Gravity from Dirac Eigenvalues

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Abstract

We study a formulation of euclidean general relativity in which the dynamical variables are given by a sequence of real numbers λ_n, representing the eigenvalues of the Dirac operator on the curved spacetime. These quantities are diffeomorphism-invariant functions of the metric and they form an infinite set of "physical observables" for general relativity. Recent work of Connes and Chamseddine suggests that they can be taken as natural variables for an invariant description of the dynamics of gravity. We compute the Poisson brackets of the λ_n's, and find that these can be expressed in terms of the propagator of the linearized Einstein equations and the energy-momentum of the eigenspinors. We show that the eigenspinors' energy-momentum is the Jacobian matrix of the change of coordinates from the metric to the λ_n's. We study a variant of the Connes-Chamseddine spectral action which eliminates a disturbing large cosmological term. We analyze the corresponding equations of motion and find that these are solved if the energy momenta of the eigenspinors scale linearly with the mass. Surprisingly, this scaling law codes Einstein's equations. Finally we study the coupling to a physical fermion field.

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1 Introduction

In this paper, we present a novel formulation of euclidean general relativity. More precisely, we study a theory that approximates general relativity at large scale and while modifying it at short scale. The theory is characterized by the fact that it is not formulated strictly as a field theory. The dynamical variables are not fields: they are an (infinite) sequence of real numbers λ_n, $n = 1, 2, 3 \ldots$. The relation between the usual representation of the gravitational field by means of the metric tensor $g_{\mu\nu}(x)$, and the representation in terms of the λ_n's is given by the fact that the λ_n's are the eigenvalues of the Dirac operator D defined on the spacetime geometry described by $g_{\mu\nu}(x)$.

Thus, the general idea is to describe spacetime geometry by giving the eigen-frequencies of the spinors that can live on that spacetime. This is in the spirit of the well known mathematical problem: "Can one hear the shape of a drum?" [1]. Namely the problem of characterizing a two-dimensional shape by means of the spectrum of the laplacian defined on that shape. (Within some approximation, this spectrum gives the Fourier decomposition of the sound emitted by a drum with that shape.)

The theory we study is implicitly contained in the recent work of Connes and Chamseddine [2]. Alain Connes' exciting and ambitious attempt to unravel a microscopic noncommutative structure of spacetime [3, 4, 5, 6] has generated, among several others, also the idea that the Dirac operator D encodes the full information about the spacetime geometry in a way usable for describing gravitational dynamics. First of all the geometry can be reconstructed from D. More precisely it can be reconstructed from the (normed) algebra generated by D and by the smooth functions f on spacetime. If x and y are spacetime points, then their geodesic distance can be expressed in terms of D [3] as

$$d(x, y) = sup_f \ \{|f(x) - f(y)| \ : \ \|[D, f]\| \leq 1\}. \tag{1}$$

The sup is over all functions f whose commutator with D has norm less than one (here f and D are viewed as operators acting on the Hilbert space of the spinors on the manifold, and the norm is the natural operator norm). Secondly, there is a natural way of giving the dynamics in terms of D: The Einstein-Hilbert action with a cosmological term is approximated by the trace of a very simple function of D [2], as we recall below. These results suggest that one can take the Dirac operator D as the object representing the dynamical field, and try to develop a dynamical theory for D.

In fact, in [2], the powerful machinery of non-commutative geometry is used to elegantly encode the Yang-Mills structure of the standard model into a non-commutative component of the spacetime geometry. Accordingly, (a generalized operator) D codes the gravitational field *and* the Yang-Mills fields, plus, as a very remarkable bonus, the Higgs fields as well. However, even independently from these results on the standard model, we think that the idea of encoding the dynamical field into D is very interesting also for gravity alone. This is the idea we pursue here. We take the purely gravitational component of the Connes-Chamseddine theory only and we remain in the regime of conventional commutative geometry.

More in detail, we consider the idea of encoding the gravitational field into D in its simplest form: we take its eigenvalues λ_n as the dynamical variables of general relativity. The main reason for which we think this is interesting is that these eigenvalues form an infinite family of *diffeomorphism invariant* observables. Objects of this kind have long been sought for describing the geometry. General relativity teaches us that fundamental physics is invariant under active diffeomorphisms. Physically, this means that there is no fixed nondynamical structure with respect to which location or motion could be defined [7, 8]. Starting with Peter Bergamann's pioneering work [7], a fully diffeomorphism invariant description of the geometry, free from the gauge redundancy of the usual formalism, has long been sought [9], but without much success so far. Such a description would be precious, in particular, for quantum gravity [10]. Now, the λ_n's are precisely diffeomorphism invariant quantities. In fact, the invariance is only under diffeomorphisms which preserve the spin structure; however, only large diffeomorphisms can change the spin structure.

As a first step towards using these ideas in classical and/or quantum theories, we derive an expression for the Poisson brackets of the Dirac eigenvalues. We obtain this result by using the covariant formulation of the phase space of general relativity described in [11] and by extending a technique developed in [12], where the eigenvalues of the three-dimensional Weyl operator, invariant under spatial diffeomorphism only, where analyzed. Surprisingly, we find that the Poisson brackets of the eigenvalues can be expressed explicitly in terms of the energy-momentum tensors of the corresponding Dirac eigenspinors. These tensors form the Jacobian matrix of the change of coordinates between metric and eigenvalues. The brackets are quadratic in these tensors, with a kernel given by the propagator of the linearized Einstein equations. The energy-momentum tensor of the Dirac eigenspinors provides therefore a key tool for analyzing the representation of spacetime geometry in terms of Dirac eigenvalues.

We also study a variant of the Chamseddine-Connes spectral action [2]. In its simplest version, this action it is a bit unrealistic because of a huge cosmological term. This term is disturbing not only phenomenologically, but also because it implies that the small-curvature geometries for which the spectral action approximates the Einstein-Hilbert action are *not* solutions of the theory. We study a version of the spectral action which eliminates the cosmological term. We close by analyzing the equations of motion, derived from this action. These are solved if the energy momenta of the high mass eigenspinors scale linearly with the mass. This scaling requirement approximates the vacuum Einstein equations. Thus, we obtain a representation of Einstein equations as a scaling law. Finally, we briefly describe the coupling of a Dirac fermion.

Our results suggest that the Chamseddine-Connes gravitational theory can be viewed as a manageable theory, possibly with powerful applications to classical and quantum gravity. The theory reproduces general relativity at low energies; it is formulated in terms of fully diffeomorphism invariant variables; and, of course, it prompts fascinating extensions of the very notion of geometry.

A condensed version of some of the results presented here has appeared in [13]. An extension of these ideas to supergravity has been considered in [14].

2 General Relativity in terms of Eigenvalues

Consider Euclidean general relativity on a compact 4d (spin-) manifold M without boundary. The metric field is $g_{\mu\nu}(x) = E^I_\mu(x) E_{\nu\,I}(x)$, with $E^I_\mu(x)$ the tetrad fields. Indices $\mu = 1,\ldots,4$ are curved while $I = 1,\ldots,4$ are internal euclidean, raised and lowered by the Kronecker metric δ_{IJ}. The spin connection $\omega^I_{\mu J}$ is defined by $\partial_{[\mu} E^I_{\nu]} = \omega^I_{[\mu J} E^J_{\nu]}$, where square brackets indicate anti-symmetrization. The dynamics is determined by the Einstein-Hilbert action

$$S_{EH}[E] = \frac{1}{16\pi G} \int d^4x \ \sqrt{g} \ R \ , \tag{2}$$

where g and R are the determinant and the Ricci scalar of the metric respectively, G is the Newton constant. We put the speed of light equal to one. The Planck constant does not enter our considerations, which are purely classical.

In spite of a widespread contrary belief, the phase space is a spacetime covariant notion: it is the space of the solutions of the equations of motion modulo gauge transformations [15]. Here, gauge transformations are 4d diffeomorphisms and local rotations of the tetrad field. Thus, the phase space Γ of general relativity is the space of the tetrad fields that solve Einstein equations, modulo internal rotations and diffeomorphisms. Equivalently, it is the space of the Ricci flat "4-geometries". We denote the space of smooth tetrad fields as $\mathcal{E}$ and the space of the orbits of the gauge group in $\mathcal{E}$ (4-geometries) as $\mathcal{G}$. By definition, observables are functions on Γ: in the ADM formalism, they are functions defined on the constraint surface that commute with *all* the constraints [8].

We now define an infinite family of such observables. Consider the Hilbert space $\mathcal{H}$ of spinor fields ψ on M. The scalar product is

$$(\psi, \phi) = \int \ d^4x \ \sqrt{g} \ \ \overline{\psi(x)} \phi(x), \tag{3}$$

with bar indicating complex conjugation, and the scalar product in spinor space being the natural one in C^4. With γ^I being the (Euclidean) hermitian Dirac matrices, the curved Dirac operator is

$$D = \imath\, \gamma^I E^\mu_I \left(\partial_\mu + \omega_{\mu\, JK}\, \gamma^J \gamma^K \right) . \tag{4}$$

The operator D is a self-adjoint operator on $\mathcal{H}$ admitting a complete set of real eigenvalues λ_n and "eigenspinors" ψ_n. The manifold M being compact, the spectrum is discrete

$$D\psi_n = \lambda_n \ \psi_n \ , \qquad n = 0, 1, 2... \tag{5}$$

The eigenvalues are labeled so that $\lambda_n \leq \lambda_{n+1}$, with repeated multiplicity. They have dimension of an inverse length.

Notice that the Dirac operator depends on the gravitational field E; so do, of course, its eigenvalues as well. We indicate explicitly this dependence by writing $D[E]$ and $\lambda_n[E]$. The latter defines a discrete family of real-valued functions on $\mathcal{E}$, $\lambda_n : E \longmapsto \lambda_n[E]$. Equivalently, we have a function λ from $\mathcal{E}$ into the space of infinite sequences R^∞

$$\begin{array}{cccc} \lambda : & \mathcal{E} & \longrightarrow & R^\infty \\ & E^I_\mu(x) & \longmapsto & \lambda_n[E]. \end{array} \tag{6}$$

The image $\lambda(\mathcal{E})$ of $\mathcal{E}$ under this map is contained in the cone $\lambda_n \leq \lambda_{n+1}$ of R^∞. The functions λ_n are invariant under 4d diffeomorphisms and under internal rotations of the tetrads. Therefore they are gauge invariant and they are well defined functions on $\mathcal{G}$. In particular, they are well defined on the phase space of general relativity Γ: thus, they are *observables* of pure general relativity.

Two metric fields with the same collection $\{\lambda_n\}$ are called "isospectral". Isometric E fields are isospectral, but the converse needs not be true [1, 16]. Therefore λ might not be injective even if restricted to $\mathcal{G}$. The λ_n's might fail to coordinatize $\mathcal{G}$: they might fail to coordinatize Γ as well, although whether or not this happens is not clear to us. However, they presumably "almost do it". Following Connes [2], it is tempting to consider the physical hypothesis that isospectral Ricci flat 4-geometries, are physical indistinguishable ("Spectral hypothesis").

3 The Poisson Brackets

A simplectic structure on Γ can be constructed in covariant form [11]. A vector field X on Γ can be written as a differential operator

$$X = \int d^4x\ X^I_\mu(x)[E]\ \ \frac{\delta}{\delta E^I_\mu(x)} \tag{7}$$

where $X^I_\mu(x)[E]$ is any solution of the Einstein equations for the tetrad field, *linearized* over the background E. The simplectic two-form Ω of general relativity is given by [11]

$$\Omega(X,Y) = \frac{1}{32\pi G}\ \int_\Sigma d^3\sigma\ n_\rho\ [X^I_\mu\ \overset{\leftrightarrows}{\nabla}_\tau\ Y^J_\nu]\ \epsilon^\tau{}_{IJv}\,\epsilon^{v\rho\mu\nu} \tag{8}$$

where $[X^I_\mu\ \overset{\leftrightarrows}{\nabla}_\tau\ Y^J_\nu] \equiv [X^I_\mu\ \nabla_\tau\ Y^J_\nu - Y^I_\mu\ \nabla_\tau\ X^J_\nu]$. From now on we put $32\pi G = 1$. Both sides of (8) are functions of E, namely scalar functions on Γ; this E is used to transform internal indices into spacetime indices. Here $\Sigma : \sigma \longmapsto x(\sigma)$ is an arbitrary three-dimensional "ADM" surface, and n_ρ its normal one-form. The coefficients of the simplectic form can be written as

$$\Omega^{\mu\nu}_{IJ}(x,y) = \int_\Sigma d^3\sigma\ n_\rho\ [\delta(x,x(\sigma))\overset{\leftrightarrows}{\nabla}_\tau\delta(y,x(\sigma))]\ \epsilon^\tau{}_{IJv}\,\epsilon^{v\rho\mu\nu}. \tag{9}$$

Then, the Poisson bracket between two functions f, g on Γ is given by

$$\{f,g\} = \int d^4x \int d^4y\ \ P^{IJ}_{\mu\nu}(x,y)\ \frac{\delta f}{\delta E^I_\mu(x)}\ \frac{\delta g}{\delta E^I_\nu(y)}. \tag{10}$$

where $P^{IJ}_{\mu\nu}(x,y)$ is the inverse of the simplectic form matrix. This is defined by

$$\int d^4y\ P^{IJ}_{\mu\nu}(x,y)\ \Omega^{\nu\rho}_{JK}(y,z) = \delta(x,z)\ \delta^\rho_\mu\ \delta^I_K. \tag{11}$$

Since the simplectic form is degenerate on the space of the fields (it is non-degenerate only when restricted to the space of equivalent classes of gauge-equivalent fields), we can only invert it on this space by fixing a gauge. Let us assume this has been done. More precisely, integrating the last equation against a vector field $F^K_\rho(z)$ that satisfies the linearized Einstein equations over E, we have

$$\int d^4y \int d^4z\ P^{IJ}_{\mu\nu}(x,y)\ \Omega^{\nu\rho}_{JK}(y,z)\ F^K_\rho(z) = \int d^4z\ \delta(x,z)\ \delta^\rho_\mu\ \delta^I_K\ F^K_\rho(z). \qquad (12)$$

Integrating over the delta functions, and using (9), we have

$$\int_\Sigma d^3\sigma\ n_\rho\ [P^{IJ}_{\mu\nu}(x,x(\sigma))\overleftrightarrow{\nabla}_\rho F^K_\tau(x(\sigma))]\ \epsilon^\rho{}_{JK\upsilon}\ \epsilon^{\upsilon\nu\tau\sigma} = F^I_\mu(x). \qquad (13)$$

This equation, where F is any solution of the linearized equations, defines P. But this equation is precisely the definition of the propagator of the linearized Einstein equations over the background E (in the chosen gauge). For instance, let us chose the surface Σ as $x^4 = 0$ and fix the gauge with

$$X^4_4 = 1, \quad X^4_a = 0, \quad X^i_4 = 1, \quad X^i_a = 0. \qquad (14)$$

where $a = 1,2,3$ and $i = 1,2,3$. Then equation (13) becomes

$$F^i_a(\vec{x},t) = \int d^3\vec{y}\ [P^{ib}_{aj}(\vec{x},t;\vec{y},0)\overleftrightarrow{\nabla}_0 F^j_b(\vec{y},0)], \qquad (15)$$

where we have used the notation $\vec{x} = (x^1,x^2,x^3)$ and $t = x^4$, and the propagator can be easily recognized.

Next, we need the Jacobian of the transformation from metric to eigenvalues. The variation of λ_n for a variation of E is computed using standard time independent quantum mechanics perturbation theory. For a self-adjoint operator D depending on a parameter v and whose eigenvalues are non-degenerate, we have

$$\frac{d\lambda_n}{dv} = (\psi_n|\frac{d}{dv}D(v)|\psi_n). \qquad (16)$$

In our situation, for generic metrics with non-degenerate eigenvalues (we refer to [17] for the general situation) we have that

$$\begin{aligned}
\frac{\delta\lambda_n}{\delta E^I_\mu(x)} &= (\psi_n|\frac{\delta}{\delta E^I_\mu(x)}D|\psi_n) && (17)\\
&= \int\sqrt{g}\ \bar{\psi}_n\frac{\delta}{\delta E^I_\mu(x)}D\psi_n \\
&= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ \bar{\psi}_n D\psi_n - \int\frac{\delta\sqrt{g}}{\delta E^I_\mu(x)}\bar{\psi}_n D\psi_n \\
&= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ \bar{\psi}_n D\psi_n - \int\frac{\delta\sqrt{g}}{\delta E^I_\mu(x)}\bar{\psi}_n\lambda_n\psi_n \\
&= \frac{\delta}{\delta E^I_\mu(x)}\int\sqrt{g}\ (\bar{\psi}_n D\psi_n - \lambda_n\bar{\psi}_n\psi_n) \\
&= T_n{}^\mu_I(x). && (18)
\end{aligned}$$

Now, $T_{n}{}^{\mu}_{I}(x)$ is nothing but the usual energy-momentum tensor of the spinor field ψ_n in tetrad notation (see for instance [18]). Indeed, the Dirac energy-momentum tensor is defined by

$$T^{\mu}_{I}(x) \equiv \frac{\delta}{\delta E^{I}_{\mu}(x)} S_{\text{Dirac}}, \tag{19}$$

where $S_{\text{Dirac}} = \int \sqrt{g}\ (\bar{\psi} D\psi - \lambda\bar{\psi}\psi)$ is the Dirac action of a spinor with "mass" λ. (Since we have not put the Planck constant in the Dirac action, λ has dimensions of an inverse length, rather than a mass.)

We have shown that the energy-momentum tensor of the eigenspinors gives the Jacobian matrix of the transformation from E to λ; namely it gives the variation of the eigenvalues for a small change in the geometry. This fact suggests that we can study the map λ locally by studying the space of the eigenspinor's energy-momenta.

By combining (10,13) and (18) we obtain our main result (in this equation we restore physical units for completeness):

$$\{\lambda_n, \lambda_m\} = 32\pi G \int d^4x \int d^4y\ T_{[n}{}^{\mu}_{I}(x)\ P^{IJ}_{\mu\nu}(x,y)\ T_{m]}{}^{\nu}_{J}(y) \tag{20}$$

which gives the Poisson bracket of two eigenvalues of the Dirac operator in terms of the energy-momentum tensor of the two corresponding eigenspinors and of the propagator of the linearized Einstein equations. The right hand side does not depend on the gauge chosen for P.

Finally, if the transformation λ between the "coordinates" $E^{I}_{\mu}(x)$ and the "coordinates" λ_n is locally invertible on the phase space Γ, we can write the simplectic form directly in terms of the λ_n's as

$$\Omega = \Omega_{mn}\ d\lambda_n \wedge d\lambda_m, \tag{21}$$

where a sum over indices is understood, and where Ω_{mn} is defined by

$$\Omega_{mn}\ T_{n}{}^{\mu}_{I}(x)\ T_{m}{}^{\nu}_{J}(y) = \Omega^{\mu\nu}_{IJ}(x,y). \tag{22}$$

Indeed, let $dE^{I}_{\mu}(x)$ be a (basis) one-form on Γ, namely the infinitesimal difference between two solutions of Einstein equations, namely a solution of the Einstein equations linearized over E. We have then

$$\begin{aligned} \Omega &= \int d^4x \int d^4y\ \Omega^{\mu\nu}_{IJ}(x,y)\ dE^{I}_{\mu}(x) \wedge dE^{J}_{\nu}(y) \\ &= \int d^4x \int d^4y \Omega_{mn}\ T_{n}{}^{\mu}_{I}(x)\ dE^{I}_{\mu}(x) \wedge T_{m}{}^{\nu}_{J}(y)\ dE^{J}_{\nu}(y) \\ &= \Omega_{mn}\ d\lambda_n \wedge d\lambda_m. \end{aligned} \tag{23}$$

An explicit evaluation of the matrix Ω_{nm} would be of great interest.

4 Action and Field Equations

We now turn to the gravitational spectral action [2]. This action contains a cutoff parameter l_0 with units of a length, which determines the scale at which the gravitational theory defined departs from general relativity. We may assume that l_0 is the Planck length $l_0 \sim 10^{-33} cm$ (although we make no reference to quantum phenomena in the present context). We use also $m_0 = 1/l_0$, which has the same dimension as D and λ_n. The action depends also on a dimensionless cutoff function $\chi(u)$, which vanishes for large u. The simplest choice for the function $\chi(u)$, which we adopt for the moment, is to take it to be smooth and monotonic on R^+ with

$$\chi(u) = \begin{cases} 1 & \text{if } u < 1-\delta \\ 0 & \text{if } u > 1+\delta \end{cases} \tag{24}$$

where $\delta << 1$. Namely $\chi(u)$ is the smoothed-out characteristic function of the $[0,1]$ interval. The spectral action is then defined as

$$S_G[D] = \kappa \; Tr \left[\chi(l_0^2 \, D^2) \right]. \tag{25}$$

κ is a multiplicative constant to be chosen to recover the right dimensions of the action and the multiplicative overall factor in (2). Notice that (25) is essentially (namely up to corrections of order δ) simply (κ times) the *number* of eigenvalues λ_n with absolute value smaller that m_0!

The action (25) approximates the Einstein-Hilbert action with a large cosmological term for "slowly varying" metrics with small curvature (with respect to the scale l_0). Indeed, the heat kernel expansion [2, 16], allows to write (see [13] for a different derivation),

$$S_G(D) = (l_0)^{-4} f_0 \kappa \int_M \sqrt{g}\, dx \; + \; (l_0)^{-2} f_2 \kappa \int_M R \sqrt{g}\, dx \; + \; \dots \; . \tag{26}$$

The functions f_0 and f_2 are defined by $f_0 = \frac{1}{4\pi^2}\int_0^\infty \chi(u) u du$ and $f_2 = \frac{1}{48\pi^2}\int_0^\infty \chi(u) du$, the integrals being of the order of unity for the choice of cutoff function made. The other terms in (26) are of higher order in l_0.

The expansion (26) shows that the action (25) is dominated by the Einstein-Hilbert action with a Planck-mass cosmological term. The presence of this term is a problem for the physical interpretation of the theory because the solutions of the equations of motions have Planck-scale Ricci scalar, and therefore they are *all* out of the regime for which the approximation taken is valid! However, the cosmological term can be canceled by replacing the function χ with $\tilde{\chi}$,

$$\tilde{\chi}(u) = \chi(u) - \epsilon^2 \chi(\epsilon u) \; , \tag{27}$$

with $\epsilon << 1$. Indeed, one finds $\tilde{f}_0 = 0$, $\tilde{f}_2 = (1-\epsilon) f_2$. The modified action becomes

$$\tilde{S}_G(D) = \frac{\tilde{f}_2 \kappa}{l_0^2} \int_M R \sqrt{g}\, dx \quad + \; \dots \; . \tag{28}$$

We obtain the Einstein-Hilbert action (2) by fixing

$$\kappa = \frac{l_0^2}{16\pi G \tilde{f}_2} \; . \tag{29}$$

If l_0 is the Planck length $\sqrt{\hbar G}$, then $\kappa = \frac{3}{2}h$, where h is the Planck constant, up to terms of order δ and ϵ. Low curvature geometries, for which the expansion (26) holds *are now* solutions of the theory. Thus we obtain a theory that genuinely approximates pure general relativity at scales large compared to l_0.

Let us now consider the equations of motion derived from this action. Following our philosophy, we want to regard the λ_n's as the gravitational variables. The action can easily be expressed in terms of these variables:

$$\widetilde{S}_G[\lambda] = \kappa \sum_n \ \widetilde{\chi}(l_0^2\lambda_n^2). \tag{30}$$

However, we cannot obtain (approximate) Einstein equations by simply varying (30) with respect to the λ_n's: we must minimize (30) on the surface $\lambda(\mathcal{E})$, not on the entire R^∞. In other words, the λ_n's are not independent variables: there are relations among them, and these relations among them code the complexity of general relativity. We shall comment on these relations at the end of the paper. We can still obtain the equations of motion by varying $\widetilde{S}_G$ with respect to the tetrad field:

$$0 = \frac{\delta \widetilde{S}_G}{\delta E^I_\mu(x)} = \sum_n \ \frac{\partial \widetilde{S}_G}{\partial \lambda_n} \ \frac{\delta \lambda_n}{\delta E^I_\mu(x)} = \sum_n \ \frac{d\widetilde{\chi}(l_0^2\lambda_n^2)}{d\lambda_n} \ T_n{}^\mu_I(x). \tag{31}$$

Defining $f(u) =: \frac{d}{du}\widetilde{\chi}(u)$, (31) becomes

$$\sum_n f(l_0^2\lambda_n^2) \ \lambda_n \ T_n{}^\mu_I(x) = 0. \tag{32}$$

These are the Einstein equations in the Dirac eigenvalues formalism.

With the choice of cutoff function mentioned above, $f(u)$ vanishes everywhere except on two narrow peaks. A negative one (width 2δ and height $1/2\delta$) centered at one; and a positive one (width $2\delta/\epsilon$ and height $\epsilon^3/2\delta$) around the arbitrary large number $1/\epsilon =: s >> 1$. The first of these peaks gets contributions from λ_n's such that $\lambda_n \sim m_0$, namely from Planck scale eigenvalues. The second from ones such that $\lambda_n \sim s m_0$. Equations (32) are solved if the contributions of the two peaks cancel. This happens if below the Planck scale the energy momentum tensor scales as

$$\lambda_{n(m_0)}\rho(1) \ T_{n(m_0)}{}^\mu_I(x) = s^{-2}\lambda_{n(sm_0)}\rho(s) \ T_{n(sm_0)}{}^\mu_I(x), \tag{33}$$

Here, $\rho(1)$ and $\rho(s)$ are the densities of eigenvalues of $l_0^2 D^2$ at the two peaks and the index $n(t)$ is defined by

$$l_0\lambda^2_{n(t)} = t. \tag{34}$$

For large n the growth of the eigenvalues of the Dirac operator is given by the Weyl formula $\lambda_n \sim \sqrt{2\pi}V^{-1/4}n^{1/4}$, where V is the volume. Using this, one derives immediately the eigenvalue densities, and simple algebra yields

$$T_n{}^\mu_I(x) = \lambda_n \ l_0 \ T_0{}^\mu_I(x) \ . \tag{35}$$

for $n >> n(m_P)$, where $T_0{}^\mu_I(x) = T_{n(m_0)}{}^\mu_I(x)$ is the energy momentum at the Planck scale. We have shown that *the dynamical equations for the geometry are solved if below* *the Planck length the energy-momentum of the eigenspinors scales as the eigenspinor's mass.* In other words, we have expressed the Einstein equations as a scaling requirement on the energy-momenta of the very-high-frequency Dirac eigenspinors.

We add a few considerations that shed some light on this scaling requirement. Notice that $T_{n\mu}^{\ I}$ is formed by a term linear in the derivatives of the spinor field and a term independent from these. The latter is a function of $(\psi, E, \partial_\mu E)$, quadratic in ψ.

$$T_{n\mu}^{\ I} = \bar{\psi}_n \gamma^I \overleftrightarrow{\partial}_\mu \psi_n + S_{n\mu}^{\ I}[\psi, E, \partial E]. \tag{36}$$

If we expand the last term around a point of the manifold with local coordinates x, covariance and dimensional analysis require that

$$S_{n\mu}^{\ I} = c_0 \lambda_n E^I_\mu + c_1 \ R^I_\mu + c_2 \ R \, E^I_\mu + O\left(\frac{1}{\lambda_n}\right). \tag{37}$$

for some fixed expansion coefficients c_0, c_1 and c_2. Here R^I_μ is the Ricci tensor.
To be convinced that terms of this form do appear, consider the following.

$$\begin{aligned} T_{n\mu}^{\ I} &= \bar{\psi}_n \gamma^I D_\mu \psi_n + \ldots \\ &= (\lambda_n)^{-1} \ \bar{\psi}_n \gamma^I \gamma^\nu D_\mu D_\nu \psi_n + \ldots \\ &= (\lambda_n)^{-1} \ \bar{\psi}_n \gamma^I \gamma^\nu [D_\mu, D_\nu] \psi_n + \ldots \\ &= (\lambda_n)^{-1} \ \bar{\psi}_n \gamma^I \gamma^\nu R_{\mu\nu} \psi_n + \ldots \\ &= (\lambda_n)^{-1} \ \bar{\psi}_n \gamma^I \gamma^\nu R^{JK}_{\mu\nu} \gamma_J \gamma_K \psi_n + \ldots \\ &= Tr \ \ \gamma^I \gamma^\nu R^{JK}_{\mu\nu} \gamma_J \gamma_K + \ldots \\ &= R^I_\mu + \ldots \end{aligned} \tag{38}$$

For sufficiently high n, the eigenspinors are locally approximated by plane waves in local cartesian coordinates. For these functions, if we double the mass the frequency doubles as well: if $\lambda_m = t \ \lambda_n$, then $\partial_\mu \psi_m = t \ \partial_\mu \psi_n$. It follows that in general the energy momentum scales as

$$T_{n\mu}^{\ I} = t \left[\bar{\psi}_n \gamma^I \overleftrightarrow{\partial}_\mu \psi_n + c_0 \lambda_n E^I_\mu \right] + \left[c_1 \ R^I_\mu + c_2 \ R \, E^I_\mu \right] + O\left(\frac{1}{\lambda_n}\right). \tag{39}$$

For large λ_n we can disregard the last term, and therefore (35) requires that the second square bracket vanishes. Taking the trace we have $R = 0$, using which we conclude $R^I_\mu = 0$, which are the vacuum Einstein equations. Thus, the equations of motion are solved if the scaling requirement on the high mass eigenspinors' energy momenta is satisfied, and this requirement, in turn, yields vacuum Einstein equations at low energy scale.

5 Matter couplings

The spinors ψ_n that appear in the previous sections do not represent physical fermions. They are mathematical quantities used to capture aspects of the pure gravitational field.

In particular, there is no sense in which they act back on the geometry. In oder to describe the physical system formed by a (classical) fermion field, say with "mass" (inverse wavelenght) m, interacting with general relativity, namely an interacting Dirac-Einstein system, we have to introduce a (physical) spinor field $\psi(x)$. The action that governs the dynamics of a fermion field and its interaction with the gravitational field is the Dirac action

$$S_{\rm Dirac}[\psi, E] = \int (\bar{\psi} D\psi - m\bar{\psi}\psi)\ \sqrt{g}\ d^4x = (\psi[D-m]\psi) \tag{40}$$

Therefore the Dirac-Einstein system is governed by the total action

$$S[D,\psi] = \widetilde{S}_G[D] + S_{\rm Dirac}[D,\psi] = \kappa\, Tr\left[\chi(l_0^2\, D^2)\right] + (\psi[D-m]\psi). \tag{41}$$

The natural thing to do in the context of the present formalism is to expand ψ in the basis formed by the ψ_n. Namely to write

$$\psi(x) = \sum_n\ a_n\ \psi_n(x) \tag{42}$$

and to describe gravity in terms of the λ_n's and the fermion in terms of its components a_n. The action becomes

$$S[\lambda_n, a_n] = \sum_n\ \left[\kappa\ \tilde{\chi}(l_0^2\lambda_n^2) + (\lambda_n - m)|a_n|^2\right]. \tag{43}$$

The equations of motion are

$$\sum_n \left[2\kappa l_0^2\ f(l_0^2\lambda_n^2)\ \lambda_n + |a_n|^2\right]\ T_{n}{}^{\mu}_{I}(x) = 0, \tag{44}$$

$$(\lambda_n - m)\ a_n = 0. \tag{45}$$

Eq. (44) corresponds to the Einstein equations with a source and (45) is the Dirac equation on a curved spacetime. Notice that the latter is algebraic, and it can be solved immediately. In order for a solution to exist there should exist an $\hat{n}$ such that

$$\lambda_{\hat{n}} = m. \tag{46}$$

The solution is

$$\begin{array}{ll} a_n = 0 & \text{for all } n \neq \hat{n}, \\ a_{\hat{n}} = a & \text{an arbitrary constant.} \end{array} \tag{47}$$

This is not surprising: a fermion field of mass m on a geometry characterized by the Dirac eigenvalues $\{\lambda_n\}$ is given precisely by the eigenspinor $\psi_{\hat{n}}$ with eigenvalue equal to m. Using the solution of the Dirac equation, (44) becomes

$$\tilde{f}_2^{-1} l_0^4 \sum_n f(l_0^2\lambda_n^2)\ \lambda_n\ T_{n}{}^{\mu}_{I}(x) = 8\pi G\ |a|^2 T_{\hat{n}}{}^{\mu}_{I}(x) \tag{48}$$

where we have used the value of κ (29). From the results of the previous section, we recognize the left hand side as the Einstein tensor; the right hand side is the energy momentum tensor of the fermion.

In the presence of matter, the scaling law (35) is altered. Using again the Weyl formula, we obtain with simple algebra

$$T_{n}{}^{\mu}_{I}(x) = \lambda_n \; l_0 \; [T_{0}{}^{\mu}_{I}(x) \; + \alpha \; |a|^2 \, T_{\hat{n}}{}^{\mu}_{I}(x)], \tag{49}$$

where

$$\alpha = \frac{16\pi^3 G \; \tilde{f}_2 l_0}{V}. \tag{50}$$

Equation (49) is the "scaling law" form of the Einstein equations, modified by the matter source term.

The extension of the theory to other conventional matter couplings should not be difficult, but we do not pursue it here.

6 Summary and perspectives

We have discussed the possibility of describing gravity by means of the Dirac operator eigenvalues. This possibility has been opened by the recent work of Connes and Chamseddine. We think that these new ideas might open a novel window over the physics of spacetime and find applications in classical and quantum gravitation. The main obstacle for a full development of this approach is its natural euclidean character, due to the fact that on a non-compact lorentzian spacetime the Dirac operator will not have discrete spectrum (but see [19] for 'lorentzian' attempts). However, the present formalism might still find a natural application in quantum or in thermal quantum physics.

We have elucidated some aspects of the dynamical structure of the theory in the λ_n variables by computing their Poisson algebra. This is given in equation (20). Perhaps a quantum theory could be constructed in a diff-invariant manner by studying representations of this algebra. At present, the Poisson algebra is not given in closed form, since the right hand side of equation (20) is not expressed in terms of the λ_n themselves. This difficulty could be faced by expanding the energy momentum tensors in terms of the eigenspinors themselves, as suggested by Hawkins [20].

We have also studied the equations of motion of (a version) of the Chamseddine-Connes spectral action. This action defines a theory that approximates general relativity at large scale, where it could be used as a tool in classical gravity. It would be interesting to explore the modifications to general relativity that it yields at short scale. We have given the expression for a fermion coupling in this formalism. We have found a puzzling and intriguing way of expressing the Einstein equations as a scaling law for the energy momenta of the ultra-high-frequency eigenspinors.

The striking feature of the formalism discussed here is that the theory is formulated in terms of diffeomorphism invariant quantities. The λ_n's are a family of diffeomorphism invariant observables in euclidean general relativity, which is presumably complete or "almost complete" (it could fail to distinguish possible isospectral and not isometric geometries). It should be possible, at least in principle, to represent "physical observations" in pure gravity as a function of the λ_n's alone. Another remarkable aspect of the spectral action is that it introduces a physical cutoff and an elementary physical length without breaking diffeomorphism invariance. The spectral action cuts off all high frequency modes, but it does so in a diffeomorphic invariant manner without introducing background structures. Since the number of the remaining modes is determined by the ratio of the spacetime volume to the Planck scale, one may expect that a theory of this sort could have infrared but not ultraviolet divergences in the quantum regime. The quantum theory based on the spectral action is therefore very much worth exploring, we think.

The key open problem, in our view, is to better understand the map λ given in (6) and its range; namely the constraints that a sequence of real numbers λ_n must satisfy, if it represents the spectrum of the Dirac operator of some geometry. This problem can be addressed locally (in phase space) by studying the *tangent* map to λ. We have show that this tangent map is given explicitly by the eigenspinor's energy-momenta. One could begin to study λ around simple geometries, such as a flat 4-torus. On a more general ground, the constraints on the λ_n's are presumably the core of the formulation of the gravitational theory that we have begun to explore here. They should be contained in Connes's axioms for D in the axiomatic definition of a spectral triple [2]. The equations in these axioms capture the notion of Riemannian manifold algebraically, and they should code the constraints satisfied by the λ_n. Finding the explicit connection between the formalism studied here and Connes axioms' equations would be of great interest.

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