THE KAC VERSION OF THE SHERRINGTON-KIRKPATRICK MODEL AT HIGH TEMPERATURES[#]

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Abstract: We study the Kac version of the Sherrington-Kirkpatrick (SK) model of a spin glass, i.e. a spin glass with long but finite range interaction on $\mathbb{Z}^d$ and gaussian mean zero couplings. We prove that for all $\beta < 1$, the free energy of this model converges to that of the SK model as the range of the interaction tends to infinity. Moreover, we prove that for all temperatures for which the infinite volume Gibbs state is unique, the free energy scaled by the square root of the volume converges to a gaussian with variance $c_{\gamma,\beta}$, where γ^{-1} is the range of the interaction. Moreover, at least for almost all values of β, this variance tends to zero as γ goes to zero, the value in the SK model. We interpret our finding as a weak indication that at least at high temperatures, the SK model can be seen as a reasonable asymptotic model for lattice spin glasses.

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1. Introduction

One of the most disputed issues in the theory of spin-glasses is the question as to what extent the results obtained for the mean-field Sherrington-Kirkpatrick (SK) model [SK] are relevant for finite dimensional short-range spin glasses (for a recent review and interesting discussions on this issue we refer to [NS1,NS2]). From the point of view of the mathematical physicist, this question is not made easier by the fact that the most interesting results on the SK model (see [MPV]) are in themselves not mathematically rigorous. On the other hand, there has been considerable progress in understanding at least the high temperature features of the SK model [ALR,FZ], and very recently some very nice probabilistic tools have been employed that make the analysis of this phase rather easy and appealing [CN,T1,T2]. One could thus ask the rather modest question to what extent these high temperature mean-field results are related to the corresponding lattice models. In standard mean field theory, the clearest interpretation of mean field results as asymptotic results for a family of lattice models if given in terms of Kac models [KUH]. The Kac version of the SK-model has been considered already in [FZ], but was not further considered in the more recent developments. The purpose of this note is to do this, and to give some quite weak results that show that in the high-temperature phase at least the SK model can be seen to some extent as a limit of a family of Kac spin glasses.

Let us define the Kac-SK model. Let $\sigma_i \in \{-1, 1\}$, $i \in \mathbb{Z}^d$ be Ising spins. Let G_{ij}, $i, j \in \mathbb{Z}^d$ be a family of i.i.d. random variables with mean zero and variance 1. To avoid complications, in this paper we only consider the case where G_{ij} are gaussian, but more general distributions could be considered. Let $J_\gamma(i)$ be a Kac-kernel, i.e. as positive function $\mathbb{Z}^d \to \mathbb{R}^+$ such that $J_\gamma(i) = \gamma^d J(\gamma i)$ where $\int d^d x J(x) = 1$ and J has compact support. Then the Hamiltonian of our model is defined, for any $\Lambda \subset \mathbb{Z}^d$ by

$$H_{\Lambda,\gamma}(\sigma)[\omega] = -\frac{1}{\sqrt{2}} \sum_{i,j\in\Lambda} \sqrt{J_\gamma(i-j)} G_{ij}\sigma_i\sigma_j \tag{1.1}$$

We define the partition function as

$$Z_{\Lambda,\gamma,\beta}[\omega] \equiv \mathbb{E}_\sigma \exp\left(-\beta H_{\Lambda,\gamma}(\sigma)[\omega]\right) \tag{1.2}$$

and the free energy as

$$f_{\Lambda,\gamma,\beta}[\omega] \equiv -\frac{1}{\beta|\Lambda|} \ln Z_{\Lambda,\gamma,\beta}[\omega] \tag{1.3}$$

Throughout the paper we will assume *periodic* boundary conditions for convenience , although this is not essential. We introduce the *quenched* and *annealed* free energies as $f^q_{\Lambda,\gamma,\beta} \equiv \mathbb{E} f_{\Lambda,\gamma,\beta}[\omega]$, respectively $f^a_{\Lambda,\gamma,\beta} \equiv -\frac{1}{\beta|\Lambda|} \ln \mathbb{E} Z_{\Lambda,\gamma,\beta}[\omega]$. As is well known, it follows from the subadditive ergodic theorem that for all $\gamma > 0$, for almost all ω,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} f_{\Lambda,\gamma,\beta}[\omega] = \lim_{\Lambda\uparrow\mathbb{Z}^d} f^q_{\Lambda,\gamma,\beta} = f_{\gamma,\beta} \tag{1.4}$$

Then our first result can be formulated as follows:

Theorem 1: *For all $\beta < 1$*

$$\lim_{\gamma\downarrow 0} f_{\gamma,\beta} = -\frac{\beta}{4}, \quad a.s. \tag{1.5}$$

Remark: From the work of [ALR] we know that the free energy of the SK model equals to $-\frac{\beta}{4}$ for $\beta \leq 1$. Theorem 1 thus says that in the high temperature regime, the free energy of the Kac-SK model converges to that of the SK model in the Lebowitz-Penrose limit. Theorem 1 extends a result of Fröhlich and Zegarlinsky [FZ] proven for β small enough to the full high-temperature region.

Our second result concerns the fluctuations of the free energy.

Theorem 2: *Let $\beta < 1$. Assume moreover that β is such that for all γ sufficiently small, the Kac-SK model has a weakly unique infinite volume Gibbs state for all $\beta' \leq \beta$. Then*

(i) If γ is small enough, there exists a constant $c_{\gamma,\beta}$ such that

$$\sqrt{|\Lambda|}\left(f_{\Lambda,\gamma,\beta} - f^q_{\Lambda,\gamma,\beta}\right) \xrightarrow{\mathcal{D}} c_{\gamma,\beta} g, \quad as\ \Lambda \uparrow \mathbb{Z}^d \tag{1.6}$$

where g is a standard gaussian random variable and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution (or "in law").

(ii) For Lebesgue almost all such β

$$\lim_{\gamma\downarrow 0} c_{\gamma,\beta} = 0 \tag{1.7}$$

Remark: We expect of course that (1.6) and (1.7) holds for all $\beta < 1$.

Remark: This result must be contrasted to the corresponding result in the SK model, where $|\Lambda|\left(f_{\Lambda,\gamma,\beta}[\omega] - f^q_{\Lambda,\gamma,\beta}\right)$ converges in distribution to a standard normal r.v.. We see that on the lattice, for all positive γ, the fluctuations of the free energy are on a much larger scale then in the SK model, but at least on this scale they tend to zero with γ. In some sense, the fluctuation result in the SK model with the $|\Lambda|$-scaling should be considered as some overly refined estimate that one happens to be able to compute in the mean field model. On the level of the "normal" fluctuations on the scale $\sqrt{|\Lambda|}$ our theorem states that the properties in the Kac model converge to those of the SK model.

Remark: The hypothesis of uniqueness of the Gibbs state is a weak point in our theorem (strictly speaking, we need less, namely only the convergence of the finite volume state with periodic (or even some other) boundary conditions) to a translation covariant infinite volume Gibbs state (see

[AW] for definition and extensive discussion of this notion). Let us recall from [FZ2] that "weak uniqueness" means that the finite volume states with any fixed ("non-random") boundary condition converge to the same infinite volume measure, almost surely. We can only assert that it holds trivially in the one dimensional model for all temperatures. In arbitrary dimensions, it is easy to adopt the proof of Fröhlich and Zegarlinski [FZ2] of weak uniqueness of the Gibbs state at high temperatures given for potentials of the form $|x|^{-\alpha d}$ to show that there is a finite β_c, independent of γ, up to which uniqueness holds. We would of course expect that $\beta_c(\gamma) \to 1$, as $\gamma \downarrow 0$, but to prove this is beyond the scope of the present note.

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1, using a beautiful idea of Talagrand. In Section 3 we prove Theorem 2, using in part the equally beautiful ideas of Comets and Neveu.

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2. Proof of Theorem 1.

The proof of Theorem 1 follows closely Talagrand's proof in the SK model that appeared in [T1]. It relies on three simple facts.

Lemma 2.1: *For any β and γ,*

$$\mathbb{E} Z_{\Lambda,\gamma,\beta} = e^{|\Lambda|\beta^2/4} \tag{2.1}$$

Proof: Just compute.$\diamondsuit$

Lemma 2.2: *For any β and γ,*

$$\frac{\mathbb{E} Z^2_{\Lambda,\gamma,\beta}}{\left[\mathbb{E} Z_{\Lambda,\gamma,\beta}\right]^2} = \exp\left(-\beta^2|\Lambda|\left(F^{\text{kac}}_{\gamma,\beta^2} + o(1)\right)\right) \tag{2.2}$$

where $F^{\text{kac}}_{\gamma,\beta}$ is the free energy of the Kac-Ising model (for a precise definition, see below), and $o(1) \downarrow 0$, as $\Lambda \uparrow \mathbb{Z}^d$. Moreover, if $\beta \leq 1$, then

$$\lim_{\gamma \downarrow 0} F^{\text{kac}}_{\gamma,\beta^2} = 0 \tag{2.3}$$

Proof: Again a trivial computation shows that

$$
\begin{aligned}
I\!\!E Z^2_{\Lambda,\gamma,\beta} &= I\!\!E_\sigma I\!\!E_{\sigma'} \exp\left(\frac{\beta^2}{4}\sum_{i,j\in\Lambda} J_\gamma(i-j)(\sigma_i\sigma_j+\sigma'_i\sigma'_j)^2\right) \\
&= \left[I\!\!E Z_{\Lambda,\gamma,\beta}\right]^2 I\!\!E_\sigma \exp\left(\frac{\beta^2}{2}\sum_{i,j\in\Lambda} J_\gamma(i-j)\sigma_i\sigma_j\right)
\end{aligned}
\tag{2.4}
$$

But the last factor is nothing but the partition function in the usual ferromagnetic Kac-Ising model at inverse temperature β^2. As is well-known (see e.g. [Tho]), by subadditivity the limit

$$
\lim_{\Lambda\uparrow \mathbb{Z}^d} -\frac{1}{\beta^2|\Lambda|}\ln I\!\!E_\sigma \exp\left(\frac{\beta^2}{2}\sum_{i,j\in\Lambda} J_\gamma(i-j)\sigma_i\sigma_j\right) = F^{\mathrm{kac}}_{\gamma,\beta^2}
\tag{2.5}
$$

exists (if the limit is taken in the sense of van Hove) for all d, all β and all positive γ. (2.3) follows from the Lebowitz-Penrose theorem that asserts that the free energy of the Kac-Ising model converges to that of the Curie-Weiss model together with the fact that the latter, given by $F^{\mathrm{CW}}_\beta = \inf_x\left(x^2/2-\beta^{-1}\ln\cosh(\beta x)\right)$, which for $\beta\leq 1$ is equal to zero. This concludes the proof of Lemma 2. $\diamondsuit$

Lemma 2.3: *For all β and all γ,*

$$
I\!\!P\left[|f_{\Lambda,\gamma,\beta}-I\!\!E f_{\Lambda,\gamma,\beta}|>x\right]\leq 2e^{-x^2|\Lambda|}
\tag{2.6}
$$

Proof: This Lemma is a simple consequence of Gaussian concentration inequalities (see [LT], Section 1.1, Eq. (1.6)) which assert that for any Lipshitz function f of gaussian r.v.'s,

$$
I\!\!P\left[|f-I\!\!E f|>x\right]\leq 2\exp\left(-\frac{x^2}{2\|f\|^2_{\mathrm{Lip}}}\right)
\tag{2.7}
$$

where $\|f\|_{\mathrm{Lip}}$ denotes the Lipshitz-norm of f. Just note that $f_{\Lambda,\gamma,\beta}$ is Lipshitz as a function of the i.i.d. gaussian random variables G_{ij}. Indeed, a simple estimate yields that

$$
|f_{\Lambda,\gamma,\beta}[\omega]-f_{\Lambda,\gamma,\beta}[\omega']|\leq \frac{1}{\sqrt{2|\Lambda|}}\sqrt{\sum_{i,j\in\Lambda} G_{ij}[\omega]-G_{i,j}[\omega']} = \frac{1}{\sqrt{2|\Lambda|}}\|J[\omega]-J[\omega']\|_2
\tag{2.8}
$$

that is $\|f_{\Lambda,\gamma,\beta}\|_{\mathrm{Lip}}\leq\frac{1}{\sqrt{2|\Lambda|}}$. Insertion into (2.7) gives (2.6).$\diamondsuit$

Remark: An elementary proof of (2.6) in the SK-case can be found in [BGP]. An estimate like Lemma 2.3 also holds in for the non-gaussian cases. We refer the interested reader to [T1,T2]. This allows to extend the validity of Theorem 1 beyond the gaussian case.

We now have all the tools ready to apply Talgrand's idea from [T1] to the Kac-SK model. The Paley-Szygmund inequality (see [T1]; the proof of this inequality is elementary) asserts that

$$\mathbb{P}\left[Z_{\Lambda,\gamma,\beta} > \frac{1}{2}\mathbb{E}Z_{\Lambda,\gamma,\beta}\right] \geq \frac{1}{4}\frac{\left[\mathbb{E}Z_{\Lambda,\gamma,\beta}\right]^2}{\mathbb{E}Z^2_{\Lambda,\gamma,\beta}} = \frac{1}{4}\exp\left(-\beta^2|\Lambda|\left[F^{\text{kac}}_{\gamma,\beta^2}+o(1)\right]\right) \tag{2.9}$$

On the other hand,

$$\begin{aligned}
&\mathbb{P}\left[Z_{\Lambda,\gamma,\beta} > \frac{1}{2}\mathbb{E}Z_{\Lambda,\gamma,\beta}\right] = \mathbb{P}\left[\ln Z_{\Lambda,\gamma,\beta} - \mathbb{E}\ln Z_{\Lambda,\gamma,\beta} > \ln \mathbb{E}Z_{\Lambda,\gamma,\beta} - \mathbb{E}\ln Z_{\Lambda,\gamma,\beta} - \ln 2\right] \\
&= \mathbb{P}\left[-f_{\Lambda,\gamma,\beta} + \mathbb{E}f_{\Lambda,\gamma,\beta} > -f^a_{\Lambda,\gamma,\beta} + f^q_{\Lambda,\gamma,\beta} - \tfrac{\ln 2}{\beta|\Lambda|}\right] \\
&\leq \exp\left(-|\Lambda|\left[f^q_{\Lambda,\gamma,\beta} - f^a_{\Lambda,\gamma,\beta} - \tfrac{\ln 2}{\beta|\Lambda|}\right]^2\right)
\end{aligned} \tag{2.10}$$

where we used that by Jenssen's inequality, $\ln \mathbb{E}Z_{\Lambda,\gamma,\beta} \geq \mathbb{E}\ln Z_{\Lambda,\gamma,\beta}$. Comparing (2.9) with (2.10) we find

$$\left[f^q_{\Lambda,\gamma,\beta} - f^a_{\Lambda,\gamma,\beta} - \tfrac{\ln 2}{\beta|\Lambda|}\right]^2 \leq \beta^2\left[F^{\text{kac}}_{\gamma,\beta^2} + o(1)\right] \tag{2.11}$$

and so, in the limit as $\Lambda \uparrow \mathbb{Z}^d$,

$$\left|f^q_{\gamma,\beta} + \frac{\beta}{4}\right| \leq \beta\sqrt{F^{\text{kac}}_{\gamma,\beta^2}} \tag{2.12}$$

Since $F^{\text{kac}}_{\gamma,\beta^2}$ tends to zero with γ if $\beta \leq 1$, the claim of Theorem 1 follows.$\diamondsuit$

Remark: Note that this proof is totally different in spirit than the usual proofs of convergence of Kac free energies to the respective mean field free energies. We do not see how such a proof could work here. For that reason, we have no analogous result at low temperatures (leaving alone the problem that the existence of the free energy in the SK model is not known rigorously at low temperatures).

3. Proof of Theorem 2

Theorem 2 is a more subtle result than Theorem 1, as can be seen by the hypothesis we need. One would expect that this theorem can be proven along the lines of the Comets-Neveu proof in the SK model, however, as we will see there are some notable differences. Moreover, Comets and Neveu make heavy use of stochastic calculus which has the advantage of being very elegant, and the disadvantage of being unfamiliar to many physicists. We have thus chosen to avoid stochastic calculus as far as possible and stick to elementary tools. While this makes our proofs in the eyes of those familiar with stochastic calculus rather clumsy, we hope in this way to convey the main ideas of this approach to a wider audience. Anyway, the translation of our proofs in the more elegant language of stochastic calculus will be trivial for the initiated. Also, we will skip most of the trivial but tedious computational details of the proofs and give only the main ideas.

The advantage of the gaussian case is that the couplings can be represented as a sum

$$G_{ij} = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} g_{ij}(k) \tag{3.1}$$

where the $g_{ij}(k)$ are all independent standard normal random variables, and n can be chosen arbitrarily. Comets and Neveu [CN] use the "infinitesimal version" of this decomposition by representing G_{ij} as a Brownian motion $B_{ij}(t)$ (at $t = 1$) which results in passing to the limit $n \uparrow \infty$. We will also let n tend to infinity, but only at a later stage of the argument. According to this representation we introduce the (decreasing family of) sigma-algebras $\mathcal{F}_k = \sigma(g(k), g(k+1), \ldots, g(n))$ that are generated by all the $g_{ij}(l)$ with $l \geq k$.

Our principle task is to compute the quantity

$$L_{\Lambda,\gamma,\beta}(t) \equiv I\!E \exp\left(t\sqrt{|\Lambda|}[f_{\Lambda,\gamma,\beta} - I\!E f_{\Lambda,\gamma,\beta}]\right) \tag{3.2}$$

To compute $L_{\Lambda,\gamma,\beta}(t)$ we use the following representation of $f_{\Lambda,\gamma,\beta} - I\!E f_{\Lambda,\gamma,\beta}$ as a martingale difference sequence, namely

$$f_{\Lambda,\gamma,\beta} - I\!E f_{\Lambda,\gamma,\beta} = \sum_{k=1}^{n} f_{\Lambda,\gamma,\beta}(k) \tag{3.3}$$

with

$$f_{\Lambda,\gamma,\beta}(k) \equiv I\!E\left[f_{\Lambda,\gamma,\beta}|\mathcal{F}_k\right] - I\!E\left[f_{\Lambda,\gamma,\beta}|\mathcal{F}_{k+1}\right] \tag{3.4}$$

The trick now is to compute first

$$\tilde{L}_{\Lambda,\gamma,\beta,n}(t) \equiv I\!E \exp\left(t\sqrt{|\Lambda|}[f_{\Lambda,\gamma,\beta} - I\!E f_{\Lambda,\gamma,\beta}] - |\Lambda|\frac{t^2}{2}F_n\right) \tag{3.5}$$

where $F_n = F_{n,\Lambda,\gamma,\beta}$ is the conditional variance of the martingale,

$$F_n \equiv \sum_{k=1}^{n} I\!E\left[f^2_{\Lambda,\gamma,\beta}(k)|\mathcal{F}_{k+1}\right] \tag{3.6}$$

Lemma 3.1: *For any $\Lambda, \beta, \gamma, t$, we have that*

$$\lim_{n\uparrow\infty} \tilde{L}_{\Lambda,\gamma,\beta,n}(t) = 1 \tag{3.7}$$

Proof: This is a fairly standard result, and we just give the main idea of the proof. One writes the right hand side of (3.5) as

$$\mathbb{E}\exp\left(t\sqrt{|\Lambda|}[f_{\Lambda,\gamma,\beta} - \mathbb{E}f_{\Lambda,\gamma,\beta}] - |\Lambda|\frac{t^2}{2}F_n\right) = \mathbb{E}\Big[\mathbb{E}\Big[\ldots\mathbb{E}\Big[e^{t\sqrt{|\Lambda|}f_{\Lambda,\gamma,\beta}(1)-\frac{t^2}{2}|\Lambda|\mathbb{E}\left[f^2_{\Lambda,\gamma,\beta}(1)|\mathcal{F}_2\right]}|\mathcal{F}_2\Big]$$
$$\times e^{t\sqrt{|\Lambda|}f_{\Lambda,\gamma,\beta}(2)-\frac{t^2}{2}|\Lambda|\mathbb{E}\left[f^2_{\Lambda,\gamma,\beta}(2)|\mathcal{F}_3\right]}|\mathcal{F}_3\Big]\ldots e^{t\sqrt{|\Lambda|}f_{\Lambda,\gamma,\beta}(n)-\frac{t^2}{2}|\Lambda|\mathbb{E}\left[f^2_{\Lambda,\gamma,\beta}(n)|\mathcal{F}_{n+1}\right]}|\mathcal{F}_{n+1}\Big] \tag{3.8}$$

and to work up the conditional expectations one by one. The point is that by rather simple estimates, one sees that

$$\left|\mathbb{E}\left[e^{t\sqrt{|\Lambda|}f_{\Lambda,\gamma,\beta}(k)-\frac{t^2}{2}|\Lambda|\mathbb{E}\left[f^2_{\Lambda,\gamma,\beta}(k)|\mathcal{F}_{k+1}\right]}|\mathcal{F}_{k+1}\right] - 1\right| \leq Cn^{-3/2} \tag{3.9}$$

where the constant C depends on t and Λ. From this (3.7) follows immediately.$\diamondsuit$

The important thing is that we can represent F_n.

Lemma 3.2: *For any Λ, β, γ*

$$\lim_{n\uparrow\infty} F_n = \int_0^1 dt\mathbb{E}\left[\frac{1}{|\Lambda|^2}\sum_{i,j\in\Lambda}\langle\langle\sigma_i\sigma_i'J_\gamma(i-j)\sigma_j\sigma_j'\rangle\rangle_{\Lambda,\gamma,\beta}|\mathcal{F}_t\right] \tag{3.10}$$

where $\langle\langle\cdot\rangle\rangle_{\Lambda,\gamma,\beta}$ denotes the expectation over two independent copies, σ and σ', of the spin variables with respect to the finite volume Gibbs measure where G_{ij} is replaced by $B_{ij}(1)$.

Proof: The main observation is that (we skip all indices referring to Λ, γ, β

$$\begin{aligned} f(k) &\equiv \mathbb{E}\left[f_{\Lambda,\gamma,\beta}|\mathcal{F}_k\right] - \mathbb{E}\left[f_{\Lambda,\gamma,\beta}|\mathcal{F}_{k+1}\right] \\ &= \mathbb{E}_{g(1)}\ldots\mathbb{E}_{g(k-1)}\mathbb{E}_{g'(k)}\Big(f(g(1),\ldots,g(k-1),g(k),\ldots,g(n)) \\ &- f(g(1),\ldots,g(k-1),g'(k),\ldots,g(n))\Big) \end{aligned} \tag{3.11}$$

where $\mathbb{E}_{g(k)}$ stands for the expectation w.r.t. all the variables $g_{ij}(k)$, and $g'(k)$ is an independent copy of the $g(k)$. Now by Taylor's formula

$$\begin{aligned} &f(g(1),\ldots,g(k-1),g(k),\ldots,g(n)) - f(g(1),\ldots,g(k-1),g'(k),\ldots,g(n)) \\ &= \sum_{i,j}\frac{\partial f}{\partial g_{ij}(k)}(g_{ij}(k) - g'_{ij}(k)) + R_2 \end{aligned} \tag{3.12}$$

where the second order is a sum over terms of the form

$$\frac{\partial^2 f}{\partial g_{ij}(k)\partial g_{lm}(k)}(g_{ij}(k) - g'_{ij}(k))(g_{ml}(k) - g'_{ml}(k)) \tag{3.13}$$

with f evaluated at some intermediate point. But all what counts is that

$$\left|\frac{\partial^2 f}{\partial g_{ij}(k)\partial g_{lm}(k)}\right| \leq n^{-1}\frac{\sqrt{J_\gamma(i-j)J_\gamma(l-m)}}{|\Lambda|} \tag{3.14}$$

and that that

$$\mathbb{E}_{g'(k)}|(g_{ij}(k)-g'_{ij}(k))(g_{ml}(k)-g'_{ml}(k))| \leq c\left[1+(|g_{ij}(k)|+|g_{lm}(k)|)+|g_{ij}(k)||g_{ml}(k)|\right] \quad (3.15)$$

with some numerical constant c. On the other hand,

$$\mathbb{E}_{g'(k)}\frac{\partial f}{\partial g_{ij}(k)}(g_{ij}(k)-g'_{ij}(k)) = \frac{1}{2\beta|\Lambda|n^{1/2}}\langle\sigma_i\sqrt{J_g(i-j)}\sigma_j\rangle_{\Lambda,\gamma,\beta}g_{ij}(k) \quad (3.16)$$

Inserting this leading term and the previous bound on R_2 into the expression for F_n we arrive, after some similar steps at the representation

$$F_n = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[\tfrac{1}{|\Lambda|^2}\sum_{i,j\in\Lambda}\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\Lambda,\gamma,\beta}|\mathcal{F}_{k+1}\right]+n^{-1/2}R(|\Lambda|,\gamma,\beta,n) \quad (3.17)$$

where $R(|\Lambda|,\gamma,\beta,n)$ is bounded uniformly in n, but not in Λ. To be rid of this remainder, we are finally obliged to follow Comets and Neveu and represent G_{ij} as the value of a Brownian motion $B_{ij}(t)$ at $t=1$. With $\mathcal{F}_t, t\in[0,1]$ denoting the corresponding filtration, the formula (3.10) follows easily from (3.17).$\diamondsuit$

The crucial point is now that under the assumption of Theorem 2, $|\Lambda|F_\infty$ converges to a constant in probility.

Lemma 3.3: *Assume that γ,β are such that almost surely, the infinite volume Gibbs state is weakly unique. Then there exists a constant $c_{\gamma,\beta}$ such that*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\Lambda,\gamma,\beta} = c_{\gamma,\beta}, \quad \textit{in Prob..} \quad (3.18)$$

Proof: Just write

$$\begin{aligned}&\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\Lambda,\gamma,\beta} = \\ &\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\infty,\gamma,\beta} \\ &+\frac{1}{|\Lambda|}\sum_{i,j\in\Lambda}\left[\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\Lambda,\gamma,\beta}-\langle\langle\sigma_i\sigma'_iJ_\gamma(i-j)\sigma_j\sigma'_j\rangle\rangle_{\infty,\gamma,\beta}\right]\end{aligned} \quad (3.19)$$

Since the unique infinite volume Gibbs state will be translation covariant, the first term converges to a constant by the ergodic theorem. Also, the finite volume states converge weakly to the infinite volume state, from which we can deduce easily that the second term converges to zero in probability. This yields the lemma. $\diamondsuit$

Now from Lemma 3.3 and Lemma 3.1 it follows easily (since $|\Lambda|F_\infty$ is uniformly bounded) that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} L_{\Lambda,\gamma,\beta}(t) = \exp\left(\frac{c_{\gamma,\beta}}{2}t^2\right) \tag{3.20}$$

which implies part (i) of Theorem 2. What is missing is to relate $c_{\gamma,\beta}$ to the difference between the quenched and annealed free energies.

At this point it turns out useful to compare our procedure above to the approach taken by Comets and Neveu. Following their approach, we would define

$$\hat{B}_{ij}(k) \equiv \frac{1}{\sqrt{n}}\sum_{l=1}^{k} g_{ij}(l) \tag{3.21}$$

which can also be represented by the brownian motion $B_{ij}(k/n)$. Correspondingly one can introduce $Z_{\Lambda,\gamma,\beta}(k)$ as the partition function of the model where G_{ij} is replaced by $\hat{B}_{ij}(k)$. It is easy to see that that $\widehat{Z}_\Lambda(k) \equiv Z_\Lambda(k)/\mathbb{E}Z_\Lambda(k)$ is a martingale in k, i.e. $\mathbb{E}\left[Z_\Lambda(k)/\mathbb{E}Z_\Lambda(k)\middle|\hat{\mathcal{F}}_{k-1}\right] = Z_\Lambda(k-1)/\mathbb{E}Z_\Lambda(k-1)$, where we have switched to the increasing sequence of sigma-algebras $\hat{\mathcal{F}}_k \equiv \sigma(g(1), g(2), \ldots, g(k))$. Now the idea is to "take the logarithm" of this martingale, that is to write

$$\widehat{Z}_\Lambda(k) = \exp\left(\sqrt{|\Lambda|}M_\Lambda(k) - \frac{1}{2}|\Lambda| < M_\Lambda > (k)\right) \tag{3.22}$$

where $M_\Lambda(k)$ is a martingale with zero mean and $< M_\Lambda > (k)$ is increasing (and called the "bracket" of the martingale $M_\Lambda(k)$), and hopefully, converging to a constant as Λ tends to infinity. In the SK-model, Comets and Neveu write this formula without the $\sqrt{|\Lambda|}$ and the $|\Lambda|$ coefficients and prove that still $< M_\Lambda > (k)$ converges in that case. Then the CLT for martingales allows them to conclude that $M_\Lambda(n)$ converges to a gaussian and this gives the desired estimate on the fluctuation of the free energy. In our case, with $\gamma > 0$, we cannot expect such a result, and convergence of $< M_\Lambda > (k)$ can only be hoped for with that normalization. On the other hand, with this normalization $M_\Lambda(k)$ has no immediate physical interpretation, in particular it is not $f_\Lambda(k) - \mathbb{E}f_\Lambda(k)$. On the other hand, the bracket of this martingale does have a nice physical interpretation, namely,

$$f_\Lambda^q(k) - f_\Lambda^a(k) = \frac{1}{2\beta}\mathbb{E} < M_\Lambda > (k) \tag{3.23}$$

which follows from (3.22) by taking the log and the expectation on both sides and recalling that $M_\Lambda(k)$ has mean zero. This appears to be a most unfortunate situation: There is a physical quantity that we know to be gaussian without control on its variance, and there is another gaussian quantity whose variance we control nicely, but we do not know what it represents. Luckily, there is a link, due to the fact that there is also a different representation of the bracket. To see how this is derived, write (3.22) in the form

$$M_\Lambda(k) = \frac{1}{\sqrt{|\Lambda|}}\ln\frac{Z_\Lambda(k)}{\mathbb{E}Z_\Lambda(k)} + \frac{\sqrt{|\Lambda|}}{2} < M_\Lambda > (k) \tag{3.24}$$

We want that $M_\Lambda(k)$ is a martingale, that is that $I\!E\left[M_\Lambda(k)|\hat{\mathcal{F}}_{k-1}\right] = M_\Lambda(k-1)$. But

$$I\!E\left[M_\Lambda(k)|\hat{\mathcal{F}}_{k-1}\right] = \frac{1}{\sqrt{|\Lambda|}} I\!E\left[\ln \widehat{Z}_\Lambda(k)|\hat{\mathcal{F}}_{k-1}\right] + \frac{\sqrt{|\Lambda|}}{2} I\!E\left[< M_\Lambda > (k)|\hat{\mathcal{F}}_{k-1}\right] \tag{3.25}$$

Now

$$\begin{aligned} I\!E\left[\ln \widehat{Z}_\Lambda(k)|\hat{\mathcal{F}}_{k-1}\right] &= I\!E\left[\ln \widehat{Z}_\Lambda(k-1)|\hat{\mathcal{F}}_{k-1}\right] + I\!E\left[\ln\left(1 + \frac{\widehat{Z}_\Lambda(k) - \widehat{Z}_\Lambda(k-1)}{\widehat{Z}_\Lambda(k-1)}\right)|\hat{\mathcal{F}}_{k-1}\right] \\ &= \ln \widehat{Z}_\Lambda(k) + \frac{1}{2} I\!E\left[\left(\frac{\widehat{Z}_\Lambda(k) - \widehat{Z}_\Lambda(k-1)}{\widehat{Z}_\Lambda(k-1)}\right)^2 |\hat{\mathcal{F}}_{k-1}\right] + R_k \\ &= \sqrt{|\Lambda|} M_\Lambda(k-1) - \frac{|\Lambda|}{2} < M_\Lambda > (k-1) \\ &+ \frac{1}{2} I\!E\left[\left(\frac{\widehat{Z}_\Lambda(k) - \widehat{Z}_\Lambda(k-1)}{\widehat{Z}_\Lambda(k-1)}\right)^2 |\hat{\mathcal{F}}_{k-1}\right] + R_k \end{aligned} \tag{3.26}$$

where R_k corresponds to the third order remainder in the Taylor expansion of the logarithm and is of order $n^{-3/2}$ and therefore can be made irrelevantly small by taking n to infinity (this is completely analogous to the estimates in the first part of this section. Thus, for $M_\Lambda(k)$ to be a martingale, we must have that

$$\begin{aligned} I\!E\left[< M_\Lambda > (k)|\hat{\mathcal{F}}_{k-1}\right] - < M_\Lambda > (k-1) &\equiv I\!E\left[< M_\Lambda > (k) - < M_\Lambda > (k-1)|\hat{\mathcal{F}}_{k-1}\right] \\ &= \frac{1}{|\Lambda|} I\!E\left[\left(\frac{\widehat{Z}_\Lambda(k) - \widehat{Z}_\Lambda(k-1)}{\widehat{Z}_\Lambda(k-1)}\right)^2 |\hat{\mathcal{F}}_{k-1}\right] + R_k \end{aligned} \tag{3.27}$$

From this we deduce the obvious solution

$$< M_\Lambda > (k) = \frac{1}{|\Lambda|} \sum_{l=0}^{k} I\!E\left[\left(\frac{\widehat{Z}_\Lambda(l) - \widehat{Z}_\Lambda(l-1)}{\widehat{Z}_\Lambda(l-1)}\right)^2 |\hat{\mathcal{F}}_{l-1}\right] + O(n^{-1/2}) \tag{3.28}$$

It remains to compute $I\!E\left[\left(\frac{\widehat{Z}_\Lambda(l) - \widehat{Z}_\Lambda(l-1)}{\widehat{Z}_\Lambda(l-1)}\right)^2 |\hat{\mathcal{F}}_{l-1}\right]$. But this goes just like in the previous case, and in complete analogy to Lemma 3.3 we obtain

Lemma 3.4:

$$< M_\Lambda > (t) \equiv \lim_{n\uparrow\infty} < M_\Lambda > ([tn]) = \int_0^t \frac{1}{|\Lambda|} \sum_{i,j\in\Lambda} \langle\langle \sigma_i \sigma'_i J_\gamma(i-j) \sigma_j \sigma'_j \rangle\rangle_{\Lambda,\gamma,\beta}(s) ds \tag{3.29}$$

By the same arguments as before, and using that a Brownian motion $B_{ij}(t)$ has the same distribution as $\sqrt{t}B_{ij}$, we get that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} I\!E < M_\Lambda > (t) = \int_0^t \sum_{i\in\Lambda} I\!E\langle\langle\sigma_0\sigma_0' J_\gamma(i)\sigma_i\sigma_i'\rangle\rangle_{\infty,\gamma,\beta\sqrt{s}} ds \tag{3.30}$$

But by (3.23), this yields

$$\int_0^t dsc_{\gamma,\beta\sqrt{s}} = \frac{\beta}{2}[f^q_{\beta\sqrt{t}} - f^a_{\beta\sqrt{t}}] \tag{3.31}$$

From this representation and Therorem 1 we get the immediate corollary:

Corollary 3.5: *Let $\beta_c < 1$ be such that the assumptions of Lemma 3.3 are satisfied for all $\beta \leq \beta_c$. Then, for Lebesgue almost all $0 \leq \beta \leq \beta_c$,*

$$\lim_{\gamma\downarrow 0} c_{\gamma,\beta} = 0 \tag{3.32}$$

Proof: Note that by construction $c_{\gamma,\beta}$ is non-negative. Moreover, by Theorem 1 its integral converges to zero as $\gamma \downarrow 0$. But than $c_{\gamma,\beta}$ itself must converge to zero except on a null set. This proves the corollary.$\diamondsuit$

This also conclude the proof of part two of Theorem 2.$\diamondsuit\diamondsuit$

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