SWKB for the Angular Momentum

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Abstract

It has been recently shown [M. Robnik and L. Salasnich, *J. Phys. A: Math. Gen.*, **30**, 1719 (1997)] that the WKB series for the quantization of angular momentum L converges to the exact value $L^2 = \hbar^2 l(l+1)$, if summed over all orders, and gives the Langer formula $L^2 = \hbar^2 (l+1/2)^2$ at the leading order. In this work we solve the eigenvalue problem of the angular momentum operator by using the supersymmetric semiclassical quantum mechanics (SWKB), and show that it gives the correct quantization already at the leading order.

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The Wentzel–Kramers–Brillouin (WKB) semiclassical method is one of the most useful techniques for solving the Schrödinger equation. It allows to obtain approximate analytic expressions for the wavefunctions and energy spectra[1)] Usually, what is used is the torus quantization, which is just the leading order of the $\hbar$-expansion of the WKB method. Higher terms can be calculated with a recursion formula in one degree systems, but are generally unknown in systems with more than one degree of freedom (see Robnik and Salasnich[2)]).

It has been observed since a long time that the WKB method, in its leading approximation, when applied to three–dimensional spherically symmetric problems yields wrong results, unless one replaces the correct value $L^2 = \hbar^2 l(l+1)$ with the Langer expression[3)] $L^2 = \hbar^2 (l+1/2)^2$. Further, when higher order terms are included, the Langer correction needs modifications at each order of approximation. It has recently been shown by Robnik and Salasnich[2)] that the Langer term and its first corrections are the first terms of an infinite series. By making a guess on the higher order terms, they could sum all the series, recovering the exact quantum result.

In the framework of the supersymmetric semiclassical quantization (SWKB), Comtet, Bandrauk and Campbell[4)] obtained at the leading order the exact quantization of the radial part of the Kepler problem by using the correct value $L^2 = \hbar^2 l(l+1)$. The aim of the present work is to complete their result. In fact, we show that also the exact quantization of the angular momentum is obtained at the first order of the SWKB expansion.

The eigenvalue equation of the angular momentum operator is

$$\hat{L}^2 Y(\theta,\phi) = \lambda^2 \hbar^2 Y(\theta,\phi) \; , \tag{1}$$

with

$$\hat{L}^2 = \hat{P}_\theta^2 + \frac{\hat{P}_\phi^2}{\sin^2(\theta)} = -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} \right) - \hbar^2 \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\phi^2} \; . \tag{2}$$

After the substitution

$$Y(\theta,\phi) = T(\theta) e^{im\phi} \; , \tag{3}$$

we obtain the equation

$$T''(\theta) + \cot(\theta)T'(\theta) + \left(\lambda^2 - \frac{m^2}{\sin^2(\theta)}\right) T(\theta) = 0 \ , \tag{4}$$

where m is the azimuthal quantum number. This equation is exactly solvable. Its eigenvalues and eigenfunctions are well known from any text of quantum mechanics (see, *e.g.* Landau and Lifshitz[5)]): the former are $\lambda^2 = l(l+1)$, $l \geq m$; the latter are the associate Legendre polynomials.

Now we will briefly outline the WKB expansion of Eq. (4). First of all, it is important to notice that in (4) does not appear $\hbar$, therefore an expansion in powers of this parameter is not possible. To override this difficulty a small parameter ϵ is introduced[6)]:

$$\epsilon^2 T''(\theta) + \epsilon^2 \cot(\theta)T'(\theta) + \left(\lambda^2 - \frac{m^2}{\sin^2(\theta)}\right) T(\theta) = 0 \ . \tag{5}$$

The parameter ϵ, which will be set to 1 at the end of the calculation, has formally the same role of $\hbar$ as ordering parameter, and is equivalent to taking the limit $\lambda^2 \to \infty$ and/or $m \to \infty$. With the further substitution

$$T(\theta) = \exp\left(\frac{1}{\epsilon}\sum_{n\geq 0} S_n \epsilon^n\right) \ , \tag{6}$$

we find

$$(S_0')^2 + \left(\lambda^2 - \frac{m^2}{\sin^2(\theta)}\right) = 0 \ , \tag{7}$$

$$\sum_{k=0}^{n} S_k' S_{n-k}' + S_{n-1}'' + \cot(\theta) S_{n-1}' = 0 \ , \quad n > 0 \ . \tag{8}$$

The exact quantization is obtained by requiring the uniqueness of the wave function:

$$\oint dS = \sum_{k=0}^{\infty} \oint dS_k = 2\pi i \ n_\theta \ , \tag{9}$$

where we have now set $\epsilon = 1$. This integral is a complex contour integral which encircles the two turning points on the real axis. Obviously, it is derived from the requirement of the uniqueness of the complex wave function T. If one stops to the leading order

$$\oint dS_0 = \oint i\sqrt{\left(\lambda^2 - \frac{m^2}{\sin^2(\theta)}\right)}\, d\theta = 2\pi i\left(n_\theta + \frac{1}{2}\right) . \tag{10}$$

The integral is easily calculated and the condition

$$\lambda = n_\theta + m + \frac{1}{2} = l + \frac{1}{2} \tag{11}$$

is obtained. Otherwise, one may go further, compute all the terms S_n and recover the exact result $\lambda^2 = l(l+1)$, as done by Robnik and Salasnich[6)].

After these preliminaries, we enter into the main topic of this paper: we shall repeat the calculation using the supersymmetric theory (for a monography about the subject of SUSY quantum mechanics see, *e.g.*, Junker[7)]).

In order to put (4) in a standard form we make the replacement

$$T(\theta) = \frac{F(\theta)}{\sqrt{\sin(\theta)}} , \tag{12}$$

and obtain

$$F''(\theta) + \left[\left(\lambda^2 + \frac{1}{4}\right) + \frac{1}{\sin^2(\theta)}\left(\frac{1}{4} - m^2\right)\right] F(\theta) = 0 . \tag{13}$$

This equation has the standard form of the one–dimensional Schrödinger equation with $\hbar = 2M = 1$. Its eigenvalues are $(\lambda^2 + 1/4)$.

To perform the SWKB for the Eq. (4) or (13), it is necessary to calculate the ground state wave–function $T_0(\theta) = \sin^m(\theta)$ and its eigenvalue $\lambda_0 = m(m+1)$. Then we can define the supersymmetric (SUSY) potential

$$\Phi(\theta) = -\frac{d\ln(F_0(\theta))}{d\theta} = -\left(m + \frac{1}{2}\right)\cot(\theta) , \tag{14}$$

with

$$F_0(\theta) = T_0(\theta)\sqrt{\sin(\theta)} . \tag{15}$$

¿From Φ the two SUSY partner potentials and Hamiltonians may be defined

$$H_{\pm} = -\frac{d^2}{d\theta^2} + V_{\pm}(\theta) \ , \tag{16}$$

$$V_{\pm}(\theta) = \Phi^2(\theta) \pm \Phi'(\theta) \ . \tag{17}$$

The following statements hold : i) the ground-state energy of H_-, E^0_-, vanishes; ii) all other eigenvalues of H_-, E_-, coincide with that of H_+; iii) the spectrum of H_- and that of (12) differ by a constant:

$$E_- = \lambda^2 + \frac{1}{4} - \left(\lambda_0^2 + \frac{1}{4}\right) \ , \tag{18}$$

where $\lambda_0 = m(m+1)$ is the eigenvalue of the ground state of Eq. (4). We do not give here the proofs since they are standard results of the SUSY theory (see Ref. 7 and references therein for details).

A further important remark is that Φ gives rise to *shape–invariant* partner potentials: V_- and V_+ depend, besides the independent variable θ, from the parameter m. A couple of V_-, V_+, depending upon an independent variable x and a set of parameters $\{a_0\}$, are called shape–invariant if the relation holds:

$$V_-(a_0, x) = V_+(a_1, x) + R(a_1) \ , \tag{19}$$

with $\{a_1\}$ a new set of parameters and R a function of a_1 but not of x. In our case we obtain

$$V_-(m, \theta) = V_+(m-1, \theta) - 2m \ , \tag{20}$$

It has been shown by Dutt, Khare and Sukhatme[8)], and by Barclay and Maxwell[9)], that for this class of potentials: i) exact quantization rules are attained at the first order, and the results may be expressed in analytical form; ii) higher order terms are identically null.

Let us apply the WKB formalism to H_- of Eq. (16); the main lines of the calculation follow Eqns. (5–10), but a careful analysis (see Ref. 7, ch.

6) shows that the quantization conditions are to be modified: to the leading order one gets

$$\int_a^b \sqrt{E_- - \Phi^2(x)}\, dx = n_\theta \pi \tag{21}$$

with a, b roots of

$$E_- - \Phi^2(x) = 0 \tag{22}$$

This formula is also referred to as CBC formula, from Comtet, Bandrauk and Campbell[4]. Notice that: i) on the left hand side there appears Φ^2 instead of the full potential V_-; ii) in the right hand side the Maslov correction (the 1/2 term) does not appear any longer.

Substituting for Φ in (21), (22) the expression (14), one easily finds

$$\int_a^b \sqrt{E_- - \Phi^2(x)}\, dx = \pi \left[\sqrt{E_- + \left(m + \frac{1}{2}\right)^2} - \left(m + \frac{1}{2}\right) \right] = n_\theta \pi \ , \tag{23}$$

with $b = -a = \arctan \sqrt{\frac{(m+\frac{1}{2})}{E_-}}$. In conclusion, we have

$$E_- = \left(n_\theta + m + \frac{1}{2}\right)^2 - \left(m + \frac{1}{2}\right)^2 \ . \tag{24}$$

Now, by using Eq. (18) with $\lambda_0 = m(m+1)$, we get

$$\lambda^2 = (n_\theta + m)(n_\theta + m + 1) \ , \tag{25}$$

which, with the position $l = n_\theta + m$, yields the result sought.

Summarizing, we have demonstrated that, by using SUSY quantum mechanics, the eigenvalue problem of the angular momentum operator can be solved exactly within the semiclassical approximation and at the lowest order. Even if these results are a consequence of the theorems of SUSY theory, we think that an explicit derivation is not devoid of interest since, notwithstanding the importance of the three–dimensional spherically symmetric problems in physics, a detailed analysis of its angular part has always been lacking in literature until the work of Robnik and Salasnich[6]. That work, however, suffered from the fact that its results were based upon some conjectures which, though fully reasonable, were not rigorously demonstrated.

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