KAM-Renormalization Group Approach to the Break-up of Invariant Tori in Hamiltonian Systems

C. Chandre, M. Govin, and H. R. Jauslin

Laboratoire de Physique, CNRS, Université de Bourgogne, BP 400, F-21011 Dijon, France

(July 15, 1997)

Abstract

We analyse the break-up of invariant tori in Hamiltonian systems with two degrees of freedom using a combination of KAM theory and renormalization group techniques. We consider a class of Hamiltonians quadratic in the action variables that is invariant under the chosen KAM transformations, following the approach of Thirring. The numerical implementation of the transformation shows that the KAM iteration converges up to the critical coupling at which the torus breaks up. By combining this iteration with a renormalization, consisting of a shift of resonances and rescalings of momentum and energy, we obtain a much more efficient method that allows to determine the critical coupling with high accuracy. This transformation is based on the physical mechanism of the break-up of invariant tori. We show that the critical surface of the transformation is the stable manifold of codimension one of a nontrivial fixed point, and we discuss its universality properties.

05.45.+b, 64.60.Ak

I. INTRODUCTION

The existence of invariant tori plays a significant role for the long-time stability of Hamiltonian systems. The KAM theorem [1–3] states that the tori with frequency vectors which satisfy diophantine conditions are stable under small perturbations. Conversely, it has been shown [4] that for large perturbations these tori do no longer exist. Greene [5–7] formulated a criterion that allows to determine the existence of a KAM torus by analysing the properties of a sequence of nearby periodic orbits, namely the resonances whose winding ratios are the continued fraction approximants of the winding ratio of the considered torus. When the amplitude of the perturbation is at its critical value, these resonances open gaps in the torus and it breaks up into a Cantor set (Aubry-Mather set) [8–11].

In order to study the self-similar scaling properties observed for the break-up of invariant tori [12–14], renormalization group ideas were proposed for two-dimensional area-preserving maps [15,16]. For Hamiltonian systems with 1.5 degrees of freedom, D. Escande and F. Doveil [17,18] set up an approximate renormalization scheme that combines KAM transformations with a rescaling of phase space.

The idea of renormalization group analysis for Hamiltonian systems is to construct a transformation $\mathcal{R}$ as a generalized canonical change of coordinates acting on some space of Hamiltonians such that the iteration of $\mathcal{R}$ converges to a fixed point. If the perturbation is smaller than the critical one, $\mathcal{R}$ must converge to some Hamiltonian H_0 for which the equations of motion show trivially the existence of a torus of a given frequency vector. All Hamiltonians attracted by this trivial fixed point have an invariant torus of that frequency: this statement can be considered as an alternative version of the KAM theorem [19]. If the perturbation is larger than critical, the system does not have a KAM torus of the considered frequency and the iteration of $\mathcal{R}$ diverges. The domain of convergence to H_0 and the domain of divergence are separated by a surface invariant under the action of $\mathcal{R}$. The main hypothesis of the renormalization group approach is that there should be another nontrivial fixed point on this critical surface that is attractive for Hamiltonians on that surface. Its existence has strong implications concerning universal properties in the mechanism of the break-up of invariant tori. The analysis of the renormalization for area-preserving maps [15] gives support to the validity of this general picture. The aim of the present work is to give similar support to this picture for Hamiltonian flows. The main ideas had been announced in Ref. [20].

The transformation we define (KAM-RG) has two main parts: a KAM iteration which is a change of coordinates that reduces the size of the perturbation from ε to ε^2, and a renormalization transformation which is a combination of a shift of the resonances and a rescaling of momentum and energy. It acts within a space of Hamiltonian systems with two degrees of freedom, quadratic in the action variables. An essential aspect of the present approach, based on a formulation of the KAM theorem by Thirring [21], is that the KAM and renormalization transformations we use map this space into itself. In order to analyse the strong coupling regime and to reach the critical coupling, the KAM-RG transformation has to converge at least up to the critical surface at which the torus breaks up. In fact, we show that the KAM iteration as well as the KAM-RG transformation converge all the way to the critical coupling. Numerically, the KAM-RG transformation is a much more efficient method to determine the critical coupling. The analysis of the KAM-RG transformation shows that the critical surface is the stable manifold of codimension one of a nontrivial fixed point.

We construct the KAM transformation by two alternative methods: by the Lie transformations and by transformations defined by a generating function. The motivation to use two different transformations is two-fold. First we verify that both approaches lead to the same results, and the Lie transformation is more efficient for numerical implementation. Further, although the two transformations lead to quantitatively different nontrivial fixed points, they have the same critical exponents, in accord with the general ideas of the renormalization group approach.

In Sec. II, we describe the renormalization transformation. In Secs. III and IV, we define the KAM iteration of the transformation by the two methods. In Sec. V, we give our numerical results, and in particular, we show evidence of the existence of an even nontrivial fixed point. In Sec. VI, we describe the behavior of the KAM-RG transformation when odd perturbations are included.

We consider the following class of Hamiltonians with two degrees of freedom, quadratic in the action variables $\boldsymbol{A} = (A_1, A_2)$ and described by three scalar functions of the angles $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$:

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2} m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + [\boldsymbol{\omega}_0 + g(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A} + f(\boldsymbol{\varphi}) \ , \tag{1.1}$$

where $\boldsymbol{\omega}_0$ is the frequency vector of the considered torus, and $\boldsymbol{\Omega} = (1, \alpha)$ is some other constant vector, not parallel to $\boldsymbol{\omega}_0$. This class of Hamiltonians has been considered by Thirring [21] in its non-degenerate version

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2} \boldsymbol{A} \cdot M(\boldsymbol{\varphi}) \boldsymbol{A} + [\boldsymbol{\omega}_0 + \boldsymbol{g}(\boldsymbol{\varphi})] \cdot \boldsymbol{A} + f(\boldsymbol{\varphi}) \ , \tag{1.2}$$

where M is a 2×2 matrix such that $\det M \neq 0$, and $\boldsymbol{g}$ a vector. The Hamiltonian (1.1) is such that

$$\det \frac{\partial^2 H}{\partial \boldsymbol{A} \partial \boldsymbol{A}} = 0, \tag{1.3}$$

i.e. it does not satisfy the twist condition, but the KAM theorem is also valid under this condition (see [22]). The advantage of the family of Hamiltonians (1.1) in the present context is that they are characterized by three scalar functions of the angles and a constant α, instead of six functions for Hamiltonians (1.2). This allows a more precise numerical treatment of the problem. The essential features are already contained in the space of Hamiltonians (1.1). In particular, the nontrivial fixed point one obtains starting with (1.2) is of the form (1.1). The functions m, g, f are represented by their Fourier series e.g.

$$f(\boldsymbol{\varphi}) = \sum_{\nu \in \mathbb{Z}^2} f_\nu e^{i\nu \cdot \varphi}. \tag{1.4}$$

The numerical implementation of the transformation requires a truncation of the Fourier series. We will approximate f by

$$f^{[\leq L]}(\boldsymbol{\varphi}) = \sum_{\nu \in \mathcal{C}_L} f_\nu e^{i\nu\cdot\varphi}, \tag{1.5}$$

where $\mathcal{C}_L = \{\boldsymbol{\nu} \in \mathbb{Z}^2 \,||\nu_1| \leq L,\, |\nu_2| \leq L\}$. We define $\langle f \rangle$ the mean value of f by

$$\langle f \rangle = \int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}}{(2\pi)^2} f(\boldsymbol{\varphi}), \tag{1.6}$$

where $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$. In the following sections, we will use the notation $\boldsymbol{\partial} f = \dfrac{\partial f}{\partial \boldsymbol{\varphi}}$ for any function of the angles.

II. RENORMALIZATION TRANSFORMATION

We construct the KAM-RG transformation combining two parts: a KAM transformation $(m, g, f, \alpha) \mapsto (m', g', f', \alpha)$ and a renormalization consisting of a shift of the resonances and a rescaling of the actions and of time $(m', g', f', \alpha) \mapsto (m'', g'', f'', \alpha')$. The renormalization scheme described in this section is for a torus of frequency $\boldsymbol{\omega}_0 = (1/\gamma, -1)$ where $\gamma = (1+\sqrt{5})/2$, but this scheme can be adapted to quadratic irrationals.

The KAM-RG transformation is composed of four steps:

1) a KAM transformation which is a change of coordinates that eliminates terms of order $\mathcal{O}(\varepsilon)$, where ε is the size of the perturbation; this transformation produces terms of the order $\mathcal{O}(\varepsilon^2)$ and does not change $\boldsymbol{\Omega} = (1, \alpha)$ (see Secs. III and IV),

2) a shift of the resonances: a canonical change of coordinates that maps the next pair of daughter resonances of the sequence of rational approximants into the two main resonances,

3) a rescaling of energy (or equivalently of time),

4) a rescaling of the action variables (which is a generalized canonical transformation).

The aim of this transformation is to treat one scale at the time. The steps 2), 3) and 4) are implemented as follows: The two main resonances $(1, 0)$ and $(1, 1)$ are replaced by the next pair of daughter resonances $(2, 1)$ and $(3, 2)$, i.e. we require that $\cos[(2, 1)\cdot\boldsymbol{\varphi}'] = \cos[(1, 0)\cdot\boldsymbol{\varphi}'']$ and $\cos[(3, 2) \cdot \boldsymbol{\varphi}'] = \cos[(1, 1) \cdot \boldsymbol{\varphi}'']$. This change is done via a canonical transformation $(\boldsymbol{A}', \boldsymbol{\varphi}') \mapsto (N^{-2}\boldsymbol{A}', N^2\boldsymbol{\varphi}')$ with $N^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This linear transformation multiplies $\boldsymbol{\omega}_0$

by γ^{-2} (since $\boldsymbol{\omega}_0$ is an eigenvector of N); therefore we rescale the energy by a factor γ^2 in order to keep the frequency fixed at $\boldsymbol{\omega}_0$. A consequence of the shift of the resonances is that $\boldsymbol{\Omega}$ is changed into $\boldsymbol{\Omega}' = (1, \alpha')$, where $\alpha' = (\alpha+1)/(\alpha+2)$.

Then we perform a rescaling of the action variables: we change the Hamiltonian H' into $\hat{H}'(\boldsymbol{A}', \boldsymbol{\varphi}') = \lambda H'\left(\frac{\boldsymbol{A}'}{\lambda}, \boldsymbol{\varphi}'\right)$ with λ such that the mean value $\langle m'' \rangle$ is equal to 1. Since the rescaling of energy and the shift N^2 transform the quadratic term of the Hamiltonian into $\gamma^2(2+\alpha)^2 m'(\boldsymbol{\varphi}')(\boldsymbol{\Omega}' \cdot \boldsymbol{A}')^2/2$, this condition leads to $\lambda = \gamma^2(2+\alpha)^2\langle m' \rangle$. This condition has the following geometric interpretation in terms of self-similarity of the resonances close to the invariant torus: The rescaling magnifies the size of the daughter resonances, and places them approximately at the location of the original main resonances. This can be seen by the following heuristic argument: In order to estimate the position of the resonances, we assume that ε is small and that $m' \approx \langle m' \rangle$. The equations of motion for H' are

$$\boldsymbol{A}' \approx const, \tag{2.1}$$

$$\dot{\boldsymbol{\varphi}}' \approx \langle m' \rangle (\boldsymbol{\Omega} \cdot \boldsymbol{A}')\boldsymbol{\Omega} + \boldsymbol{\omega}_0. \tag{2.2}$$

The position of the resonance $\boldsymbol{\nu}$ is given by the condition $\boldsymbol{\nu} \cdot \dot{\boldsymbol{\varphi}}' = 0$, i.e. it is located at $\boldsymbol{A}'$ such that

$$\boldsymbol{\Omega} \cdot \boldsymbol{A}' \approx -\frac{1}{\langle m' \rangle} \frac{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}}{\boldsymbol{\Omega} \cdot \boldsymbol{\nu}}. \tag{2.3}$$

The linear change of coordinates N^2 gives the new position of the resonance $\boldsymbol{\nu}$

$$\boldsymbol{\Omega}' \cdot \boldsymbol{A}' \approx -\frac{1}{\langle m' \rangle \gamma^2 (2+\alpha)^2} \frac{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}}{\boldsymbol{\Omega}' \cdot \boldsymbol{\nu}}. \tag{2.4}$$

Thus, the rescaling of the actions $\boldsymbol{A}' \mapsto \boldsymbol{A}'' = \boldsymbol{A}'/\lambda$ with $\lambda = \gamma^2(2+\alpha)^2\langle m' \rangle$ places the resonances at the location of the original resonances.

In summary, the renormalization rescales m, g, f and $\boldsymbol{\Omega} = (1, \alpha)$ into

$$m''(\boldsymbol{\varphi}) = \frac{m'\left(N^{-2}\boldsymbol{\varphi}\right)}{\langle m' \rangle}, \tag{2.5}$$

$$g''(\boldsymbol{\varphi}) = \gamma^2(2+\alpha) g'\left(N^{-2}\boldsymbol{\varphi}\right), \tag{2.6}$$

$$f''(\boldsymbol{\varphi}) = \gamma^4(2+\alpha)^2\langle m'\rangle f'\left(N^{-2}\boldsymbol{\varphi}\right), \tag{2.7}$$

$$\alpha' = \frac{1+\alpha}{2+\alpha}. \tag{2.8}$$

The iteration of the transformation (2.8) converges to $\alpha_* = \gamma^{-1}$. It means that $\boldsymbol{\Omega}$ converges under successive iterations to $\boldsymbol{\Omega}_* = (1, 1/\gamma)$, which is orthogonal to $\boldsymbol{\omega}_0$ and is the unstable eigenvector of N^2 with the largest eigenvalue γ^2.

We remark that this renormalization scheme can also be implemented on a more general class of Hamiltonians quadratic in the action variables considered by Thirring [21]:

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2}\boldsymbol{A}\cdot M(\boldsymbol{\varphi})\boldsymbol{A} + [\boldsymbol{\omega}_0 + \boldsymbol{g}(\boldsymbol{\varphi})]\cdot \boldsymbol{A} + f(\boldsymbol{\varphi}), \tag{2.9}$$

where M is a 2×2 matrix and $\boldsymbol{g}$ a vector. This class of Hamiltonians is also invariant under KAM transformations (see Secs. III and IV). The renormalization described above changes the direction of $\boldsymbol{g}$ and M. The vector $\boldsymbol{g}$ is renormalized into $\mathcal{R}(\boldsymbol{g}) = \gamma^2 N^2 \boldsymbol{g}$. The iteration of $\mathcal{R}$ converges to the unstable eigenvector of N^2: $\boldsymbol{g} \to g\boldsymbol{\Omega}_*$. The matrix M is renormalized into $\mathcal{R}(M) = N^2 M N^2/(N^2 M N^2)_{11}$. This transformation has only one stable fixed point $M_* = \boldsymbol{\Omega}_* \otimes \boldsymbol{\Omega}_*$. Thus the iteration of the renormalization transformation on Hamiltonians of the form (2.9) (which can satisfy the twist condition or not) converges to a twistless Hamiltonian (1.1) with $\boldsymbol{\Omega} = \boldsymbol{\Omega}_*$.

III. LIE APPROACH TO THE KAM TRANSFORMATION

The Poisson bracket of two functions of $\boldsymbol{\varphi}$ and $\boldsymbol{A}$ is given by

$$\{f, g\} = \frac{\partial f}{\partial \boldsymbol{\varphi}}\cdot\frac{\partial g}{\partial \boldsymbol{A}} - \frac{\partial f}{\partial \boldsymbol{A}}\cdot\frac{\partial g}{\partial \boldsymbol{\varphi}}. \tag{3.1}$$

We will work with Lie transformations $\mathcal{U}_S : (\boldsymbol{\varphi}, \boldsymbol{A}) \mapsto (\boldsymbol{\varphi}', \boldsymbol{A}')$

$$\boldsymbol{A}' = e^{-\hat{S}(A,\varphi)}\boldsymbol{A} \equiv \boldsymbol{A} - \{S, \boldsymbol{A}\} + \frac{1}{2!}\{S, \{S, \boldsymbol{A}\}\}\dots \tag{3.2}$$

$$\boldsymbol{\varphi}' = e^{-\hat{S}(A,\varphi)}\boldsymbol{\varphi} \equiv \boldsymbol{\varphi} - \{S, \boldsymbol{\varphi}\} + \frac{1}{2!}\{S, \{S, \boldsymbol{\varphi}\}\}\dots \tag{3.3}$$

generated by functions S linear in the action variables, of the form

$$S(\boldsymbol{A},\boldsymbol{\varphi}) = Y(\boldsymbol{\varphi})\boldsymbol{\Omega}\cdot\boldsymbol{A} + Z(\boldsymbol{\varphi}) + a\boldsymbol{\Omega}\cdot\boldsymbol{\varphi}\ , \tag{3.4}$$

characterized by two scalar functions Y, Z, and a constant a. The expression of the Hamiltonian in these new variables is obtained by the following equation [23,24]

$$\begin{aligned} H'(\boldsymbol{A}',\boldsymbol{\varphi}') &= e^{+\hat{S}(A,\varphi)}H(\boldsymbol{A},\boldsymbol{\varphi})\mid_{(A',\varphi')} \\ &\equiv H + \{S,H\} + \frac{1}{2!}\{S,\{S,H\}\}\ldots \end{aligned} \tag{3.5}$$

A consequence of the linearity of S in $\boldsymbol{A}$ is that the Hamiltonian H' is again quadratic in the actions, and of the form

$$H'(\boldsymbol{A}',\boldsymbol{\varphi}') = \frac{1}{2}m'(\boldsymbol{\varphi}')(\boldsymbol{\Omega}\cdot\boldsymbol{A}')^2 + [\boldsymbol{\omega}_0 + g'(\boldsymbol{\varphi}')\boldsymbol{\Omega}]\cdot\boldsymbol{A}' + f'(\boldsymbol{\varphi}'). \tag{3.6}$$

We notice that $\boldsymbol{\Omega}$ is unchanged by this transformation. Following the approach of Thirring [21], we consider the scalar functions g and f of order $\mathcal{O}(\varepsilon)$, and m of order one. We determine S such that g' and f' are of order $\mathcal{O}(\varepsilon^2)$. The nth iteration of this transformation will produce g and f of order $\mathcal{O}(\varepsilon^{2^n})$ and m of order one. The idea is that $m(\boldsymbol{\varphi})$ does not need to be eliminated. In order to show the existence of a torus of frequency $\boldsymbol{\omega}_0$ that is located at $\boldsymbol{A}=0$, it suffices that the iteration reduces the Hamiltonian (1.1) into one with $f=0$, $g=0$ but $m(\boldsymbol{\varphi})\neq 0$. This is an immediate consequence of the equations of motion associated to $H = m(\boldsymbol{\varphi})(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2/2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A}$. The fact that $m(\boldsymbol{\varphi})$ does not need to be eliminated is what allows to work with canonical transformations that are linear in the actions. This has the important practical advantage that the KAM transformation leaves invariant the subspace of Hamiltonians quadratic in the actions [Eq. (1.1) or Eq. (1.2)]. The expressions of g' and f' up to the order $\mathcal{O}(\varepsilon^2)$ are

$$g'(\boldsymbol{\varphi}') = g(\boldsymbol{\varphi}') + \boldsymbol{\omega}_0\cdot\boldsymbol{\partial}Y + m(\boldsymbol{\varphi}')(\boldsymbol{\Omega}\cdot\boldsymbol{\partial}Z + a\Omega^2) + \mathcal{O}(\varepsilon^2), \tag{3.7}$$

$$f'(\boldsymbol{\varphi}') = f(\boldsymbol{\varphi}') + \boldsymbol{\omega}_0\cdot\boldsymbol{\partial}Z + a\boldsymbol{\omega}_0\cdot\boldsymbol{\Omega} + \mathcal{O}(\varepsilon^2). \tag{3.8}$$

Thus in order to eliminate the terms of order $\mathcal{O}(\varepsilon)$, we determine:

1) $Z(\boldsymbol{\varphi})$ such that the term independent of the action variables is of the order $\mathcal{O}(\varepsilon^2)$. The function Z must satisfy the equation

$$f + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z = const, \tag{3.9}$$

which has the solution

$$Z(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} f_\nu e^{i\nu\cdot\varphi}. \tag{3.10}$$

The mean value of Z is not determined by Eq. (3.9). We choose it equal to zero.

2) Next, we determine a and $Y(\boldsymbol{\varphi})$ such that the linear term in the action variables becomes of the form $[\boldsymbol{\omega}_0 + \mathcal{O}(\varepsilon^2)\boldsymbol{\Omega}] \cdot \boldsymbol{A}'$. This leads to the condition

$$g + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + ma\Omega^2 = 0, \tag{3.11}$$

which has the solution

$$a = -\frac{\langle g \rangle + \langle m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z \rangle}{\Omega^2 \langle m \rangle}, \tag{3.12}$$

and

$$Y(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} \left(g_\nu + (m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z)_\nu + m_\nu a \Omega^2 \right) e^{i\nu\cdot\varphi} \ . \tag{3.13}$$

The transformed Hamiltonian (3.6) is constructed by defining $H^{(0)} = H$ and $H^{(i)}$ for $i = 1, 2, \ldots$ by the recursive relation

$$\begin{aligned} H^{(i+1)}(\boldsymbol{A}, \boldsymbol{\varphi}) &= \{ S(\boldsymbol{A}, \boldsymbol{\varphi}), H^{(i)}(\boldsymbol{A}, \boldsymbol{\varphi}) \} \\ &= \frac{1}{2} m^{(i+1)}(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + g^{(i+1)}(\boldsymbol{\varphi})\boldsymbol{\Omega} \cdot \boldsymbol{A} + f^{(i+1)}(\boldsymbol{\varphi}), \end{aligned} \tag{3.14}$$

which leads to

$$H' = \sum_{i=0}^{\infty} \frac{H^{(i)}}{i!}. \tag{3.15}$$

This can be expressed in terms of the image of the three scalar functions (m, g, f) given by the following equations:

$$(m', g', f') = \left(\sum_{i=0}^{\infty} \frac{m^{(i)}}{i!}, \sum_{i=0}^{\infty} \frac{g^{(i)}}{i!}, \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!} \right), \tag{3.16}$$

$$(m^{(0)}, g^{(0)}, f^{(0)}) = (m, g, f), \tag{3.17}$$

$$m^{(1)} = 2m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} m, \tag{3.18}$$

$$g^{(1)} = g\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} g + m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + ma\Omega^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y, \tag{3.19}$$

$$f^{(1)} = -Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} f + g\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + ga\Omega^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z, \tag{3.20}$$

$$m^{(i+1)} = 2m^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} m^{(i)}, \tag{3.21}$$

$$g^{(i+1)} = g^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y - Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} g^{(i)} + m^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + m^{(i)}a\Omega^2, \tag{3.22}$$

$$f^{(i+1)} = -Y\boldsymbol{\Omega} \cdot \boldsymbol{\partial} f^{(i)} + g^{(i)}\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z + g^{(i)}a\Omega^2, \tag{3.23}$$

for $i \geq 1$.

IV. GENERATING FUNCTION APPROACH TO THE KAM TRANSFORMATION

In this section we describe another construction of the KAM transformation. The transformation is now taken as a canonical transformation $\mathcal{U}_S : (\boldsymbol{\varphi}, \boldsymbol{A}) \mapsto (\boldsymbol{\varphi}', \boldsymbol{A}')$ defined by a generating function [25] characterized by two scalar functions Y, Z, of the angles, and a constant a, of the form

$$F(\boldsymbol{A}', \boldsymbol{\varphi}) = (\boldsymbol{A}' + a\boldsymbol{\Omega}) \cdot \boldsymbol{\varphi} + Y(\boldsymbol{\varphi})\boldsymbol{\Omega} \cdot \boldsymbol{A}' + Z(\boldsymbol{\varphi}), \tag{4.1}$$

leading to

$$\boldsymbol{A} = \frac{\partial F}{\partial \boldsymbol{\varphi}} = \boldsymbol{A}' + \boldsymbol{\Omega} \cdot \boldsymbol{A}' \boldsymbol{\partial} Y + a\boldsymbol{\Omega} + \boldsymbol{\partial} Z, \tag{4.2}$$

$$\boldsymbol{\varphi}' = \frac{\partial F}{\partial \boldsymbol{A}'} = \boldsymbol{\varphi} + Y(\boldsymbol{\varphi})\boldsymbol{\Omega}. \tag{4.3}$$

Inserting Eq. (4.2) into Eq. (1.1), we obtain the expression of the Hamiltonian in the mixed representation of new action variables and old angle variables

$$\tilde{H}(\boldsymbol{A}', \boldsymbol{\varphi}) = \frac{1}{2}\tilde{m}(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A}')^2 + [\boldsymbol{\omega}_0 + \tilde{g}(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A}' + \tilde{f}(\boldsymbol{\varphi}) \ , \tag{4.4}$$

with

$$\tilde{m} = (1 + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y)^2 m, \tag{4.5}$$

$$\tilde{g} = g + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + mb + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y \,(g + mb)\,, \tag{4.6}$$

$$\tilde{f} = f + \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z + \frac{1}{2} m b^2 + gb, \tag{4.7}$$

where $b(\boldsymbol{\varphi}) = a\Omega^2 + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z$. We notice that the KAM transformation does not change $\boldsymbol{\Omega}$. We determine the generating function (4.1) such that $\mathcal{H} \circ \mathcal{U}_F$ vanishes to the first order in ε. This leads to the conditions

$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Z + f = const, \tag{4.8}$$

$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y + g + m \left(a\Omega^2 + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z\right) = 0. \tag{4.9}$$

We recall that the functions g and f are of order $\mathcal{O}(\varepsilon)$ and m is of order one; as a consequence Y, Z and a are of order $\mathcal{O}(\varepsilon)$. We notice that these equations are the same as Eqs. (3.9) and (3.11) which determine the generator of the Lie transformation. The present transformation and the Lie transformation described in Sec. III are canonical transformations with identical linear part [i.e. $\mathcal{O}(\varepsilon)$ part] but different nonlinear terms (of the higher order in ε). The main practical difference with the Lie transformation is that Eq. (4.3) which determines the new angles has to be inverted. Equations (4.8) and (4.9) are solved by representing them in Fourier space. They define the generating function F as

$$Z(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} f_\nu e^{i\nu\cdot\varphi}, \tag{4.10}$$

$$a = -\frac{\langle g \rangle + \langle m \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z \rangle}{\Omega^2 \langle m \rangle}, \tag{4.11}$$

$$Y(\boldsymbol{\varphi}) = \sum_{\nu \neq 0} \frac{i}{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}} \left(g_\nu + (m\boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z)_\nu + m_\nu a \Omega^2\right) e^{i\nu\cdot\varphi}. \tag{4.12}$$

Thus the scalar functions of $\tilde{H}$ become

$$\tilde{m} = (1 + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y)^2 m, \tag{4.13}$$

$$\tilde{g} = -\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Y, \tag{4.14}$$

$$\tilde{f} = \frac{1}{2}(g - \boldsymbol{\omega}_0 \cdot \boldsymbol{\partial} Y)\left(a\Omega^2 + \boldsymbol{\Omega} \cdot \boldsymbol{\partial} Z\right). \tag{4.15}$$

We notice that $\tilde{m}$, $\tilde{g}$ and $\tilde{f}$ are given by products and sums of functions whose Fourier coefficients are explicitely known. We expand these functions in Fourier series e.g. $\tilde{m}(\boldsymbol{\varphi}) = \sum_\nu \tilde{m}_\nu e^{i\nu\cdot\varphi}$.

The expression of the Hamiltonian in the new angle variables requires the inversion of Eq. (4.3). The Jacobian of this transformation is

$$\left|\det\left(\frac{\partial\varphi'_{\mathrm{k}}}{\partial\varphi_{\mathrm{l}}}\right)\right| = |1 + \boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y|\,. \tag{4.16}$$

The scalar functions m', g' and f' are respectively $\tilde{m}$, $\tilde{g}$ and $\tilde{f}$ expressed in the new angle variables e.g. $m'(\boldsymbol{\varphi}') = \tilde{m}[\boldsymbol{\varphi}(\boldsymbol{\varphi}')]$. The Fourier coefficients of m' are determined by the following integrals

$$m'_\nu = \int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}'}{(2\pi)^2} m'(\boldsymbol{\varphi}')\, e^{-i\nu\cdot\varphi'}. \tag{4.17}$$

With the change of variables $\boldsymbol{\varphi}' \mapsto \boldsymbol{\varphi}$, we can write

$$m'_\nu = \int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}}{(2\pi)^2} \left|\det\left(\frac{\partial\varphi'_{\mathrm{k}}}{\partial\varphi_{\mathrm{l}}}\right)\right| \tilde{m}(\boldsymbol{\varphi})\, e^{-i\nu\cdot(\varphi + Y(\varphi)\Omega)}. \tag{4.18}$$

Therefore m'_ν, g'_ν and f'_ν can be expressed in the following way:

$$m'_\nu = \sum_{\nu'} \tilde{m}_{\nu'} C_{\nu'\nu}, \quad g'_\nu = \sum_{\nu'} \tilde{g}_{\nu'} C_{\nu'\nu}, \quad f'_\nu = \sum_{\nu'} \tilde{f}_{\nu'} C_{\nu'\nu}, \tag{4.19}$$

where

$$C_{\nu'\nu} = \int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}}{(2\pi)^2} |1 + \boldsymbol{\Omega}\cdot\boldsymbol{\partial}Y| e^{i(\nu'-\nu)\cdot\varphi} e^{-i\nu\cdot\Omega Y(\varphi)}. \tag{4.20}$$

In order to compute $C_{\nu'\nu}$, we choose the Gauss quadrature that approximates an integral as a sum over a lattice of constant step:

$$\int_{\mathbb{T}^2} \frac{d^2\boldsymbol{\varphi}}{(2\pi)^2}\, G(\boldsymbol{\varphi}) = \lim_{M\to\infty} \frac{1}{(M+1)^2} \sum_{i,j=0,\ldots,M} G\left(\frac{2\pi}{M+1}i, \frac{2\pi}{M+1}j\right). \tag{4.21}$$

This quadrature is exact if all the non-zero Fourier modes of the considered function G are inside $\mathcal{C}_M$. For instance in the case of the computation of $C_{\nu'\nu}$ for the identity transformation ($Y = 0$), since $\boldsymbol{\nu}' - \boldsymbol{\nu} \in \mathcal{C}_{2L}$, one needs to take $M \geq 2L$. For the general case, if we expand the function to be integrated as a power series of ε, we notice that the algorithm with $M=(k+1)L$ gives an approximation up to $\mathcal{O}(\varepsilon^k)$. For instance this algorithm with $k=5$ gives an accurate approximation of the exact transformation and allows to compute all its properties.

V. DETERMINATION OF THE CRITICAL COUPLING; FIXED POINT OF THE KAM-RG TRANSFORMATION

We start with the same initial Hamiltonian as in Refs. [18,26]

$$H(\boldsymbol{A},\boldsymbol{\varphi})=\frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2+\boldsymbol{\omega}_0\cdot\boldsymbol{A}+\varepsilon f(\boldsymbol{\varphi})\ , \tag{5.1}$$

where $\boldsymbol{\Omega}=(1,0)$, $\boldsymbol{\omega}_0=(1/\gamma,-1)$, $\gamma=(1+\sqrt{5})/2$ and a perturbation

$$f(\boldsymbol{\varphi})=\cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi})+\cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi}), \tag{5.2}$$

where $\boldsymbol{\nu}_1=(1,0)$ and $\boldsymbol{\nu}_2=(1,1)$. We perform an iteration of the KAM transformation described in Sec. III or in Sec. IV. The two methods give qualitatively the same results. The algorithm using the Lie transformation is numerically more efficient. The following results are those obtained by the Lie transformation. We represent all the functions by their Fourier series truncated by retaining only the coefficients in the square $\mathcal{C}_L$ which contains $(2L+1)^2$ Fourier coefficients. For fixed L we take successively larger couplings ε and determine whether the KAM iteration converges to a Hamiltonian with $f=0$, $g=0$, or whether it diverges ($f,g\to\infty$). By a bisection procedure, we determine the critical coupling $\varepsilon_c(L)$. We repeat the calculation with larger numbers of Fourier coefficients, to obtain a more accurate approximation. In Fig. 1, we show $\varepsilon_c(L)$ i.e. the dependence of the critical coupling on the number of Fourier coefficients retained. We observe that $\varepsilon_c(L)$ decreases with L in a stepwise manner. It stays essentially constant except at the values of L where a new rational approximant of the frequency $\boldsymbol{\omega}_0$ is included, corresponding to a resonance at the next smaller scale. The size of the jumps diminishes approximately geometrically, and we can extrapolate to obtain the value $\varepsilon_c(L) \to 0.0276$. This value is close to the critical coupling $\varepsilon_c = 0.0275856$ obtained by the Greene criterion [5,26] which is surmised to yield the exact value. This gives a numerical evidence that the KAM iteration can be expected to converge in the whole domain of existence of the torus.

In Figs. 1 and 2, we show the values of the critical coupling $\varepsilon_c(L)$, calculated by the KAM-RG transformation which is a combination of a Lie transformation (Sec. III) and a renormalization transformation (Sec. II). We obtain $\varepsilon_c \in [0.027585, 0.027595]$, which is in very good agreement with the value $\varepsilon_c = 0.0275856$ obtained with the Greene criterion. We observe that the KAM-RG transformation gives very high precision already with few Fourier coefficients e.g. $\varepsilon_c(L=5) = 0.0276633$.

The improvement with respect to the KAM iteration is not only quantitative; the disappearence of the steps is a strong evidence that the KAM-RG transformation we have constructed captures the essential physical mechanism of the break-up of the tori.

By iterating the KAM-RG transformation starting from a point on the critical surface, we observe that the process converges to a nontrivial critical point H_*, which we characterize by the Fourier coefficients of the three functions f_*, g_*, m_* and $\mathbf{\Omega}_* = (1, \gamma^{-1})$. Figure 3 shows the weight of the Fourier coefficients of f_*. We observe that the nonzero coefficients are strongly concentrated on a band around the direction $D_\perp$ perpendicular to the line of resonances, i.e. to the frequency vector $\boldsymbol{\omega}_0$. $D_\perp$ is the expansive direction of the map $\boldsymbol{\nu} \mapsto N^2\boldsymbol{\nu}$. The decrease of the size of the coefficients along $D_\perp$ is quite slow. The Fourier coefficients of g_* and m_* have a similar overall behavior, but they decay faster in the $D_\perp$ direction (see Figs. 4 and 5). By linearizing the KAM-RG transformation around the fixed point H_*, we calculate the critical exponents. There is only one with modulus greater than one, denoted by δ. This implies that the critical surface, which is the stable manifold of H_*, is of codimension one. The value we obtain for the relevant critical exponent is $\delta \in [2.67, 2.68]$ which is quite close to the one obtained by MacKay for area-preserving maps [15] ($\delta = 2.65$), and to the one obtained by Escande *et al.* with the approximate scheme ($\delta = 2.75$) [27].

For the scaling factor at the nontrivial fixed point, we obtain numerically $\lambda_* = 17.944$, which is very close to γ^6 (the scaling factor at the trivial fixed point). This value can be compared with the one given for area-preserving maps $\lambda_* = 18.827$ obtained in Refs. [12,13,15].

We remark that if we consider the more general starting Hamiltonian of the form (1.2) $H = \boldsymbol{A}\cdot M\boldsymbol{A}/2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A} + f$ with the twist condition $\det M \neq 0$, the KAM-RG iteration has the same nontrivial fixed point as we have found for twistless Hamiltonians (1.1).

We have also performed this analysis using the KAM transformation constructed with a generating function as described in Sec. IV. The results (using fewer Fourier coefficients than the Lie transformation) are qualitatively similar. The nontrivial fixed point is quantitatively different, but the critical exponents are the same. This is what is usually expected in a renormalization group approach.

VI. SYMMETRIES OF THE TRANSFORMATION: GENERALIZATION TO NON-EVEN PERTURBATIONS

In the previous section, we found numerically the existence of a nontrivial fixed point for the KAM-RG transformation starting with an even initial perturbation on the critical surface

$$f_0(\boldsymbol{\varphi}) = -M\cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi}) - P\cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi}), \tag{6.1}$$

where $M, P \geq 0$ and $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ correspond to the two main resonances. For a perturbation containing also odd terms $\sin(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi})$ and $\sin(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi})$, the KAM-RG transformation, acting on the critical surface, does not necessarily converge to a fixed point but to a periodic or even a nonperiodic orbit (we shall see that these attracting orbits are those of the Arnold's cat map). Already if we start with the even perturbation (6.1) with $M, P \leq 0$, we obtain a cycle of period three as it has also been encountered in area-preserving maps [28]. This can be understood by the symmetries of the transformation [29]. Starting with the two main resonances $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$, the most general perturbation can be written as

$$f_0(\boldsymbol{\varphi}) = -M_e\cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi}) - P_e\cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi})$$

$$
\begin{aligned}
&-M_o \sin(\boldsymbol{\nu}_1 \cdot \boldsymbol{\varphi}) - P_o \sin(\boldsymbol{\nu}_2 \cdot \boldsymbol{\varphi}), \\
&= -M \cos(\boldsymbol{\nu}_1 \cdot (\boldsymbol{\varphi} + \boldsymbol{\theta})) - P \cos(\boldsymbol{\nu}_2 \cdot (\boldsymbol{\varphi} + \boldsymbol{\theta})),
\end{aligned} \tag{6.2}
$$

where $M = \sqrt{M_e^2 + M_o^2}$, $P = \sqrt{P_e^2 + P_o^2}$,
and $\boldsymbol{\theta} = (-\arctan(M_o/M_e), \arctan(M_o/M_e) - \arctan(P_o/P_e))$. The question is to analyse the effect of a shift of the angles on the transformation. We define a shift operator by

$$\mathcal{T}_\theta : \boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} + \boldsymbol{\theta}. \tag{6.3}$$

The KAM transformation commutes with $\mathcal{T}_\theta$, as can be easily verified from Eq. (3.5) or Eqs. (4.2)-(4.7). The action of $\mathcal{T}_\theta$ on the shift of the resonances N^2 is described by the following intertwining relation:

$$\mathcal{R} \circ \mathcal{T}_\theta = \mathcal{T}_{N^2\theta} \circ \mathcal{R}, \tag{6.4}$$

where $\mathcal{R}$ denotes the KAM-RG transformation. Applying this relation to the fixed point $H_*(\boldsymbol{\varphi})$, we have $\mathcal{R}H_*(\boldsymbol{\varphi} + \boldsymbol{\theta}) = H_*(\boldsymbol{\varphi} + N^2\boldsymbol{\theta})$. The following map (which is Arnold's cat map [30])

$$\boldsymbol{\theta} \mapsto N^2\boldsymbol{\theta} \bmod 2\pi, \tag{6.5}$$

gives the nature of the orbit to which the transformation converges starting with the initial perturbation (6.2) on the critical surface. For instance, if we start with $\boldsymbol{\theta} = \pi\boldsymbol{\nu}_1$ which corresponds to the perturbation (6.1) with $M, P \leq 0$, the transformation converges to a cycle of period three because $N^6\boldsymbol{\theta} = \boldsymbol{\theta} \bmod 2\pi$. More precisely this 3-cycle is the periodic sequence

$$\{H_*(\boldsymbol{\varphi} + \pi\boldsymbol{\nu}_1), H_*(\boldsymbol{\varphi} + \pi(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2)), H_*(\boldsymbol{\varphi} + \pi\boldsymbol{\nu}_2)\}. \tag{6.6}$$

For any orbit of the map (6.5), there is a fixed set that plays the same role as the nontrivial fixed point for the even perturbation (6.1). These invariant sets belong to the same universality class as the fixed point, and in particular, they have the same critical exponents.

We can define a modified renormalization transformation such that the KAM-RG transformation converges to the fixed point H_* for all initial perturbations of the form (6.2) on the critical surface, by adding an initial shift of the angles $\mathcal{T}_{-\theta}$.

VII. CONCLUSION

The results show that the KAM-RG transformation is a powerful tool to describe the break-up of invariant tori. In particular, the transformation describes with high accuracy the critical surface which is the stable manifold of a nontrivial fixed point (or more generally, of a nontrivial fixed set related to this nontrivial fixed point by symmetries).
We have implemented the KAM-RG transformation for the torus with frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$. The extension to other frequencies that are quadratic irrationals is relatively clear. The case of a general irrational frequency will involve qualitatively new features. The KAM-RG transformation we described was for systems with two degrees of freedom, but the extension to three [31] or higher dimensional systems [19] should be accessible.

ACKNOWLEDGMENTS

We would like to thank G. Gallavotti who initiated this approach in a manuscript with G. Benfatto (1987, unpublished). Our discussions with them were very helpful for this project. We would like also to thank A. Celletti for providing the program to determine the critical coupling by Greene's criterion. We acknowledge useful discussions with R. S. MacKay and H. Koch. Support from the EC contract ERBCHRXCT94-0460 for the project "Stability and universality in classical mechanics" and from the Conseil Régional de Bourgogne is acknowledged.

FIGURES

FIG. 1. Critical coupling $\varepsilon_c(L)$ as a function of L, the size of the cell $\mathcal{C}_L$ containing $(2L+1)^2$ Fourier coefficients. The upper curve corresponds to the KAM transformation, and the lower one to the KAM-RG transformation.

FIG. 2. Critical coupling $\varepsilon_c(L)$ of the KAM-RG transformation as a function of L, the size of the cell $\mathcal{C}_L$ containing $(2L+1)^2$ Fourier coefficients.

FIG. 3. Weight of the Fourier coefficients of f_* indicated by grey-levels: a) white: $< 10^{-10}$, b) $[10^{-10}, 10^{-7}]$, c) $[10^{-7}, 10^{-5}]$, d) $[10^{-5}, 10^{-3}]$, e) black: $[10^{-3}, 10^{-2}]$.

FIG. 4. Weight of the Fourier coefficients of g_* indicated by grey-levels: a) white: $< 10^{-10}$, b) $[10^{-10}, 10^{-7}]$, c) $[10^{-7}, 10^{-5}]$, d) $[10^{-5}, 10^{-3}]$, e) black: $[10^{-3}, 10^{-2}]$.

FIG. 5. Weight of the Fourier coefficients of m_* indicated by grey-levels: a) white: $< 10^{-10}$, b) $[10^{-10}, 10^{-7}]$, c) $[10^{-7}, 10^{-5}]$, d) $[10^{-5}, 10^{-3}]$, e) black: $[10^{-3}, 1]$.

FIG. 1

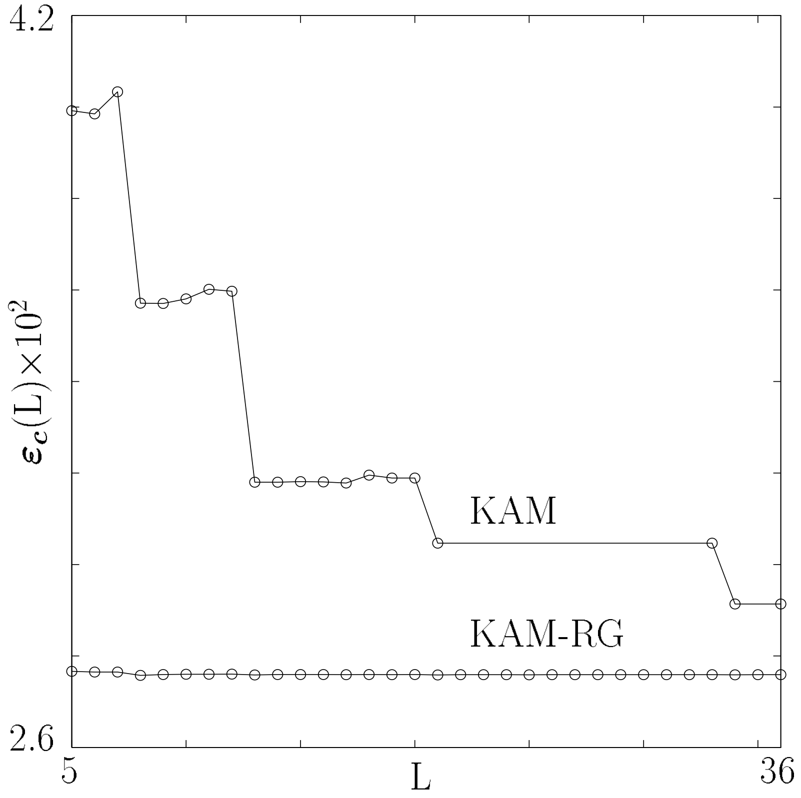


FIG. 2

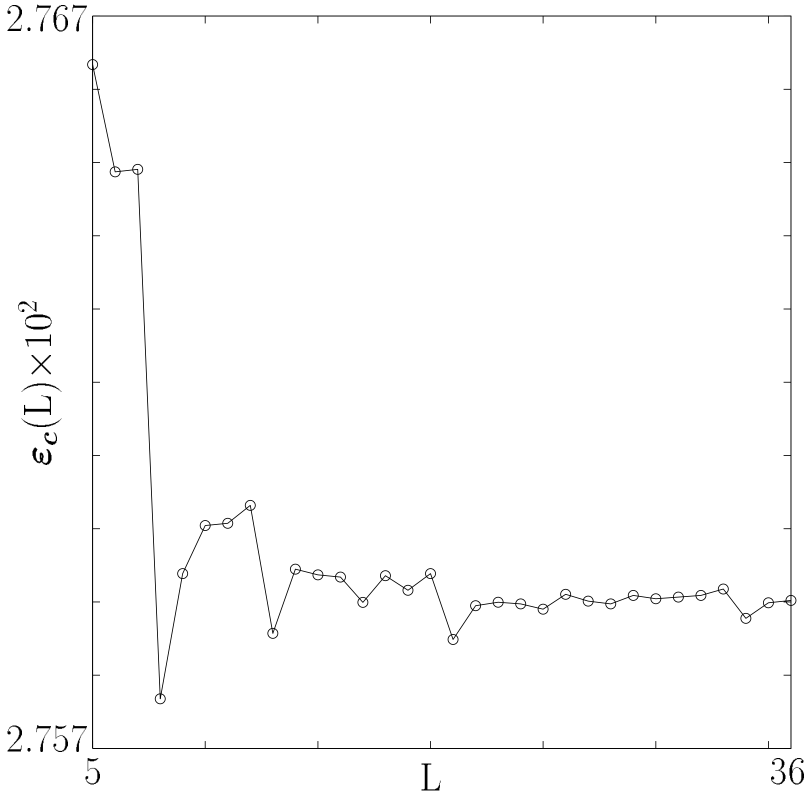


FIG. 3

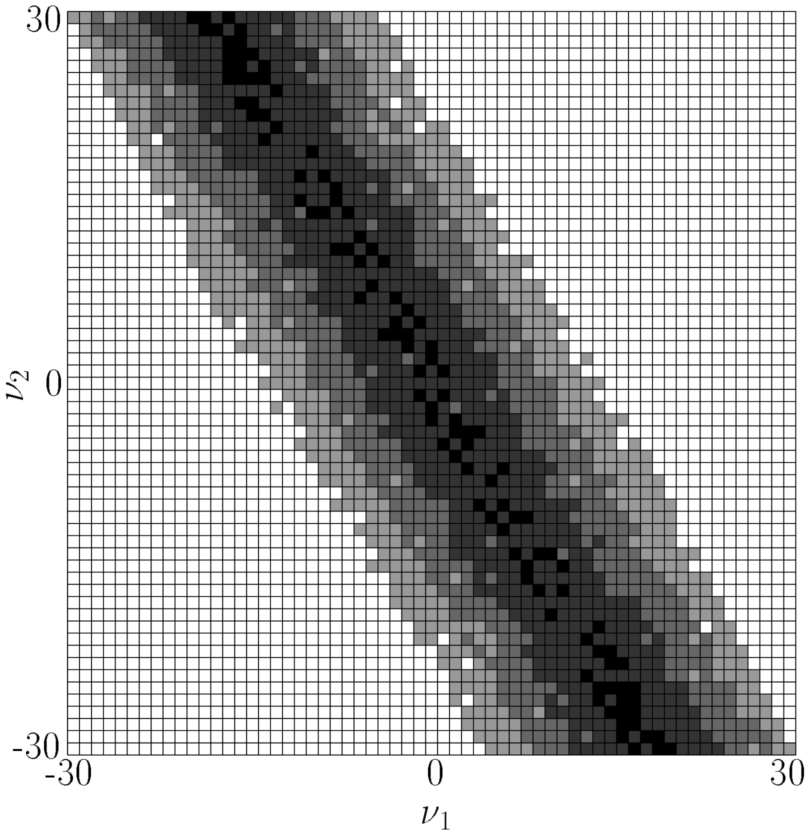


FIG. 4

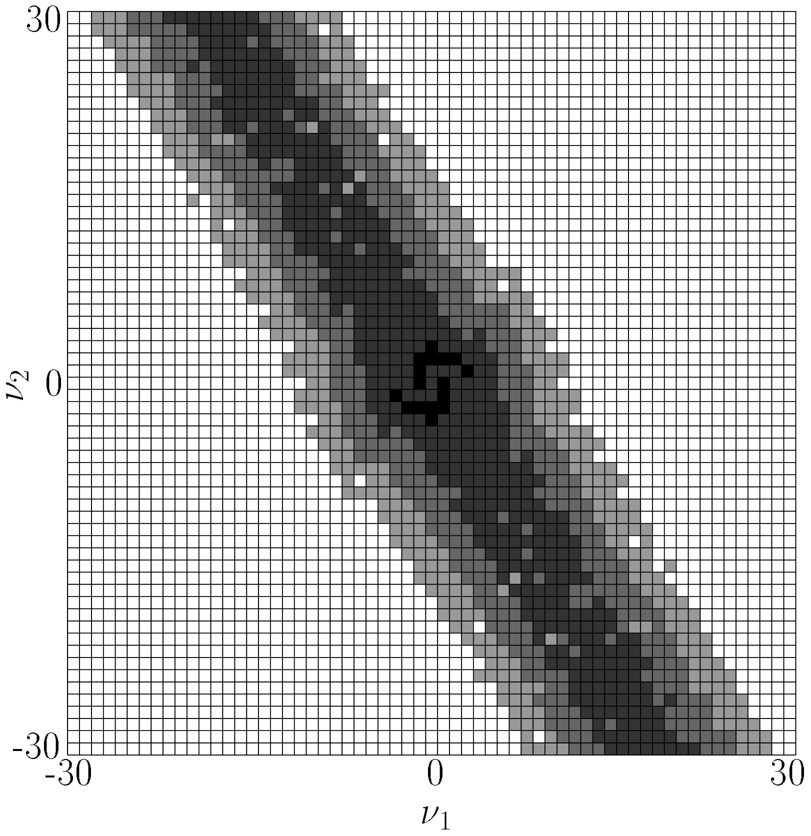
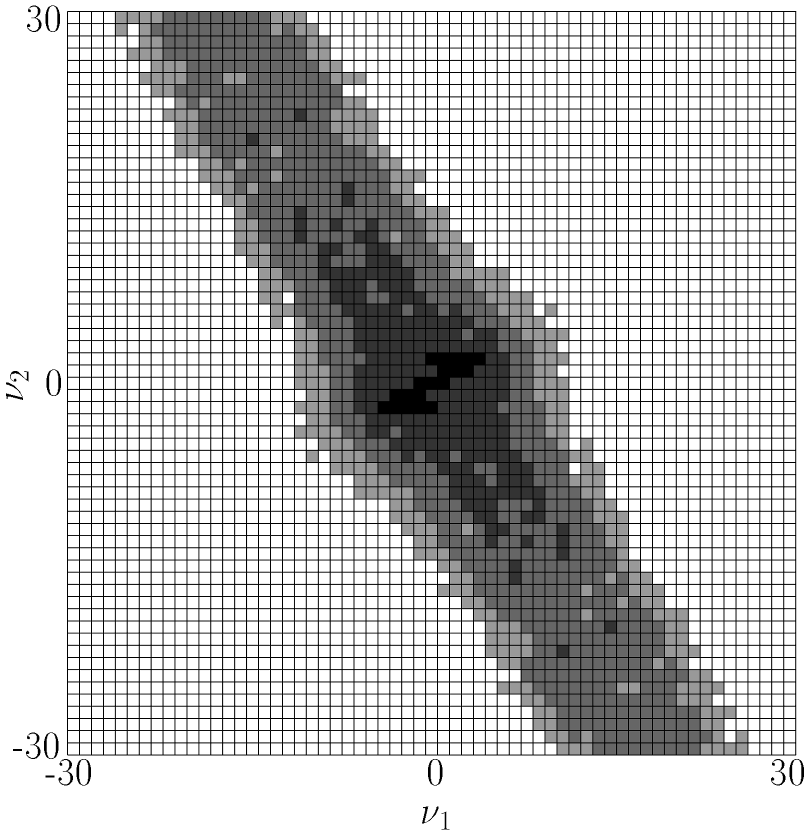


FIG. 5



REFERENCES

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **98**, 527 (1954).

[2] V. I. Arnold, Russ. Math. Surv. **18**, 9 (1963).

[3] J. Moser, Nach. Akad. Wiss. Göttinger, Math. Phys. Kl.,1 (1962).

[4] R. S. MacKay and I. C. Percival, Commun. Math. Phys. **98**, 469 (1985).

[5] J. M. Greene, J. Math. Phys. **20**, 1183 (1979).

[6] C. Falcolini and R. de la Llave, J. Stat. Phys. **67**, 609 (1992).

[7] R. S. MacKay, Nonlinearity **5**, 161 (1992).

[8] S. Aubry and P. Y. Le Daeron, Physica D **8**, 381 (1983).

[9] J. N. Mather, Topology **21**, 457 (1982).

[10] J. Moser, SIAM Review **28**, 459 (1986).

[11] J. Denzler, J. Appl. Math. and Phys. **38**, 791 (1987).

[12] L. P. Kadanoff, Phys. Rev. Lett. **47**, 1641 (1981).

[13] S. J. Shenker and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).

[14] D. Rand, S. Ostlund, J. Sethna, and E. D. Siggia, Phys. Rev. Lett. **49**, 132 (1982).

[15] R. S. MacKay, Physica D **7**, 283 (1983).

[16] R. S. MacKay, *Renormalisation in Area-preserving Maps* (World Scien-

tific, 1993).

[17] D. F. Escande and F. Doveil, J. Stat. Phys. **26**, 257 (1981).

[18] D. F. Escande, Phys. Rep. **121**, 165 (1985).

[19] H. Koch, archived in mp_arc@math.utexas.edu, #96-383 (1996).

[20] M. Govin, C. Chandre, and H. R. Jauslin, preprint archived in mp_arc@math.utexas.edu, #97-309 (1997).

[21] W. Thirring, *A Course in Mathematical Physics I: Classical Dynamical Systems* (Springer Verlag, Berlin, 1992), p. 153.

[22] G. Gallavotti, Commun. Math. Phys. **164**, 145 (1994).

[23] A. Deprit, Celest. Mech. **1**, 12 (1969).

[24] G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, Nuovo Cimento **79**B, 201 (1984).

[25] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1980).

[26] A. Celletti and L. Chierchia, Commun. Math. Phys. **118**, 119 (1988).

[27] D. F. Escande, M. S. Mohamed-Benkadda, and F. Doveil, Phys. Lett. A **101**, 309 (1984).

[28] J. M. Greene and J. Mao, Nonlinearity **3**, 69 (1990).

[29] R. S. MacKay, in *Proceedings of the International Conference on Dy-*

namical Systems and Chaos, Tokyo, edited by Y. Aizawa, S. Saito and K. Shiraiwa (World Scientific, 1995), p. 34.

[30] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics* (Benjamin, New-York, 1968).

[31] R. S. MacKay, J. D. Meiss, and J. Stark, Phys. Lett. A **190**, 417 (1994).