to \infty$, where as usual, $|\vec{k}| = |k_1| + |k_2|$.

It's easy to see that a particular example of such a series is the Melnikov function in (91), when $n = 3$.

We will prove that for all small enough positive values of ε ($0 < \varepsilon \leq \varepsilon_0 \ll 1$), except for a set of asymptotically as $\varepsilon \to 0$ zero measure, consisting of some small neighborhoods of the sequence of its so-called "critical values", the latter sequence being nowhere dense and accumulating at zero, the exponents $E_{\vec{k}}$ in the Fourier coefficients have an absolute minimum for some $\vec{k}^*(\vec{\omega}, \varepsilon, \sigma_0)$, (and, of course, $-\vec{k}^*$ by symmetry). The location of this maximum depends on the particular frequency vector $\vec{\omega}$, namely $|\vec{k}^*| \sim \mathcal{A}(\vec{\omega}, \sigma_0)\varepsilon^{-\frac{1}{4}}$, for some function $\mathcal{A}$ to be described, which can be computed, if one follows the developed procedure.

The absolute value of the corresponding term in the Fourier series will be shown to differ from the sum of the absolute values of the rest of the terms by a quantity large enough for it to yield the leading-order asymptotics for the splitting distance function with exponentially small error. The proof will hold true for those values of ε which are away from the aforementioned critical values: otherwise the minimum in consideration turns out to be not sharp enough for a single term in (104) to dominate.

Furthermore, this only term does not suffice to somehow characterize the quantity Υ, for its sole contribution into it is obviously going to be zero (simply because the zeroes of $\sin(\vec{k}^* \cdot \vec{\alpha})$ are not isolated). Thus, we'll be able to locate the second smallest exponent with an index $\vec{k}^{**}$, and to prove that these two terms together will give the leading-order answer for the splitting size, as is stated by Theorem 2.3.

We'll outline the procedure, which prescribes how $\vec{k}^*$ and $\vec{k}^{**}$ can be found, which can be implemented numerically. For specifically chosen frequency vectors with well-known approximation properties this will be fairly straightforward.

At last, the result we claim will hold true for an asymptotically full measure set of "almost" non-resonant frequencies in the sense that they do not come to the resonances closer than ε^{δ_2}, the latter being such that $0 < \delta_2 \ll \frac{1}{2}$.

Moreover, near the "critical values" of ε, some "switching of the modes" occurs, namely when ε is being decreased and it passes the critical value, $\vec{k}^*$ quickly changes to a vector with a bigger norm, so for ε near the critical value there are at least two modes whose contributions are comparable, and we will two or more terms vying to dominate the whole sum; this would be much harder to describe rigorously.

Our principal tool will be a simple number theoretical lemma, called the Transition lemma, the centerpiece of the algorithm of finding $\vec{k}^*$ and $\vec{k}^{**}$.

Without loss of generality we shall assume that $|\omega_1| > |\omega_2|$ (they can't be equal or zero by the diophantine condition (14)), and denote

$$\omega_+ = |\omega_1|,$$

which we assume to be ε-independent. Moreover, for simplicity let's suppose that both ω_1 and ω_2 are positive (the modifications one will have to make to remove this assumption will be simple bookkeeping).

For the scalar product $\vec{k}\cdot\vec{\omega}$ then we get:

$$\vec{k}\cdot\vec{\omega} = k_1\omega_+ + k_2\omega_2 = \omega_+\left(\vec{k}_1 + k_2\frac{\omega_2}{\omega_+}\right) \equiv \omega_+(-p+q\beta), \tag{106}$$

where we've denoted $\beta = \frac{\omega_2}{\omega_+}$, $p = -k_1$, $q = k_2$ following the standard notation of the approximation theory. So for $\vec{k} \in Z^2$ the question of asymptotic behavior of the series (104) can be explored in terms of elementary Continued fractions, at least for β with well predictable approximation properties. Besides, as we've mentioned it's enough to consider only $q \geq 0$ by symmetry.

6.4.1 Minimum background and notation from Continued fractions theory

In the following section we shall give several definitions and facts from the theory of Continued fractions, as a reference we would suggest Schmidt [1980].

For each irrational number β there exists a unique expansion into a *Continued fraction*

$$\beta = [a_0, a_1, a_2, ..],$$

where $[a_0, a_1, a_2, ..]$ is an infinite sequence of integers, all of which are positive, except for possibly a_0, which equals the integer part of β, and if zero, can be omitted in the Continued fraction. Besides, a finite sequence $[a_0, a_1, a_2, .., a_n]$ gives a unique Continued fraction expansion of a rational if we deem the finite Continued fractions $[a_0, a_1, .., a_n]$ with $a_n > 1$ and $[a_0, a_1, .., a_n - 1, 1]$ equivalent.

The following relation explains the term "Continued fraction," whether the sequence under consideration is finite or infinite:

$$\begin{aligned} \beta &= [a_0, a_1, a_2, ..] &&= a_0 + \tfrac{1}{[a_1, a_2, ..]} \\ &= a_0 + \tfrac{1}{a_1 + \frac{1}{[a_2, a_3, ..]}} &&= [a_0, a_1, .., a_{n-1}, [a_n, a_{n+1}, ..]]. \end{aligned} \tag{107}$$

We call a fraction $\frac{p}{q}$, where p, q are integers with $q > 0$ a *best approximation*, or a *convergent* to β if for any integer $0 < q' \leq q$ and all p', not equal to p if $q' = q$, one has:

$$\min_{p'} |q'\beta - p'| > |q\beta - p|.$$

Then for every β, rational or irrational, there is a sequence $\{q_n\}$ of positive integers with $q_0 = 1$ and a sequence $\{p_n\}$ of integers such that $\frac{p_n}{q_n}$ are the best approximations to β, these sequences being finite for a rational β and infinite if it is irrational. With some abuse of terminology we will most of the time apply the term "convergents" solely to the members of the sequence $\{q_n\}$, whereas the members of the sequence $\{\frac{p_n}{q_n}\}$ will be referred to as "best approximations".

In the standard notation

$$\|q\beta\| = \min_{p} |q\beta - p|.$$

For any real β the Continued fraction expansion above uniquely encodes the sequences $\{p_n\}$ and $\{q_n\}$ by the recursion rule:

$$\begin{aligned} p_{-2} = 0, &\quad p_{-1} = 1, \\ q_{-2} = 1, &\quad q_{-1} = 0, \end{aligned} \tag{108}$$

and for $n \geq 0$

$$
\begin{aligned}
p_n &= a_n p_{n-1} + p_{n-2}, \\
q_n &= a_n q_{n-1} + q_{n-2}.
\end{aligned}
\tag{109}
$$

By the Dirichlet theorem (see Schmidt [1980]), every best approximation $\frac{p_n}{q_n}$ to β satisfies

$$
\|q_n\beta\| = |q_n\beta - p_n| \leq \frac{1}{q_{n+1}}. \tag{110}
$$

Besides, there is another well-known fact about the consecutive best approximations, that we'll state as a Proposition:

Proposition 6.3 *For $n = 0, 1..$ the quantities $q_n\beta - p_n$ and $q_{n+1}\beta - p_{n+1}$ have the opposite signs and*

$$
p_n q_{n+1} - q_n p_{n+1} = \pm 1.
$$

Proof: See Schmidt [1980], pp. 10-11. □

A number β is called *badly approximable* if there exists a constant $0 < c(\beta) < 1$, such that for every $q > 0$ one has

$$
\|q\beta\| > \frac{c(\beta)}{q}. \tag{111}
$$

Proposition 6.4 *β is badly approximable if and only if all the terms in its Continued fraction expansion $\beta = [a_0, a_1, ..]$ are uniformly bounded:*

$$
a_i \leq \bar{a}, \quad i = 0, 1, 2, ...
$$

Proof: See Schmidt [1980]. In fact, it follows from the two following important relations, which are the consequences of (107), (109) and can be found in Schmidt [1980] respectively on pp. 10, 23, stating that for $n \geq 2$

$$
\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, .., a_1], \tag{112}
$$

and for $n \geq 0$

$$
\left|\beta - \frac{p_n}{q_n}\right| = \frac{1}{q_n^2\left([a_{n+1}, a_{n+2}, ..] + \frac{1}{[a_n, a_{n-1}, .., a_1]}\right)}, \tag{113}
$$

which gives the exact value for the difference between the irrational itself and its n-th best approximation. Moreover, (107) tells us how to evaluate the right-hand part of (113), so that we can always take in the definition (111) of badly approximable numbers

$$
c(\beta) = \frac{1}{\max_{i>0} a_i + 2} \quad \square \tag{114}
$$

In particular, this Proposition together with (107) imply that there is a continuum of badly approximable numbers (the argument is essentially the same as in the well-known proof of Cantor theorem on uncountability of real numbers).

In addition given

$$
\bar{a} = \max_{i>0} a_i \geq 1, \tag{115}
$$

all the 2-vectors $\vec{\omega} = \omega_+(1, \beta)$ will be diophantine with $\varpi = 1$ and $\gamma = \frac{\omega_+}{4+2\bar{a}}$ in the standard notation of (14), which implies that for a fixed $\bar{a}$ they form a zero measure set.

Although in our analysis we will be mostly dealing with badly approximable numbers, the ultraviolet cutoff that we are going to apply to the series (104), as we have done previously, will eventually enable us to extend our result to the set of frequencies of asymptotically (as $\varepsilon \to 0$) full measure by letting $\bar{a}$ in (115) go to infinity as $\varepsilon \to 0$ and using instead of (115) the following less stringent definition:

$$\bar{a}(\varepsilon) = \max_{0<i\leq n_\varepsilon} a_i, \tag{116}$$

for some large enough n_ε which tends to infinity as $\varepsilon \to 0$.

Besides, we'll be using another simple fact (usually less emphasized in the literature), stated in the following Proposition.

Proposition 6.5 *If $n \geq 1$, then for any $q: \ q_{n-1} < q < q_n$ one has:*

$$\|q\beta\| \geq \|q_{n-1}\beta\| + \|q_n\beta\|. \tag{117}$$

Proof: Let $q_{n-1} < q < q_n$. For any integer p, we will define η, ξ by the equations:

$$\begin{array}{rcl} \eta p_n + \xi p_{n-1} & = & p, \\ \eta q_n + \xi q_{n-1} & = & q. \end{array}$$

It follows from Proposition 6.3 that the determinant of the matrix of this system of equations equals ± 1, so η, ξ are integers.

If $\xi = 0$, then $(p, q) = \eta \cdot (p_n, q_n)$, which is impossible by the choice of q.

If $\eta = 0$, then the choice of q will require $\xi \geq 2$, because we get $(p, q) = \xi \cdot (p_{n-1}, q_{n-1})$, and then $\|q\beta\| \geq 2\|q_{n-1}\beta\| \geq \|q_{n-1}\beta\| + \|q_n\beta\|$, as stated in (117).

If both η and ξ are nonzero, then by the choice of q they must have the opposite signs. Hence, $\eta \cdot (\beta q_n - p_n)$ and $\xi \cdot (\beta q_{n-1} - p_{n-1})$ will be of the same sign, and therefore by Proposition 6.3 for any p:

$$|\beta q - p| = |\eta| \cdot \|q_n\beta\| + |\xi| \cdot \|q_{n-1}\beta\|.$$

Hence we will always have

$$\|q\beta\| = \min_p |\beta q - p| \geq \|q_{n-1}\beta\| + \|q_n\beta\|,$$

which is the restatement of (117). $\square$

If β is badly approximable, then by (110) and (111)

$$\|q_n\beta\| \geq \frac{c(\beta)}{q_n} \geq c(\beta)\|q_{n-1}\beta\|,$$

so Proposition 6.5 implies that for $q_{n-1} < q < q_n$

$$\|q\beta\| \leq (1 + c(\beta))\|q_{n-1}\beta\| \leq \left(1 + \frac{1}{\bar{a}+2}\right)\|q_{n-1}\beta\|, \tag{118}$$

with $\bar{a}$ defined in (115).

6.4.2 Preliminary simplifications

If we rewrite (104) in the notation of (106) as two separate sums

$$
\begin{aligned}
\mathcal{M}(\vec{\alpha}) &= \Sigma_1 + \Sigma_2 \\
&\equiv \textstyle\sum_{p,q\in Z,\ q>0,\ |q\beta-p|<\frac{1}{2}} \exp\left(-(p+q)\sigma_0 - \frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}) \\
&+ \textstyle\sum_{p,q\in Z,\ q\geq 0,\ |q\beta-p|\geq\frac{1}{2}} \exp\left(-(p+q)\sigma_0 - \frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}),
\end{aligned} \tag{119}
$$

where $\vec{k}=(-p,q)$, then for any positive δ_0 and ε small enough, the absolute value of the second sum will be bounded as:

$$
|\Sigma_2| \leq \exp\left(-\frac{\pi\omega_+}{4\sqrt{\varepsilon}}\right) \sum_{\vec{k}\in Z^2} \left|\vec{A}_{\vec{k}}\right| e^{-|\vec{k}|\sigma_0} \leq \exp\left(-\frac{\pi\omega_+}{8\sqrt{\varepsilon}}\right), \tag{120}
$$

where we've used the assumption on the coefficients $\vec{A}_{\vec{k}}$ and the well known equality $\sum_{\vec{k}\in Z^l} e^{-|\vec{k}|\sigma_0} = \coth^l\left(\frac{\sigma_0}{2}\right)$.

This estimate will turn out to be exponentially small compared to the asymptotics of the first sum Σ_1, to which we'll further restrict our attention.

The first sum Σ_1 in (119) has in fact only one index, since for each integer $q>0$ there is only one p such that $|q\beta-p|<\frac{1}{2}$. To this sum we can apply the ultraviolet cutoff, when q becomes as large as $q_\varepsilon=[\varepsilon^{-\frac{1}{2}+\delta_0}]$, where $[\cdot]$ means an integer part, and $0<\delta_0\ll\frac{1}{2}$ to be specified. We write

$$
\begin{aligned}
\Sigma_1 &= \Sigma_1^< + \Sigma_1^> \\
&\equiv \left(\textstyle\sum_{p,q\in Z,\ 1\leq q\leq q_\varepsilon,\ |q\beta-p|<\frac{1}{2}} + \sum_{p,q\in Z,\ q>q_\varepsilon,\ |q\beta-p|<\frac{1}{2}}\right) \\
&\quad \exp\left(-(p+q)\sigma_0 - \omega_+\frac{\pi|q\beta-p|}{2\sqrt{\varepsilon}}\right) \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}),
\end{aligned} \tag{121}
$$

where again and henceforward $\vec{k}=(-p,q)$.

It's easy to see (likewise in the preceding sections) that the ultraviolet part of the sum can be bounded as:

$$
|\Sigma_1^>| \leq \exp\left(-\varepsilon^{-\frac{1}{2}+\delta_0+\bar{\delta}}\right), \tag{122}
$$

for any small $\bar{\delta}>0$ if ε is small enough.

Moreover, one can write

$$
\begin{aligned}
\Sigma_1^< &= \bar{\Sigma}_1^< + \tilde{\Sigma}_1^< \\
&\equiv \textstyle\sum_{n\geq 0,\ q_n\leq q_\varepsilon} e^{-(p_n+q_n)\sigma_0 - \frac{\pi\omega_+\|q_n\beta\|}{2\sqrt{\varepsilon}}} A_{\vec{k}^n} \sin(\vec{k}^n\cdot\vec{\alpha}) \\
&+ \textstyle\sum_{n\geq 0,\ q_{n-1}<q<\min(q_n,q_\varepsilon)} e^{-(p+q)\sigma_0 - \frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}} \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha}),
\end{aligned} \tag{123}
$$

with $\vec{k}^n=(-p_n,q_n)$ in the first sum and $\vec{k}=(-p,q)$ in the second one, the first one including the *convergents* q_n to β only, and the second one all the q's that sit in between (recall that for every $q>0$ there is one only p such that $|\beta q-p|<\frac{1}{2}$).

The ultraviolet cutoff that we have previously made, enables us to define a "cutoff index" n_ε in (116) as:

$$
n_\varepsilon \equiv \max\{j>0:\ q_j\leq q_\varepsilon\}. \tag{124}
$$

Suppose q_{n-1}, q_n stand for the two consecutive convergents to β. Then in $\tilde{\Sigma}_1^<$ we can estimate the absolute value of the sum of all the terms, corresponding to $q_{n-1} < q < q_n$, using Proposition 6.5 and its consequence (118) as follows:

$$\begin{aligned}&\left|\sum_{q_{n-1}<q<\min(q_n,q_\varepsilon)} e^{-(p+q)\sigma_0-\frac{\pi\omega_+|q\beta-p|}{2\sqrt{\varepsilon}}} \vec{A}_{\vec{k}} \sin(\vec{k}\cdot\vec{\alpha})\right| \\ &\leq q_\varepsilon(\max_{q\leq q_\varepsilon}|\vec{A}_{\vec{k}}|)\exp[-(|p_{n-1}|+q_{n-1})\sigma_0]\exp\left[-\left(1+\tfrac{1}{\bar{a}+2}\right)\tfrac{\pi\omega_+\|q_{n-1}\beta\|}{2\sqrt{\varepsilon}}\right].\end{aligned} \tag{125}$$

Then the ratio of the absolute value of the sum, estimated in (125) to a single coefficient, corresponding to the $(n-1)$th convergent in the first sum in (123) will be at most

$$q_\varepsilon \frac{\max_{q\leq q_\varepsilon}|\vec{A}_{\vec{k}}|}{\min_{q\leq q_\varepsilon}|\vec{A}_{\vec{k}}|}\exp\left(-\frac{1}{\bar{a}+2}\frac{\pi\omega_+\|q_{n-1}\beta\|}{2\sqrt{\varepsilon}}\right) \leq \exp\left(-\frac{\pi\omega_+}{4(\bar{a}+2)^2\varepsilon^{\delta_0}}\right),$$

where we have used (111) and (114) to evaluate $\|q_{n-1}\beta\|$.

This in turn implies that in the Fourier norm "with zero width"

$$\left|\tilde{\Sigma}_1^<\right|_0 \leq \exp\left(-\frac{\omega_+}{4(\bar{a}+2)^2\varepsilon^{\delta_0}}\right)\cdot|\bar{\Sigma}_1^<|_0\,. \tag{126}$$

The coefficient in the right-hand part of (126) will also be exponentially small for any $\delta_0 > 0$ provided that ε is small enough. Moreover, one can see that we can let $\bar{a}$ in (116) "slightly" depend on ε if this still guarantees that the factor in (126) is exponentially small and the estimates (120), (122) are exponentially small in comparison with the asymptotics that we are going to find. In our further estimate we will always keep $\bar{a}$, since we finally expect to let it grow as $\varepsilon \to 0$.

So, we are basically left with $\bar{\Sigma}_1^<$ only to evaluate. Our main purpose, fulfilled in the succeeding section will be to show that it can be represented by its one single term plus an exponentially small error for the vast majority of the values of ε small enough.

6.4.3 The Transition lemma

The focus of the whole discussion in this section is the Transition lemma, our principal number-theoretical tool to determine the asymptotics of the series (104), which states the simple fact that almost always there is a single term that accounts for its leading-order behavior, except for those values of ε that sit in the "critical intervals" near the "critical values" , when the minimum of the exponent is not sharp enough to insure this.

After completing the proof we'll make the necessary generalizations (in particular, we'll analyze what happens when ε passes through one of the aforementioned, but not yet defined, critical values; we will also let the parameter $\bar{a}$ in (116) depend on ε).

First, we notice that from our convenience assumption $\beta > 0$ follows the fact that in its Continued fraction $a_0 = 0$ and all the numerators p_n in the sequence $\{\frac{p_n}{q_n}\}$ of the best approximations to β for $n \geq 1$ will be strictly positive ($p_0 = a_0 = 0$).

So further on we will be dealing with the series

$$\Sigma \equiv \sum_{0\leq n\leq n_\varepsilon} \exp\left(-k_n\sigma_0 - \frac{\pi\omega_+\|q_n\beta\|}{2\sqrt{\varepsilon}}\right) A_{\vec{k}^n}\sin(-p_n\alpha_1 + q_n\alpha_2), \tag{127}$$

where the summation is taken over all the convergents under the cutoff. Here $\vec{k}^n = (-p_n, q_n),\ k_n = p_n + q_n$.

Clear enough, $k_{-2} = k_{-1} = 1$, and for $n \geq 0$ the k_n's abide the same recursion relations as (109) for p_n, q_n, namely

$$k_n = a_n k_{n-1} + k_{n-2}. \tag{128}$$

Our next step will be to find tight upper and lower bounds for the exponents in the series (127), using (113). With some abuse of notation we will still apply the term "badly approximable" to all the numbers that satisfy

(116). We will think of $\bar{a}$ as a constant and in the end will simply check to what extent all the key estimates can digest the possible dependence of $\bar{a}$ on ε.

Once again, we assume that

$$\beta = [a_1, a_2, ...],$$

is given by an infinite Continued fraction, such that all the a_n's are uniformly bounded by some integer constant $\bar{a}$, for the moment independent of ε:

$$a_n \leq \bar{a}, \quad n = 1, 2, .., n_\varepsilon. \tag{129}$$

The exponents will equal

$$E_n = k_n \sigma_0 + \frac{\pi \omega_+ \|q_n \beta\|}{2\sqrt{\varepsilon}}. \tag{130}$$

The key quantity we will have to deal with is going to be

$$\mathcal{D}_n^1 = E_n - E_{n-1}, \quad n \geq 1, \tag{131}$$

but in certain cases it will not be enough, and we'll have to compare the exponents whose indices differ by 2 and consider a quantity

$$\mathcal{D}_n^2 = E_n - E_{n-2} = \mathcal{D}_n^1 + \mathcal{D}_{n-1}^1, \quad n \geq 2. \tag{132}$$

We shall try to express $\mathcal{D}_n^1$ and $\mathcal{D}_n^2$ in terms of as few parameters as possible, the range thereof depending on $\bar{a}$, and then prove that for ε small enough typically (in the sense of avoiding the sequence of relatively small intervals near the critical values for ε to be defined shortly) for any badly approximable β, as far as the shape of the sequence $\{E_n\}$ is concerned, we encounter either of the two situations described below.

- *Case 1 (V-shape):* If after a certain finite index N_0, the same for all β: $a_n > 1$, $n \geq N_0$, or if $a_n = 1\ \forall n \geq N_0$, for $n > N_0$ (the latter possibility pertains to numbers *equivalent* in the sense of their Continued fraction expansion to the "golden mean"), the sequence $\{E_n\}$ has a unique absolute minimum (V-shape) (depending on β, ε).
- *Case 2 (V,W-shape):* If a badly approximable β does not fall into the above category, then depending on ε small enough and β, the sequence $\{E_n\}$ has either one absolute minimum (V-shape), as described above, or two local minima for some $n = n_*$, $n = n_* + 2$, (W-shape), out of which we will be able to find the absolute minimum implementing the quantity $\mathcal{D}_{n_*+2}^2$. The necessary conditions for these occurrences will be considered further in details.

Thus, despite the presence of the small divisors in E_n, their sequence usually behaves quite nicely, first decreasing monotonically, and then monotonically increasing in case of V-shape, which sometimes can bifurcate into W-shape, characterized by the presence of two, but never more, neighboring local minima as ε changes.

Finally, we will see that we can let the parameter $\bar{a}$ from (115) depend on ε and extend the domain of β to the set of asymptotically full measure.

The only two simple formulas that we shall invoke will be (112) and (113). By the latter we have

$$\|q_n \beta\| = \frac{1}{q_n \left([a_{n+1}, a_{n+2}, ..] + \frac{1}{[a_n, a_{n-1}, .., a_1]}\right)} \equiv \frac{1}{q_n} \cdot \frac{1}{z_n + x_n}, \tag{133}$$

where we've defined for $n = 1, 2, ..$

$$\begin{aligned} x_n &\equiv \tfrac{1}{[a_n, a_{n-1}, .., a_1]} = \tfrac{q_{n-1}}{q_n}, \\ z_n &\equiv [a_{n+1}, a_{n+2}, ..]. \end{aligned} \tag{134}$$

By their definitions, the above parameters x_n, z_n are the functions of a large (infinite in case of z_n) number of integers, but (107) actually tells us that they really "strongly" depend only on a_n and a_{n+1} respectively, so one "almost" has $x_{n+1} \simeq \frac{1}{z_n} \simeq \frac{1}{a_{n+1}}$. Because of that, during the proof we'll always deal with the upper or lower bounds for x_n, z_n, which will be expressed in terms of no more than two integer parameters: a_n, a_{n-1} for the former and a_{n+1}, a_{n+2} for the latter.

Using (107) we can easily derive the following recursion relations for the parameters x_n, z_n which will subsequently be rather useful:

$$\begin{aligned} x_n &= \tfrac{1}{a_n + x_{n-1}}, \\ z_n &= a_{n+1} + \tfrac{1}{z_{n+1}}. \end{aligned} \tag{135}$$

By (129) we can easily establish for all n the general uniform bounds for x_n, z_n for all n in terms of the parameter $\bar{a}$ from (115):

$$\frac{1}{\bar{a}+1} < x_n < \frac{\bar{a}+1}{\bar{a}+2}, \tag{136}$$

$$\frac{\bar{a}+2}{\bar{a}+1} < z_n < \bar{a}+1. \tag{137}$$

The relative error in these estimates (the relative tolerance) will be at least $O(\bar{a}^{-1})$, where we assume $\bar{a}$ to be large enough.

These are the most general bounds that one can get; in particular cases, if we know the terms a_n and a_{n+1} for some n in the Continued fraction, we can use the recursion relation (135) to obtain rather precise upper and lower bounds for x_n and z_n by plugging in these formulae respectively a_n or a_{n+1} and the above lower and upper bounds for x_{n-1} and z_{n+1}.

Rewriting (133) with $n-1$ instead of n, using (107), we get:

$$\|q_{n-1}\beta\| = \frac{1}{q_{n-1}} \cdot \frac{1}{[a_n, a_{n+1}, ..] + \frac{1}{[a_{n-1}, a_{n-2}, .., a_1]}}$$

$$= \frac{1}{q_{n-1}} \cdot \frac{1}{\frac{1}{[a_{n+1}, ..]} + a_n + \frac{1}{[a_{n-1}, a_{n-2}, .., a_1]}},$$

so due to (112) and the definitions (134) of x_n, z_n, we end up with:

$$\|q_{n-1}\beta\| = \frac{1}{q_n} \cdot \frac{z_n}{z_n + x_n}. \tag{138}$$

The recursion relation (128) implies by induction that

$$\frac{k_n}{k_{n-1}} = a_n + \frac{1}{k_{n-1}/k_{n-2}} = ... = [a_n, a_{n-1}, ..a_1, a_0 + 1],$$

since (108) tells us that $k_{-1} = k_{-2} = 1, k_0 = a_0 + 1$. Recall that $a_0 = 0$ in our exposition, since $0 < \beta < 1$.

Still aiming to decrease the number of parameters, we write

$$\frac{k_n}{q_n} = 1 + \frac{p_n}{q_n} = 1 + \beta + \left(\frac{p_n}{q_n} - \beta\right) = 1 + \beta + O(q_n^{-2}), \tag{139}$$

where we've used (110).

Besides, comparing $x_n^{-1} = [a_n, a_{n-1}, ..a_1]$ and $x_n'^{-1} \equiv [a_n, a_{n-1}, .., a_1, 1]$, we can easily see that their difference is $O(q_{n-1}^{-2})$, since

$$x'_n k_n = k_{n-1} = (1+\beta)q_{n-1} + O(q_{n-1}^{-1}) = x_n k_n + O(q_{n-1}^{-1}). \tag{140}$$

Collecting all the building blocks (133), (138), (139), we eventually arrive at the following expression:

$$\mathcal{D}^1_n = (1+\beta)\sigma_0(1-x_n)q_n - \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q_n}\frac{z_n-1}{z_n+x_n} + O(q_{n-1}^{-1}). \tag{141}$$

The term $O(q_{n-1}^{-1})$ can be written explicitly as

$$\sigma_0\left[(x_n - x_n')(1+\beta)q_n + (-1)^{n+1}(1-x_n')\frac{1}{q_n}\frac{1}{x_n+z_n}\right], \tag{142}$$

by (139), (140) and Proposition 6.3.

Besides from (133) and (139) we may write:

$$E_n = \sigma_0(1+\beta)q_n + \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q_n}\frac{1}{x_n+z_n} + O(q_n^{-1}). \tag{143}$$

In the forthcoming analysis we shall always deal with q_{n-1} very large (approximately $O(\varepsilon^{-\frac{1}{4}})$), consequently it's possible to reckon without the last small term in (141) if we are able to estimate its impact upon the main estimates that we are going to perform.

We will consider a function

$$\mathcal{D}^1(q,x,z) = (1+\beta)\sigma_0(1-x)q - \frac{1}{q}\frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{z-1}{z+x},$$

of one continuous variable q and two parameters x, z, varying within the intervals, specified by (136), (137).

Given some values of the parameters x, z we will compute the sole zero of this function of q on the positive semiaxis. Thus, given x and z, for the values of q to the left of this zero, the function $\mathcal{D}^1(q,x,z)$ will return strictly negative values, conversely, if q is to the right of this zero, the function will be returning strictly positive values.

The condition $\mathcal{D}^1(q,x,z) = 0$ then can be rewritten as:

$$q = \mathcal{Q}^1(x,z) = \sqrt{\frac{\omega_+\pi}{2\sqrt{\varepsilon}\sigma_0(1+\beta)}}\cdot\left(\frac{z-1}{(z+x)(1-x)}\right)^{\frac{1}{2}} \equiv \mathcal{K}(\varepsilon)\sqrt{\mathcal{F}(x,z)}, \tag{144}$$

where

$$\mathcal{F}(x,z) = \frac{z-1}{(z+x)(1-x)} \tag{145}$$

and

$$\mathcal{K}(\varepsilon) = \sqrt{\frac{\pi\omega_+}{2\sqrt{\varepsilon}\sigma_0(1+\beta)}} \equiv \varepsilon^{-\frac{1}{4}}K^*, \tag{146}$$

thus also defining an $\bar{a}, \varepsilon$-independent constant K^*.

Given n, we will always be comparing the convergent q_n with the value $\mathcal{Q}^1(x_n, z_n)$: if the former exceeds the latter (by a quantity, which is sufficient to disregard the error in (143)), then obviously $\mathcal{D}^1_n$, defined by (131) is positive, and otherwise negative. This comparison will be the main operation performed through the proof of the ensuing Transition lemma, being in a certain sense the main point of the whole discussion.

For convenience we'll introduce a little formalism.

We will further refer to some $n_*(\varepsilon)$ as a *transition index* and to the convergent q_{n_*} as a *transition point* if $\mathcal{D}^1_{n_*-1}$ and $\mathcal{D}^1_{n_*}$ have strictly opposite signs. Henceforward, saying "a point" we will almost always refer to the q_n's.

If n_* is a transition index, this would mean in terms of the function $\mathcal{Q}^1(x,z)$ defined by (144) that with sufficient tolerance to disregard the error in (131), the following conditions are satisfied: either

$$q_{n_*-1} < \mathcal{Q}^1(x_{n_*-1}, z_{n_*-1}), \quad q_{n_*} > \mathcal{Q}^1(x_{n_*}, z_{n_*})$$

or

$$q_{n_*-1} > \mathcal{Q}^1(x_{n_*-1}, z_{n_*-1}), \quad q_{n_*} < \mathcal{Q}^1(x_{n_*}, z_{n_*}).$$

We assign a sign index $\chi(n)$ to each term of the sequence $\{q_n\}$ by the following rule:

$$\chi(n) = \text{sign}(q_n - \mathcal{Q}^1(x_n, z_n)).$$

Thus, the transition index corresponds to a sign change in the sequence $\{\chi(n)\}$.

Taking the partial derivatives of the function $\mathcal{F}(x,z)$ from (145) in x, z we find

$$\begin{aligned} \partial_x \mathcal{F}(x,z) &= \tfrac{z-1}{(x+z)^2(1-x)^2}(z-1+2x) > 0, \\ \partial_z \mathcal{F}(x,z) &= \tfrac{1}{(1-x)(x+z)^2}(x+1) > 0, \end{aligned}$$

if x and z lie in the intervals, specified by (136), (137).

This enables us to limit the range of $\mathcal{F}(x,z)$ simply by taking the lower bounds for x, z for the lower bound and the upper bounds for x, z for the upper bound on $\mathcal{F}(x,z)$, and consequently on $\mathcal{Q}^1(x,z)$. We get:

$$\frac{1}{\bar{a}+2} < \mathcal{F}(x,z) < \bar{a}, \tag{147}$$

We recall that the bounds (136), (137) for x and z have been determined with the relative tolerance $O(\bar{a}^{-1})$. Therefore, it's easy to see that the above interval for the values of $\mathcal{F}$ is $(1 + O(\bar{a}^{-1}))$ times wider than its true range for the admissible values of x and z. Then with the same tolerance we can write:

$$\mathcal{K}(\varepsilon)\sqrt{\frac{1}{\bar{a}+2}} < \mathcal{Q}^1(x,z) < \mathcal{K}(\varepsilon)\sqrt{\bar{a}}. \tag{148}$$

We need to keep a close eye on the tolerance to make sure that the true zeroes of the quantity $\mathcal{D}^1_n$ (considered as a function of a continuous variable q and parameters x, z, whose range is specified by (136), (137)), also fall into the interval (148). In general, this will be our common strategy of neutralizing the remainder in (143), (141), incorporated in several more cases below. Our inability to compute the exact values for the parameters x_n, z_n, which necessitates resorting to the estimates instead, becomes an advantage, for we will always insure that the tolerance of "measuring" x_n and z_n, and the functions of these parameters that we deal with, is much larger than the possible damage due to the error terms.

We shall estimate the possible error in the location of the true zero of the quantity $\mathcal{D}^1_n$, with q_n considered as a continuous variable, resulting from the impact of the error term $O(q_{n-1}^{-1})$ in (141) by observing that

$$\frac{d}{dq}\mathcal{D}^1(\mathcal{Q}^1(x_n, z_n), x_n, z_n) = 2(1+\beta)\sigma_0(1-x_n),$$

whereas the error term is

$$O(q_{n-1}^{-1}) = O\left(\frac{1}{q_n x_n}\right),$$

with q_n satisfying (148). This implies that the true zero of the quantity $\mathcal{D}^1_n$ from (141) will be separated from $\mathcal{Q}^1(x_n, z_n)$ by at most a distance, which is $O(\bar{a}^{\frac{3}{2}}\varepsilon^{\frac{1}{4}})$ (see (148) and (136)). We require that this distance not exceed the tolerance in the width of (148), which works to our advantage. Indeed, due to the $O(\bar{a}^{-1})$ relative tolerance, the possible zeroes of $\mathcal{D}^1$ will sit at least $O(\varepsilon^{-\frac{1}{4}}\bar{a}^{-\frac{3}{2}})$ away from the boundary inside the interval (148).

Thus, if we require that

$$\sqrt{\varepsilon}\bar{a}^3 \ll 1, \tag{149}$$

this will be sufficient to insure that all the true zeroes of the function $\mathcal{D}^1_n$ will also respect the bounds (148).

We see that given a small ε, there is some integer $s(\varepsilon)$ and a finite number $r+1$ of convergents $q_s, q_{s+1}, .., q_{s+r-1}, q_{s+r}$ falling into the interval, given by (148); then $s, .., s+r+1$ become the only possible candidates for the transition indices, and for all the convergents q_{s-i}, preceding q_s, we will have $\mathcal{D}^1_{s-i} < 0$, $i = 1, 2, .., s-1$, and so their χ-index will equal -1, whereas for all the convergents, following q_{s+r}, we will have $\mathcal{D}^1_{s+r+i} > 0$, $i = 1, 2, ..$, and the χ-index equal $+1$. Therefore, there always exists at least one transition point with a positive index.

Suppose now, that ε is such that the equality $\mathcal{Q}^1(x_n, z_n) = q_n$ never occurs. Otherwise we say that ε attains one of its *critical values.*

In fact, we aim to insure that $\mathcal{D}^1_n(q_n, x_n, z_n) \neq 0$ (otherwise we say that ε attains its *true critical value*); this can happen only if q_n lies in the interval, specified by (148).

Suppose, for some $\delta_1 > 0$, the value of ε is such that

$$|\mathcal{Q}^1(x_n, z_n) - q_n| \geq 2q_n\varepsilon^{\delta_1}. \tag{150}$$

Otherwise we say that ε falls into a *critical interval.* We cannot take δ_1 arbitrarily large, since the true critical value can be separated from a (computable) critical value by $O(\bar{a}^{\frac{3}{2}}\varepsilon^{\frac{1}{4}})$, so we must require

$$\bar{a}^{\frac{3}{2}}\varepsilon^{\frac{1}{4}} \ll q_n\varepsilon^{\delta_1},$$

what can be expressed in terms of $\varepsilon, \bar{a}$ only regarding (148):

$$\bar{a}^2\varepsilon^{\frac{1}{2}-\delta_1} \ll 1. \tag{151}$$

Roughly speaking (when $\bar{a}$ is $O(1)$), from (148) we can see that the aforesaid equality can occur only when $q_n \sim \varepsilon^{-\frac{1}{4}}$, which will also be a measure of the distance between q_n and q_{n-1}, so if we choose $0 < \delta_1 < \frac{1}{4}$, then the critical intervals will have a small measure. We'll give it a strict formulation in the following Proposition. Moreover, if we avoid the critical intervals for the values of ε, then the difference between the exponents E_n and E_{n-1} is going to be of the order $O(\varepsilon^{-\frac{1}{4}+\delta_1})$.

Proposition 6.6 *The sequence $\{\varepsilon_n\}$ of the (true) critical values for ε is nowhere dense as $n \to \infty$ and limits at zero, the relative measure of the critical intervals being bounded as $O(\bar{a}^2\varepsilon^{\delta_1})$, provided that δ_1 satisfies (151).*

Proof: Let's fix some small number ε_*. If $\frac{\varepsilon_*}{16} \leq \varepsilon \leq \varepsilon_*$, then by (148), the range of $\mathcal{Q}^1(x, z)$ will be bounded between

$$\frac{K^*}{\sqrt{\bar{a}+2}}\varepsilon_*^{-\frac{1}{4}} < \mathcal{Q}^1(x, z) < 2K^*\sqrt{\bar{a}}\varepsilon_*^{-\frac{1}{4}}.$$

Suppose, the convergents $q_s, .., q_{s+N_*-1}$ fall into this interval. ¿From (136), the maximum number N_* of such convergents, which are the only ones for which the equality in question can occur, can be found from the following condition:

$$\left(\frac{\bar{a}+2}{\bar{a}+1}\right)^{N_*} \leq 2\sqrt{\bar{a}(\bar{a}+2)} \leq 2(\bar{a}+1),$$

for min $\frac{q_n}{q_{n-1}} = \frac{\bar{a}+2}{\bar{a}+1}$. so $N_* \leq 2(\bar{a}+1)^2$. In fact, Lochak [1993] can provide a much better estimate for this number.

Suppose, ε_{s+n} for $n = 0, .., N_*-1$ are the critical values for ε when

$$\mathcal{Q}^1(x_{s+n}, z_{s+n}) = \varepsilon_{s+n}^{-\frac{1}{4}}K^*\sqrt{\mathcal{F}(x_{s+n}, z_{s+n})} = q_{s+n},$$

then clearly $\varepsilon_{s+n} \geq \frac{\varepsilon_*}{16}$, $\forall n$. By (150), around each ε_{s+n} we have to cut out an interval $\Delta\varepsilon_{s+n}$ to satisfy

$$q_{s+n}(1 + 2\varepsilon^{\delta_1}) \leq K^*(\varepsilon_{s+n} - \Delta\varepsilon_{s+n})^{-\frac{1}{4}}\sqrt{\mathcal{F}(x_{s+n}, z_{s+n})}$$

and

$$q_{s+n}(1 - 2\varepsilon^{\delta_1}) \leq K^*(\varepsilon_{s+n} + \Delta\varepsilon_{s+n})^{-\frac{1}{4}}\sqrt{\mathcal{F}(x_{s+n}, z_{s+n})},$$

what can be done for ε_* small enough if

$$\frac{\Delta\varepsilon_{s+n}}{\varepsilon_{s+n}} = 16\varepsilon^{\delta_1}, \tag{152}$$

provided that (151) is satisfied, meaning that q_{s+n} and $\mathcal{Q}^1(x_n, z_n)$ are very close to each other.

Since the length of the interval of the values of ε in consideration is $\frac{15}{16}\varepsilon_*$ and for all $n = 0, .., N^* - 1$ we have $\varepsilon_{s+n} \geq \frac{\varepsilon_*}{16}$, then the relative measure of N^* critical intervals will be $O(\bar{a}^2\varepsilon^{\delta_1})$.

The statement that the sequence of the (true) critical values accumulates at zero as $n \to \infty$ is quite obvious, just as the one that it is nowhere dense. In particular, the latter follows from the fact that having ε vary in a small fixed interval, there will be a finite only number of convergents, for which, the critical values can lie inside this interval. $\square$

Given a critical value of ε, we call its critical multiplicity the number of equalities $\mathcal{Q}^1(x_n, z_n) = q_n$ that take place. It's very unlikely that the multiplicity of a critical value be greater than one (a maximum value for it being the number of convergents sitting inside the interval, defined by (148)); if ε lies outside the union of all the critical intervals, its critical multiplicity is certainly zero.

Before we proceed we want to point out one more time that if the condition (149) holds, then the sequence $\{E_n\}$ decreases monotonically if $n < s$ and grows monotonically if $n > s + r$.

Now we shall fix ε outside the union of all the critical intervals and rescale the q_n's and the function $\mathcal{Q}^1(x, z)$ by the factor $\mathcal{K}(\varepsilon)$.

We will define the range of the rescaled function $\mathcal{Q}^1(x, z)$ where x, z satisfy (136), (137) as an interval $\mathcal{I}_{\bar{a}}$:

$$\mathcal{I}_{\bar{a}} \equiv \left(\frac{1}{\sqrt{\bar{a}+2}}, \sqrt{\bar{a}}\right) \tag{153}$$

Before we transit to our main result concerning the number of transition points, we'll have to express the quantity $\mathcal{D}_n^2$ in (132) in terms of some parameters (a convenient set will be q_{n-1}, x_n, z_n) as we've done it with $\mathcal{D}_n^1$.

Using the recursion relations (135) we derive:

$$\begin{aligned} x_n &= \tfrac{1}{a_n + x_{n-1}}, \\ z_{n-1} &= a_n + \tfrac{1}{z_n}, \\ q_n &= q_{n-1}(a_n + x_{n-1}). \end{aligned}$$

Substituting these relations into (151) we get:

$$\begin{aligned} \mathcal{D}_n^1 &= (1+\beta)\sigma_0(a_n - 1 + x_{n-1})q_{n-1} - \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q_{n-1}}\frac{z_n - 1}{z_n x_n^{-1} + 1} + O(q_{n-1}^{-1}), \\ \mathcal{D}_{n-1}^1 &= (1+\beta)\sigma_0(1 - x_{n-1})q_{n-1} - \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q_{n-1}}\frac{z_n(a_n - 1) + 1}{z_n x_n^{-1} + 1} + O(q_{n-2}^{-1}), \\ \mathcal{D}_n^2 &= (1+\beta)\sigma_0 a_n q_{n-1} - \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q_{n-1}}\frac{a_n z_n}{z_n x_n^{-1} + 1} + O(q_{n-2}^{-1}). \end{aligned} \tag{154}$$

In the same manner as we've done with $\mathcal{D}_n^1$, we throw away the supposedly small term $O(q_{n-2}^{-2})$ (which is not difficult to write up explicitly), and consider a function

$$\mathcal{D}^2(q, x, z) = (1+\beta)\sigma_0 q - \frac{\pi\omega_+}{2\sqrt{\varepsilon}}\frac{1}{q}\frac{z}{1 + z/x}$$

of a continuous variable q and parameters x, z which vary within the range specified by (136), (137); this function is zero if:

$$q = \mathcal{Q}^2(x, z) = \mathcal{K}(\varepsilon)\sqrt{\mathcal{G}(x, z)}, \tag{155}$$

where $\mathcal{K}(\varepsilon)$ is the same as in (146) and

$$\mathcal{G}(x,z) = \frac{z}{1+z/x}. \tag{156}$$

It's easy to see that within the range of x, z the partial derivatives of $\mathcal{G}$ never change their sign, namely $\partial_x \mathcal{G}(x,z) > 0$ and $\partial_z \mathcal{G}(x,z) > 0$.

The range of $\mathcal{G}$, fixed a_n turns out to be:

$$\frac{1}{a_n+2} < \mathcal{G}(x,z) < \frac{1}{a_n}, \tag{157}$$

with the relative tolerance in these inequalities being $O(\bar{a}^{-1})$.

In particular, for we will need it most, for $a_n = 1$ we can upgrade (157) to

$$\frac{\bar{a}+2}{3\bar{a}+4} < \mathcal{G}(x,z) < \frac{\bar{a}+1}{\bar{a}+3}, \tag{158}$$

Then from (132) if $q_{n-1} > \mathcal{Q}^2(x_n, z_n)$, then $E_n > E_{n-2}$, and if $q_{n-1} < \mathcal{Q}^2(x_n, z_n)$, then $E_n < E_{n-2}$, provided that the former inequalities are fulfilled with enough tolerance to neutralize the error term in (154). In order to do this, we assume that likewise with the first-order differences, ε is such that the equality $q_{n-1} = \mathcal{Q}^2(x_{n-1}, z_n)$ never occurs and proceed as we've done before for the one-step difference $\mathcal{D}^1_n$.

Quantitatively, we do the same thing adding to the set of the (true) critical values values of ε those, when $\mathcal{Q}^2(x_n, z_n) = q_{n-1}$, cutting out the critical intervals around these values of ε as we have done in (150). If the condition (149) is fulfilled, then all the true zeroes of the quantity $\mathcal{D}^2_n$ (considered as the function of the continuous variable q and the parameters x, z, whose range is given by (136), (137)) also lie within the limits, given by (155) and (157). Obviously, the estimate for the relative measure of the critical intervals of Proposition (6.6) still holds, (one can repeat the argument preceding this Proposition with (154), (155), (157) to verify this).

Through the end of this section (unless specified) we shall be dealing with the sequence $\{q_n\}$, rescaled by the factor $\mathcal{K}(\varepsilon)$.

Now we will formulate and prove two simple technical Propositions which will be very helpful in our search for transition points and quest how the sequence $\{E_n\}$ may behave for n such that $q_n \in \mathcal{I}_{\bar{a}}$.

Proposition 6.7 *Given the integers n_1, n_2, such that $n_2 > n_1$, if*

$$n_2 = n_1 + 1,\ a_{n_2} \geq 2,$$

or

$$n_2 \neq n_1 + 1,\ a_{n_2} \geq 1,$$

then the necessary condition for $\chi(n_2) = -1$ is

$$q_{n_1} q_{n_2} < \frac{\bar{a}}{\bar{a}+1}.$$

Remark: The only possibility which is not covered by this Proposition, as well as the next one, is when simultaneously $n_2 = n_1 + 1$ AND $a_{n_2} = 1$.

Proof: We have $\mathcal{F}(x_{n_2}, z_{n_2}) = \frac{z_{n_2}-1}{(z_{n_2}+x_{n_2})(1-x_{n_2})}$. The maximum of z_{n_2}, is always smaller than $\bar{a}+1$ by (137).

First, we'll show that under the constraints of this Proposition the maximum of x_{n_2} will be always smaller than $1 - \frac{q_{n_1}}{q_{n_2}}$ (which would be invalid if simultaneously $n_2 = n_1 + 1$ and $a_{n_2} = 1$).

Indeed, if $a_{n_2} \geq 2$ then always $\frac{q_{n_1}}{q_{n_2}} < \frac{1}{2}$, so $x_{n_2} = \frac{q_{n_1}}{q_{n_2}} < \frac{1}{2} < 1 - \frac{q_{n_1}}{q_{n_2}}$.

Otherwise, if $n_2 > n_1 + 1$, then there is an integer n': $n_1 < n' < n_2$, such that $q_{n_2} \geq q_{n'} + q_{n_1}$ (the equality occurs if $n_2 = n_1 + 2$ and $a_{n_2} = 1$, such that:

$$x_{n_2} \leq \frac{1}{1+\frac{q_{n_1}}{q_{n'}}} = 1 - \frac{q_{n'}}{q_{n_1}+q_{n'}} \leq 1 - \frac{q_{n_1}}{q_{n_2}}.$$

In both cases, taking the above upper bounds for x_{n_2} and z_{n_2} we obtain the estimate:

$$\mathcal{F}(x_{n_2}, z_{n_2}) < \frac{\bar{a}}{\bar{a}+2-\frac{q_{n_1}}{q_{n_2}}} \frac{q_{n_2}}{q_{n_1}} < \frac{\bar{a}}{\bar{a}+1} \frac{q_{n_2}}{q_{n_1}},$$

consequently

$$\mathcal{Q}^1(x_{n_2}, z_{n_2}) < \sqrt{\frac{\bar{a}}{\bar{a}+1} \cdot \frac{q_{n_2}}{q_{n_1}}}.$$

If $\chi(n_2) = -1$, then $q_{n_2} < \mathcal{Q}^1(x_{n_2}, z_{n_2})$, so the necessity of the condition $q_{n_1} q_{n_2} < \frac{\bar{a}}{\bar{a}+1}$ follows. □

Proposition 6.8 *Given integers n_1, n_2, such that $n_2 > n_1$, if*

$$n_2 = n_1 + 1, \ a_{n_2} \geq 2,$$

or

$$n_2 \neq n_1 + 1, \ a_{n_2} \geq 1,$$

then the necessary condition for $\chi(n_1) = 1$ is

$$q_{n_1} q_{n_2} > \frac{\bar{a}+1}{\bar{a}}.$$

Proof: Exactly in the same fashion as the previous proposition. We have $\mathcal{F}(x_{n_1}, z_{n_1}) = \frac{z_{n_1}-1}{(z_{n_1}+x_{n_1})(1-x_{n_1})}$. Substituting the minimum of x_{n_1}, which is greater than $\frac{1}{\bar{a}+1}$, and the minimum of z_{n_1}, which is greater than $1 + \frac{q_{n_1}}{q_{n_2}}$, (which would be invalid if simultaneously $n_2 = n_1 + 1$ and $a_{n_2} = 1$) we get:

$$\mathcal{F}(x_{n_1}, z_{n_1}) > \frac{\bar{a}+1}{\bar{a}} \frac{q_{n_1}}{q_{n_2}} \frac{1}{1 + \frac{1}{\bar{a}+1} + \frac{q_{n_1}}{q_{n_2}}} > \frac{\bar{a}+1}{\bar{a}} \frac{q_{n_1}}{q_{n_2}},$$

consequently

$$\mathcal{Q}^1(x_{n_1}, z_{n_1}) > \sqrt{\frac{\bar{a}}{\bar{a}+1} \cdot \frac{q_{n_2}}{q_{n_1}}}.$$

If $\chi(n_1) = 1$, then $q_{n_1} > \mathcal{Q}^1(x_{n_1}, z_{n_1})$, so the necessity of the condition $q_{n_1} q_{n_2} > \frac{\bar{a}+1}{\bar{a}}$ follows. □

These two Propositions make the proof of our main result, which we call the Transition lemma, almost trivial, although the fact, stated in this Lemma, essentially tells us that if ε is away from the union of the critical intervals, we can always find a unique absolute minimum of the sequence $\{E_n\}$.

Lemma 6.1 (The Transition lemma) *Assume, ε lies outside the union of all the critical intervals, and the convergents $q_s, .., q_{s+r}$ lie inside the interval $\mathcal{I}_{\bar{a}}$.*

If for some $n_* \in \{s, .., s+r-1\}$ one has

$$a_{n_*+2} = 1,$$

$$\chi(n) = -1, \ s \leq n \leq n_*,$$

$$\frac{1}{1 - x_{n_*+1}} < q_{n_*+1}^2 (1 + z_{n_*+2}(1 + x_{n_*+1})) < \frac{z_{n_*+2} - 1}{x_{n_*+1}}, \tag{159}$$

then q_{n_*+1}, q_{n_*+3} are the only two transition points with the positive labels, and q_{n_*+2} is the only transition point with the negative label. In particular, the necessary condition for this to occur is

$$z_{n_*+2} > \frac{1}{1 - x_{n_*+1}}. \tag{160}$$

Otherwise, there is one and only one transition point q_{n_*+1} with $n_* \in \{s-1, .., s+r\}$, and $\chi(n_* + 1) = +1$.

Proof: Once again, we notice that all the points to the left of the interval $\mathcal{I}_{\bar{a}}$ carry negative labels and all the points to the right of it carry positive labels. That's why there is no point with the positive label inside $\mathcal{I}_{\bar{a}}$, or the only such a point is the last one, then the lemma holds trivially yielding V-shape.

Otherwise, suppose, q_{n_1} is the first point inside the interval $\mathcal{I}_{\bar{a}}$, such that $\chi(n_1) = +1$, then by definition it will be a transition point carrying the positive label.

Also suppose that for $n_2 > n_1$ the point q_{n_2} also lies inside this interval and $\chi(n_2) = -1$, for otherwise q_{n_1} would be the only transition point, and again we are in the Case 1 (V-shape).

Then applying Propositions 6.7 and 6.8, unless simultaneously $n_2 = n_1 + 1$ AND $a_{n_2} = 1$, we must have

$$\frac{\bar{a}}{\bar{a}+1} > q_{n_1} q_{n_2} > \frac{\bar{a}+1}{\bar{a}},$$

which is a contradiction.

So the *only* possibility to have two such points inside $\mathcal{I}_{\bar{a}}$ will be the simultaneous fulfillment of two conditions:

$$\begin{array}{rcl} n_2 & = & 1 + n_1, \\ a_{n_2} & = & 1, \end{array} \tag{161}$$

namely q_{n_1} and q_{n_2} must be neighbors, and the spacing between them must be pretty small, because $a_{n_2} = 1$ implies $q_{n_2} = q_{n_1} + q_{n_1 - 1}$. By definition, q_{n_1} will be a transition point with a positive label, and q_{n_2} will be a transition point with a negative label. Moreover, there will be no points with negative labels to the right of q_{n_2}, because if some q_{n_3} such that $n_3 > n_2$ had $\chi(n_3) = -1$, then a pair of points n_3 and n_1 would satisfy the conditions of Propositions 6.7, 6.8, whose simultaneous application would lead to an absurd statement

$$\frac{\bar{a}}{\bar{a}+1} > q_{n_1} q_{n_3} > \frac{\bar{a}+1}{\bar{a}}.$$

Hence, the only possibility to have more than one transition point is when for some $n_* \in \{s-1, .., s+r-2\}$ the point q_{n_*+1} is the leftmost one with a positive index, and the point q_{n_*+2} has a negative index; then all the points to the right of q_{n_*+2} will have positive indices, as it has just been shown.

To have simultaneously $\chi(n_* + 1) = 1$ and $\chi(n_* + 2) = -1$ we must have

$$\mathcal{F}(x_{n_*+1}, z_{n_*+1}) < q^2_{n_*+1} < q^2_{n_*+2} = \frac{q^2_{n_*+1}}{x^2_{n_*+2}} < \mathcal{F}(x_{n_*+2}, z_{n_*+2}).$$

By the recursion relations (135) $x_{n_*+2} = \frac{1}{1+x_{n_*+1}}$ and $z_{n_*+1} = 1 + \frac{1}{z_{n_*+2}}$. We substitute this in (145) to express $\mathcal{F}(x_{n_*+1}, z_{n_*+1})$, and $\mathcal{F}(x_{n_*+2}, z_{n_*+2})$ in in terms of x_{n_*+1}, z_{n_*+2}, we easily see that the last expression of (159) is just the previous inequality rewritten in details, and thus the sufficient condition for us to encounter the W-shape.

In particular, the last expression in (159) implies that

$$z_{n_*+2} > \frac{1}{1 - x_{n_*+1}}$$

is the necessary condition for the W-shape to take place. This enables us to improve the lower bound for the rescaled value $\mathcal{Q}^2(x_{n_*+2}, z_{n_*+2})$ instead of (158) for $a_{n_*+2} = 1$ up to

$$\frac{1}{\sqrt{2}} < \mathcal{Q}^2(x_{n_*+2}, z_{n_*+2}) < \sqrt{\frac{\bar{a}+2}{\bar{a}+4}}. \tag{162}$$

The proof of the Transition Lemma will be complete now if we recall that if the conditions (149), (150), and (151) are satisfied, then the neglected error terms in (141) and (154) cannot be of any harm. □

Suppose, we are in the case of W-shape, so the sequence $\{E_n\}$ has two local minima for $n = n_*$ and $n = n_* + 2$. One of the necessary conditions is $a_{n_*+2} = 1$, and we can use the function $\mathcal{D}^2$ to find the absolute minimum. Namely, we will have to compare q_{n_*+1} and $\mathcal{Q}^2(x_{n_*+2}, z_{n_*+2})$, the latter necessarily falling into the limits of (162). Being always away from the critical intervals, we don't have to worry about the errors. The following Corollary states that if one of the indices n_* or $n_* + 2$ corresponds to the absolute minimum, then the other yields the second smallest member of the sequence $\{E_n\}$.

Corollary 6.1 *Suppose, n_*+2 is the transition index, labeled -1 and ε is small enough and away from the union of all the critical intervals. Then $E_{n_*+1} < E_{n_*-1}$ and $E_{n_*+1} < E_{n_*+3}$.*

Proof: First, we notice that if there is a transition index, labeled -1, then we encounter W-shape by the Transition lemma. This means that there are two local minima of the sequence $\{E_n\}$, occurring for n equal n_* and n_*+2, and all the necessary conditions (159) must hold. In particular, this implies that $E_{n_*+1} > E_{n_*}$ and $E_{n_*+1} > E_{n_*+2}$, therefore if we prove the fact stated in this Corollary, this would mean that once we encounter W-shape, E_{n_*} and E_{n_*+2} are always the two smallest terms in the sequence $\{E_n\}$.

Suppose, $E_{n_*-1} < E_{n_*+1}$. To rule this out, we compute the quantity $\mathcal{G}(x_{n_*+1}, z_{n_*+1})$ from (156) and compare it with q_{n_*}. Under the assumption that $E_{n_*-1} < E_{n_*+1}$ we shall have $q_{n_*}^2 > \mathcal{G}(x_{n_*+1}, z_{n_*+1})$. We'll show that this contradicts (159).

We compute $\mathcal{G}(x_{n_*+1}, z_{n_*+1})$ in terms of x_{n_*+1} and z_{n_*+2}, using the recursion relations (135) and the necessary condition $a_{n_*+2} = 1$ to substitute $z_{n_*+1} = 1 + \frac{1}{z_{n_*+2}}$. This yields

$$q_{n_*}^2 > \frac{x_{n_*+1}(z_{n_*+2}+1)}{1 + z_{n_*+2}(1 + x_{n_*+1})}.$$

or, since $q_{n_*} = x_{n_*+1} q_{n_*+1}$,

$$q_{n_*+1}^2 > \frac{1}{x_{n_*+1}} \frac{z_{n_*+2}+1}{1 + z_{n_*+2}(1 + x_{n_*+1})},$$

whereas (159) implies that

$$q_{n_*+1}^2 < \frac{1}{x_{n_*+1}} \frac{z_{n_*+2}-1}{1 + z_{n_*+2}(1 + x_{n_*+1})},$$

which is in the obvious contradiction with the previous statement.

Now, suppose $E_{n_*+3} < E_{n_*+1}$. Then we shall have $q_{n_*+2}^2 < \mathcal{G}(x_{n_*+3}, z_{n_*+3})$. This yields

$$\frac{q_{n_*+2}^2}{x_{n_*+3}} < \frac{z_{n_*+3}}{z_{n_*+3} + x_{n_*+3}} < 1.$$

On the other hand, since $\chi(n_*+1) = +1$, we can apply Proposition 6.7 to argue that $q_{n_*+1} q_{n_*+3} > 1$, which implies that

$$\frac{q_{n_*+2}^2}{x_{n_*+3}} > \frac{1}{x_{n_*+2}} > 1,$$

which contradicts the previous statement. Thus, the Corollary is proved. □

We can prove another Corollary, which will illustrate that the situation with W-shape does happen indeed.

Corollary 6.2 *The necessary condition for*

$$\begin{array}{rcl} a_{n_*+2} & = & 1, \\ \chi(n_*+1) & = & -1, \\ \chi(n_*+2) & = & +1 \end{array}$$

is

$$z_{n_*+2} < \frac{1}{1 - x_{n_*+1}}.$$

Remark: Compare with the necessary condition (160) for W-shape to take place. If the latter is satisfied, for no values of ε is V-shape possible with the absolute minimum at n_*+1. But when ε is decreased smoothly, the absolute minimum shall move to the right towards larger values of n. Thus most likely one will encounter W-shape for a certain interval of the values of ε.

Proof: Using the recursion relations (135) and what is given, we can write

$$q^2_{n_*+1} \quad < \quad \mathcal{F}(x_{n_*+1}, z_{n_*+1}) = \frac{1}{(1+z_{n_*+2}(1+x_{n_*+1}))(1-x_{n_*+1})},$$

$$q^2_{n_*+2} = (1+x_{n_*+1}))^2 q^2_{n_*+1} \quad > \quad \mathcal{F}(x_{n_*+2}, z_{n_*+2}) = \frac{(z_{n_*+2}-1)(1+x_{n_*+1}))^2}{(1+z_{n_*+2}(1+x_{n_*+1}))} \frac{1}{x_{n_*+1}}.$$

These two statements will lead to the contradiction unless the necessary condition, claimed in this Corollary, holds true. □

By the Transition lemma, the only two possible shapes for the sequence $\{E_n\}$ are such that it has either one or two local minima, and we can detect the absolute minimum if ε is outside the union of all the critical intervals. Suppose, given ε, the absolute minimum occurs for $n = n_*$. If the convergents $q_s, .., q_{s+r}$ sit inside the interval $\mathcal{I}_{\bar{a}}$, then $n_* \in \{s-1, s, .., s+r\}$. Obviously, since this is a minimum $\chi(n_*) = -1$, $\chi(n_*+1) = 1$; for all $s > n_*$, except for possibly $s = n_* + 2$, in the case when n_* corresponds to the left prong of W-shape, when $\chi(n_*+2) = -1$.

Now we can do better than (148) to get the estimates on the location of the convergent q_{n_*}, corresponding to the absolute minimum of the sequence $\{E_n\}$. A rough shot would be

$$\frac{1}{(\bar{a}+1)\sqrt{\bar{a}+2}} < q_{n_*} < \sqrt{\bar{a}}.$$

but the following two simple complementary to each other Propositions show that this estimate can be significantly improved.

Proposition 6.9 *Suppose, (149) is satisfied. If q_{n_*} corresponds to the absolute minimum of the sequence $\{E_n\}$, then $q_{n_*} < 2$.*

Proof: We notice that if q_{n_*} corresponds to the absolute minimum of the sequence $\{E_n\}$, then $\chi(n_*) = -1$.

First, we assume that $a_{n_*} \geq 2$. In this case we have:

$$q^2_{n_*} < \mathcal{F}(x_{n_*}, z_{n_*}) < 2,$$

where we have used that if $a_{n_*} \geq 2$, then $x_{n_*} < \frac{1}{2}$, and (145).

Otherwise, suppose $a_{n_*} = 1$. Then either for all $n < n_*$ we have $\chi(n) = -1$ (if we have V-shape or W-shape when n_* corresponds to the left local minimum), or n_* corresponds to the right local minimum of W-shape.

In the latter case we would have $q^2_{n_*-1} < \mathcal{G}(x_n, z_n) < \frac{\bar{a}+2}{\bar{a}+4} < 1$ from (157), which means that $q_{n_*} = \frac{q_{n_*-1}}{x_{n_*}} < 2q_{n_*-1} < 2$, for $x_{n_*} > \frac{1}{2}$.

So, suppose, all the points to the left of q_{n_*} carry negative indices (also including q_{n_*} itself) and $a_{n_*} = 1$. Then we must have simultaneously $q^2_{n_*} < \mathcal{F}(x_{n_*}, z_{n_*})$ and $q^2_{n_*-1} = (q_{n_*} x_{n_*})^2 < \mathcal{F}(x_{n_*-1}, z_{n_*-1})$. Under the assumption that $a_{n_*} = 1$, we can write $z_{n_*-1} = 1 + \frac{1}{z_{n_*}}$ and $x_{n_*-1} = \frac{1}{x_{n_*}} - 1$ and then use (145) to obtain the following two inequalities that must be satisfied simultaneously:

$$q^2_{n_*} < \frac{z_{n_*}-1}{z_{n_*}+x_{n_*}} \frac{1}{1-x_{n_*}} < \frac{1}{1-x_{n_*}},$$

and

$$q^2_{n_*} < \frac{1}{z_{n_*}+x_{n_*}} \frac{1}{2x_{n_*}-1} < \frac{1}{2x_{n_*}-1}.$$

Together these inequalities result in $q^2_{n_*} < 3$.

At the end we notice than if (149) is satisfied, then the tolerance for all the inequalities exceeds any possible error that can come from the influence of the neglected terms in (141), (154). □

The second Proposition is essentially the one that we have just proved turned upside down.

Proposition 6.10 *Suppose, (149) is satisfied. If q_{n_*} corresponds to the absolute minimum of the sequence $\{E_n\}$, then $q_{n_*+1} > \frac{1}{2}$.*

Proof: First, obviously $\chi(q_{n_*+1}) = +1$.

If $a_{n_*+2} \geq 2$, then $z_{n_*+1} > 2$, and from (145) we see that since $\chi(n_* + 1) = 1$, we must have $q^2_{n_*+1} > \mathcal{F}(x_{n_*+1}, z_{n_*+1}) > \frac{1}{3}$.

Otherwise, suppose $a_{n_*+2} = 1$. Suppose, we deal with W-shape and q_{n_*} corresponds to the left local minimum. Then we must have $q^2_{n_*+1} > \mathcal{G}(x_{n_*+2}, z_{n_*+2}) > \frac{1}{2}$ by (162).

Otherwise, all the points to the right of q_{n_*+1} must have positive indices. Then we must have simultaneously $q^2_{n_*+1} > \mathcal{F}(x_{n_*+1}, z_{n_*+1})$ and $q^2_{n_*+2} = \left(\frac{q_{n_*+1}}{x_{n_*+2}}\right)^2 > \mathcal{F}(x_{n_*+2}, z_{n_*+2})$. Under the assumption that $a_{n_*+2} = 1$, we can write $z_{n_*+1} = 1 + \frac{1}{z_{n_*+2}}$ and $x_{n_*+2} = \frac{1}{1+x_{n_*+1}}$ and then use (145) to obtain the following two inequalities that must be satisfied simultaneously:

$$q^2_{n_*+1} > \frac{1}{z_{n_*+2}},$$

and

$$q^2_{n_*+1} > \frac{z_{n_*+2} - 1}{z_{n_*+2} + 1},$$

which entails $q^2_{n_*+1} > \frac{1}{3} > \frac{1}{4}$.

At the end we notice than if (149) is satisfied, then the tolerance for all the inequalities exceeds any possible error that can come from the influence of the neglected terms in (141), (154). □

The above two Propositions imply that

$$\mathcal{K}(\varepsilon)\frac{1}{2(\bar{a}+1)} < q_{n_*} < 2\mathcal{K}(\varepsilon). \tag{163}$$

We suggest a couple of remarks illustrating some facts that can be derived from the preceding results.

Remark 1: When none of $a_{s+1}, .., a_{s+r}$ equals 1, then we always have only one transition point, so the sequence $\{E_n\}$ has V-shape, first decreasing monotonically until it reaches the absolute minimum for $n = n_*$ and afterwards increasing. Moreover, if β is such a number that in its Continued fraction one never occurs, then instead of a bound (153) for the range of the rescaled function $\mathcal{Q}^1(x, z)$ we will have much better bounds, almost as tight as $\frac{1}{\sqrt{2}} < \mathcal{Q}^1(x, z) < \sqrt{2}$, because in this case always $x < \frac{1}{2}$, $z > 2$. Which implies that the interval $\mathcal{I}_{\bar{a}}$ cannot contain more than one member of the sequence $\{q_n\}$, as for all n one has $q_{n+1} > 2q_n$ if $a_{n+1} \geq 2$. So in this case the sequence $\{E_n\}$ behaves very nicely.

Remark 2: If $\beta = \frac{\sqrt{5}-1}{2} = [1, 1, 1, ...]$, namely the so-called "golden number", then for all n the values of the parameters z_n are the same, and of x_n almost the same (different at $O(q^{-2}_{n-1})$), so the values of the function $\mathcal{Q}^1(x_n, z_n)$ for all n fall into a very narrow interval compared to the spacing between the consecutive convergents q_{n-1} and q_n. So for any (small) ε we will be able to find some $n_*(\varepsilon)$ such that at least $q_{n_*-1} < \mathcal{Q}^1(x_n, z_n) < q_{n_*+1}$ for all n, which implies the existence of only one transition point (either q_{n_*} or q_{n_*+1}). Besides, it's easy to check that the necessary condition (160) will not be satisfied for this number if ε is small enough, so V-shape is the only possibility.

We can estimate from below the absolute values of the differences $|\mathcal{D}^1_{n_*}| = |E_{n_*-1} - E_{n_*}|$ and $|\mathcal{D}^1_{n_*+1}| = |E_{n_*+1} - E_{n_*}|$ using (150).

By definition of $\mathcal{Q}^1$, for any n, the substitution of a triple $(\mathcal{Q}^1(x_n, z_n), x_n, z_n)$ into (141), where the error term has been thrown away, gives us zero. Then (141) with the triple (q_n, x_n, z_n) substituted in it can be estimated from below using (150).

For any $n = s-1, ..., s+r$ such that $\chi(n) = -1$, we have $q_n < \mathcal{Q}^1(x_n, z_n)$, and moreover, we assume that (150) is satisfied, so

$$q_n \leq \frac{\mathcal{Q}^1(x_n, z_n)}{1 + 2\varepsilon^{\delta_1}}.$$

For simplicity we will write $\mathcal{Q}^1_n = \mathcal{Q}^1(x_n, z_n)$. Substituting the above inequality as an equality in (141) we easily get:

$$
\begin{aligned}
|\mathcal{D}_n^1| &\geq (1+\beta)\sigma_0(1-x_n)\tfrac{\mathcal{Q}_n^1}{1+2\varepsilon^{\delta_1}} \\
&- \tfrac{\omega+\pi}{2\sqrt{\varepsilon}}\tfrac{1+2\varepsilon^{\delta_1}}{\mathcal{Q}_n^1}\tfrac{z_n-1}{z_n+x_n} + O(q_{n-1}^{-1}) \\
&\geq \tfrac{\sigma_0}{2}\tfrac{\varepsilon^{-\frac{1}{4}+\delta_1}}{(\bar{a}+2)^{\frac{3}{2}}},
\end{aligned}
\tag{164}
$$

provided that

$$
\varepsilon^{\frac{1}{2}-\delta_1}(\bar{a}+2)^2 \ll 1,
$$

which is not surprisingly the restatement of (151). The latter condition is easily obtained by observing that $O(q_{n-1}^{-1}) = O(\frac{1}{x_n q_n})$; when we compare the latter to the first summand in (164) we use the fact that $\frac{1}{x_n}$ and $\frac{1}{x_n(1-x_n)}$ can both be of the maximum order $O(\bar{a})$. In addition, to obtain (164) we have used (151) and certainly the fact that $\mathcal{Q}_n^1$ zeroes the principal part of the above expression.

In the same fashion for any n such that $\chi(n) = +1$, the quantity $\mathcal{D}_n^1$ will be positive, moreover

$$
q_n \geq \frac{\mathcal{Q}_n^1}{1-2\varepsilon^{\delta_1}},
$$

so repeating what we have just done we again arrive at the same estimate

$$
|\mathcal{D}_n^1| \geq \frac{\sigma_0}{2}\frac{\varepsilon^{-\frac{1}{4}+\delta_1}}{(\bar{a}+2)^{\frac{3}{2}}},
$$

under the condition (151). This implies that if the sequence $\{E_n\}$ has the absolute minimum for $n = n_*$, then

$$
E_{n_*-1} - E_{n_*},\ E_{n_*+1} - E_{n_*} \geq \frac{\sigma_0}{2}\frac{\varepsilon^{-\frac{1}{4}+\delta_1}}{(\bar{a}+2)^{\frac{3}{2}}}.
$$

Exactly in the same groove using the quantity $\mathcal{Q}^2(x_n, z_n)$ from (156) and the relation ((154) we can compare the exponents E_{n-2} and E_n for $n = s+1, .., s+r+1$, if we recall that by our rule of avoiding the critical intervals, either

$$
q_{n-1} \geq \frac{\mathcal{Q}^2(x_n, z_n)}{1-2\varepsilon^{\delta_1}},
$$

or

$$
q_n \leq \frac{\mathcal{Q}^2(x_n, z_n)}{1+2\varepsilon^{\delta_1}}.
$$

Substituting the above inequalities as equalities into (154), we easily get the following, provided that (151) holds:

$$
|\mathcal{D}_n^2| \geq \frac{\sigma_0}{2\sqrt{\bar{a}+2}}\varepsilon^{-\frac{1}{4}+\delta_1}, \tag{165}
$$

which is larger than the right-hand side of (164).

At last, suppose, ε is away from the union of all the critical intervals, and n_* yields the absolute minimum of the sequence $\{E_n\}$.

If we encounter V-shape, then using the function $\mathcal{D}^2(x_{n_*+1}, z_{n_*+1})$ we can find the second smallest exponent $E_{n_{**}}$ with n_{**} equal either $n_* - 1$ or $n_* + 1$.

If we deal with W-shape, then by Corollary 6.1, the second smallest exponent will be given by the other local minimum, which will correspond to either $n_{**} = n_* - 2$, or $n_{**} = n_* + 2$ (depending on which prong of the "W" the absolute minimum is achieved).

In both cases the n_*th exponent will be smaller than all the rest by at least a quantity in the right hand side of (164). As for $E_{n_{**}}$, it will in turn be smaller than all the other exponents by at least a quantity, in the right-hand part of (164).

Furthermore, we want to provide an estimate similar to (163) for the convergent, yielding the second minimum exponent.

If we have V-shape with the minimum, corresponding to q_{n_*}, then from (157) we can derive that the necessary condition for the second minimum exponent to be labeled by n_*+1 is that $q_{n_*}^2 < \frac{1}{a_{n_*}}$, therefore (rescaled) $q_{n_*+1} < \sqrt{\bar{a}+1}$. On the other hand, if the second minimum exponent corresponds to q_{n_*-1}, then we must have $q_{n_*}^2 > \frac{1}{\bar{a}+2}$. Thus,

$$\mathcal{K}(\varepsilon)(\bar{a}+2)^{-\frac{3}{2}} < q_{n_{**}} < \mathcal{K}(\varepsilon)(\bar{a}+2)^{\frac{1}{2}}, \tag{166}$$

and the same (in fact, tighter) bounds will hold for W-shape.

6.4.4 Conclusion of the proof of Theorem 2.3

Finally, using the results of the preceding discussion, we turn to the conclusion of the proof of Theorem 2.3. At first we will be concerned with the smallness conditions.

Suppose that

$$\bar{a} = \bar{a}_0 \varepsilon^{-\delta_2},$$

where $\delta_2 > 0$ is rather small, and $\bar{a}_0$ is a constant, independent of ε.

To be sure that (149) and (151) are satisfied if ε is small enough, we do the following: for any $\delta_3 > 0$ and ε small enough we require

$$\begin{array}{lllll} \frac{1}{2} & & -3\delta_2 & -\delta_3 & \geq 0, \\ \frac{1}{2} & -\delta_1 & -2\delta_2 & -\delta_3 & \geq 0. \end{array} \tag{167}$$

Now we have to recall (120), (122), and (126), and also the fact that the number of convergents under the cutoff is definitely smaller than $\frac{1}{\sqrt{\varepsilon}}$.

In the following relations the first inequality is to insure that the cutoff parameter q_ε, that we have introduced before getting (121), lies to the right of the interval where q_{n_*} and $q_{n_{**}}$ can be located, with the upper bound from (166). The second inequality insures that the factor in the right-hand side in (126) can be rewritten as $\exp(-\varepsilon^{-\delta_3})$. The third one will be the restatement of (164), provided that (167) holds:

$$\begin{array}{lllll} \frac{1}{4} & -\delta_0 & -\frac{1}{2}\delta_2 & -\delta_3 & \geq 0, \\ & \delta_0 & -2\delta_2 & -2\delta_3 & \geq 0, \\ \frac{1}{4} & -\delta_1 & -\frac{3}{2}\delta_2 & -2\delta_3 & \geq 0, \end{array} \tag{168}$$

Then, following our strategy of making estimates, we can rewrite the sum (127) as follows:

$$\begin{array}{rcl} \Sigma & = & 2A_{\vec{k}^*} \exp\left(-k_*\sigma_0 - \frac{\omega + \|q_*\beta\|\pi}{2\sqrt{\varepsilon}}\right) \sin(\vec{k}^* \cdot \vec{\alpha}) \\ & + & 2A_{\vec{k}^{**}} \exp\left(-k_{**}\sigma_0 - \frac{\omega + \|q_{**}\beta\|\pi}{2\sqrt{\varepsilon}}\right) \sin(\vec{k}^{**} \cdot \vec{\alpha}) \\ & \times & \left[1 + O\left(e^{-\varepsilon^{-\delta_3}}\right)\right], \end{array} \tag{169}$$

where $q_* = q_{n_*}$, p_* is the unique and positive numerator in the n_*th best approximation to β, which can be computed from the Continued fraction for β using the recursion rule (109), $\vec{k}^* = (-p_*, q_*)$, and $k_* = p_* + q_*$ and similarly for the second smallest term marked by double asterisks.

Using (143), we can also estimate the minimum and the maximum for the exponents E_{n_*} and $E_{n_{**}}$ when n_* and n_{**} are such that q_{n_*} lies inside the interval, specified by (163) and $q_{n_{**}}$ assumes the bounds (166).

In addition, we know that $E_{n_{**}}$ exceeds E_{n_*}. While evaluating these exponents, we will keep in mind that provided that (149) is satisfied, the error term in (143) can be ignored.

By (143) without the error term, E_n reaches its minimum as a continuous function of q when

$$q = \mathcal{K}(\varepsilon)(x_n + z_n)^{-\frac{1}{2}},$$

which equals

$$\left(\frac{2\sigma_0(1+\beta)\pi\omega_+}{\sqrt{\varepsilon}(x_n+z_n)}\right)^{\frac{1}{2}}.$$

We use the upper bound $x_n + z_n = \bar{a} + 2$ from (136) and (137) and notice that $0 < \beta < 1$, which implies

$$E_{n_*} \geq \sqrt{\frac{2\pi\omega_+\sigma_0}{(\bar{a}_0+2)}}\, \varepsilon^{-\frac{1}{4}+\frac{1}{2}\delta_2}. \tag{170}$$

To compute the maximum of the exponents E_{n_*} and $E_{n_{**}}$ we substitute the (rescaled) upper bounds $q_{n_*} = 2$ from (163) and $q_{n_{**}} = \sqrt{\bar{a}+2}$ from (166) into the first term of (143). As for the second term, we notice that $\frac{1}{q_n}\frac{1}{(z_n+x_n)} < \frac{1}{q_n}\frac{2}{z_n+1} < \frac{2}{q_{n+1}}$, so we can use the lower bound $\frac{1}{2}$ for q_{n_*+1} from Proposition 6.10 for E_{n_*} and the lower bound for $q_{n_{**}+1}$ for $E_{n_{**}}$; the latter can be easily seen to be $(\bar{a}+2)^{-\frac{1}{2}}$. Thus, we obtain:

$$\begin{aligned} E_{n_*} &\leq 8\sqrt{\pi\omega_+\sigma_0}\varepsilon^{-\frac{1}{4}}, \\ E_{n_{**}} &\leq 8\sqrt{\pi\omega_+\sigma_0(\bar{a}_0+2)}\varepsilon^{-\frac{1}{4}-\frac{1}{2}\delta_2}. \end{aligned} \tag{171}$$

where we have finally substituted 1 for β.

That's why the estimates (120) and (122) will be exponentially small in comparison with $|\Sigma|$ and negligible in the error analysis if we let $\bar{\delta} = \delta_3$ and require

$$\frac{1}{4} - \delta_0 - \frac{1}{2}\delta_2 - 3\delta_3 \geq 0.$$

It remains to say that if (151), or (167) is satisfied, then according to Proposition 6.6, the relative measure of the union of all the critical intervals will be $O(\varepsilon^{\delta_1 - 2\delta_2})$.

The next thing to show will be that after we let the parameter $\bar{a}$ depend on ε, the frequencies to which this Theorem can be applied, form a set of asymptotically full measure.

It's easy to see that our argument will not work for numbers $\bar{\beta}$ which are too close to rationals $\frac{p}{q}$ for $q \leq q_\varepsilon = [\varepsilon^{-\frac{1}{2}-\delta_0}]$. So, for all these rationals on the unit interval we will have to cut out their small neighborhoods, containing numbers $\bar{\beta}$ such that $|\bar{\beta} - \frac{p}{q}| \leq \frac{1}{(\bar{a}+2)q^2}$. For each $q \leq q_\varepsilon$ there are at most q choices for the numerator p. That's why the overall length of all the intervals to be cut out is bounded from above by the quantity

$$\frac{1}{\bar{a}}\sum_{q=1}^{q_\varepsilon}\frac{1}{q} \leq \varepsilon^{\frac{3\delta_2}{4}}$$

with our choice of $\bar{a} \sim \varepsilon^{-\delta_2}$. On the other hand, in what remains there are points not satisfying the diophantine condition (14) with $b = \frac{1}{2}$ (see (15)) and $\varpi = 2 - 2\delta > 1$; the relative measure of the latter won't exceed $\varepsilon^{\frac{1}{4}}$. Thus, the relative measure of the frequencies to which either the KAM theorem, or the argument of this chapter cannot be applied, will be certainly bounded from above by $\varepsilon^{\frac{\delta_2}{2}}$.

Eventually, suppose $*$ is the index for the minimum exponent term and $**$ is the index for the second minimum exponent term in the series (104). We know that $\vec{\alpha} = \vec{0}$ is its true zero (in fact, there are more true zeroes as is stated by Theorem 2.1). The representation (169) hints us that there are many other zeroes of (104), which will be close to the points where the two closed trajectories on a 2-torus, given by $\vec{k}^* \cdot \vec{\alpha} = 0$ and $\vec{k}^{**} \cdot \vec{\alpha} = 0$ intersect. We will be able to prove it, by showing that the maximum error in (169) turns out to be exponentially small in comparison with the estimate for the quantity Υ that we are going to obtain further. The number of these additional zeroes will be at least $p_* q_*$, where $\frac{p_*}{q_*}$ is the n_*th best approximation to β. The latter product can be bounded from below by $\beta\varepsilon^{-\frac{1}{2}+2\delta_2+!elta_3}$, where we have used (163). The same argument will be true for the intersections of the curves $\vec{k}^* \cdot \vec{\alpha} = \pi$ and $\vec{k}^{**} \cdot \vec{\alpha} = 0$, etc., so we can multiply this estimate for the number of zeroes by at least 4.

One can easily evaluate Υ from its definition (21) applied to (169) as follows:

$$\begin{aligned}\Upsilon &= 4\mathcal{E}_*\mathcal{E}_{**}|k_1^* k_2^{**} - k_1^{**}k_2^*||A_1^* A_2^{**} - A_1^{**}A_2^*| \\ &\times \left(1 + O\left(e^{-\varepsilon^{-\delta_3}}\right)\right),\end{aligned}$$

where the multiplier 4 comes from parity.

The splitting distance is given by the Melnikov function and a relatively small remainder, see (88). Besides, the contribution of the second and higher order terms due to the Fourier modes with either k_1 or k_2 equal zero will be $\sim e^{-C\varepsilon^{-\frac{1}{2}}}$, hence, negligible. Therefore, if the above expression does not turn into zero for the Melnikov function, the remainder can be relegated into the error term. Apparently, this error won't influence the foregoing argument for the number of the new zeroes.

Since $\vec{k} = (-p, q)$, in case of V-shape when $n_{**} = n_* \pm 1$ we use Proposition 6.3 to claim that $|k_1^* k_2^{**} - k_1^{**}k_2^*| = 1$. In case of W-shape the two local minima are indexed by n_* and $n_* + 2$, and moreover, $a_{n_*+2} = 1$. Using the recursion rules (109), and Proposition 6.3 we see that in this case also $|k_1^* k_2^{**} - k_1^{**}k_2^*| = 1$.

As for the Melnikov function, in the notation of (104) we have:

$$\vec{A}_{\vec{k}} = \vec{k}\frac{\pi|\vec{k}\cdot\vec{\omega}|}{\varepsilon}\sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}} \prod_{l=1}^{j-1}\left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon} + 4l^2\right),$$

therefore, using the same reasoning we conclude that

$$|A_1^* A_2^{**} - A_1^{**}A_2^*| = \left(\frac{\pi}{\varepsilon}\right)^2 \prod_{\vec{k}\in\{\vec{k}^*, \vec{k}^{**}\}} |\vec{k}\cdot\vec{\omega}| \sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}} \prod_{l=1}^{j-1}\left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon} + 4l^2\right).$$

The latter quantity is obviously nonzero.

At this point, we notice that our choice of a positive β meant nothing, except for the fact that in its Continued fraction $a_0 = 0$, which was insignificant. The other thing would be to change the multiplier $(1+\beta)$ for $(1 + |\beta|)$ in formulas for E_n, $\mathcal{D}_n^{1,2}$, $\mathcal{K}$, etc. This concludes the proof of Theorem 2.3. □

6.4.5 Shape evolution

In this section we demonstrate how the shape of the sequence $\{E_n\}$ changes as ε goes to zero.

Suppose, that ε is away from the union of all the critical intervals, and $\vec{k}^*$ is the index of a mode, corresponding to the convergent $q_* = q_{n_*}$, giving the leading-order behavior of the Fourier series (104). Moreover, suppose we deal with more common V-shape and the second minimum exponent is indexed by $n_* - 1$. Anent the critical values, it is very unlikely that they can have multiplicities higher than 1, so for simplicity we assume that all the critical values are simple.

When ε is decreased and it first comes close to the critical value for the quantity $\mathcal{D}^2_{n_*+1}$ (because of (132) it happens before the critical value for the quantity $\mathcal{D}^2_{n_*+1}$ is reached) , and when it passes this critical value, the second minimum exponent becomes the one indexed by $n_* + 1$. Further, ε passes the critical value for the quantity $\mathcal{D}^2_{n_*+1}$, and after it does this, the absolute minimum will be given by E_{n_*+1} and the second minimum exponent will be E_{n_*}. Eventually, suppose we are in the situation stipulated by Proposition 6.2, but the necessary condition for V-shape is not satisfied. This means that before ε passes through the critical value for $\mathcal{D}^1_{n_*+1}$ it passes the one for $\mathcal{D}^1_{n_*+2}$, and another local minimum gets born, then the one for $\mathcal{D}^2_{n_*+2}$, so the right local minimum becomes the absolute minimum. Then, eventually, !*ilon* passes through the critical value for $\mathcal{D}^1_{n_*+1}$, and the left local minimum gets annihilated, so W-shape transforms into V-shape.

We see that although the class of frequencies that we consider is very large, the arithmetics is quite regular and predictable, in particular as far as the mutual location of the critical values is concerned. We believe, this accounts for some "jumping" phenomena, observed numerically in Simó [1994].

7 Appendix. Proof of the Extension Lemma

As we have already mentioned, we are not pursuing optimality as far as the value of μ_0 is concerned, and most of our following estimates finally leading to the smallness condition of the Extension lemma can probably be improved.

During the proof we'll be concerned with the unstable whisker only, that's why we will further drop the superscript u.

As one can see from the unperturbed solution (13) the function $y_0(\tau)$ has simple poles at

$$\tau = \frac{\iota}{\sqrt{\varepsilon}}\left(\frac{\pi}{2} + m\pi\right) \tag{172}$$

for $m \in Z$. We denote

$$p_m = \frac{1}{\sqrt{\varepsilon}}\left(\frac{\pi}{2} + m\pi\right).$$

We recall that by Theorem 3.1, for $\Re\tau \leq -T^*$ and $|\Im\vec{\alpha}| \leq \sigma_1 = \sigma_0 - \frac{1}{2}\sqrt{\varepsilon}$, the phase trajectories on the perturbed unstable whisker are $O\left(\frac{\mu}{\mu_0}\right)$ close to the corresponding unperturbed ones.

Recall the definition of the variations (51), for which we will also use the following notation:

$$\delta\tilde{\Gamma}(\tau,\vec{\alpha},t_0) = (\xi,\eta,\vec{\varphi},\vec{\zeta}) = \tilde{\Gamma}(\tau,\vec{\alpha},t_0) - \tilde{\Gamma}_0(\tau,\vec{\alpha},t_0),$$

The momenta components $(\eta,\vec{\zeta})$ are the time-derivatives of the coordinate components $(\xi,\vec{\varsigma})$, the evolution of the latter being described by the second order system of ODE's (52).

We will further insure by our choice of μ_0 that all the variations remain small during their time-evolution.

We follow Delshams and Seara [1992] in choosing the fundamental solution of the linearized system (54). Namely we take $\xi_1(\tau) = \frac{1}{\cosh\sqrt{\varepsilon}\tau}$ for the first solution; as for the second one (which we will still denote $\xi_2(\tau)$ independently from Section 5), we do the following: for $p_m - \frac{\pi}{2\sqrt{\varepsilon}} \leq \Im\tau < p_m + \frac{\pi}{2\sqrt{\varepsilon}}$ we choose it in such a way that it is regular at ιp_m, namely:

$$\xi_2(\tau) = \xi_1(\tau)\int_{\iota p_m}^{\tau}\frac{ds}{\xi_1^2(s)} = \frac{2\sqrt{\varepsilon}(\tau-\iota p_m) + \sinh 2\sqrt{\varepsilon}\tau}{4\sqrt{\varepsilon}\cosh\sqrt{\varepsilon}\tau}. \tag{173}$$

Thus, $\tau = \iota p_m$ is a double zero for $\xi_2(\tau)$. Therefore, it will be enough to prove the lemma for $\tau \in \Sigma'$, where the latter is defined as

$$\Sigma' = \{\tau \in C:\ |\Re\tau| \leq T^*,\ 0 \leq \Im\tau \leq p_*\},$$

with p_* given by (22), for it will all boil down to properly choosing p_m depending on the imaginary part of τ.

We will also define a subset Σ'' of Σ' as follows:

$$\Sigma'' = \{\tau \in C:\ |\Re\tau| \leq T,\ 0 \leq \Im\tau \leq p_*\}.$$

It follows from (173) that for $\tau \in \Sigma'$ we can estimate

$$|\xi_2(\tau)|, |\dot{\xi_2}(\tau)| \leq K_0\varepsilon^{-\frac{5}{4}} \tag{174}$$

for some constant $K_0 > 0$.

Just like (57) (see the notation of (13), (53), and the convention of Section 5, according to which we omit the parameter dependencies), we study the equivalent system of integral equations (this time over a finite time interval):

$$\begin{aligned}\xi(\tau,\tau_0) &= \xi^{in}(\tau,\tau_0) + \textstyle\int_{\tau_0}^{\tau} G(\tau,s)[\varepsilon h(s,\xi) - \mu g(x_0(s)+\xi(s),\vec{\varphi}_0(s)+\vec{\varsigma}(s))ds,\\ \vec{\varsigma}(\tau,\tau_0) &= \vec{\varsigma}(\tau_0) + (\tau-\tau_0)\vec{\zeta}(\tau_0) - \mu\textstyle\int_{\tau_0}^{\tau}(\tau-s)\vec{f}(x_0(s)+\xi(s),\vec{\varphi}_0(s)+\vec{\varsigma}(s))ds,\end{aligned} \tag{175}$$

where

$$G(\tau,\tau_0)=\xi_2(\tau)\xi_1(\tau_0)-\xi_1(\tau)\xi_2(\tau_0), \tag{176}$$

and

$$\xi^{in}(\tau,\tau_0)=-\frac{\partial}{\partial\tau_0}G(\tau,\tau_0)\xi(\tau_0)+G(\tau,\tau_0)\eta(\tau_0). \tag{177}$$

The integration in (175) is carried out along horizontal lines in the complex plane, in other words, $\Im\tau=\Im\tau_0=-\Im t_0$.

It's useful to notice that the "worst" possible values for the arguments of $G(\tau,\tau_0)$ when $\tau,\tau_0\in\Sigma'$ are $\tau_0=\pm T^*, \tau=\imath p_*=\imath(\frac{\pi}{2\sqrt{\varepsilon}}-\sqrt{\varepsilon})$. Thus, for $\tau,\tau_0\in\Sigma'$, increasing K_0 if necessary, we have:

$$\max_{\tau,\tau_0\in\Sigma'}\left(|G(\tau,\tau_0)|,\left|\frac{\partial}{\partial\tau}G(\tau,\tau_0)\right|\right)\le K_0\varepsilon^{-\frac{11}{4}}. \tag{178}$$

We'll use the method of iteration to ascertain the existence and uniqueness of the solution of (175) and, consequently, (11).

Mathematically speaking, we will consider a Banach space $\mathcal{B}_1$ of functions of a real variable t, analytic in a complex parameter t_0 such that $\tau\in\Sigma'$ for $\tau=t-t_0$, also analytic in $\vec{\alpha}$ for $|\Im\vec{\alpha}|\le\sigma_*$, equipped with the *sup*-norm. We'll show that for $|\mu|\le\mu_0$, the latter to be defined, the nonlinear operator, defined by (175), is uniformly bounded and is a contraction operator, proving the Existence and Uniqueness of the solution of (52) in $\mathcal{B}_1$.

We'll start the iterations by letting $\xi^0,\eta^0,\vec{\varsigma}^0,\vec{\zeta}^0$ be identically zero. Then we can write down the explicit expressions for the first iterates.

$$\begin{aligned}\xi^1&=\xi^{in}-\mu\int_{\tau_0}^{\tau}G(\tau,s)g(x_0(s),\vec{\alpha}+\vec{\omega}(s+t_0))ds,\\ \vec{\varsigma}^1&=\vec{\varsigma}(\tau_0)+\vec{\zeta}(t_0)(\tau-\tau_0)-\mu\int_{\tau_0}^{\tau}(\tau-s)\vec{f}(x_0(s),\vec{\alpha}+\vec{\omega}(s+t_0))ds.\end{aligned} \tag{179}$$

Recall that $\vec{\varphi}_0=\alpha+\omega t$

¿From the KAM theorem the values of $\xi,\eta,\vec{\zeta},\vec{\varsigma}$ at "time" $\tau_0=-T^*$ for the whole admissible range of parameters are $O\left(\frac{\mu}{\mu_0}\right)$, the latter satisfying (35). Then from (178) we'll have

$$|\xi^{in}|,\ |\dot{\xi}^{in}| \quad=\quad O\left(\frac{\mu}{\mu_0\varepsilon^{\frac{11}{4}}}\right). \tag{180}$$

Moreover, since

$$\begin{aligned}\sin x_0(\tau)&=\frac{2\sinh\sqrt{\varepsilon}\tau}{\cosh^2\sqrt{\varepsilon}\tau},\\ \cos x_0(\tau)&=1-\frac{2}{\cosh^2\sqrt{\varepsilon}\tau},\end{aligned} \tag{181}$$

then once again increasing K_0 if necessary we have for $\tau\in\Sigma'$

$$\begin{aligned}|\sin x_0(\tau)|,|\cos x_0(\tau)|&\le K_0(1+|\xi_1^2(\tau)|)\\ |g(x_0(\tau),\vec{\varphi}_0(\tau))|&\le K_0(1+|\xi_1^{2\nu_0}(\tau)|)\\ |\vec{f}(x_0(\tau),\vec{\varphi}_0(\vec{\alpha},\vec{\omega},t))|&\le K_0\frac{1}{\sqrt{\varepsilon}}(1+|\xi_1^{2\nu_0}(\tau)|)\end{aligned} \tag{182}$$

for $\vec{\alpha}\in W_{\sigma_1}T^{n-1}$. The factor $\frac{1}{\sqrt{\varepsilon}}$ is due to the fact that initially we have an estimate for the norm of the perturbation $F(x,\vec{\varphi})$ in (7) only.

Now, having the expressions (179) for the first iterates we need several estimates for the integrals involved. They will easily follow from the expressions for ξ_1 and ξ_2, and the following essentially obvious formula that we borrow from Delshams and Seara [1992].

Consider two complex variables w_1,w_2 such that $\Im w_1=\Im w_2$, $\Re w_2>\Re w_1$ and also a real number p. Denote for $l=1,2,...$

$$\rho^{-l}_{[p,w_1,w_2]} \equiv \sup_{\Re w_1 \le \Re s \le \Re w_2,\ \Im s = \Im w_1} \frac{1}{|s - \iota p|^l}.$$

For $l = .., -1, 0$ we'll let $\rho^{-l}_{[p,w_1,w_2]} = 1$. Then the following Proposition is easy to verify:

Proposition 7.1 *For $l = 1, 2..$ and the constants $0 < K_l \le \pi$ one has*

$$\int_{w_1}^{w_2} \frac{ds}{|s - \iota p|^{l+1}} \le K_l \rho^{-l}_{[p,w_1,w_2]}.$$

Proof: We observe that

$$\int_{w_1}^{w_2} \frac{ds}{|s - \iota p|^{l+1}} \le \frac{1}{r^l} \int_{-\infty}^{x} \frac{dz}{(z^2+1)^{\frac{l+1}{2}}} \le \frac{1}{r^l}\pi \quad \forall l \ge 1,$$

where $r = p - \Im w_1$, $x = \frac{\Re w_1}{r}$, since π is the value for the latter integral for $x = +\infty$ and $l = 1$.

If $\Re w_1 \le 0 \le \Re w_2$, then in the definition of $\rho^{-l}_{[p,w_1,w_2]}$ the supremum is reached at $\Re s = 0$, so in this case $\rho^{-l}_{[p,w_1,w_2]} = \frac{1}{r^l}$, and the statement holds trivially.

If $x \le 0$, then for an odd $l:\ l = 2m+1$ one has

$$\int \frac{dx}{(x^2+1)^{m+1}} = \frac{x}{2m+1} \sum_{k=1}^{m} \frac{K_{m,k}}{(1+x^2)^{m+1-k}} + \frac{(2m-1)!!}{2^m m!} \arctan x,$$

for some positive constants $K_{m,k}$ Which implies that

$$\begin{aligned} |s - \iota p|^{\frac{2m+1}{2}} \textstyle\int_{\tau_0}^{\tau} \frac{d\sigma}{|s-\iota p|^{m+1}} &\le (x^2+1)^{\frac{2m+1}{2}} \tfrac{(2m-1)!!}{2^m m!} (\arctan x + p) \\ &\le \tfrac{(2m-1)!!}{2^m m!} p, \end{aligned}$$

since, as we've assumed $x \le 0$. The constant before p can be easily shown to be not greater than 1 by induction in m.

Furthermore, for an even $l:\ l = 2m$ we have

$$\int \frac{dx}{(x^2+1)^{\frac{2m+1}{2}}} = x \sum_{k=1}^{m} K^1_{m,k} \frac{1}{(x^2+1)^{m - \frac{2k-1}{2}}}$$

for some set of positive constants $K^1_{m,k}$, which can be found from the recursion rule:

$$\int \frac{dx}{(x^2+1)^{\frac{2m+1}{2}}} = \frac{x}{2m-1} \frac{1}{(x^2+1)^{\frac{2m-1}{2}}} + \frac{2(m-1)}{2m-1} \int \frac{dx}{(x^2+1)^{\frac{2m-1}{2}}}.$$

This implies that

$$|s - \iota p|^m \int_{\tau_0}^{\tau} \frac{d\sigma}{|s - \iota p|^{\frac{2m+1}{2}}} \le K_{m,m}$$

if $x \le 0$. The constant $K^1_{m,m}$ can be easily shown to be not greater than 1 by induction in m, starting from the fact that $\int \frac{dx}{\sqrt{(x^2+1)^3}} = \frac{x}{\sqrt{x^2+1}}$, so $K^1_{1,1} = 1$.

Finally, the case $\Re w_1 \ge 0$ is the same as the case when $\Re w_2 \le 0$, since the integrand is an even function.
$\square$

Using this result along with (176), one can easily establish the following bounds: for easier cross-reference we compile them in the following Proposition.

Proposition 7.2 *Given $\tau_0, \tau \in \Sigma''$ such that $\Re\tau > \Re\tau_0$, $\Im\tau_0 = \Im\tau = -\Im t_0$. and $\vec\alpha$ such that $|\Im\vec\alpha| \le \sigma_1$, there exist independent of ε, μ positive constants K_1, K_2, K_3, K_4, such that the following once differentiable in τ estimates take place:*

$$\begin{aligned}
\left|\int_{\tau_0}^{\tau} G(\tau,s) g(x_0(s), \vec\alpha + \vec\omega(s+t_0) ds\right| &\le K_1 \tfrac{1}{\sqrt{\varepsilon}}\xi_1(\tau)\left(\rho^{-2\nu_0+3}_{[\frac{\pi}{2},\sqrt{\varepsilon}\tau_0,\sqrt{\varepsilon}\tau]} + 1\right)\\
&+ K_2 \tfrac{1}{\sqrt{\varepsilon}}\xi_2(\tau)\left(\rho^{-2\nu_0}_{[\frac{\pi}{2},\sqrt{\varepsilon}\tau_0,\sqrt{\varepsilon}\tau]} + 1\right),\\
\left|\int_{t_0}^{t} (\tau - s)\vec f\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) ds\right| &\le K_3 \tau \tfrac{1}{\varepsilon}\left(\rho^{-2\nu_0+1}_{[\frac{\pi}{2},\sqrt{\varepsilon}\tau_0,\sqrt{\varepsilon}\tau]} + 1\right)\\
&+ K_4 \tfrac{1}{\varepsilon\sqrt{\varepsilon}}\left(\rho^{-2\nu_0+2}_{[\frac{\pi}{2},\sqrt{\varepsilon}\tau_0,\sqrt{\varepsilon}\tau]} + 1\right),
\end{aligned}$$

Proof: We note that $g(x, \vec\varphi)$ is a trigonometric polynomial in x of degree ν_0. The proof follows if we simultaneously apply (176), (182) and Proposition 7.1, where we change $\sqrt{\varepsilon}\tau \to \tau$ under the integral signs.□

Again, let's redefine $K_0 = \max(K_0, K_1, K_2, K_3, K_4)$. Proposition 7.2 admits an easy corollary that yields the estimates in powers of ε. If $|\Re\tau| \le \varepsilon^{\frac14}$ and $\frac{\pi}{2\sqrt{\varepsilon}} - \varepsilon^{\frac14} \le \Im\tau \le p_*$, one can use the fact that ξ_2 is small near ιp, since it has a second-order zero at this point, namely $\xi_2 \le K_0\varepsilon$, again increasing K_0 if necessary. Otherwise, if either $|\Re\tau| \ge \varepsilon^{\frac14}$ or $\Im\tau \le \frac{\pi}{2\sqrt{\varepsilon}} - \varepsilon^{\frac14}$, we clearly have $\rho^{-l}_{[\frac{\pi}{2},\sqrt{\varepsilon}\tau_0,\sqrt{\varepsilon}\tau]} \le \varepsilon^{-\frac{3l}{4}}$.

This leads to the following simple fact:

Proposition 7.3 *For $\tau_0, \tau, \vec\alpha$ as in Proposition 7.2 the following estimates take place for $\nu_0 \ge 1$:*

$$\begin{aligned}
\left|\int_{\tau_0}^{\tau} G(\tau,s) g\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) ds\right| &\le K_0\varepsilon^{-2\nu_0+\frac12},\\
\left|\int_{\tau_0}^{\tau} \tfrac{\partial}{\partial\tau} G(\tau,s) g\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) d\sigma\right| &\le K_0\varepsilon^{-2\nu_0+\frac12},\\
\left|\int_{\tau_0}^{\tau} (\tau - s)\vec f\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) ds\right| &\le K_0\varepsilon^{-2\nu_0-\frac12},\\
\left|\int_{t_0}^{t} \vec f\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) d\sigma\right| &\le K_0\varepsilon^{-2\nu_0}.
\end{aligned}$$

The next Proposition gives us the bounds for the same integrals away from the pole when the reason for concern is the growth of ξ_2.

Proposition 7.4 *Given τ_0, τ from the same connected component of $\Sigma' \setminus \Sigma''$ such that $\Re\tau_0 < \Re\tau$, $\Im\tau_0 = \Im\tau$, and $\vec\alpha$ such that $|\Im\vec\alpha| \le \sigma_1$, the following estimates take place for ε small enough.*

$$\begin{aligned}
\left|\int_{\tau_0}^{\tau} G(\tau,s) g\left(x_0(s), \vec\alpha + \vec\omega(s+t_0)\right) ds\right| &\le K_0\varepsilon^{-1},\\
\left|\int_{\tau_0}^{\tau} \tfrac{\partial}{\partial t} G(\tau,s) g(x_0(s), \vec\alpha + \vec\omega(s+t_0)) ds\right| &\le K_0\varepsilon^{-1},\\
\left|\int_{t_0}^{t} (\tau - s)\vec f(x_0(s), \vec\alpha + \vec\omega(s+t_0)) ds\right| &\le K_0\varepsilon^{-2},\\
\left|\int_{t_0}^{t} \vec f(x_0(s), \vec\alpha + \vec\omega(s+t_0)) ds\right| &\le K_0\varepsilon^{-2}.
\end{aligned}$$

Proof: Obvious from (182) and the definition of T^*, if one does not hesitate to increase K_0. $\square$

Now we can work out the bounds for the first iterates. We fix $\tau_0 = -T^*$.

It's easy to see that in the worst possible case the first iterates (179) can be bounded as follows:

$$\begin{aligned} |\xi^1|,\ |\eta^1| &\leq K_0|\mu|\left(\mu_0^{-1}\varepsilon^{-\frac{11}{4}} + \varepsilon^{-2\nu_0+\frac{1}{2}}\right), \\ |\vec{\varsigma}^1|,\ |\vec{\zeta}^1| &\leq K_0|\mu|\left(\mu_0^{-1}\varepsilon^{-1} + \varepsilon^{-2\nu_0-1}\right), \end{aligned}$$

where μ_0 comes from (35). Therefore, we can redefine

$$\mu_0 = \varepsilon^{\max\left(\frac{n}{2}+\frac{13}{4}+\varpi+2a+\delta,\ 2\nu_0+\frac{1}{2}+\delta\right)} \tag{183}$$

to insure that the first iterates are smaller than 1 for $\tau \in \Sigma'$, $|\Im\vec{\alpha}| \leq \sigma_1$.

We now define the iterative process as:

$$\delta\tilde{\Gamma}^{l+1}(\tau, \vec{\alpha}, t_0) = \bar{\mathcal{I}}\delta\tilde{\Gamma}^l,\ l = 0, 1, .., \tag{184}$$

where $\bar{\mathcal{I}}$ is the non-linear operator, corresponding to the system of integral equations (175).

First of all for

$$h(\tau, \xi) = [\sin x_0(\tau)(\cos\xi - 1) + \cos x_0(\tau)(\xi - \sin\xi)]$$

we will clearly have:

$$\begin{aligned} |h(\tau, \xi^l) - h(\tau, \xi^{l-1})| &\leq K_0|\xi_1^2(\tau)|(|\xi^{l-1}| + |\xi^l|) \cdot |\xi^l - \xi^{l-1}| \\ |h(\tau, \xi)| &\leq K_0|\xi_1^2(\tau)||\xi|^2. \end{aligned} \tag{185}$$

Therefore, repeating the estimates that we have just performed, we can convince ourselves that there exists a ball in $\mathcal{B}_1$ near zero of the radius $\mu_0\sqrt{\varepsilon}$, the norm of $\bar{\mathcal{I}}$ is uniformly bounded by 1, where μ_0 comes from (183).

Besides if $\delta\tilde{\Gamma}^{l-1}, \delta\tilde{\Gamma}^l$ are the two subsequent iterates (lying in the ball above), then one can write up the following estimates for $|\Im\vec{\alpha}| \leq \sigma_2$:

$$\begin{aligned} \left|g(x_0 + \xi^l, \vec{\varphi}_0 + \vec{\varsigma}^l) - g(x_0 + \xi^{l-1}, \vec{\varphi}_0 + \vec{\varsigma}^{l-1})\right| &\leq \varepsilon^{-\frac{1}{2}}K_0(1 + |\xi_1|)^{2\nu_0} \\ &\times \left(|\xi^l - \xi^{l-1}| + |\vec{\varsigma}^l - \vec{\varsigma}^{l-1}|\right), \end{aligned} \tag{186}$$

and

$$\begin{aligned} \left|\vec{f}(x_0 + \xi^l, \vec{\varphi}_0 + \vec{\varsigma}^l) - \vec{f}(x_0 + \xi^{l-1}, \vec{\varphi}_0 + \vec{\varsigma}^{l-1})\right| &\leq K_0\varepsilon^{-1}(1 + |\xi_1|)^{2\nu_0} \\ &\times \left(|\xi^l - \xi^{l-1}| + |\vec{\varsigma}^l - \vec{\varsigma}^{l-1}|\right). \end{aligned} \tag{187}$$

Using (175), (185), (186), (187), and Proposition 7.3, we can write, having possibly increased the constant K_0:

$$\begin{aligned} |\xi^{l+1} - \xi^l| &\leq K_0\varepsilon[(|\xi^l| + |\xi^{l-1}|)|\xi^l - \xi^{l-1}| \\ &+ \tfrac{|\mu|}{\mu_0\sqrt{\varepsilon}}(|\xi^l - \xi^{l-1}| + |\vec{\varsigma}^l - \vec{\varsigma}^{l-1}|)] \end{aligned}$$

and

$$|\vec{\varsigma}^{l+1} - \vec{\varsigma}^l| \leq K_0\tfrac{|\mu|}{\mu_0\sqrt{\varepsilon}}[|\xi^l - \xi^{l-1}| + |\vec{\varsigma}^l - \vec{\varsigma}^{l-1}|].$$

The same estimates hold for the derivatives η and $\vec{\zeta}$.

This implies that for ε small enough and $|\mu| \leq \mu_0\sqrt{\varepsilon}$, where μ_0 is given by (183) $\bar{\mathcal{I}}$ is a contraction operator (δ in (183) takes care of all the constants, as usual), and therefore, the solution on the unstable manifold exists as is unique, as stated in the Lemma.

Finally, we notice that the first iterates were $O\left(\frac{\mu}{\mu_0}\right)$, therefore if we multiply μ_0 by $\varepsilon^{\frac{1}{2}}$, it will insure that $\vec{\varsigma}$ will never become larger than $\frac{1}{4}\sqrt{\varepsilon}$, so we never find ourselves outside the analyticity domain of $\vec{f}$ (owing to δ in the expression for μ_0). This completes the proof of the Extension lemma.□

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