A New Representation For (1+1)-Dimensional Landau-Lifshitz Model

S.B. Rutkevich

Institute of Physics of Solids and Semiconductors,
P. Brovki St. 17, 220072 Minsk, Belarus
E-mail: lttt@ifttp.basnet.minsk.by

ABSTRACT

Recently an exactly solvable classical Hamiltonian model was obtained, which appears in analysis of the correlation function in a free-fermion model. It is proved here, that the above mentioned Hamiltonian model is equivalent to the Landau-Lifshitz model of the one-dimensional biaxial ferromagnet.

It is widely excepted that correlation functions in quantum integrable models are closely related with some classical Hamiltonian models, which in turn can be solved exactly by the inverse scattering method (see [1, 2, 3]). Recently such a relationship was established for a certain free-fermion model [4], which is equivalent to the XY-spin chain model in the double scaling limit [1]. It was shown that the two-point correlation function in the free-fermion model is related with the classical integrable model defined by the Hamiltonian [4]:

$$\mathcal{H} = \frac{1}{2}\int \mathrm{d}x \Big\{ \pi^2(x)[u'^2(y) + (1/4)R(u)] + \frac{[u''(x) + (1/8)\, dR(u)/du]^2}{[u'^2(x) + (1/4)R(u)]} - k^2\, u^2(x) \Big\} \tag{1}$$

where

$$R(u) = (1 - u^2)(1 - k^2 u^2), \tag{2}$$

$u(x)$ and $\pi(x)$ are the canonical coordinate and momentum functions, and k is the elliptic module.

A natural question arises in connection with Hamiltonian (1): whether it describes a new exactly solvable model, or it corresponds to some well-known model, written in unusual variables, however. The answer on this question is obtained in the present letter. We prove that above classical model is canonically equivalent to the complex generalization of the Landau-Lifshitz model of the biaxial one-dimensional ferromagnet.

Landau-Lifshitz model is defined by the Hamiltonian density [6]

$$H_{LL}(x) = \frac{1}{2}\left[\left(\frac{\partial \mathbf{S}}{\partial x}\right)^2 - J(\mathbf{S})\right] \tag{3}$$

where $\mathbf{S}(x) = \Big(S_1(x), S_2(x), S_3(x)\Big)$ is a real vector on a unit sphere $\mathbf{S}(x) \in \mathrm{S}^2 \subset \mathbf{R}^3$:

$$\mathbf{S}^2(x) = \sum_{a=1}^{3} S_a^2(x) = 1, \tag{4}$$

$J(\mathbf{S})$ is the diagonal quadratic form

$$J(\mathbf{S}) = J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2$$
$$J_1 \le J_2 \le J_3, \qquad J_1 + J_2 + J_3 = 0 \tag{5}$$

The equation of motion is Hamiltonian

$$\dot{\mathbf{S}} = \{\mathcal{H}_{LL}, \mathbf{S}\} \tag{6}$$

where $\mathcal{H}_{LL}$ is the Hamiltonian: $\mathcal{H}_{LL} = \int \mathrm{d}x\, H_{LL}(x)$. The Poisson structure is induced by the Poissin brackets

$$\{S_a(x), S_b(y)\} = -\epsilon_{abc} S_c(x) \delta(x - y) \tag{7}$$

Above model is known as a "uiversal" (in a certain sense) integrable model, which contains sine-Gordon and nonlinear Schrödinger models as limit cases. We shall consider the

[1] The double scaling limit of the XY-model was introduced by Vaidya and Tracy [5], and studied in detail by Jimbo *et al* [1].

comlex generalization of the model (3-7) supposing , that $\mathbf{S}(x)$ is a vector on a *complex* unit sphere (4): $\mathsf{S}^2 \subset \mathbf{C}^3$.

First, let us introduce variables $\psi(x)$ and $\psi^*(x)$ on the sphere by the relations

$$\psi = \frac{S_1 + iS_2}{(1 - S_3)^{1/2}}, \quad \psi^* = \frac{S_1 - iS_2}{(1 - S_3)^{1/2}}$$

Since the Poisson bracket on these functions is canonical

$$\{\psi(x), \psi^*(y)\} = -i\delta(x - y)$$

apart from the factor $-i$, the action functional $\mathcal{S}$ for the model (3-7) can be written as

$$\mathcal{S} = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}x L(x, t)$$

with the Lagrangian density $L(x, t)$ given by

$$L = -i\psi^* \dot{\psi} - H_{LL}$$

Next, rewrite this Lagrangian density in terms of another pair of variables $w(x)$ and $w^*(x)$ which give the stereographic projection of the sphere S^2:

$$S_1 + iS_2 = \frac{2w}{1 + ww^*}, \quad S_1 - iS_2 = \frac{2w^*}{1 + ww^*}, \quad S_3 = \frac{1 - ww^*}{1 + ww^*} \tag{8}$$

or, equivalently

$$\psi = \left[\frac{2w}{w^*(1 + ww^*)}\right]^{1/2}, \quad \psi^* = \left[\frac{2w^*}{w(1 + ww^*)}\right]^{1/2} \tag{9}$$

The resulting expression takes the form

$$L \simeq -\frac{2i\dot{w}}{w(1 + ww^*)} - 2\frac{(w'/w)'}{(1 + ww^*)} + 2\frac{w^*(w')^2}{w(1 + ww^*)^2} - \frac{1}{2}\frac{(J_2 - J_1)(w^2 + w^{*2}) + 6J_3ww^*}{(1 + ww^*)^2} \tag{10}$$

¿From now on the equivalence $\simeq$ implies that we ignore terms having the structure of full derivatives in time or in coordinate. It shoulg be pointed out, that variables w and w^* which were complex conjugate to each other when $\mathbf{S}(x) \in \mathbf{R}^3$, become independent ones in the considered complex case $\mathbf{S}(x) \in \mathbf{C}^3$. Varying the action corresponding to the Lagrangian density (10) with respect to w^*, one obtains

$$(-i\dot{w} - w'')(1 + ww^*) + 2w^*(w')^2 - \frac{J_2 - J_1}{2}(w^3 - w^*) + \frac{3}{2}J_3w(1 - ww^*) = 0 \tag{11}$$

This representation of the Landau-Lifshitz equation has been widely used in study of its soliton and multisoliton solutions (see [7] and references therein).

We would like to note the following. Equality (11) is a complicated nonlinear partial differential equation with respect to the function $w(x)$. However, it becomes an elementary linear algebraic equation when regarded with respect to the function $w^*(x)$. Eliminating $w^*(x)$ from the Lagrangian density (10) by use of (11), we find after some algebra

$$L \simeq -\frac{1}{2}\,\frac{\dot{u}^2 - [u''(x) + (1/2)\,(dP(u)/du)]^2}{[u'^{\,2}(x) + P(u)]} + (J_2 - J_1)\,u^2 \tag{12}$$

Here we have denoted

$$P(u) = \frac{J_2 - J_1}{4}\,(1 + u^4) - \frac{3J_3}{2}\,u^2, \quad u = 1/w$$

Rescaling of variables

$$u \to \sqrt{k}\,u, \quad x \to \sqrt{\frac{k}{J_2 - J_1}}\,x, \quad t \to \frac{k}{J_2 - J_1}\,t, \quad L \to -\frac{J_2 - J_1}{k}\,L$$

with k obeying the relation $k^2 - \frac{6J_3}{J_2 - J_1}k + 1 = 0$ tranforms (12) to

$$L \simeq \frac{1}{2}\left\{\frac{\dot{u}^2 - [u''(x) + (1/8)\,dR(u)/du]^2}{[u'^{\,2}(x) + (1/4)R(u)]} + k^2\,u^2\right\} \tag{13}$$

where we have used notation (2) for $R(u)$. This is exactly the Lagrangian density corresponding to the model (1). Thus, we have proved the equivalence of the complex generalization of the Landau-Lifshitz model with the Hamiltonian model (1) obtained in reference [4].

In conclusion we present another form of the Lagrangian density (13):

$$L \simeq \frac{1}{2}\left\{\frac{\dot{z}^2 - z''^{\,2}}{1 + z'^{\,2}} - (1 + 3z'^{\,2})\wp(z)\right\} \tag{14}$$

Variables u and z are related by the formula $u = \mathrm{sn}[(z - 2K)/2]$. Here sn denotes elliptic sinus with elliptic module k. The function $u(z)$ has primitive periods $8K$ and $4iK'$ in the complex z-plane. Primitive periods of the elliptic Weierstrass function $\wp(z)$ in (14) are reduced by half and equel $4K$ and $2iK'$.

It is surprising, that the Landau-Lifshitz model allows such a compact representation (14).

I wish to thank A.S. Kovalev and I.M. Babich for helpful discussion.

References

[1] M. Jimbo, T. Miwa, Y. Mori and M. Sato, *Physica* **1D** (1980) 80

[2] V. E. Korepin, A. G. Izergin, and N. M. Bogoliubov, 1993 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press)

[3] F.H.L. Eßler, H. Frahm, A.B. Its and V.E. Korepin, *J. Phys. A: Math. Gen* **29** (1996) 5916

[4] S.B. Rutkevich, *J. Phys. A: Math. Gen* **30** (1997) 3883

[5] H.G. Vaidya and C.A. Tracy, *Phys. Letts.* **68A** (1978) 378

[6] L.D. Faddeev and L.A. Takhtajan, 1986 *Hamiltonian Method in the Theory of Solitons* (Berlin: Springer)

[7] A.M. Kosevich, B.A. Ivanov and A.S. Kovalev, *Phys. Rep.* **194** (1990) 117