Dynamical Localization for Discrete and Continuous Random Schrödinger Operators

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Abstract

We show for a large class of random Schrödinger operators H_ω on $\ell^2({}^\nu)$ and on $L^2({}^\nu)$ that dynamical localization holds, *i.e.* that, with probability one, for a suitable energy interval I and for q a positive real,

$$\sup_t r^q(t) \equiv \sup_t < P_I(H_\omega)\psi_t,\ |X|^q P_I(H_\omega)\psi_t > < \infty.$$

Here ψ is a function of sufficiently rapid decrease, $\psi_t = e^{-iH_\omega t}\psi$ and $P_I(H_\omega)$ is the spectral projector of H_ω corresponding to the interval I. The result is obtained through the control of the decay of the eigenfunctions of H_ω and covers, in the discrete case, the Anderson tight-binding model with Bernouilli potential (dimension $\nu = 1$) or singular potential ($\nu > 1$), and in the continuous case Anderson as well as random Landau Hamiltonians.

1 Introduction

We show for a large class of random Schrödinger operators H_ω on $\ell^2({}^\nu)$ and on $L^2({}^\nu)$ that dynamical localization holds, *i.e.* that, with probability one, for a suitable energy interval I and $q > 0$,

$$\sup_t r^q(t) \equiv \sup_t < P_I(H_\omega)\psi_t,\ |X|^q P_I(H_\omega)\psi_t > < \infty.$$

Here ψ is a function of sufficiently rapid decrease, $\psi_t = e^{-iH_\omega t}\psi$, $P_I(H_\omega)$ is the spectral projector of H_ω corresponding to the interval I. The result covers all random Schrödinger operators for which exponential localization has been proved, including operators with Bernouilli potentials in dimension 1 and random Landau Hamiltonians, for example.

The strategy of the proof is as follows. First recall that exponential localization, *i.e.* pure point spectrum and exponentially decaying eigenfunctions, is by now a well established property of random Schrödinger operators in many situations. On the other hand, it is also known that exponential localization does not systematically entail dynamical localization [10]. The authors of [10] point out that, to obtain dynamical localization, some control is needed on the location and the size of the boxes outside of which the eigenfunctions "effectively" decrease exponentially. This is precisely what is achieved for random Schrödinger operators in the present paper (Theorem 3.1 and Theorem 4.2). Our proof here uses the ideas of Von Dreifus and Klein [13], and in particular those of the proof of their Theorem 2.3. We proceed as follows: once exponential localization has been proved, and using the fact that the spectrum is now known to be discrete, one can exploit the result of the multi-scale analysis a second time to get better (and sufficient) control on the eigenfunction decay. We first deal with the discrete case (Sections 2 and 3). In section 2 we start by proving (along the lines of [10]) a sufficient condition (see (2.1)) on the eigenfunctions of a Hamiltonian H which implies dynamical localization. In section 3 we give the proof of the announced result for the discrete Anderson model.

The continuous case is dealt with in sections 4 and 5. Exponential localization for Schrödinger operators has recently been carried over to the continuum by Combes and Hislop [6] and by Klopp [19]. The case of random Landau Hamiltonians is dealt with by Combes, Hislop and Barbaroux [3] [7], by Wang [23] and by Dorlas, Macris and Pulé [12]. All those papers use an adaptation to the continuous case of the multi-scale analysis originally developed for discrete Schrödinger operators ([16] [17] [13] or see [5]). This reduces the proof of exponential localization to the verification of two hypotheses: a Wegner estimate and an estimate allowing the "initialization" of the multi-scale analysis. Our central result here (Theorem 4.2) shows that those two hypotheses actually imply dynamical localization. We give some applications in section 5.

To put our results in perspective, we recall that Del Rio, Jitomirskaya, Last and Simon [10] used bounds of Aizenman [1] to give a simple proof (avoiding the multi-scale analysis) of dynamical localization for the discrete Anderson model with a potential with bounded density. But the bounds of [1] do not seem to carry over to the continuous case, nor to Bernouilli and other singular potentials in the discrete case. To deal with these cases, we were therefore obliged to return to the (rather painful) multi-scale analysis.

A further application of our results to the random dimer model [15] will be given elsewhere [9].

2 Eigenfunction decay and dynamical localization

In this section we give, for a class of self-adjoint operators H with pure point point spectrum, defined on either $l^2(\,^\nu)$ or $L^2(\,^\nu)$, a sufficient condition on the eigenfunctions (see (2.1)) that guarantees dynamical localization. Our strategy for proving dynamical localization for random Schrödinger operators is then to prove that a property much stronger than this condition is indeed satisfied (Theorem 3.1, Theorem 4.2).

Let H_0 be the following operator on $L^2(\,^\nu)$:

$$H_0 = H_1 \oplus H_2,$$

where $H_1 = p_1^2 + p_2^2$, $p_1 = \partial_{x_1} + Bx_2/2$, $p_2 = \partial_{x_2} - Bx_1/2$, $B \geq 0$, and $H_2 = \sum_3^\nu p_i^2$, $p_i = \partial_{x_i}$. One can also write $H_0 = (P - A)^2$, where A is the vector potential $B/2(x_2, -x_1)$, written in the symmetric gauge, associated to the constant magnetic field $\overrightarrow{B} = B\vec{e}_{x_3}$.

Theorem 2.1 *Let H be a self-adjoint operator on $\mathcal{H} = l^2(\,^\nu)$ or $L^2(\,^\nu)$ with pure point spectrum on some interval $I \subset$. Let φ_n be its eigenfunctions with corresponding eigenvalues $E_n \in I$. In the case $\mathcal{H} = L^2(\,^\nu)$, suppose that I is compact and that H has the form $H_0 + V$, H_0 as described above, $V \in L^\infty(\,^\nu)$. Suppose moreover that*

$$\exists \gamma > 0, \gamma' \in]0, \gamma/2[\text{ and sites } (x_n) \text{ s.t.} \forall\, n,\ |\varphi_n(x)| < C_\gamma e^{\gamma'|x_n|} e^{-\gamma|x-x_n|}. \quad (2.1)$$

Let $q > 0$ and $\psi \in \mathcal{H}$ decaying exponentially at a rate $\theta > 2\gamma'$. Then there exists a constant $C_\psi = C_\psi(I, \gamma, \gamma', \theta, q)$ such that:

$$\forall\, t \geq 0,\ ||\,|X|^{q/2} e^{-iHt} P_I(H)\psi||^2 \leq C_\psi. \quad (2.2)$$

This simple result relies on ideas of section 7 in [10]. Note that (2.1) says roughly that the eigenfunctions are localized inside boxes of size $|x_n|/2$ around "centers" x_n. This is stronger than exponential localization of H on I, which only means that

$$\exists \gamma > 0 \text{ such that } \forall n,\ \exists C_n > 0,\ |\varphi_n(x)| \leq C_n e^{-\gamma|x|}, \quad (2.3)$$

but weaker than what the authors of [10] called SULE (Semi-Uniformly Localized Eigenfunctions).

We choose to present the proof of this theorem in the continuous case, since the first part of the proof (Lemma 2.2) is a little bit more technical in this situation. In order to prove Theorem 2.1 we need control on the growth of the $|x_n|$ in n, which is given by the following preliminary lemma:

Lemma 2.2 *Let H be as in the proposition and $\delta > 0$. Then one can order the $|x_n|$ in increasing order, and there exists a constant $C = (4\max(B,1))^{-1}$ such that for n large enough (depending on δ):*

$$|x_n| \geq C n^{1/(\nu+\delta)}.$$

Proof of Lemma 2.2: Essentially we follow the ideas of the proof of Theorem 7.1 of [10]. We recall that the energies E_n that we consider belong to I. Let $\delta > 0$ be given and let $\delta' > 0$ so that $\delta > \delta'(\nu - 1)$. Let $L > 0$ be given, define $J = [0, L^{\delta'}]$, and write $\chi_{2L}(x)$ for the characteristic function of the ball of radius $2L$ and centered at 0.

Suppose that $|x_n| < L$. Then, for such n and for some constant C_1, one has:

$$\begin{aligned} &< \varphi_n, (1-\chi_{2L}(X))\chi_J(H_0)\chi_{2L}(X)\varphi_n > \\ &\quad \leq \; ||(1-\chi_{2L}(X))\varphi_n||_{L^2} ||\chi_{2L}(X)\varphi_n||_{L^2} \\ &\quad \leq \; C_\gamma e^{(\gamma'+\gamma)|x_n|} \left|\left|(1-\chi_{2L}(X))e^{-\gamma|x|}\right|\right|_{L^2} \\ &\quad \leq \; C_1 e^{-(\gamma-\gamma')L}. \end{aligned} \tag{2.4}$$

Secondly, using $1 - \chi_J(y) \leq y L^{-\delta'}$, $y \geq 0$, and $H\varphi_n = E_n\varphi_n$:

$$\begin{aligned} < \varphi_n, (1-\chi_J(H_0))\varphi_n > \; &\leq \; (< \varphi_n, H\varphi_n > + ||V||_\infty)\, L^{-\delta'} \\ &\leq \; C(I, ||V||_\infty) L^{-\delta'}. \end{aligned} \tag{2.5}$$

So, using (2.4) and (2.5),

$$\begin{aligned} & tr(\chi_{2L}(X)\chi_J(H_0)\chi_{2L}(X)) \qquad (2.6) \\ &\quad \geq \sum_{n|\, E_n\in I,\, |x_n|\leq L} < \varphi_n, \chi_{2L}(X)\chi_J(H_0)\chi_{2L}(X)\varphi_n > \\ &\quad \geq \; \sharp\{n|\, E_n \in I,\, |x_n| \leq L\} \left(1 - 3C_1 e^{-(\gamma-\gamma')L} - C(I, ||V||_\infty) L^{-\delta'}\right) \\ &\quad \geq \; \frac{1}{2}\sharp\{n|\, E_n \in I,\, |x_n| \leq L\} \text{ for } L \geq L_0 = L_0(\gamma, \gamma', I, ||V||_\infty, \delta'). \qquad (2.7) \end{aligned}$$

The next step is then to bound the trace class norm of the operator $Q = \chi_{2L}(X)\chi_J(H_0)\chi_{2L}(X)$. Let's study the case $B \neq 0$. Since $\{u_1^2 + u_2^2 \leq L\} \subset \{u_1^2 \leq L, u_2^2 \leq L\}$, and denoting by $\chi_{2L}^{(d)}(x)$ the characteristic function of the d-dimensional ball of radius $2L$ centered at 0, remark that

$$\begin{aligned} tr(Q) \;&\leq\; tr\left(\chi_{2L}(X)(\chi_J(H_1)\otimes\chi_J(H_2))\chi_{2L}(X)\right) \\ &=\; tr\left((\chi_{2L}^{(2)}(X)\chi_J(H_1)\chi_{2L}^{(2)}(X)) \otimes (\chi_{2L}^{(\nu-2)}(X)\chi_J(H_2)\chi_{2L}^{(\nu-2)}(X))\right) \\ &\equiv\; tr\left(Q_1 \otimes Q_2\right). \end{aligned} \tag{2.8}$$

The operator Q_2 is the product of two Hilbert-Schmidt operators with respective kernel $\chi_{2L}^{(\nu-2)}(x)\left(\mathcal{F}^{-1}g\right)(x-y)$ and $\chi_J(x)\left(\mathcal{F}^{-1}\chi_{2L}^{(\nu-2)}\right)(x-y)$ [21], where $\mathcal{F}^{-1}g$ denotes the inverse Fourier transform of $g(x)=\chi_J\circ s(x)$, with $s(x_1,...,x_{\nu-2})=\sum_1^{\nu-2}x_i^2$. So, denoting by $||C||_1$ the trace class norm of an operator C defined on $\mathcal{H}$, one has:

$$\begin{aligned} ||Q_2||_1 &\leq ||\chi_{2L}(X)\chi_J(H_2)||_{HS}||\chi_J(H_2)\chi_{2L}(X)||_{HS} \\ &\leq ||(\chi_{2L}^{(\nu-2)}(x)||_{L^2}^2||\chi_J\circ s(x)||_{L^2}^2 \leq (4L)^{\nu-2}L^{\delta'(\nu-2)}. \end{aligned} \tag{2.9}$$

We turn now to the magnetic part Q_1. It is well known that the spectrum of our H_1 consists of eigenvalues $(2n+1)B$, $n=0,1,2,....$ The corresponding projectors are operators with kernel [20],

$$P_n(x,x')=e^{i\frac{B}{2}x\wedge x'}p_n\left(\sqrt{B}(x-x')\right), \tag{2.10}$$

where

$$p_n((x_1,x_2))=\frac{1}{2\pi}\int e^{ikx_1}h_n(k+x_2/2)h_n(k-x_2/2)dk,$$

and $h_n(k)$ are the normalised Hermite functions. Note that we gave the expression of P_n in the symmetric gauge, and not in the Landau gauge as in [20]. Write now $P_J=\sum_{n|\,E_n\in J}P_n$, the projector corresponding to the eigenvalues belonging to J. Then $Q_1=\left(\chi_{2L}^{(2)}(X)P_J\right)\left(P_J\chi_{2L}^{(2)}(X)\right)$, which are two Hilbert-Schmidt operators with the same norm (because of (2.10)). And one has, for each n,

$$\begin{aligned} &||\chi_{2L}^{(2)}(X)P_n||_{HS}^2 \\ &\quad= \int_x \chi_{2L}^{(2)}(x)\int_{x'}\left|p_n\left(\sqrt{B}(x-x')\right)\right|^2 dxdx' \\ &\quad\leq (4BL)^2\int_{x_2}\left|\left|\mathcal{F}\left(h_n(.+\frac{x_2}{2})h_n(.-\frac{x_2}{2})\right)(x_1)\right|\right|_{L^2}^2 dx_2 \\ &\quad= (4BL)^2\int_{x_2,k}\left|h_n(k+\frac{x_2}{2})\right|^2\left|h_n(k-\frac{x_2}{2})\right|^2 dx_2\,dk \\ &\quad= (4BL)^2||h_n||_{L_2}^4 \end{aligned}$$

So, since $||h_n||_{L^2}=1$, and using $\sharp\{n\geq 0,(2n+1)B\leq L^{\delta'}\}\leq L^{\delta'}/2B$ if $L^{\delta'}\geq B$, one has

$$||\chi_{2L}^{(2)}(X)P_J||_{HS}^2\leq 8BL^{2+\delta'}.$$

And then, using (2.8) and (2.9),

$$tr(Q)\leq CL^{\nu+\delta}, \tag{2.11}$$

taking $\delta > \delta'(\nu-1)$, and $C = 4\max(B,1)$. Note here that if $B=0$ then the free Hamiltonian H_0 has the form $\sum_1^\nu \partial_i^2$. Hence the analysis made previously for Q_2 is valid for such a Q and (2.11) holds.

In order to finish the argument, note that, together with (2.7) and (2.8), (2.11) tells us that, for $L \geq L_0$, $N(L) \equiv \sharp\{n|,\ E_n \in I, |x_n| \leq L\}$ is finite. Order then the eigenfunctions in such a way that $|x_n|$ increases. So, $N(|x_n|) = n$, and if $n \geq N_0 \equiv N(L_0)$, one has, with $L = |x_n|$:

$$|x_n| \geq Cn^{1/(\nu+\delta)}, \text{ for } n > N_0. \tag{2.12}$$

The main difference between the continuous and discrete cases comes from this Lemma, in the sense that on $L^2(^\nu)$ one has to control the behaviour of the eigenfunctions in the momentum variables: that was achieved through the use of $\chi_J(H_0)$. In the discrete case, (2.11) is replaced by the trivial equality $tr(\chi_{2L}) = (4L+1)^\nu$. This also explains why no specific form for H is needed on $l^2(^\nu)$. It is easy, then, to rewrite the proof of Lemma 2.2 in this case (see also [10]). □

Proof of Theorem 2.1: Let $\psi \in \mathcal{H}$ such that, for some constant $C(\psi) > 0$ and $\theta > 2\gamma'$, $|\psi(x)| < C(\psi)e^{-\theta|x|}$. We have to bound $||X^{q/2}P_I(H)e^{-iHt}\psi||$, for $q > 0$, $t > 0$. Let $0 < \epsilon < \inf(\gamma/2 - \gamma', \theta/2 - \gamma')$. Then

$$\begin{aligned}
&||X^{q/2}P_I(H)e^{-iHt}\psi||^2 \\
\leq &\sum_{n|\ E_n \in I} |<\varphi_n, \psi>|^2 ||X^{q/2}\varphi_n||^2 \\
\leq &\sum_{n|\ E_n \in I} (C(\psi)C_\gamma)^2 e^{-4\epsilon|x_n|} \\
&\qquad \int dx \int dy \left(|y|^q e^{-2\theta|x|} e^{-2(\gamma-\gamma'-\epsilon)|x-y|} e^{2(\gamma'+\epsilon)(2|x_n|-|x-x_n|-|y-x_n|)}\right) \\
\leq &\ C_3 \left(\sum_{n|\ E_n \in I} e^{-4\epsilon|x_n|}\right) \int \left(\int |y|^q e^{-2(\gamma-2(\gamma'-\epsilon))|x-y|} dy\right) e^{4(\gamma'+\epsilon)|x|-2\theta|x|} dx.
\end{aligned}$$

This last line is uniformly bounded by a constant $C_\psi(\gamma, \gamma', I, \theta, \epsilon)$, according to Lemma 2.2 and because of the choice of ϵ. □

3 The discrete Anderson model

We consider the self-adjoint operator H_ω

$$H_\omega = -\Delta + V_\omega,$$

where Δ is the discrete Laplacian on $\ell^2(^\nu)$ and V_ω $(\omega \in \Omega)$ is a random potential, the $(V_\omega(x))_{x\in^\nu}$ being i.i.d. random variables. Their common probability measure μ is assumed to be non degenerate, *i.e.* not concentrated on a single point. The conditions that we impose on μ are:

$$\begin{array}{l} \text{if } \nu = 1 : \exists\, \eta > 0\ ,\ \int |v|^\eta \, d\mu(v) < \infty; \\ \text{if } \nu \geq 2 : \mu \text{ is } \alpha\text{-Hölder continuous.} \end{array} \tag{3.1}$$

Let us recall how the disorder $\delta(\mu)$ of a α-Hölder continuous measure μ is defined:

$$\delta(\mu)^{-1} = \inf_{\tau > 0} \sup_{|b-a|<\tau} |b-a|^{-\alpha} \mu([a,b]).$$

Theorem 3.1 *Let H_ω be the Anderson Hamiltonian, and μ the common probability measure of the $V_\omega(x)$, $x \in {}^\nu$, not concentrated in a single point and satisfying condition (3.1). Let I be a compact interval and $\epsilon > 0$.*

- *If $\nu = 1$, define $\Gamma \equiv \gamma(I) \equiv inf\{\gamma(E),\ E \in I\}$, where $\gamma(E)$ is the Lyapunov exponent at energy E. Suppose $\Gamma = \gamma(I) > 0$.*
- *If $\nu > 1$, pick $\Gamma > 0$. Suppose the disorder $\delta(\mu)$ is taken sufficiently high.*

Then, almost surely, H_ω has pure point spectrum on I, and there exist centers $x_{n,\omega}$ associated to the eigenfunctions $\varphi_{n,\omega}$ with energy $E_{n,\omega} \in I$ such that: $\forall\ \gamma_0 \in]0, \Gamma[$, there exists a constant $C(\omega, \epsilon, \gamma_0)$ such that

$$\forall x \in {}^\nu,\ \ |\varphi_{n,\omega}(x)| \leq C(\omega, \epsilon, \gamma_0) e^{\gamma_0 |x_{n,\omega}|^\epsilon} e^{-\gamma_0 |x - x_{n,\omega}|}. \tag{3.2}$$

Evidently, one can also write a "low energy" version of this theorem. As an immediate consequence of Theorem 2.1 and Theorem 3.1 we have:

Corollary 3.2 *Let H_ω be as in Theorem 3.1, and $P_I(H_\omega)$ the spectral projection on the compact interval I. Then for $q > 0$ and $\psi \in \ell^2({}^\nu)$ decaying exponentially with rate $\theta > 0$, $||\,|X|^q P_I(H_\omega) e^{-iH_\omega t} \psi||^2$ is bounded uniformly in t almost surely.*

Comparing (3.2) to (2.3) and to (2.1), one notices that now the size of the boxes in which the eigenfunctions "live" can grow at most as $|x_{n,\omega}|^\epsilon$. One expects this can be improved to a polynomial bound (but no more - see [10]). Supposing μ has a bounded density with compact support, the polynomial bound follows from the proof of Theorem 7.6 in [10].

The proof of Theorem 3.1 is based on the ideas of [13], and in particular on the proof of Theorem 2.3 in [13]. The strategy is the following: since the hypotheses of Theorem 3.1 imply exponential localization, we know that there exist "centers" $x_{n,\omega}$ where the eigenfunction $\varphi_{n,\omega}$ is maximal, and one can then exploit the result of the multi-scale analysis a second time to improve the control of the decay of the eigenfunctions. As already pointed out, this proof has the advantage of yielding the result for singular potentials and in particular for Bernouilli potentials in dimension 1. In addition, the proof extends to continuous random Schrödinger operators, as shown in sections 4 and 5.

To make this paper self-contained, we start by recalling the elements from [13] that we need. First of all, $\Lambda_L(x)$ denotes the cube of side $L/2$ centered in x and $\partial\Lambda_L(x)$ its boundary. $H_{\Lambda_L(x),\omega}$ is the restriction of the operator H_ω to the cube $\Lambda_L(x)$ with Dirichlet boundary conditions, and $G_{\Lambda_L(x)}(E,.,.)$ is its resolvent. Given $L_0 > 1$, $\alpha \in]1,2[$, we define L_k ($k \in$) recursively via $L_{k+1} = L_k^\alpha$. Given in addition an integer $b \geq 2$, we define

$$A_{k+1}(x_o) = \Lambda_{2bL_{k+1}}(x_o) \backslash \Lambda_{2L_k}(x_o).$$

Note that we do not indicate the dependence of L_k and A_k on L_0, α and b since these quantities will at any rate be fixed later on. We further need the following definition:

Definition 3.3 *Let $\gamma > 0$ and an energy $E \in$ be given. A cube $\Lambda_L(x)$ is said to be (γ, E)-regular if $E \notin \sigma(H_{\Lambda_L(x)})$ and if for all $y \in \partial\Lambda_L(x)$,*

$$|G_{\Lambda_L(x)}(E,x,y)| \leq e^{-\gamma L/2}.$$

Otherwise $\Lambda_L(x)$ will be called (γ, E)-singular.

Note that this definition is ω-dependent, but we follow the usual practice by not indicating this. For $x_o \in {}^{\nu}$, $E_k(x_o)$ is defined to be the following set:

$$\{\omega |\, \exists\, E \in I,\ \exists\, x \in A_{k+1}(x_o),\ \Lambda_{L_k}(x_o) \; and \; \Lambda_{L_k}(x) \; are \; (\gamma, E) - singular\}.$$

Finally, we recall a well known identity. Let $x \in {}^{\nu}$, $E \notin \sigma(H_{\Lambda_L(x)})$, and $\varphi \in \ell^2({}^{\nu})$ so that $H\varphi = E\varphi$ be given, then:

$$\varphi(x) = \sum_{(y,y') \in \partial\Lambda_L(x)} G_{\Lambda_L(x)}(E,x,y)\varphi(y'). \tag{3.3}$$

Here (with some abuse of notation) $(y, y') \in \partial\Lambda_L$ means y and y' are nearest neighbours with $y \in \Lambda_L(x)$ and $y' \notin \Lambda_L(x)$. $\partial\Lambda^+_{L_k}(x)$ will denote the points y' just outside $\Lambda_{L_k}(x)$.

In order to prove Theorem 3.1, we start with the following three lemmas:

Lemma 3.4 *Let $p > \nu$ and $\alpha \in]1, 2 - 2\nu/(p + 2\nu)[$ and $b > 1$ be given. Assume the hypotheses of Theorem 3.1 are satisfied. Then for any $\gamma \in]0, \Gamma[$ there exists $L_0 = L_0(p, \nu, \gamma, b, \delta(\mu)) > 1$ such that:*

$$\forall\, k \in\ ,\ \forall\, x \in {}^{\nu},\ \ (E_k(x)) \leq \frac{(2bL_{k+1}+1)^\nu}{(L_k)^p}.$$

Lemma 3.5 *Let $\gamma > 0$ be fixed. There exists a constant $L_*(\nu, \gamma)$ so that, if H is a Schrödinger operator and $\varphi \in \ell^2({}^{\nu})$ an eigenvector of H with eigenvalue E, and if $x_* \in {}^{\nu}$ satisfies $|\varphi(x_*)| = sup\{|\varphi(x)|,\ x \in {}^{\nu}\}$, then $\Lambda_L(x_*)$ is (γ, E)-singular, for all $L \geq L_*(\nu, \gamma)$.*

Lemma 3.6 *If ν, s_ν, and γ are some positive constants, then $\forall\ \eta \in]0,1[$ there exists $L(\eta,\ \gamma,\ \nu)$ such that, if $L \geq L(\eta,\ \gamma,\ \nu)$:*

$$\forall\ x,\ x_o \in {}^\nu,\ \left(s_\nu L^{\nu-1} e^{-\gamma L/2}\right)^{\frac{|x-x_o|}{L/2+1}} \leq e^{-\gamma\eta|x-x_o|}. \tag{3.4}$$

The first lemma follows immediately from the Appendix and from Theorem 2.2 of Von Dreifus and Klein [13] and constitutes the core of the proof of exponential localization in [13]. We will prove an analog of it for continuous random Schrödinger operators in the Appendix.

The second lemma says roughly that if φ is an eigenvector of H with eigenvalue E, then E must be close to the spectrum of $H_{\Lambda_L(x)}$ provided L is big enough and $\Lambda_L(x)$ is centered on a maximum of $|\varphi_{n,\omega}|$. It is quite simple, but central for what follows.

Obviously the third lemma doesn't need a proof. We have stated it separately in order to make clear later on that $L(\eta,\gamma,\nu)$ only depends on the model parameters and not on the particular eigenfunction we consider. Note that $L(\eta,\gamma,\nu)$ behaves like $(1/\gamma)$ at a positive power.

Proof of Lemma 3.5: Let φ be as in the lemma: $\varphi \in \ell^2({}^\nu)$, so x_* exists. Suppose that $\Lambda_L(x_*)$ is $(\gamma, E) - regular$, and apply the identity (3.3) at the point x_*. Then for some $y' \in \partial\Lambda_L^+(x)$:

$$\begin{aligned} |\varphi_{n,\omega}(x_*)| &\leq s_\nu L^{\nu-1} e^{-\gamma L/2} |\varphi_{n,\omega}(y')| \\ &\leq s_\nu L^{\nu-1} e^{-\gamma L/2} |\varphi_{n,\omega}(x_*)|, \end{aligned} \tag{3.5}$$

where s_ν is a constant depending only on the dimension. Now let $L_*(\gamma,\ \nu)$ be a positive real such that $s_\nu L^{\nu-1} e^{-\gamma L/2} < 1$ for $L \geq L_*$. Then, for such L, (3.5) is impossible, and $\Lambda_L(x_*)$ cannot be (γ, E)-regular any more, that is: $\Lambda_L(x_*)$ is (γ, E)-singular for $L \geq L_*(\gamma, \nu)$. □

Proof of Theorem 3.1: Under the hypotheses of the theorem, H_ω has $-\,a.s.$ exponential localization on I (see [4] and [13]). This means that there exists $\Omega_0 \subset \Omega$, $\mu(\Omega_0) = 1$ so that for all $\omega \in \Omega_0$, $\sigma_c(H_\omega) \cap I = \emptyset$ and for all eigenvalue $E_{n,\omega} \in I$, the corresponding eigenfunction $\varphi_{n,\omega}$ is ℓ^2 and satisfies (2.3).

The aim is therefore to control the constant $C_{n,\omega}$ of (2.3) and more precisely to show that $x_{n,\omega}$ can be chosen so that this $C_{n,\omega}$ grows slower than an exponential in $|x_{n,\omega}|^\epsilon$. In order to prove Theorem 3.1, we wish to use equation (3.3) repeatedly, and on a scale L_k for suitably large k, to estimate the value of $\varphi_{n,\omega}(x)$ when x belongs to $A_{k+1}(x_{n,\omega})$ for suitably chosen $x_{n,\omega}$. To do this, one has to work "outward" from $x \in A_{k+1}(x_{n,\omega})$ to the boundary of $A_{k+1}(x_{n,\omega})$, making sure that the boxes of size L_k to which one applies (3.3) are regular.

After these preliminaries, let's start the proof properly speaking, which will consist of three steps. Firstly, let I, $\Gamma > 0$ and γ_0 be as in the theorem. Pick $\epsilon > 0$ and $\gamma \in]\gamma_0, \Gamma[$.

First step: Let $p > \nu$, $\alpha \in]1, 2 - 2\nu/(p + 2\nu)[$ and $b > 1$ be given. With the L_k, $k \geq 0$, defined in Lemma 3.4, consider

$$F_k = \bigcup_{|x_o| \leq (L_{k+1})^{1/\epsilon}} E_k(x_o).$$

Lemma 3.4 then implies that for some constant $C(\epsilon, \nu, b)$

$$(F_k) \leq C(\epsilon, \nu, b) L_k^{-p+\nu\alpha(1+1/\epsilon)}.$$

Hence, since p can be chosen larger than $2\nu(1 + 1/\epsilon)$, one has $\sum_{k=0}^{\infty} (F_k) < \infty$. The Borel-Cantelli lemma then implies that:

$$\left(\lim_{m\to\infty} \bigcup_{k \geq m} F_k \right) = 0,$$

so that the set

$$\Omega_1 = \{\omega \in \Omega | \exists\ \tilde{k}_1 = \tilde{k}_1(\omega, \epsilon, p, \gamma)\ such\ that\ \forall\ k \geq \tilde{k}_1,\ \omega \notin F_k\}$$

has full measure. This ends the probabilistic part of the proof. Note that the choice of ϵ puts a lower bound on p. This in turn forces the disorder to be high *via* Lemma 3.4.

Second step: Now pick an ω in $(\Omega_0 \cap \Omega_1)$, which will be kept fixed throughout the rest of the proof. Let $E_{n,\omega} \in I$, $\varphi_{n,\omega}$ its eigenfunction, and let $x_{n,\omega}$ be a point where $|\varphi_{n,\omega}(x)|$ is maximal. Note that such a point exists since $\omega \in \Omega_0$ and therefore $\varphi_{n,\omega} \in \ell^2(^\nu)$. Let

$$k_1 = k_1(\omega, \epsilon, p, \gamma, x_{n,\omega}) = max(\tilde{k}_1, k_0(\epsilon, x_{n,\omega})), \tag{3.6}$$

where for all $y \in {}^\nu$, the integer $k_0(\epsilon, y)$ is defined as follows:

$$k_0(\epsilon, y) = min\{k \geq 0\ such\ that\ |y|^\epsilon < L_{k+1}\}. \tag{3.7}$$

Hence $\forall k \geq k_1$, $\omega \notin E_k(x_{n,\omega})$. Indeed, if $k \geq k_1$, then $\omega \notin F_k$ and $|x_{n,\omega}|^\epsilon < L_{k+1}$. This implies that $\omega \notin E_k(x_{n,\omega})$, and consequently that $\forall k \geq k_1$, $\forall E \in I$ and $\forall y \in A_{k+1}(x_{n,\omega})$ either $\Lambda_{L_k}(x_{n,\omega})$ or $\Lambda_{L_k}(y)$ is $(\gamma, E) - regular$. We can then apply Lemma 3.5 to conclude that there exists an integer $k_2 = max(\tilde{k}_2, k_0(\epsilon, x_{n,\omega}))$ where $\tilde{k}_2$ depends on the same parameters as $\tilde{k}_1$ and not on n, so that

$$\forall k \geq k_2,\ \forall E_{n,\omega} \in I,\ \forall y \in A_{k+1}(x_{n,\omega}),\ \ \Lambda_{L_k}(y) \text{ is } (\gamma, E_{n,\omega}) - regular.$$

Last step: We now finish the argument along the lines of the proof of Theorem 2.3 in [13] ; ω is still fixed in $(\Omega_0 \cap \Omega_1)$. For $k \geq k_2$ and for $x \in A_{k+1}(x_{n,\omega})$, $\Lambda_{L_k}(x)$ being $(\gamma, E_{n,\omega}) - regular$, we can apply relation (3.3):

$$|\varphi_{n,\omega}(x)| \leq s_\nu L_k^{\nu-1} e^{-\gamma L_k/2} |\varphi_{n,\omega}(x')|,$$

where $x' \in \partial\Lambda^{+}_{L_k}(x)$ is chosen so that

$$|\varphi_{n,\omega}(x')| = sup\{|\varphi_{n,\omega}(y)|,\ y \in \partial\Lambda^{+}_{L_k}(x)\}.$$

As long as x' is still in $A_{k+1}(x_{n,\omega})$ we can use (3.3) again, so that we can repeat this step at least $d(x, \partial A_{k+1}(x_{n,\omega}))/(L_k/2+1)$ times. Hence for $k \geq k_2$ and $x \in A_{k+1}(x_{n,\omega})$,

$$|\varphi_{n,\omega}(x)| \leq \left(s_\nu L_k^{\nu-1} e^{-\gamma L_k/2}\right)^{\dfrac{d(x,\partial A_{k+1}(x_{n,\omega}))}{L_k/2+1}}.$$

Comparing this to (3.4), we see that in order to get a useful estimate we need a lower bound on $d(x, \partial A_{k+1}(x_{n,\omega}))$.

For that purpose we cover $^{\nu}$ with new annular regions $\tilde{A}_{k+1}(x_{n,\omega})$ defined as follows. Pick $\rho \in]0,1[$ so that $\rho\gamma > \gamma_0$ and choose the integer b introduced at the beginning of the proof so that $b > (1+\rho)/(1-\rho)$. Then set

$$\tilde{A}_{k+1}(x_{n,\omega}) = \Lambda_{[2bL_{k+1}/(1+\rho)]}(x_{n,\omega}) \backslash \Lambda_{2L_k/(1-\rho)}(x_{n,\omega}) \subset A_{k+1}(x_{n,\omega}).$$

Note that if $x \in \tilde{A}_{k+1}(x_{n,\omega})$ then $d(x, \partial A_{k+1}(x_{n,\omega})) \geq \rho|x - x_{n,\omega}|$. Hence, repeating (3.3) $\rho|x - x_{n,\omega}|$ times, one has that for all $k \geq k_2$ and for all $x \in \tilde{A}_{k+1}(x_{n,\omega})$,

$$|\varphi_{n,\omega}(x)| \leq \left(s_\nu L_k^{\nu-1} e^{-\gamma L_k/2}\right)^{\frac{\rho|x-x_n|}{L_k/2+1}},$$

or, applying Lemma 3.6, and choosing $\eta \in]0,1[$ such that $\gamma_0 = \rho\eta\gamma$ we conclude that there exists an integer $k_3 = max(\tilde{k}_3, k_0(\epsilon, x_{n,\omega}))$ where $\tilde{k}_3$ depends again not on n, such that:

$$\forall\, k \geq k_3,\ and \,\forall\, x \in \tilde{A}_{k+1}(x_{n,\omega}),\ \ |\varphi_{n,\omega}(x)| \leq e^{-\gamma_0|x-x_n|}. \tag{3.8}$$

But now note that for all $x \in {}^{\nu}$, and provided $|x - x_{n,\omega}| > L_0/(1-\rho)$ there exists a k so that $x \in \tilde{A}_{k+1}(x_{n,\omega})$. This means that there exists an integer $\overline{k} = max(\tilde{k}_4, k_0(\epsilon, x_{n,\omega}))$, $\tilde{k}_4$ depending once again not on n, such that (3.8) holds for all $x \in {}^{\nu}$ satisfying $|x - x_{n,\omega}| > L_{\overline{k}}$. Hence, using that $|\varphi_{n,\omega}(x)| \leq 1$ for all $x \in {}^{\nu}$:

$$\forall x \in {}^{\nu},\ |\varphi_{n,\omega}(x)| \leq C(\omega, \epsilon, \gamma_0) e^{\gamma_0 L_{\overline{k}}} e^{-\gamma_0|x-x_{n,\omega}|}. \tag{3.9}$$

So far, we have only proved that the eigenfunctions decay exponentially, but we are now in a position to control the n-dependence of the constant $e^{\gamma_0 L_{\overline{k}}}$ as follows. Note that the only n-dependence of $\overline{k}$ comes from $k_0(\epsilon, x_{n,\omega})$. Suppose $sup\{|x_{n,\omega}|, E_{n,\omega} \in I\} < \infty$, then $\overline{k}$ can be chosen n-independently, so that we actually obtain a uniform localization (called ULE in [10]), and *a fortiori* condition (2.1) of Theorem 2.1. But Lemma 2.2 contradicts this first possibility. So, in fact, $sup\{|x_{n,\omega}|, E_{n,\omega} \in I\} = \infty$, and, for n sufficiently large, one has:

$$\overline{k} = k_0(\epsilon, x_{n,\omega})$$

i.e., with (3.7),

$$L_{\overline{k}} \leq |x_{n,\omega}|^{\epsilon}.$$

Inserting this in (3.9) yields the announced result. □

4 The continuous case

In this section, our goal is to obtain the analog of Theorem 3.1 and Corollary 3.2 for continuous random Schrödinger operators. The result is stated in Theorem 4.2 below. In section 5, we will present some models where the hypotheses of this theorem are satisfied.

We consider random Schrödinger operators on $L^2({}^{\nu})$ of the following type ($\nu \geq 1$):

$$H_\omega = H_0 + \sum_{i \in {}^{\nu}} \lambda_i(\omega) u(x-i). \tag{4.1}$$

Here

i) $H_0 = (i\nabla - A)^2 + V_{per}$, where A is a vector potential of a constant magnetic field $\overrightarrow{B}$=rot(A), and V_{per} a periodic potential.

ii) The variables $\lambda_i(\omega)$, $i \in {}^{\nu}$ are independent and identically distributed, with common distribution μ.

iii) The function $u(x)$ belongs to $C_0^2({}^{\nu})$, with $supp\, u \subset [-R, R]^{\nu}$.

To state the hypotheses, we need to recall some notations and simple facts. We introduce $|x| = \max\{|x_i|,\ i = 1, ..., \nu\}$, and denote by $\Lambda_L(x)$ the cube

$$\Lambda_L(x) = \{y \in {}^{\nu} |\, |y - x| < L/2\}.$$

Moreover $\delta > 0$ being fixed (independently of L), $\tilde{\Lambda}_L(x)$ is the subset

$$\tilde{\Lambda}_L(x) = \{y \in \Lambda_L(x)\, such\, that\, L/2 - \delta < |x - y| < L/2\};$$

$\tilde{\chi}_{L,x}$ will denote its characteristic function.We denote furthermore by $\chi_{L,x}$ a function in $C^2({}^{\nu})$ with support in $\Lambda_L(x)$ and satisfying, $0 \leq \chi_{L,x} \leq 1$, $\chi_{L,x} \equiv 1$ on the subcube $\Lambda_L(x) \backslash \tilde{\Lambda}_L(x)$ so that $\nabla \chi_{L,x}$ lives in $\tilde{\Lambda}_L(x)$. Note that we will often drop the ω or x-dependence of the objects introduced in order to alleviate the notations.

We furthermore define local Hamiltonians $H_{\Lambda_L(x),\omega}$ as follows. When $A = 0$ and $V_{per} = 0$, $H_{\Lambda_L(x),\omega}$ is the restriction of the operator H_ω to the cube $\Lambda_L(x)$ with Dirichlet boundary conditions. When $A \neq 0$ or $V_{per} \neq 0$,

$$H_{\Lambda_L(x),\omega} = H_0 + \sum_{i \in \Lambda_L \cap {}^{\nu}} \lambda_i(\omega) u(x-i).$$

We denote by $W_{L,x}$ the first order differential operator $W_{L,x} \equiv [H_0, \chi_{L,x}]$ and write $R_{\Lambda_L(x)}(E)$ the resolvent of $H_{\Lambda_L(x),\omega}$. We then always have the geometric resolvent equation: if $\Lambda_l \subset \Lambda_L \subset {}^\nu$ and if $E \notin \sigma(H_{\Lambda_L})$ and $E \notin \sigma(H_{\Lambda_l})$ then

$$\chi_l R_{\Lambda_L}(E) = R_{\Lambda_l}(E)\chi_l + R_{\Lambda_l}(E) W_l R_{\Lambda_L}(E). \tag{4.2}$$

Definition 4.1 *Let $\gamma > 0$ and an energy $E \in$ be given. A cube $\Lambda_L(x)$ is said to be (γ, E)-regular if $E \notin \sigma(H_{\Lambda_L(x)})$ and if:*

$$||\chi_{4\delta,x} R_{\Lambda_L(x)}(E) W_{L,x}|| \leq e^{-\gamma L/2}.$$

Otherwise $\Lambda_L(x)$ will be called (γ, E)-singular.

We now state the result. Given a compact interval I, and reals $\gamma_0 > 0,\ p > \nu$, $L_0, \tilde{L} > 1$, we introduce
Hypothesis [H1](γ_0, I, p, L_0):

$$(\forall\, E \in I,\ \Lambda_{L_0}\, is\, (\gamma_0, E) - regular) > 1 - \frac{1}{L_0^p}.$$

Hypothesis [H2]$(I, \tilde{L})$ (Wegner): *there exists C_W so that for all $\tilde{I} \subset I_1 \equiv \{E \,|\, d(E, I) < 1\}$ and $L > \tilde{L}$,*

$$(Tr(E_{\Lambda_L(0)}(\tilde{I}))) < C_W |\tilde{I}|\, |\Lambda_L|.$$

Note that, using Chebyshev's inequality and [H2]$(I, \tilde{L})$, one also has, for $L > \tilde{L}$, $0 < \eta < 1$ and $E \in I$,

$$(d(E, \sigma(H_{\Lambda_L(\omega)})) < \eta) < C_W |\Lambda_L| \eta. \tag{4.3}$$

This is the so-called "Wegner Estimate". We will need both [H2] and (4.3) for the proof of Proposition 4.3.

Theorem 4.2 *Let $\epsilon > 0$. Suppose that for some interval I and reals $\gamma_0 > 0$, $p > 2\nu(1 + 1/\epsilon)$, the hypotheses [H1](γ_0, I, p, L_0) and [H2]$(I, \tilde{L})$ hold for $L_0 > \tilde{L}$ large enough. Then with probability one there exist points $x_{n,\omega}$, associated to eigenfunctions $\varphi_{n,\omega}$ with energy $E_{n,\omega} \in I$, so that: $\forall\, \gamma \in]0, \gamma_0[$ and for some constant $C_\omega = C(\omega, \epsilon, \gamma, \gamma_0, I, L_0)$, one has, for all $x \in {}^\nu$,*

$$|\varphi_{n,\omega}(x)| \leq C_\omega e^{\gamma |x_{n,\omega}|^\epsilon} e^{-\gamma |x - x_{n,\omega}|}. \tag{4.4}$$

Moreover, if $q > 0$ and $\psi \in L^2({}^\nu)$ decays exponentially with mass $\theta > 0$, then, with probability 1, there exists a constant $C_{\psi,\omega}$ such that,

$$||\, |X|^q P_I(H(\omega)) e^{-iH(\omega)t} \psi||^2 \leq C_{\psi,\omega}.$$

Analysing the proofs of the previous two sections, one sees that the only missing ingredient for the proof of Theorem 4.2 is an analog of Lemma 3.4, stated as Proposition 4.3 below. Indeed, the arguments of sections 2 and 3 are readily transcribed to the continuous case, provided one makes the following adaptation. First, one replaces equation (3.3) by the following equality: if $\varphi \in L^2(R^\nu)$ satisfies $H\varphi = E\varphi$ for some E, then for $\Lambda_L(x) \subset {}^\nu$ so that $E \notin \sigma(H_{\Lambda_L(x)})$:

$$\chi_{4\delta,x}\varphi = \chi_{4\delta,x} R_{\Lambda_L(x)}(E) W_{L,x}\varphi.$$

Secondly, one defines $x_*(\varphi)$, the analog of x_* in Lemma 3.5, as follows. If φ belongs to $L^2({}^\nu)$ and $H\varphi = E\varphi$, then

$$\sup_{y \in 4\delta^\nu} \left\{ \int_{\Lambda_{4\delta}(y)} |\varphi(x)|^2 dx \right\} = \int_{\Lambda_{4\delta}(x_*(\varphi))} |\varphi(x)|^2 dx.$$

So, writing $x_{n,\omega} \equiv x_*(\varphi_{n,\omega})$, and redefining the annular $A_{k+1}(x_{n,\omega})$ as $\Lambda_{2bL_{k+1}}(x) \backslash \Lambda_{2L_k+2R}(x_{n,\omega})$, one obtains the bound written in (4.4), but for $||\chi_{4\delta,x}\varphi_{n,\omega}||$ rather than for $|\varphi_{n,\omega}(x)|$. Then to get the pointwise estimate (4.4) apply Theorem 2.4 of [8], or decompose $||e^{\gamma|x_{n,\omega}|^\epsilon} e^{-\gamma|x-x_{n,\omega}|} \varphi_{n,\omega}||_{L^2}^2$ on boxes of size 4δ and centered on $4\delta k$, $k \in {}^\nu$, and apply Theorem IX.26 of [21].

It therefore remains to state and prove the analog of Lemma 3.4. Following [13] let's denote by $R(\gamma, L, x, y)$ the set

$$R(\gamma, L, x, y) \equiv \{\omega \in \Omega |\, \forall\, E \in I,\ \Lambda_L(x)\, or\, \Lambda_L(y)\, is\, (\gamma, E) - regular\}.$$

As in the discrete case, for all $\alpha \in]1, 2[$ and $L_0 > 1$ we define the sequence $(L_k)_{k\in}$ by $L_{k+1} = L_k^\alpha$.

Proposition 4.3 *For any $\gamma \in]0, \gamma_0[$, $p > \nu$ and $\alpha \in]1, 2-2\nu/(p+2\nu)[$ there exist $L_* = L_*(\gamma, I, \alpha)$ such that if [H1](γ_0, I, p, L_0) and [H2](I,$\tilde{L}$) hold for $L_0 > L_*$, $L_0 > \tilde{L}$, then for all $k \geq 0$:*

$$|x - y| > L_k + 2R \implies (R(\gamma, L_k, x, y)) > 1 - \frac{1}{L_k^{2p}}.$$

The proof of Proposition 4.3 follows upon adapting the arguments of the proof of Theorem 2.2 in [13] to the continuum. Various authors [6, 19] have written up versions of the multi-scale analysis for continuous Schrödinger operators, but, to our knowledge, the version we need is not available in the literature. Netherthless, nobody seems to doubt that any such argument can be carried over from the discrete to the continuous case. Since multi-scale arguments are in addition to this painful, we have chosen to put the proof of Proposition 4.3 in the appendix, while making an effort to give a clear, complete and relatively simple argument.

5 Applications

We briefly indicate two applications of Theorem 4.2 to an Anderson model on $^{\nu}$ [6] and to random Schrödinger operators with a magnetic field.

The Anderson tight-binding model

Here the free Hamiltonian H_0 of equation (4.1) is $-\Delta$. We suppose, following [6], that

$$u(x) > \chi_{3/2}(x),$$

where $\chi_{3/2}$ is the characteristic function of the cube $\Lambda_{3/2}(0)$. Putting together Proposition 4.5 and Theorem 5.1 of [6], one has immediately from Theorem 4.2:

Theorem 5.1 *Suppose that μ has a L^∞ density $g(\lambda)$ with support $[0, \lambda_{max}]$ and disorder $\delta_0 = ||g||_\infty^{-1} > 0$ then, for energy $E_A > 0$ fixed and disorder δ_0 high enough, or for disorder $\delta_0 > 0$ fixed and energy E_A low enough, the conclusions of Theorem 4.2 hold on $[0, E_A]$.*

The Landau Hamiltonian

Here the Hamiltonian has the general form described in equation (4.1), *i.e.* $A \neq 0$ and $V_{per} = 0$. Although the result is still valid in arbitrary dimension under further assumptions (see [2]) we prefer to state the application in the well-known two dimensional version [3] [7] [23]. In that case, the vector potential A is given by

$$A = \frac{B}{2}(x_2, -x_1),$$

with $B > 0$. Recall that the spectrum of the free Landau Hamiltonian H_0 consists of a sequence of eigenvalues

$$E_n(B) = (2n+1)B, \ n \in .$$

We suppose that $u > 0$, $supp\, u \subset B(0, 1/\sqrt{2})$, and that there exist C_0 and $r_0 > 0$ such that $u_{|B(0,r_0)} > C_0$. We suppose that the common measure μ (of the $\lambda_i(\omega)$) has a bounded density function $g \in C_0^2()$, g being even and positive for almost every $\lambda \in \ supp\, g$.

Under those assumptions it is well-known that $M_0 = \sup\{|V_\omega(x)|, \ x, \omega\} < +\infty$. Let's define the following bands:

$$I_0(B) = [-M_0, B - \epsilon_0(B)],$$
$$I_{n+1}(B) = [E_n(B) + \epsilon_n(B)), E_{n+1}(B) - \epsilon_n(B)]$$

with some $\epsilon_n(B) > 0$. It follows from [7] and Theorem 4.2:

Theorem 5.2 *Let V be as described above. Then for B high enough, there exist some $\epsilon_n(B) = O(B^{-1})$ such that the conclusions of Theorem 4.2 hold on each interval $I_n(B)$, $n \in$.*

A similar result holds for the model treated by Wang [23], where u can be negative and its support is included in $B(0, r)$, $0 < r < 1$.

A Appendix

We turn to the proof of Proposition 4.3. Let us point out that our definition of a $(\gamma, E)-regular$ box (Definition 4.1) differs slightly from the one in [6]. This difference will allow us to free ourselves from most of the difficulties due to the use of an auxiliary lattice in [6] and [19]. We need the following two concepts:

Definition A.1 *Let $r > 0$. Two boxes Λ_1 and Λ_2 will be called r non-overlapping iff $d(\Lambda_1, \Lambda_2) > 2r$.*

Note that, if Λ_1 and Λ_2 are $R\, non-overlapping$, then, since supp $u \subset [-R, R]^\nu$, two events depending respectively on the λ_i with i in Λ_1 and in Λ_2 are necessarily independent.

Definition A.2 *Let $\beta \in]0, 1[$ be given. A box Λ_L will be called non resonant at energy E (we'll write $E - NR$) if*

$$||R_{\Lambda_L}(E)|| \leq 2e^{L^\beta}$$

This means, in other words, that $d(E, \sigma(H_{\Lambda_L})) > (1/2)e^{-L^\beta}$.

Remark that the commutator W_L does not appear in Definition A.2 as it did in Definition 4.1. In fact one can replace W_L with the characteristic function $\tilde{\chi}_{L,x}$ defined above (see [6] for the Anderson case and [7] Lemma 5.1 and lines (5.27 - 5.29) if $A \neq 0$) as follows. There exists a constant $C(\delta, I)$ with:

$$||\chi_{4\delta,x} R_{\Lambda_L(x)}(E) W_{L,x}|| \leq C(\delta, I) ||\chi_{4\delta,x} R_{\Lambda_L(x)}(E) \tilde{\chi}_{L,x}||. \tag{A.1}$$

Remark then that this last bound tells us that

$$\Lambda_L \, is \, E - NR \Longrightarrow ||\chi_{4\delta} R_{\Lambda_L}(E) W_L|| \leq 2C(\delta, I) e^{L^\beta} \tag{A.2}$$

An essential ingredient of the proof of Proposition 4.3 is the following deterministic lemma.

Lemma A.3 *Let $L = l^\alpha$ with $\alpha \in]1, 2[$ and $x \in {}^\nu, l > 12R$. Denote by s_ν the number of faces of a cube in dimension ν. Assume that for some*

$$\gamma > \left(27 l^\beta + 4(\nu - 1) ln(l/\delta)) + 4 ln(2 s_\nu)\right) / l,$$

with δ and β defined previously, and for some energy E
i) $\Lambda_L(x)$ is $E - NR$;
ii) Each box of size $4j(l+R)$, j=1,2,3, centered in $x + l^\nu$ and contained in $\Lambda_L(x)$ is E-NR;
iii) Among all the $(\gamma, E)-singular$ boxes of size l contained in $\Lambda_L(x)$, there are no more than three that are two by two R non-overlapping.

Then $\Lambda_L(x)$ *is* $(\gamma', E) - regular$ *with*

$$\gamma' = \gamma\left(1 - \frac{27}{l^{\alpha-1}}\right) - \frac{2}{l^{\alpha(1-\beta)})} - \frac{ln\left(2s_\nu C(l/\delta)^{\nu-1}\right)}{l}, \tag{A.3}$$

with $C = C(\delta, I)$ *defined in (A.1).*

In [6], Combes and Hislop have proved a simpler version of this result. In fact, they have adapted to the continuous case a simplified version of [13] which is contained in chapter IX of [5] (see also [14]). But, as in the discrete case, this simplified version does not seem to suffice to obtain the results of this paper, since we need to obtain regular boxes at any size with good probability, uniformly in a compact interval of energy, and no longer at some fixed energy E. So we turn to [13] and adapt it to the continuous case.

Proof of Lemma A.3: The aim is to bound $||\chi_{4\delta,x}R_{\Lambda_L(x)}(E)\ W_{L,x}||$. Using first inequality (A.1), we are reduced to control $||\chi_{4\delta,x}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||$. This is achieved in (A.12) below. We recall that $\delta > 1$ has been chosen small, so, without loose of generality one can suppose $l > 3\delta$.

In order to achieve our goal, we will recursively construct inside $\Lambda_L(x)$ a chain of n boxes $\Lambda_l(v_k)$, $k = 0, ..., n-1$, being most of the time $(\gamma, E) - regular$, and starting at $v_0 \equiv x$. At each step of this process, we will use the geometric resolvent equation as follows.

Let $l' > 3\delta$ and consider any box $\Lambda_{l'}(z) \subset \Lambda_L(x)$ with $d(\Lambda_{l'}(z), \tilde{\Lambda}_L(x)) > 0$. For $E \notin \sigma(H_{\Lambda_{l'}(z)}) \cup \sigma(H_{\Lambda_L(x)})$, the resolvent identity (4.2) gives:

$$\chi_{4\delta,z}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x} = \chi_{4\delta,z}R_{\Lambda_{l'}(z)}(E)W_{l',z}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}.$$

The support of $W_{l',z}$ can be covered by a family of boxes $\Lambda_{4\delta}(v) \subset \Lambda_L(x)$, indexed by points v that satisfy $|v - z| = l'/2$ and so that the sum over all the corresponding characteristic functions $\chi_{4\delta,v}$ is equal to 1 on $SuppW_{l',z}$ (Note that $s_\nu(1+l'/3\delta)^{\nu-1} < s_\nu(l'/\delta)^{\nu-1}$ such boxes suffice). Clearly, there exists one of those v for which

$$\begin{aligned} &||\chi_{4\delta,z}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}|| \\ &\qquad \leq s_\nu(l'/\delta)^{\nu-1}||\chi_{4\delta,z}R_{\Lambda_{l'}(z)}(E)W_{l',z}||\ ||\chi_{4\delta,v}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||. \end{aligned} \tag{A.4}$$

Suppose now in addition that $\Lambda_{l'}(z)$ is $(\gamma, E) - regular$. Then we immediately have:

$$||\chi_{4\delta,z}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}|| \leq s_\nu(l'/\delta)^{\nu-1}e^{-\gamma l'/2}||\chi_{4\delta,v}R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||. \tag{A.5}$$

Apply now this argument to $\Lambda_l(x)$, and set $v_0 = x, v_1 = v$. Repeat the process as long as $\Lambda_l(v_k)$ is $(\gamma, E) - regular$ and stays away from the boundary of $\Lambda_L(x)$. Clearly, there exists some $k^* \geq 0$ so that for all $0 \leq k < k^*$, $\Lambda_l(v_k)$

is $(\gamma, E) - regular$ and $\Lambda_l(v_k) \subset \Lambda_L(x)$, $d(\Lambda_l(v_k), \tilde{\Lambda}_L(x)) > 0$, whereas one of these conditions fails for $\Lambda_l(v_{k^*})$. As a result, if $k^* > 0$, we have

$$||\chi_{4\delta,x} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}|| \leq (s_\nu (l'/\delta)^{\nu-1})^{k^*} e^{-\gamma k^* l/2} ||\chi_{4\delta,v_{k^*}} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||. \tag{A.6}$$

If $k^* = 0$, this equation holds trivially. The important point here is that we gained a factor $e^{-\gamma k^* l/2}$: if there were no $(\gamma, E) - singular$ boxes Λ_l inside $\Lambda_L(x)$, this would end the proof. Indeed, in that case, the process could only end when $\Lambda_l(v_{k^*})$ gets too close to the boundary of $\Lambda_L(x)$, implying $k^* \geq (L/l)$, so that (A.6) immediately yields the result upon using hypothesis (i).

Of course, there may be $(\gamma, E) - singular$ boxes in $\Lambda_L(x)$ and we now use hypothesis (iii) of the lemma to control the case in which the above process stops because $\Lambda_l(v_{k^*})$ is $(\gamma, E) - singular$ and v_{k^*} is at a distance greater than $12(l+R)$ from the boundary of $\Lambda_L(x)$. Using hypothesis (iii) and drawing a few pictures one easily convinces oneself that one can pack all the singular boxes of size l in $t \leq 3$ slightly bigger and disjoint boxes $\Lambda_{l_i} \subset \Lambda_L(x)$, centered in $x + l^\nu$, and so that each box $\Lambda_l(z)$, where z belongs to the edge of one of those Λ_{l_i}, is $(\gamma, E) - regular$. More precisely, the l_i are taking on one of the values $4j(l+R)$, $1 \leq j \leq 3$, they satisfy $\sum_{i=1}^t l_i \leq 12(l+R) \equiv l_0$, and the two following facts are simultaneously true:

$$(1) \qquad \begin{pmatrix} d(z, \partial\Lambda_L(x)) \geq l/2 \\ z \in \Lambda_L(x) \backslash \bigcup_{i=1}^r \Lambda_{l_i} \end{pmatrix} \Longrightarrow (\Lambda_l(z)\, is\, (\gamma, E) - regular)\,. \tag{A.7}$$

$$(2) \qquad \text{if } z \text{ belongs to the edge of one of the } \Lambda_{l_i}, \text{ then } z \notin \textstyle\bigcup_{i=1}^r \Lambda_{l_i}. \tag{A.8}$$

So, if $\Lambda_l(v_{k^*})$ is $(\gamma, E) - singular$, there exists $i \in \{1, .., t\}$ so that $\Lambda_l(v_{k^*}) \subset \Lambda_{l_i}$. Define a new family of points v on the boundary of Λ_{l_i}, such that the boxes $\Lambda_{4\delta}(v)$ cover $supp\, W_{l_i}$. Then use the equivalent of (A.4) with $\Lambda_{l'}(z) = \Lambda_{l_i}$ and $z = v_{k^*}$. This produces some v_{k^*+1} that belongs to the edge of Λ_{l_i}, and consequently (A.7)-(A.8) implies that $\Lambda_l(v_{k^*+1})$ is $(\gamma, E) - regular$, provided $d(v_{k^*+1}, \partial\Lambda_L(x)) > l/2$. By checking how v_{k^*+1} is positioned with respect to v_{k^*} one sees that the latter condition is satisfied because v_{k^*} is at least at a distance $12(l+R)$ from $\partial\Lambda_L(x))$. Use now Hypothesis (ii) and apply once again (A.4) to $\Lambda_l(v_{k^*+1})$, to obtain that for some v_{k^*+2} on the edge of $\Lambda_l(v)$, with $|v_{k^*} - v_{k^*+2}| \leq l_i$:

$$||\chi_{4\delta,v_{k^*}} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}|| \leq 2s_\nu^2 \left(\frac{l_0 l}{\delta^2}\right)^{\nu-1} e^{-\gamma l/2 + l_i^\beta} ||\chi_{4\delta,v_{k^*+2}} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||. \tag{A.9}$$

Using the preliminary condition on γ, this leads to

$$||\chi_{4\delta,v_{k^*}} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}|| \leq ||\chi_{4\delta,v_{k^*+2}} R_{\Lambda_L(x)}(E)\tilde{\chi}_{L,x}||. \tag{A.10}$$

This is the way in which we get past a singular box $\Lambda_l(v_{k^*})$ far from the edge of $\Lambda_L(x)$. We have now completely described the recursive construction of the

v_k and it is clear that the process grinds to a standstill only when, for some n, $\Lambda_l(v_n)$ is too close to $\partial\Lambda_L(x)$. From (A.10) one sees that, when meeting a singular box, we do not gain a factor $\exp^{-\gamma l/2}$, so that we have to assure ourselves this does not happen to often before the process ends. We therefore need to count how many of the boxes $\Lambda_l(v_k)$, $0 \leq k \leq n$ are regular. Since $|v_{k+1} - v_k| = l/2$ in that case and $|v_{k+2} - v_k| \leq l_i$ if not, it is not hard to see that the process cannot stop before $n = n^* = n_1^* + 2t$ where

$$n_1^* = \left[\frac{L/2 - \sum_{i=1}^t l_i}{l/2} \right],$$

or

$$[L/l] - 27 \leq n_1^* \leq L/l. \tag{A.11}$$

Hence, using (i) of the lemma, one has

$$||\chi_{4\delta,x} R_{\Lambda_L(x)}(E) \tilde{\chi}_{L,x}|| \leq \left(s_\nu (l/\delta)^{\nu-1} e^{-\gamma l/2} \right)^{n_1^*} 2e^{L^\beta}. \tag{A.12}$$

Putting together relations (A.1) and (A.12) and using (A.11) as well as the definition of γ' stated in the lemma leads to the desired result. $\square$

Proof of Proposition 4.3: Take $\alpha \in]1, 2 - 2\nu/(p+2\nu)[$ and $\gamma \in]0, \gamma_0[$. Let $L \equiv L_{k+1} = L_k^\alpha$, and use equation (A.3) to produce a sequence of exponents $\gamma_k \in]0, \gamma_0]$. It will be enough to show that

a) $\forall k \geq 0,\ \gamma \leq \gamma_k \leq \gamma_0$;

b) $\forall k \geq 0 :\ |x - y| > L_{k+1} + 2R \Longrightarrow (R(\gamma_k, L_{k+1}, x, y)) > 1 - 1/L_{k+1}^{2p}$.

To prove (a), choose $L_0 > 0$: the sequence $(\gamma_k)_{k\geq 0}$ produced by repeatedly using (A.3) decreases, so for all $k \geq 0$, $\gamma_{k+1} \leq \gamma_k \leq \gamma_0$. Then, using equation (A.3), it is clear there exists $L_* = L_*(\gamma, \gamma_0, \beta, \alpha, \nu)$ so that, if $L_0 > L_*$, the sequence $(\gamma_k)_{k\geq 0}$ satisfies

$$\begin{aligned} 0 < \sum_{k=0}^{\infty} (\gamma_k - \gamma_{k+1}) &\leq 15\gamma_0 \sum_{k=0}^{\infty} L_k^{1-\alpha} + \sum_{k=0}^{\infty} L_k^{-\frac{1}{2} min(1, \alpha(1-\beta))} \\ &\leq \gamma_0 - \gamma \end{aligned}$$

Hence (a) follows, and we turn to (b), which clearly follows from:

$$|x - y| > L_{k+1} + 2R \Longrightarrow$$
$$\left(\begin{array}{l} \forall E \in I, \textit{the hypotheses of Lemma A.3 with } \gamma = \gamma_k \\ \textit{and } L = L_{k+1} \textit{ are satisfied for either point } x \textit{ or } y \end{array} \right) > 1 - 1/L_{k+1}^{2p}. \tag{A.13}$$

Firstly, since for all $k \geq 0$, $\gamma_k \geq \gamma$, provided L_0 is large enough, one has:

$$\forall k \geq 0,\ \gamma_k > \frac{1}{L_k} \left(27 L_k^\beta + 4(\nu - 1) ln(L_k/\delta)) + 4 ln(2 s_\nu) \right)$$

Let's now define $I_l = \cap\{E \in \ , \ d(E,I) \leq e^{-l^\beta}/2\}$ and $\sigma'(H_{\Lambda_l}) = \sigma(H_{\Lambda_l}) \cap I_l$. It is easy to estimate the probability that the distance between the respective spectrum of two R non-overlapping boxes Λ_{l_1} and Λ_{l_2}, $(l_1, l_2 > \tilde{L}$, is greater than η, $0 < \eta < 1$. Using first (4.3) and then Hypothesis [H2], one has, with some abuse of notations:

$$\begin{aligned}
(d(\sigma'(H_{\Lambda_{l_1}}), \sigma'(H_{\Lambda_{l_2}})) < \eta) &\leq \int \sum_{E \in I \cap \sigma'(H_{\Lambda_{l_2}})} {}_{\Lambda_{l_1}} \big(d(\sigma'(H_{\Lambda_{l_1}}), E) < \eta\big)\, d\omega_2 \\
&\leq C_W |\Lambda_{l_1}| \eta \, (Tr(E_{\Lambda_{l_2}}(I_l))) \\
&\leq C_W^2 (|I| + 1) |\Lambda_{l_1}| \, |\Lambda_{l_2}| \eta \\
&= C_{W,I} |\Lambda_{l_1}| \, |\Lambda_{l_2}| \eta. \qquad \text{(A.14)}
\end{aligned}$$

Hence, for all k, and writing for convenience $L \equiv L_{k+1}$ and $l = L_k$: if $|x - y| > L + 2R$, it follows from (A.14) that

$$\begin{aligned}
&(\exists\, u \in (x + l^\nu) \cap \Lambda_L(x),\ v \in (y + l^\nu) \cap \Lambda_L(y) \ and \ l_1,\ l_2 = \ L \ or \\
&\quad 4j(l + R)l,\ j = 1, .., 3 \ with \ \Lambda_{l_1}(u) \subset \Lambda_L(x) \ and \ \Lambda_{l_2}(v) \subset \Lambda_L(y), \\
&\quad with\, d(\sigma'(H_{\Lambda_{l_1}}), \sigma'(H_{\Lambda_{l_2}})) < \eta) \leq C_{W,I} (L/l)^{2\nu} |\Lambda_L|^2 \eta. \qquad \text{(A.15)}
\end{aligned}$$

But consider this elementary exercise in logic: let A_i and B_j, i and $j = 1, .., J$ be $2J$ intervals, then

$$\begin{aligned}
&(\forall\, i, j = 1, ..., J,\ d(A_i, B_j) > \eta) \\
&\iff \ (\forall\, E \in \ \forall\, i, j = 1, ..., J,\ (d(E, A_i) > \eta/2 \, or \, d(E, B_j) > \eta/2)) \\
&\iff \ \left(\begin{array}{c} \forall\, E \in \ ,\ either\, (\forall\, i = 1, ..., J,\ d(E, A_i) > \eta/2) \\ or\ (\forall\, j = 1, ..., J,\ d(E, B_j) > \eta/2) \end{array} \right).
\end{aligned}$$

This, combined with inequality (A.15) and $\eta = e^{-L_k^\beta}$, gives for all $k \geq 0$, and if $|x - y| > L_{k+1} + 2R$ that

$$\left(\begin{array}{l} \forall\, E \in I,\ (i) \, and \, (ii) \, of \, Lemma \, A.3 \, with \, \gamma = \gamma_k \\ and \, L = L_{k+1} \, are \, satisfied \, for \, either \, point \, x \, or \, y \end{array} \right) > 1 - 1/L_{k+1}^{2p+1}. \qquad \text{(A.16)}$$

Let's finish the proof: for L_0 large enough, Hypothesis [H1](γ, I, p, L_0) gives (A.13) at rank 0. Suppose it is true at rank k: points (i) and (ii) of Lemma A.3, with $\gamma = \gamma_k$ and $L = L_{k+1}$, are satisfied for either points x or y, with probability evaluated line (A.16). Now,

$$\begin{aligned}
&(for \ any \ E \in I, (iii) \ of \ Lemma \ A.3 \ holds) \\
&\quad = \ 1 - (\exists\, E \in I \, s.t. \ there \ are \ at \ least \ 4 \ R \, non-overlapping \\
&\qquad\qquad (\gamma_k, E) - singular \ boxes \ \Lambda_{L_k} contained \, in \, \Lambda_{L_{k+1}}(x)) \\
&\quad \geq \ 1 - (\exists\, E \in I \, s.t. \ there \ are \ at \ least \ 2 \ R \, non-overlapping
\end{aligned}$$

$$
\begin{aligned}
&(\gamma_k, E) - singular\ boxes\ \Lambda_{L_k} contained\, in\, \Lambda_{L_{k+1}}(x))^2 \\
\geq\ & 1 - \left(\frac{(L_{k+1}/L_k + 1)^{2\nu}}{L_k^{2p}} \right)^2 , \qquad \text{(A.17)}
\end{aligned}
$$

where we obtained the last inequality using the recurrence hypothesis. Hence, since $\alpha < 2 - 2\nu/(p + 2\nu)$, combining (A.16), (A.17), there exists a constant $L_* = L_*(p, \gamma, \gamma_0, \nu)$ such that if $L_0 > L_*$ and $|x - y| > L_{k+1} + 2R$:

$$
(R(\gamma_k, L_{k+1}, x, y)) > 1 - \frac{1}{L_{k+1}^{2p}}.
$$

Use now that $\gamma_k > \gamma$, and Proposition 4.3 is proved. □

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