CONVEX HAMILTONIANS WITHOUT CONJUGATE POINTS

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Introduction.

Let M be a closed connected riemannian manifold and T^*M its cotangent bundle. By a convex Hamiltonian on T^*M we shall mean a C^3 function $H : T^*M \to \mathbb{R}$ satisfying the following conditions:

(a) *Convexity:* For all $q \in M$, $p \in T_q^*M$, the Hessian matrix $\frac{\partial^2 H}{\partial p_i \, \partial p_j}(q,p)$ (calculated with respect to linear coordinates on T_q^*M) is positive definite.

(b) *Superlinearity:*

$$\lim_{|p|\to\infty} \frac{H(q,p)}{|p|} = +\infty \,.$$

The *Hamiltonian equation* for H is defined as

$$\begin{aligned} q' &= H_p \,, \\ p' &= -H_q \end{aligned} \tag{1}$$

where H_q and H_p are the partial derivatives with respect to q and p. Observe that the Hamiltonian function H is constant along the solutions of (1). Its level sets $\Sigma = H^{-1}\{e\}$ are called *energy levels* of H. Then the compacity of M and the superlinearity hypothesis imply that the energy levels are compact. Since the Hamiltonian vectorfield (1) is Lipschitz, the solutions of (1) are defined on all $\mathbb{R}$. Denote by ψ_t the corresponding *Hamiltonian flow* on T^*M.

Our main interest in this paper are *disconjugate* orbits of the Hamiltonian flow, i.e. orbits without conjugate points. Let $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $V(\theta) = \ker(d\pi)$. Two points θ_1, $\theta_2 \in T^*M$ are said *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(V(\theta_1)) \cap V(\theta_2) \neq \{0\}$.

In this work we generalize some results of the theory of geodesics without conjugate points. A first step is to construct the *Green subspaces* for disconjugate orbits:

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Proposition A.

Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two φ-invariant Lagrangian subbundles $\mathbb{E}$, $\mathbb{F} \subset T(T^*M)$ along the orbit of θ given by

$$\mathbb{E}(\theta) = \lim_{t\to+\infty} d\psi_{-t}\big(V(\psi_t(\theta))\big) \,,$$

$$\mathbb{F}(\theta) = \lim_{t\to+\infty} d\psi_{t}\big(V(\psi_{-t}(\theta))\big) \,.$$

Moreover, $\mathbb{E}(\theta)\cup\mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta)\cap V(\theta) = \mathbb{F}(\theta)\cap V(\theta) = \{0\}$, $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ and $\dim\mathbb{E}(\theta) = \dim\mathbb{F}(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the Hamiltonian vectorfield and $\Sigma = H^{-1}\{e\}$.

These bundles where constructed for disconjugate geodesics of riemannian metrics by Green [16] and of Finsler metrics by Foulon [15]. In the Finsler case this is done by defining a new connection $\widetilde{\nabla}$ such that the solutions γ of the Euler-Lagrange equation satisfy $\widetilde{\nabla}_{\dot\gamma}(\dot\gamma) \equiv 0$. This approach leads to extensions of many important theorems in Riemannian geometry to Finsler geometry. See Chern [6] and the references therein for a survey and an introduction to this field.

If the energy level Σ does not have conjugate points, the Green bundles $\mathbb{E}$ and $\mathbb{F}$ can be defined on all $\theta \in \Sigma$. Nevertheless, in general they are neither continuous (cf. [4]) nor transversal in $T\Sigma$ (i.e. $\dim \mathbb{E}\cap\mathbb{F} = 1$).

An example of the relationship between the transversality of the Green subspaces and hyperbolicity appears in the following

Proposition B.

Let Γ be a periodic orbit of ψ_t without conjugate points. Then Γ is hyperbolic (on its energy level) if and only if $\mathbb{E}(\theta)\cap\mathbb{F}(\theta) = \langle X(\theta)\rangle$ for some $\theta \in \Gamma$, where $\langle X(\theta)\rangle$ is the 1-dimensional subspace generated by the Hamiltonian vectorfield $X(\theta)$. In this case $\mathbb{E}$ and $\mathbb{F}$ are its stable and unstable subspaces.

We can compare proposition A with the following theorem by G. Paternain and M. Paternain [29].

0.1. Theorem. [Paternain] [29]

Suppose that e is a regular value of H, and that $\psi_t|_\Sigma$ admits a continuous invariant Lagrangian subbundle E. Then,

(a) $E(\theta)\cap V(\theta) = \{0\} \qquad \forall\theta \in \Sigma$.
(b) $\pi(\Sigma) = M$.
(c) Σ *contains no conjugate points.*

It is not difficult to see (e.g. [21]) that when the flow $\psi_t|_\Sigma$ is Anosov, the stable and unstable subbundles of $T\Sigma$ are Lagrangian. Paternain's theorem then implies that Anosov Lagrangian flows do not have conjugate points. This result was proven for geodesic flows by Klingenberg [20]. An extension of Klingenberg result appears in Mañé [21].

Define

$$B(\theta) = \left\{ \xi \in T_\theta\Sigma \mid \sup_{t\in\mathbb{R}} |d\psi_t(\theta)\,\xi| < +\infty \right\}.$$

An important property (cf. proposition 1.11) in the proof of proposition B is that if the orbit of θ is disconjugate then $B(\theta) \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$. This in turn implies that $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ and $\mathbb{E}(\theta)\cup\mathbb{F}(\theta) \subsetneq T_\theta\Sigma$ (cf. corollary 1.12). The global version of proposition B is given by

Theorem C.
Let $\Sigma = H^{-1}\{e\}$ be a regular energy level without conjugate points.
Then the following statements are equivalent:

(a) $\psi_t|_\Sigma$ *is Anosov.*
(b) *For all $\theta\in\Sigma$, $\mathbb{E}(\theta)$ and $\mathbb{F}(\theta)$ are transversal in $T_\theta\Sigma$.*
(c) *For all $\theta\in\Sigma$, $\mathbb{E}(\theta)\cap\mathbb{F}(\theta) = \langle X(\theta)\rangle$.*
(d) *If $\theta\in\Sigma$, $v\in T_\theta\Sigma$, $v\notin\langle X(\theta)\rangle$ then* $\displaystyle\sup_{t\in\mathbb{R}} |d\psi_t(\theta)\cdot v| = +\infty$.

Where $\langle X(\theta)\rangle \subset T_\theta\Sigma$ is the 1-dimensional subspace generated by the vectorfield $X(\theta)$ of ψ.

This theorem was proven by Eberlein (cf. [10]) for geodesic flows. An interesting ingredient on its proof is the characterization of recurrent quasi-hyperbolic $\mathbb{R}$-actions of vector bundles. Given a vector bundle $\pi : \boldsymbol{E} \to B$ consider a continuous $\mathbb{R}$-action $\Lambda : \mathbb{R} \to \mathrm{Isom}(\boldsymbol{E},\boldsymbol{E})$ where $\Lambda_t : \boldsymbol{E}\to\boldsymbol{E}$ is a bundle map which is a linear isomorphism on each fiber and $\Lambda_{s+t} = \Lambda_s\circ\Lambda_t$. The action Λ induces a continuous flow $\phi_t : B \hookleftarrow$ such that $\phi_t\circ\pi = \pi\circ\Lambda_t$. We say that Λ is *quasi-hyperbolic* if

$$\sup_{t\in\mathbb{R}} |\Lambda_t(\xi)| = +\infty \qquad \text{for all } \xi\in\boldsymbol{E},\ \xi\neq 0\,,$$

and we say that it is *hyperbolic* if there exists a continuous splitting $\boldsymbol{E} = E^s\oplus E^u$ and $C>0$, $\lambda>0$ such that

(i) $|\Lambda_t(\xi)| \le C\,e^{-\lambda t}$ for all $t>0$, $\xi\in E^s$.
(ii) $|\Lambda_{-t}(\xi)| \le C\,e^{-\lambda t}$ for all $t>0$, $\xi\in E^u$.

It is easy to see that hyperbolic actions are quasi-hyperbolic. The converse is false by counterexamples in Robinson [32] and Franks & Robinson [11]. But it is true with an additional recurrency hypothesis, (that in our case is always true because the Lagrangian flow preserves the Liouville measure).

Define the *non-wandering* set $\Omega(\phi)$ of ϕ as the set of points $b \in B$ such that for every neighbourhood U of b there exists $T > 0$ such that $\phi_T(U) \cap U \neq 0$. The following theorem is a slight modification of a theorem in Freire [12], written there for quasi-Anosov flows (i.e when $\Lambda_t = d\phi_t$ and $\boldsymbol{E}$ is a continuous invariant subbundle of TB transversal to the vectorfield X, see theorem 3.1). For completeness of the exposition we present a proof of the following theorem in section §3.

0.2. Theorem. *Let $\pi : \boldsymbol{E} \to B$ be a continuous vector bundle and $\Lambda : \mathbb{R} \to$ $\mathrm{Isom}(\boldsymbol{E}, \boldsymbol{E})$ a continuous $\mathbb{R}$-action with induced flow ϕ_t: $\pi \circ \Lambda_t = \phi_t \circ \pi$. If Λ is quasi-hyperbolic and $\Omega(\phi|_B) = B$, then Λ is hyperbolic.*

The reason to use this version of Freire's result is that our Hamiltonian flows are not of contact type and a priori they may not preserve any continuous bundle transversal to the vectorfield $X(\theta)$. Indeed, in apendix I we show an example of a convex Hamiltonian without conjugate points where this phenomenon occurs. The solution is to study the behaviour of an (orbitwise) reparametrization of the flow which preserves a continuous tranversal bundle $\boldsymbol{E} \subset T(T^*M)$. This reparametrization can not be made global but instead we use the quasi-hyperbolicity of the action $\Lambda_t(\theta) = P\big(\psi_t(\theta)\big) \circ d\psi_t(\theta) \circ P(\theta)$ where $P(\theta) : T_\theta T^*M \to \boldsymbol{E}(\theta)$ is the projection along the direction of the vectorfield $X(\theta)$.

Generic Lagrangians.

A C^4 function $L : TM \to \mathbb{R}$ is called a (convex autonomous) *Lagrangian* if it satisfies the following properties:

(a) *Convexity:* For all $x \in M$, $v \in T_xM$, the Hessian matrix $\frac{\partial^2 L}{\partial v_i\, \partial v_j}(x, v)$ (calculated with respect to linear coordinates on T_xM) is positive definite.

(b) *Superlinearity:*
$$\lim_{|v|\to\infty} \frac{L(x,v)}{|v|} = +\infty\,,$$
uniformly on $x \in \mathbb{R}$.

Observe that by the compactness of M, the *Euler-Lagrange* equation, that in local coordinates is:
$$\frac{d}{dt}\,\frac{\partial L}{\partial v}(x, \dot{x}) = \frac{\partial L}{\partial x}(x, \dot{x})\,, \tag{2}$$
generates a complete flow $\varphi : TM \times \mathbb{R} \to TM$ defined by
$$\varphi_t(x_0, v_0) = \big(x(t), \dot{x}(t)\big),$$
where $x : \mathbb{R} \to M$ is the maximal solution of (2) with initial condition $\dot{x}(0) = v_0$.

A Lagrangian flow is conjugated to a Hamiltonian flow by the Legendre transform $\mathcal{F}_L : TM \to T^*M$ defined by

$$\mathcal{F}_L(x,v) = \left(x, \frac{\partial L}{\partial v}(x,v) \right) .$$

The corresponding Hamiltonian is given by $H(q,p) = E(\mathcal{F}_L^{-1}(q,p))$, where $E : TM \to \mathbb{R}$ is the *energy* function defined by

$$E(x,v) := \left. \frac{\partial L}{\partial v} \right|_{(x,v)} \cdot v - L(x,v) \ . \tag{3}$$

Conversely, given a convex Hamiltonian H on T^*M, the corresponding convex Lagrangian is given by the Legendre transform of the Hamiltonian $\mathcal{F}_H : T^*M \to T^{**}M \approx TM$ which can be seen as

$$L(x,v) = \max_p \{ \, p(v) - H(x,p) \, \} = (p \, H_p - H) \circ \mathcal{F}_H \, .$$

Observe that item (b) in theorem 0.1 implies that the energy e of an Anosov level for the Lagrangian flow satisfies

$$e > e_0 := - \max_{p \in M} L(p,0) \, ,$$

because e_0 is the minimal energy level e for which $\pi(\Sigma) = M$. Define

$$c(L) := - \min \left\{ \int_{TM} L \, d\mu \;\middle|\; \mu \text{ is a } \varphi - \text{invariant probability} \right\} .$$

It is proven in [25], [8], that $c(L) \geq e_0$.

We say that a φ-invariant probability μ is *minimizing* if

$$c(L) = - \int L \, d\mu \, ,$$

and we say that it is *uniquely ergodic* if it is the only invariant probability measure supported inside $\text{supp}(\mu)$.

Action minimizing curves do not contain conjugate points. This fact is known since Morse earlier works (cf. [27]). We present a simple proof of this fact in section §4 and we use proposition B to prove the following theorem, which was stated as theorem III in Mañé's unfinished work [25].

Theorem D.

There exists a generic set $\mathcal{O} \subseteq C^\infty(M,\mathbb{R})$ such that for all $\phi \in \mathcal{O}$ the Lagrangian $L + \phi$ has a unique minimizing measure and this measure is uniquely ergodic. When this measure is supported on a periodic orbit or a fixed point, this orbit (point) Γ is hyperbolic and its stable and unstable manifolds intersect transversally $W^s(\Gamma) \pitchfork W^u(\Gamma)$.

On Mañé [25], it is conjectured that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.

The first statement of theorem D was proved by Mañé in [23]. Here we prove the generic hyperbolicity of the minimizing periodic orbit and the transversality property $W^s(\Gamma) \pitchfork W^u(\Gamma)$. The hyperbolicity of Γ was announced in Mañé [25] and the transversality $W^s(\Gamma) \pitchfork W^u(\Gamma)$ was conjectured in Mañé [24]. We use the Clebsch transformation [7] to derive formulas for the index form on an orbit without conjugate points. Then we use similar arguments to the Rauch comparison theorem to obtain the transversality of the Green bundles of the periodic orbit. Finally, proposition B implies the hyperbolicity of the periodic orbit.

The metric entropy.

The machinery developed for the proof of theorem C can be used to obtain formulas for the metric entropy of a Hamiltonian flow without conjugate points. Oseledec's theory [28] gives a splitting

$$T_\theta\Sigma = E^s(\theta) \oplus E^c(\theta) \oplus E^u(\theta)$$

for a set of points $\theta \in \Lambda \subseteq \Sigma$ of total measure, such that

$$\begin{aligned}
&\lim_{t\to\pm\infty} \tfrac{1}{t} \log |d\psi_t(\theta)\,\xi| < 0 \quad \text{for} \quad \xi \in E^s(\theta)\,,\\
&\lim_{t\to\pm\infty} \tfrac{1}{t} \log |d\psi_t(\theta)\,\xi| = 0 \quad \text{for} \quad \xi \in E^c(\theta)\,,\\
&\lim_{t\to\pm\infty} \tfrac{1}{t} \log |d\psi_t(\theta)\,\xi| > 0 \quad \text{for} \quad \xi \in E^u(\theta)\,.
\end{aligned}$$

Following Freire and Mañé [13], we see in proposition 1.14 that

$$\begin{aligned}
E^u(\theta) &\subseteq \mathbb{F}(\theta) \subseteq E^u(\theta) \oplus E^c(\theta)\,,\\
E^s(\theta) &\subseteq \mathbb{E}(\theta) \subseteq E^s(\theta) \oplus E^c(\theta).
\end{aligned}$$

Given a riemannian metric on M, define the *horizontal subspace* as the kernel of the connection map. Then there exists symmetric linear isomorphisms $\mathbb{U} : H(\theta) \to V(\theta)$ such that $\mathbb{F}(\theta) = \operatorname{graph}(\mathbb{U}(\theta))$ for all $\theta \in \Sigma$. It satisfies the Ricatti equation (7)

$$\dot{\mathbb{U}} + \mathbb{U}\,H_{pp}\,\mathbb{U} + \mathbb{U}\,H_{pq} + H_{qp}\,\mathbb{U} + H_{qq} = 0,$$

where H_{pp}, H_{qp}, H_{pq} and H_{qq} are linear maps which coincide with the second derivatives of H in local coordinates, and $\dot{\mathbb{U}}$ is the covariant derivative of $\mathbb{U}$ defined as

$$\dot{\mathbb{U}}(\theta)\,v = \lim_{h\to 0} \frac{1}{h}\big[\tau_h\,\mathbb{U}(\psi_h(\theta))\,v - \mathbb{U}(\theta)\,v\big] \in V(\theta) \approx T^*_\theta M,$$

where $\tau_h : T^*_{\pi\psi_h(\theta)}M \to T^*_{\pi(\theta)}M$ is the parallel transport.

For $\theta \in \Lambda$ define the unstable Jacobian by

$$\chi(\theta) := \lim_{T\to+\infty} \tfrac{1}{T} \log \left|\det \, d\psi_T|_{E^u(\theta)}\right| .$$

We prove in proposition 1.14 that

$$\chi(\theta) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \operatorname{tr}\left[H_{pq} + H_{pp}\, \mathbb{U}\right] (\psi_t(\theta))\, dt \,,$$

where tr ( ) is trace. From Ruelle's inequality [33], for any invariant measure μ for $\psi_t|_\Sigma$, we obtain that the metric entropy satisfies

$$h_\mu(\psi|_\Sigma) \le \int_\Sigma \chi \, d\mu \le \int_\Sigma \operatorname{tr}[H_{pq} + H_{pp}\, \mathbb{U}]\, d\mu \,.$$

Pesin's formula states that for a smooth invariant measure μ of a $C^{1+\alpha}$ diffeomorphism $f : N \hookleftarrow$, we have that

$$h_\mu(f) = \int_N \chi\, d\mu \,.$$

In our case we obtain

Theorem E.

Let m be the Liouville measure on an energy level Σ without conjugate points. Then

$$h_m(\psi|_\Sigma) = \int_\Sigma tr[H_{pq} + H_{pp}\, \mathbb{U}]\ dm \,.$$

This theorem was proved for geodesic flows by Freire and Mañé [13], Finsler metrics by Foulon [15] and mechanical Lagrangians by Innami [19].

The exponential map.

The quasi-hyperbolicity in the case $\mathbb{E}(\theta) \pitchfork \mathbb{F}(\theta)$ is obtained from proposition 1.10 which states the horizontal growth of vertical vectors, i.e.

$$\lim_{t\to+\infty} \left\| d\pi \circ d\psi_t|_{V(\theta)} \right\| = +\infty.$$

The bounds obtained in proposition 1.10 are not uniform, by an example in Ballman, Brin & Burns [4]. In section §6 we give a uniform lower bound for $\left\| d\pi \circ d\psi_t|_{V(\theta)} \right\|$, $\theta \in \Sigma$, $|t| \ge 1$. This bound was given by Freire and Mañé [13] in the case of geodesic flows.

Define the exponential map of an energy level Σ as $\exp_q : T_q^*M \to M$, $\exp_q(t\theta) = \pi(\psi_t(\theta))$, where $\theta \in \Sigma \cap T_q^*M$. This map is not differentiable at $\mathbf{0} \in T_q^*M$, but its Lagrangian version is differentiable with $d\exp_q(\mathbf{0}) = I$. An important question is wether the lift of this map to the universal cover, $\exp_q : T_q^*M \to \tilde{M}$, is a diffeomorphism in an energy level without conjugate points and, in particular, if the universal cover $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$. We show in corollary 1.18 that if the energy level has no conjugate points then $\exp_q$ is a local diffeomorphism.

Let $\Theta = p\,dq$ be the Liouville's 1-form (4) in T^*M. Given a regular energy level Σ and $q \in \pi(\Sigma)$, let $\Sigma_q := (\pi|_\Sigma)^{-1}\{q\}$. Since $\tilde{M}$ is simply connected, in order to obtain a global diffeomorphism $\exp_q : T_q^*M \to \tilde{M}$ it is sufficient to show that $\exp_q$ is a covering map. For this it is enough the condition $\Theta(X) > 0$. Indeed, in section §6 we prove the following

Theorem F.
Let $H : T^*M \to \mathbb{R}$ be convex and $\Sigma = H^{-1}\{e\}$ a regular energy level. Suppose that $q \in \pi(\Sigma)$ satisfies

(a) *The positive orbit of all $\theta \in \Sigma_q$ has no conjugate points.*
(b) $\inf\big\{\, \big|\Theta\big(X(\vartheta)\big)\big| \;\big|\; \vartheta \in \psi_{\mathbb{R}^+}(\Sigma_q)\,\big\} > 0.$

Then the (Lagrangian) exponential map $\exp_q : T_q^*\tilde{M} \to \tilde{M}$ associated to (ψ_t, Σ) is a diffeomorphism.

We give an example (cf. ex. I.4) in apendix I in which $\exp_q$ is not surjective and where all the orbits in Σ of ψ_t starting at $\pi^{-1}\{q\}$ have no conjugate points. This shows that condition (a) is not sufficient in theorem F. By theorem X in Mañé [25] (see also [8]), this can not happen for high energy levels. Example 1 in apendix I also shows that the norm of the derivative $\|d_\theta(\exp_q)\|$ may not be bounded below on an energy level without conjugate points. We don't know if the exponential map is a diffeomorphism when the whole energy level has no conjugate points.

The condition $\Theta(X) = p\,H_p > 0$ is written $\Theta(X) = v\,L_v > 0$ in Lagrangian form. It is satisfied by Finsler Lagrangians, $L(x,v) = \frac{1}{2}\,\|v\|_x^2$, mechanical Lagrangians $L(x,v) = \frac{1}{2}\langle v,v\rangle_x - \phi(x)$ (with $\Theta(X) = 1$, $H(q,p) = \frac{1}{2}\langle p,p\rangle_q + \phi(q)$) and all convex Hamiltonians on sufficiently high energy levels. It fails for magnetic Lagrangians $L(x,v) = \frac{1}{2}\langle v,v\rangle_x - \phi(x) + \eta_x(v)$ (with $d\eta \neq 0$) on lower energy levels. If one adds a closed 1-form to a Lagrangian, the Euler-Lagrange equation and hence the Lagrangian flow does not change. This may help to make a given Lagrangian flow satisfy condition (b).

The obstruction to obtain a bound on $\|d_{\psi_t(\theta)}(\exp_q)\|$ is the angle $\sphericalangle\big(X(\psi_t(\theta)), d\psi_t(V(\theta)\cap T_\theta\Sigma)\big)$. If the orbit of θ is disconjugate this angle is always nonzero by proposition 1.16. On Example 3 in apendix I this angle reaches zero *before* the first conjugate point appear. Moreover, there are crossings of solutions on Σ starting at $\pi(\theta)$ before the first conjugate point.

On section §1 we prove proposition A, theorem E and we show the proof of item (c) in theorem 0.1. In section §2 we study the transversal behaviour of the Hamiltonian flow and prove proposition B. On section §3 we show the proof of theorem 0.2 and we prove theorem C. On section §4 we give a characterization of the index form and on section §5 we prove theorem D. On section §6 we give a uniform lower bound for $d\pi \circ d\psi_t|_{V(\theta)}$ and prove theorem F. Finally, we add some examples in the apendix.

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1. The Green Bundles.

For proofs of the basic results of this section see Abraham & Marsden [1]. There is a canonical symplectic structure on T^*M given by a closed non-degenerate 2-form $\omega = d\Theta$, where Θ is the Liouville's 1-form, defined by

$$\Theta_\theta(\xi) = \theta(d\pi(\theta)\,\xi)\,, \tag{4}$$

where $\pi : T^*M \to M$ is the canonical projection. The Hamiltonian vectorfield X of a function $H : T^*M \to \mathbb{R}$ is defined by $\omega(X, \cdot) = -dH$. The Hamiltonian flow ψ_t preserves the function H and the 2-form ω. The level sets of H are called *energy levels*. Given a local chart $q : U \subseteq M \to \mathbb{R}^n$ it induces a *natural chart* $(q,p) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ of T^*M. In natural charts the Liouville's 1-form is written as $\Theta = p\,dq$. Hence in natural charts $\omega = dp \wedge dq$ and the Hamiltonian equations are given by

$$\dot q = H_p \quad , \quad \dot p = -H_q\,. \tag{5}$$

In general, a set of local coordinates (q,p) of T^*M is called *symplectic* if the canonical 2-form is written as $\omega = dp \wedge dq$. A subspace $E \subset T(T^*M)$ is said *Lagrangian* if $\dim E = n = \dim M$ and $\omega(x,y) = 0$ for all x, $y \in E$.

Fix a riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two Lagrangian subspaces: the vertical subspace $V(\theta) = \ker(d\pi(\theta))$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M \times T^*_{\pi(\theta)}M$, $\xi \mapsto (d\pi(\theta)\,\xi\,,\, \nabla_\theta(\pi\,\xi))$, we can identify $H(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $V(\theta) \approx \{0\} \times T^*_{\pi(\theta)}M \approx T_{\pi(\theta)}M$. If we choose local coordinates along $t \mapsto \pi\,\psi_t(\theta)$ such that $t \mapsto \frac{\partial}{\partial q_i}(\pi\,\psi_t(\theta))$ are parallel vectorfields, then this identification becomes $\xi \leftrightarrow (dq(\xi),\, dp(\xi))$. Let $E \subset T_\theta T^*M$ be an n-dimensional subspace such that $E \cap V(\theta) = \{0\}$. Then E is a graph of some linear map $S : H(\theta) \to V(\theta)$. It can be checked that E is Lagrangian if and only if in symplectic coordinates S is symmetric.

Proof of Proposition A.

Take $\theta \in T^*M$ and $\xi = (h, v) \in T_\theta T^*M = H(\theta) \oplus V(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Consider a variation

$$\alpha_s(t) = \big(q_s(t), p_s(t)\big)$$

such that for each $s \in]-\varepsilon, \varepsilon[$, α_s is a solution of the Hamiltonian H such that $\alpha_0(0) = \theta$ and $\frac{d}{ds}\alpha_s(\theta)|_{s=0} = \xi$. Writing $d\phi_t(\xi) = \big(h(t), v(t)\big)$, we obtain the Hamiltonian Jacobi equations

$$\begin{aligned} \dot{h} &= \quad H_{pq}\, h + H_{pp}\, v\,, \\ \dot{v} &= -H_{qq}\, h - H_{qp}\, v\,, \end{aligned} \tag{6}$$

where the covariant derivatives are evaluated along $\pi(\alpha_o(t))$, and H_{qq}, H_{qp}, H_{pp} and H_{qq} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)$, $\left(\frac{\partial^2 H}{\partial q_i\, \partial p_j}\right)$, $\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right)$ and $\left(\frac{\partial^2 H}{\partial p_i\, \partial q_j}\right)$. Moreover, since the Hamiltonian H is convex, then H_{pp} is positive definite.

We derive now the Ricatti equation. Let E be a Lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon, \varepsilon[$ we have that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$. Then we can write $d\psi_t(E) = \text{graph } S(t)$, where $S(t) : H(\psi_t\, \theta) \to V(\psi_t\, \theta)$ is a symmetric map. That is, if $\xi \in E$ then

$$d\psi_t(\xi) = \big(h(t), S(t)\, h(t)\big)\,.$$

Using equation (6) we have that

$$\dot{S}h + S(H_{pq}h + H_{pp}Sh) = -H_{qq}h - H_{qp}Sh\,.$$

Since this holds for all $h \in H(\psi_t(\theta))$ we obtain the Ricatti equation:

$$\dot{S} + SH_{pp}S + SH_{pq} + H_{qp}S + H_{qq} = 0\,. \tag{7}$$

Take the symmetric map $S(t)$ for which $d\psi_t(V(\theta)) = \text{graph}(S(t))$, $t > 0$. Let $Y(t) : T_{\pi(\theta)}M \approx V(\theta) \to H(\psi_t\theta) \approx T_{\pi\psi_t\,\theta}M$, $t \in \mathbb{R}$, be the family of isomorphisms $Y(t)w = d\pi \cdot d\psi_t(\theta)\,(0, w)$, where $(0, w) \in H(\theta) \oplus V(\theta)$. Then for $t > 0$, $Y(t)$ satisfies

$$\dot{Y} = (H_{pq} + H_{pp}\, S)\ Y \tag{8}$$

$$Y(0) = 0 \quad , \quad \dot{Y}(0) = H_{pp}\,,$$

where $\dot{Y}(0)$ is calculated from (8) using that

$$\lim_{t\to 0} S(t), Y(t) = I, .$$

Given $u \in T_{\pi(\theta)}M$, define

$$\begin{aligned} h_1(t) &= Y(t)\, u\,, \\ v_1(t) &= S(t)\, Y(t)\, u\,, \quad t > 0\,. \end{aligned}$$

Then (h_1, v_1) is a solution of equation (6) with $h_1(0) = 0$, $v_1(0) = u$. Let $\xi \in T_\theta T^*M$ and let $(h(t), v(t)) = d\psi_t(\xi) \in H \oplus V$. Since $d\psi_t$ preserves the symplectic form, we have that

$$\langle h(t), v_1(t)\rangle - \langle v(t), h_1(t)\rangle = \langle h(0), u\rangle$$

Hence

$$\langle Y^*(t)\, S(t)\, h(t)\,,\, u\rangle - \langle Y^*(t)\, v(t)\,,\, u\rangle = \langle h(0)\,,\, u\rangle$$

for all $u \in T_{\pi(\theta)}M$. Therefore

$$v(t) = S(t)\, h(t) - (Y^*(t))^{-1}\, h(0)\,. \tag{9}$$

Since

$$\dot h = H_{pq}\, h + H_{pp}\, v$$

we get

$$\dot h = (H_{pq} + H_{pp}S)h - H_{pp}(Y^*)^{-1}h(0) \tag{10}$$

Differentiating $Y^{-1}\, h$ and using equation (8), we obtain

$$h(t) = Y(t)\, Y(c)^{-1}\, h(c) + Y(t) \int_t^c Y^{-1}(s)\, H_{pp}\, (Y^*(s))^{-1}\, h(0)\; ds\,, \text{ for } 0 < t < c. \tag{11}$$

Now consider the map $d\psi_t$ restricted to the subspace $d\psi_{-c}(V(\psi_c\,\theta))$, $c > 0$. Its projection $Z_c(t)$ to the horizontal subspace is given by the solutions of (6) with $h(c) = 0$. Then $Z_c(t)$ is the family of isomorphisms given by

$$Z_c(t) = Y(t) \int_t^c Y^{-1}(s)\, H_{pp}\, (Y^*(s))^{-1}\; ds\ ,\ \ 0 < t < c\,, \tag{12}$$

because $Z_c(t)$ satisfies (6) for $0 < t < c$, $Z_c(c) = 0$ and

$$Z_c(0) = \lim_{t\to 0^+} Z_c(t) = I\,.$$

Observe that from its definition, the linear map $Z_c(t)$ is defined for all $t \in \mathbb{R}$ and the no-conjugate points condition implies that $Z_c(t)$ is an isomorphism for all $t \neq c$. Moreover, since $Z_c(t)$ is a matrix solution of the Jacobi equation (6), then $\dot Z_c(0)$ exists, and taking the limit in (12), we have that the linear isomorphism

$$\dot Z_d(0) - \dot Z_c(0) = H_{pp}(\theta) \int_c^d Y(s)^{-1}\, H_{pp}\ (Y^*(s))^{-1}\; ds \tag{13}$$

is symmetric and the second factor on the right is positive definite for all $0 < c < d$. In particular

$$Z_d(t) - Z_c(t) = Y(t)\, H_{pp}(\theta)^{-1} \left[\dot Z_d(0) - \dot Z_c(0)\right]\ ,\quad 0 < c < d\,. \tag{14}$$

For $\varepsilon > 0$ define

$$Z_{-\varepsilon}(t) := Y(t)\, N_c + Z_c(t) \quad ,\quad t \in \mathbb{R}, \tag{15}$$

where $N_c := -Y(-\varepsilon)^{-1} Z_c(-\varepsilon)$. We have that $Z_{-\varepsilon}(t)$ is the solution of the Jacobi equation (6) satisfying $Z_{-\varepsilon}(-\varepsilon) = 0$, $Z_{-\varepsilon}(0) = I$ and

$$\dot{Z}_{-\varepsilon}(0) - \dot{Z}_c(0) = H_{pp}(\theta)\, N_c\,. \tag{16}$$

1.1. Claim. *$Y^{-1}(t)\, Z_c(t)$ is symmetric for all $t \neq 0$. In particular N_c is symmetric for all $c > 0$.*

Proof: Using (10), we have that

$$\begin{aligned} \dot{Z}_c &= A\, Z_c + H_{pp}\, (Y^*)^{-1}\,, \\ \dot{Y} &= A\, Y + \quad 0 \qquad , \quad t \neq 0\,, \end{aligned}$$

where $A = H_{pq} + H_{pp}\, S$. And then

$$\dot{Z}_{-\varepsilon} = A\, Z_{-\varepsilon} + H_{pp}(Y^*)^{-1}\,.$$

Using (9) we have that the functions

$$\begin{aligned} h_2(t) &= Z_{-\varepsilon}(t)\, u\,, \\ v_2(t) &= S(t)\, h_2(t) - \big(Y^*(t)\big)^{-1}\, u\,, \end{aligned} \tag{17}$$

are solutions of (6) with $h_2(-\varepsilon) = 0$, $h_2(0) = u$. Similarly, the functions

$$\begin{aligned} h_3(t) &= Z_c(t)\, w\,, \\ v_3(t) &= S(t)\, h_3(t) - \big(Y^*(t)\big)^{-1}\, w\,, \end{aligned} \tag{18}$$

are solutions of (6) with $h_3(0) = w$, $h_3(c) = 0$.

Since the flow preserves the symplectic form, we have that the following expressions do not depend on t for all u, $w \in T_{\pi(\theta)}M$:

$$\langle\, h_2(t), v_3(t)\,\rangle - \langle\, v_2(t), h_3(t)\,\rangle$$
$$\langle\, Z_{-\varepsilon}\, u\,,\, (S Z_c - (Y^*)^{-1})\, w\,\rangle - \langle\, (S Z_{-\varepsilon} - (Y^*)^{-1})\, u\,,\, Z_c\, w\,\rangle$$
$$Z_c^* S Z_{-\varepsilon} - Y^{-1} Z_{-\varepsilon} - Z_c^* S Z_{-\varepsilon} + Z_c^*(Y^*)^{-1} = Z_c^*(Y^*)^{-1} - Y^{-1} Z_{-\varepsilon}\,.$$

Using formula (15), this is equal to

$$(Y^{-1} Z_c)^* - Y^{-1} Z_c - N_c = -N_c^*\,, \tag{19}$$

where the right hand side corresponds to its value when $t = -\varepsilon$. Since $Z_c(c) = 0$, evaluating the equation at $t = c$, we get that N_c is symmetric for all $c > 0$. Using (19) again, we obtain that $Y^{-1} Z_c$ is symmetric for all $t \neq 0$.

1.2. Claim. *N_c is positive definite for all $c > 0$.*

Proof: It is enough to show it for N_c^{-1}. We have that

$$\frac{d}{dt} - Z_c(t)^{-1}\,Y(t)\Big|_{t=0} = Z_c^{-1}\,\dot Z_c\,Z_c^{-1}\,Y - Z_c^{-1}\,\dot Y\big|_{t=0} = -H_{pp}\,.$$

Therefore $-Z_c(t)^{-1}Y(t)$ is positive definite for small $t<0$. By claim 1.1, this matrix is symmetric for all $t<0$. Since for $t<0$ its determinant never vanishes, it is positive definite for all $t<0$, in particular for $t=-\varepsilon$, N_c^{-1} is positive definite.

□

Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \rhd B$ if $H_{pp}(\theta)^{-1}\,(A-B)$ is positive definite. From (13) we have that the family $\mathcal{F}=\{\dot Z_c(0)-\dot Z_1(0)\,|\,c>1\,\}$ is monotone increasing. Moreover, by (16) and claim 1.2, we have that

$$\begin{aligned}\dot Z_c(0)-\dot Z_1(0) &\lhd H_{pp}\,N_c + \Big(\dot Z_c(0)-\dot Z_1(0)\Big)\\ &\lhd \dot Z_{-\varepsilon}(0)-\dot Z_1(0)\,.\end{aligned}$$

So that the family $\mathcal{F}$ is bounded above. It can be seen that this implies the existence of a lower upper bound which is the limit of the family. We obtain the first part of

1.3. Claim.

(a) $\lim_{c\to+\infty}\dot Z_c(0)-\dot Z_1(0)=Q$ *exists and is symmetric.*
(b) $\lim_{c\to+\infty}Z_c(t)=D(t)$ *exists (uniformly for bounded t intervals).*

Moreover,

$$\begin{aligned}&\mathbf{h}(t)=D(t)\,,\\ &\mathbf{v}(t)=S(t)\,D(t)+\big(Y^*(t)\big)^{-1}\,,\; t\neq 0\end{aligned}$$

is a matrix solution of (6) *such that* $D(0)=I$, $\dot D(0)=Q+\dot Z_1(0)$ *and* $\det D(t)\neq 0$ *for all* $t\in\mathbb{R}$.

Part (b) is a consequence of (14) and the continuous dependence of the solutions of equation (6) on the initial data.

We have that the subspace

$$\mathbb{E}_\theta(0) := \big\{\,\big(u,\mathbf{v}(0)\,u\big)\,\big|\,u\in T_{\pi(\theta)}M\,\big\}\subset H(\theta)\oplus V(\theta)$$

is the limit of the subspaces

$$E_c(\theta) := d\psi_{-c}\big(V(\psi_c(\theta))\big) = \big\{\,\big(h_3(\theta),v_3(\theta)\big)\,\big|\,u\in T_{\pi(\theta)}M\,\big\}$$

where $h_3(t)$ and $v_3(t)$ are given by (18). Moreover

$$\mathbb{E}_\theta(t) := d\psi_t\big(\mathbb{E}_\theta(0)\big) = \big\{\,\big(\mathbf{h}(t)\,u,\mathbf{v}(t)\,u\big)\,\big|\,u\in T_{\pi(\theta)}M\,\big\}$$

satisfies $\mathbb{E}_\theta(t) \cap V(\psi_t\theta) = \{0\}$ and

$$\mathbb{E}_\theta(t) = \lim_{c\to+\infty} d\psi_{t-c}\big(V(\psi_c(\theta))\big) = \lim_{d\to+\infty} d\psi_{-d}\big(V(\psi_d(\psi_t\theta))\big) = \mathbb{E}_{\psi_t\theta}(0)\,.$$

Since the vertical subspace is Lagrangian, then $K_\theta(c)$ is Lagrangian. By the continuity of the symplectic form, the subspaces $\mathbb{E}_\theta(t)$ are Lagrangian.

To obtain the "unstable" Green bundle $\mathbb{F}(\theta)$, observe that the flow of the Hamiltonian $\overline{H}(\theta) = -H(\theta)$ is the flow ψ_t of H with the time reversed. In this case $\overline{H}_{pp}$ is negative definite, but similar arguments apply to obtain the subbundle $\mathbb{F}$. Applying the lemma to $\overline{H}$, we obtain the subbundle $\mathbb{F}$. Moreover, in this case the family $\mathcal{G} = \{\,\dot{Z}_{-c}(0) - \dot{Z}_{-1}(0)\,|\, c > 1\,\}$ is monotone decreasing and bounded below by $\dot{Z}_\varepsilon(0) - \dot{Z}_{-1}(0)$. This finishes the proof of the first part of proposition A, we defer the proof of the second statement to corollary 1.12.

$\square$

Let

$$\begin{aligned}\mathbb{S}(t) &:= \boldsymbol{v}(t)\;\boldsymbol{h}(t)^{-1}\\ &= S(t) + Y^*(t)^{-1}\,D(t)^{-1}.\end{aligned} \tag{20}$$

Then $\mathbb{S}(t)$ is a symmetric solution of the Ricatti equation (7), which is defined on all $t \in \mathbb{R}$. The symmetry of $\mathbb{S}(t)$ can be checked either because $\mathbb{E}(t) = \operatorname{graph}(\mathbb{S}(t))$ is a Lagrangian subspace or because

$$Y(t)^{-1}\,D(t) = \lim_{c\to\infty} Y(t)^{-1}\,Z_c(t)$$

is symmetric and this implies that $(Y^*)^{-1}\,D^{-1}$ is symmetric.

Let $\mathbb{S}(\theta)$, $\mathbb{U}(\theta)$ be the the symmetric solutions of the Ricatti equation (7) corresponding to the green bundles $\mathbb{E}(\theta) = \mathrm{Graph}(\mathbb{S}(\theta))$ and $\mathbb{F}(\theta) = \mathrm{Graph}(\mathbb{U}(\theta))$. Let $K_c(\theta) : H(\theta) \to V(\theta)$ be the symmetric linear map such that $\mathrm{Graph}(K_c(\theta)) = d\psi_{-c}(V(\psi_c(\theta)))$. Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \succ B$ if $A - B$ is positive definite.

1.4. Proposition. *For all $\varepsilon > 0$,*

(a) *If $d > c > 0$ then $K_{-\varepsilon} \succ K_d \succ K_c$.*
(b) *If $d < c < 0$ then $K_\varepsilon \prec K_d \prec K_c$.*
(c) $\displaystyle\lim_{d\to+\infty} K_d = \mathbb{S}$, $\displaystyle\lim_{d\to-\infty} K_d = \mathbb{U}$.
(d) $\mathbb{S} \preccurlyeq \mathbb{U}$.

Proof:

Let $(Z_c(t),\, V_c(t))$ be the matrix solution of the Jacobi equation (6) with $Z_c(0)=I$, $Z_c(c)=0$, $c>0$. From (9) we have that

$$V_c(t) = S(t)\, Z_c(t) - (Y^*(t))^{-1}. \tag{21}$$

Hence, using (14) and (13) we have that

$$V_d(t) - V_c(t) = S(t)\ [Z_d(t) - Z_c(t)] \tag{22}$$

$$= S(t)\, Y(t) \int_c^d Y^{-1}\, H_{pp}\, (Y^*)^{-1}\, ds. \tag{23}$$

When $t\to 0$ we have that $S(t)\, Y(t) \to I$. Therefore

$$V_d(0) - V_c(0) = \int_c^d \big(Y^{-1}\, H_{pp}\, (Y^*)^{-1}\big)\, ds. \tag{24}$$

Let $K_c(t) = V_c(t)\, Z_c(t)^{-1}$ be the corresponding solution of the Ricatti equation (7). Since $Z_c(0)=I$, we have that $K_c(0)=V_c(0)$. By (24), $K_d(0)-K_c(0)$ is positive definite if $d>c>0$, hence the sequence $K_d(0)$, $d>0$ is strictly increasing.

From (17) we have that equation (22) also holds for $c=-\varepsilon$. Using (15), we have that

$$V_d(t) - V_{-\varepsilon}(t) = S(t)\ (Z_d(t) - Z_{-\varepsilon}(t))$$
$$= S(t)\,(-Y(t)\, N_d)\,.$$

Hence $V_d(0) - V_{-\varepsilon}(0) = -N_d$ is negative definite. Since $Z_{-\varepsilon}(0)=I$, then $K_{-\varepsilon}(0) = V_{-\varepsilon}(0)$. Therefore $K_{-\varepsilon} \succ K_d$ for all $d>0$.

This completes the proof of item (a). Item (c) has already been proven on claim 1.3 above. Item (b) is proven similarly using the concave Hamiltonian $\overline{H} = -H$ and item (d) is a corollary of items (a), (b) and (c).

□

1.5. Remark. If a bounded open segment $\{\,\psi_t(\theta)\,|\,a^- < t < a^+\,\}$ has no conjugate points, the arguments above apply to obtain limit solutions $\lim_{c\to a^\pm} Z_c(t) = D^\pm(t)$, $D^\pm(0)=I$, $\det D(t)\neq 0$, Lagrangian subspaces $E^\pm(\theta) = \lim_{t\to a^\pm} d\psi_{-t}(V(\psi_t\,\theta)) = d\psi_{a^\pm}\big(V(\psi_{a^\pm}(\theta))\big)$ and the monotonicity properties of proposition 1.4. Nevertheless, in general $E^\pm(\theta)\not\subset T_\theta\Sigma$, see remark 1.17

1.6. Remark. The existence of the Green bundles, the solutions $\boldsymbol{h}$, $\boldsymbol{v}$, $\mathbb{U}$, $\mathbb{S}$ and the monotonicity properties in proposition 1.4 do not need the uniform boundedness of the operators H_{pp}, H_{pq}, H_{qq}. But in the unbounded case we can not guarantee neither that $X(\theta)\in\mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ nor $\mathbb{E}(\theta)\cup\mathbb{F}(\theta)\subset T_\theta\Sigma$.

1.7. Proposition.

(a) *Any symmetric solution* $S_\theta(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ *of the Ricatti equation* (7) *which is defined on* $t > 0$ *is uniformly bounded on* $t \geq 1$*, i.e. there exists* $A > 0$ *such that*

$$\|S_\theta(t)\| < A \quad \textit{for all} \quad \theta \in \Sigma,\ t > 1.$$

(b) *Any measurable symmetric solution of the Ricatti equation defined on the energy level* Σ *is uniformly bounded.*

An inmediate consequence is the following

1.8. Corollary.

(a) *The solution* $S_\theta(t)$ *of equation* (8) *given by* $\mathrm{Image}(Y_\theta(t)\,, S_\theta(t)\, Y_\theta(t)) = d\psi_t\, V(\theta)$ *is uniformly bounded for* $|t| > 1$.

(b) *The solutions* $\mathbb{S}_\theta(t)$, $\mathbb{U}_\theta(t)$ *corresponding to the Green bundles are uniformly bounded on* Σ.

We shall need the following lemma, whose proof is elementary:

1.9. Lemma. *The function* $w(r) = R\, \coth\,(R\,t - d)$, $R > 0$, *satisfies*

(i) $\dot{w} + w^2 - R^2 = 0$.
(ii) $\dot{w} < 0$, $w\,(-t + \frac{d}{R}) = -w\,(t + \frac{d}{R})$.
(iii) $\lim_{t\to+\infty} w(t) = R$, $\lim_{t\to-\infty} w(t) = -R$.
(iv) $\lim_{t\to(\frac{d}{R})^+} w(t) = +\infty$, $\lim\limits_{t\to(\frac{d}{R})^-} w(t) = -\infty$.

Proof of proposition 1.7:

We first prove (a). By the spectral theorem, it is enough to prove that $x^*\, S_\theta(t)\, x$ is uniformly bounded for $\|x\| = 1$ and $t > 1$. Since $H_{pq}^* = H_{qp}$ and $S^* = S$, we have that

$$\dot{S} + S^*\, H_{pp}\, S + H_{pq}^*\, S + S^*\, H_{pq} + H_{qq} = 0.$$

Let $C := (H_{pp})^{-1}\, H_{pq}$, $D := H_{qq} - C^* H_{pp} C - \dot{C}$ and $V_\theta(t) := S_\theta(t) + C(\psi_t(\theta))$. Then

$$\dot{V} + V^*\, H_{pp}\, V + D = 0\,.$$

Since $C(\theta)$ and $D(\theta)$ are continuous, then they are uniformly bounded on the energy level. Hence it is enough to prove that $x^* V_\theta(t)\, x$ is uniformly bounded on $t > 1$ and $\theta \in \Sigma$. Observe that $V_\theta(t)$ is not necessarily symmetric. Let $M > 0$ be such that

$$y^* H_{pp}(\theta)\, y \geq \frac{1}{M}\, \|y\|^2 \qquad \text{for all} \quad \theta \in \Sigma. \tag{25}$$

Let $R > 0$ be such that

$$|\, y^* D(\theta)\, y\,| < M\, R^2 \qquad \text{for all} \quad \theta \in \Sigma, \quad \|y\| = 1. \tag{26}$$

We claim that

$$|\, y^* V_\theta(t)\, y\,| \le M\,R\,\coth(R) \qquad \text{for all} \quad \|y\|=1, \quad t>1, \quad \theta\in\Sigma.$$

For, let $\theta_0 \in \Sigma$, $\|x\|=1$. Write $V(t) := V_\theta(t)$, $H_{pp}(t) := H_{pp}(\psi_t\,\theta)$. Suppose that there exists $t_o \in \mathbb{R}$ such that

$$x^* V(t_0)\, x =: M\,\alpha > M\,R\,\coth(R).$$

There exists $d_0 \in \mathbb{R}$ such that $R\,\coth\left(R\,t_0 - d_0\right) = \alpha$. Observe that $d_0 > 0$ and $t_0 > \frac{d_0}{R} > 0$. Write $w(t) := R\,\coth\left(R\,t - d_0\right)$. Then $w(t)$ is a solution of $\dot w + w^2 - R^2 = 0$ for $t > \frac{d_0}{R}$. In particular,

$$M\,\dot w + M\,w^2 - M\,R^2 = 0.$$

Let $f(t) := x^*\,V(t)\,x - M\,w(t)$. Then $f(t_0) = 0$ and

$$f'(t) + \left(x^*\,V(t)^* H_{pp}(t)\,V(t)\,x - M\,w(t)^2\right) + \left(x^* D(t)\,x + M\,R^2\right) = 0\,. \tag{27}$$

Using Schwartz inequality and (25), we have that

$$\begin{aligned} M\,w(t_0)^2 = M\,\alpha^2 &= \frac{1}{M}\,(M\,\alpha)^2 = \frac{1}{M}\,\langle x, V(t_0)\,x\rangle^2 \\ &\le \frac{1}{M}\,|V(t_0)\,x|^2 \le \big(V(t_0)\,x\big)^* H_{pp}(t_0)\,\big(V(t_0)\,x\big) \end{aligned}$$

$$M\,w(t_0)^2 \le x^*\,V(t_0)^* H_{pp}(t_0)\,V(t_0)\,x \tag{28}$$

Then (28), (26) and (27) imply that $f'(t_0) < 0$. The same argument can be applied each time that $f(t) = 0$. Therefore $x^*\,V(t)\,x \le M\,w(t)$ for all $t > t_0$ and

$$x^*\,V(t)\,x \ge M\,w(t) \quad \text{for all } \tfrac{d_0}{R} < t < t_0.$$

Then

$$\lim_{t\to\left(\frac{d_0}{R}\right)^+} x^*\,V(t)\,x \ge \lim_{t\to\left(\frac{d_0}{R}\right)^+} M\,w(t) = +\infty\,.$$

Since $\frac{d_o}{R} > 0$, this contradicts the existence of $V(t)$ for $t > 0$. Hence such t_0 does not exists and

$$x^*\,V(t)\,x \le M\,R\,\coth(R) \qquad \text{for all } t > 0.$$

Now suppose that there exists $t_1 \in \mathbb{R}$, $\|z\| = 1$ such that

$$x^*\,V(t_1)\,x < -M\,R\,.$$

We compare $v(t) = x^*\,V(t)\,x$ with $M\,w_1(t)$, where $w_1(t) = R\,\coth\left(R\,t - c_0\right)$ is such that $M\,w_1(t_1) = v(t_1)$. Observe that $w_1(t)$ is defined for $t < \frac{c_0}{R}$ and in this case $c_0 > 0$ and $\frac{c_0}{R} > t_1 > 0$. The same argument as above shows that $v(t) \le M\,w_1(t)$ for $t_1 \le t < \frac{c_0}{R}$. Since $\lim_{t\to\left(\frac{c_0}{R}\right)^-} w_1(t) = -\infty$, this contradicts the fact that $V(t)$ is defined for all $t \in \mathbb{R}$.

The prove of (b) is similar to that of (a). In this case we can change the $MR\coth(R)$ bound by MR. We allow $t_0 \in \mathbb{R}$ and $d_0 \in \mathbb{R}$ to be nonpositive if necessary. The second inequality has also the same proof, but here we allow $t_1 \in \mathbb{R}$ and $c_0 \in \mathbb{R}$. $\square$

The following proposition states that $\lim_{t\to\pm\infty} \left\| d\pi \circ d\psi_t(\theta)|_{V(\theta)} \right\| = +\infty$ for all $\theta \in \Sigma$. This limit is not uniform in $\theta \in \Sigma$. A uniform lower bound for this norm is given in section §6.

1.10. Proposition. *For all $R > 0$ there exists $T > T(R,\theta) > 0$ such that $|Y_\theta(t)\, v| > R\,|v|$ for all $|t| > T$ and all $v \in T_\theta M \setminus \{0\}$.*

Proof:
Let

$$M(t) := \int_t^\infty Y^{-1}(s)\, H_{pp}(s)\, \left(Y^*(s)\right)^{-1}\, ds\,.$$

From (12), using that $D(t) = \lim_{c\to\infty} Z_c(t)$, we have that

$$D(s) = Y(s)\, M(s) \qquad \text{for} \quad s > 0\,. \tag{29}$$

Consider the solutions $S(t)$, $\mathbb{S}(t)$ of the Ricatti equation given on (8) and (20). From (20) and (29) we have that

$$\mathbb{S}(t) - S(t) = Y^*(t)^{-1}\, M(t)^{-1}\, Y(t)^{-1} \quad \text{for} \quad t > 0.$$

Since the solutions $S(t)$, $\mathbb{S}(t)$ are defined on all $t > 0$, by proposition 1.7, there exist $t_0 > 0$ and $k > 0$ such that $\|S(t)\| < k$ and $\|\mathbb{S}(t)\| < k$ for all $t > t_0$. Then for $|x| = 1$ and $t > t_0$,

$$\left|\left\langle\, M(t)^{-1}\, Y(t)^{-1}\, x\,,\; Y(t)^{-1}\, x\,\right\rangle\right| = |\langle\, S(t)\, x\,,\; x\,\rangle| + |\langle\, \mathbb{S}(t)\, x\,,\; x\,\rangle| \le 2\,k\,.$$

Let $\lambda(t)$ be the largest eigenvalue of $M(t)$, $t > 0$. Since $M(t)$ is positive definite then $\lambda(t) > 0$ and $\|M(t)\| = \lambda(t)$. Moreover,

$$2\,k \ge \left|\left\langle\, M(t)^{-1}\, Y(t)^{-1}\, x\,,\; Y(t)^{-1}\, x\,\right\rangle\right| \ge \frac{1}{\lambda(t)}\, \left|\, Y(t)^{-1}\, x\,\right|^2\,,$$

$$\left|\, Y(t)^{-1}\, x\,\right| \le \sqrt{2\,k}\; \|M(t)\|^{\frac{1}{2}} \qquad \text{for all} \quad |x| = 1,\; t > t_0\,.$$

Then if $|v| = 1$, we have that

$$|Y(t)\, v| \ge \frac{1}{\|Y(t)^{-1}\|} \ge \frac{1}{\sqrt{2\,k}\; \|M(t)\|^{\frac{1}{2}}} \qquad \text{for} \quad t > t_0\,.$$

Since $M(t) \to 0$ when $t \to +\infty$, given $R > 0$ there exists $T > t_0$ such that $(2\,k\; \|M(t)\|)^{\frac{1}{2}} < \frac{1}{R}$ for all $t > T$ and hence $|Y(t)\, v\,| > R$ for all $t > T$ and $|v| = 1$.

Using the Hamiltonian $\overline{H} := -H$, we get the result for $t < -T$. $\square$

For $\theta \in TM$, define

$$B(\theta) := \Big\{ \xi \in T_\theta T^*M \;\Big|\; \sup_{t\in\mathbb{R}} |d\psi_t(\theta)\cdot\xi| < +\infty \Big\}.$$

1.11. Proposition. *If the ψ-orbit of $\theta \in TM$ does not contain conjugate points, then $B(\theta) \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$.*

An example showing that in general $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) \not\subset B(\theta)$ is given in Ballmann, Brin & Burns [4].

Proof: Let $\xi \in B(\theta)$ and let $\zeta_\Gamma \in E_T(\theta) := d\psi_{-T}\left(V\left(\psi_T(\theta)\right)\right)$ be such that $d\pi(\theta)\cdot\zeta_T = d\pi(\theta)\cdot\xi$, in particular $\xi - \zeta_T \in V(\theta)$. Since $\lim_{T\to+\infty} E_T(\theta) = \mathbb{E}(\theta)$ and $\mathbb{E}(\theta) \pitchfork V(\theta)$ then there exists $\zeta := \lim_{T\to+\infty}\zeta_T \in \mathbb{E}(\theta)$. Moreover, $d\pi\circ d\psi_T(\xi - \zeta_T) = d\pi\circ d\psi_T(\xi)$. Since $d\pi\circ d\psi_T(\xi)$ is bounded on $T > 0$, by proposition 1.10, $\lim_{T\to+\infty}(\xi - \zeta_T) = 0$. Since $\zeta_T \to \zeta \in \mathbb{E}(\theta)$, then $\xi \in \mathbb{E}(\theta)$.

Similarly if $\eta_T \in F_T(\theta) := d\psi_T\left(V\left(\psi_{-T}(\theta)\right)\right)$ is such that $d\pi(\theta)\cdot\eta_T = d\pi(\theta)\cdot\xi$. Then $\lim_{T\to+\infty}\eta_T \in \mathbb{F}(\theta)$ and $\lim_{T\to+\infty}(\xi - \eta_T) = 0$.

□

1.12. Corollary. *If the orbit of θ has no conjugate points then*

(a) *$\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$.*

(b) *$\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subseteq T_\theta\Sigma$.*

Proof:

The property $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ is a direct consequence of proposition 1.11. It remains to show that $\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subset T_\theta\Sigma$. Observe that since $-dH = \omega(X,\cdot)$, then

$$T_\theta\Sigma = \{\, \xi \in T_\theta T^*M \mid \omega(X(\theta),\xi) = 0 \,\}.$$

Since $\mathbb{F}(\theta)$ is Lagrangian and $X(\theta) \in \mathbb{F}(\theta)$, then $\omega(X(\theta),\xi) = 0$ for all $\xi \in \mathbb{F}(\theta)$. Hence $\mathbb{F}(\theta) \subseteq T_\theta\Sigma$. Similarly $\mathbb{E}(\theta) \subseteq T_\theta\Sigma$.

□

1.13. Corollary. *If M is connected and Σ is a regular energy level without conjugate points, then $\pi(\Sigma) = M$.*

Proof: For $\theta \in \Sigma$, $d\pi(\theta) : \mathbb{F}(\theta) \to T_{\pi(\theta)}M$ is an isomorphism. By corollary 1.12, $\mathbb{F}(\theta) \subset T_\theta\Sigma$. Hence $\pi : \Sigma \to M$ is an open map. Since M is connected then $\pi(\Sigma) = M$. □

The metric entropy.

Let Λ be the set of total measure having the Oseledec's splitting $T_\theta T^*M = E^s(\theta)\oplus E^c(\theta) \oplus E^u(\theta)$.

1.14. Proposition. *For all $\theta \in \Lambda$*

(a) $E^s(\theta) \subseteq \mathbb{E}(\theta) \subseteq E^s(\theta) \oplus E^c(\theta)$.

(b) $E^u(\theta) \subseteq \mathbb{F}(\theta) \subseteq E^u(\theta) \oplus E^c(\theta)$.

(c) $\displaystyle \lim_{T\to+\infty} \frac{1}{T}\int_0^T \operatorname{tr}\,\big[H_{pq} + H_{pp}\,\mathbb{U}\big]\big(\psi_t(\theta)\big)\;dt.$

Proof:

We only proof

$$E^u(\theta) \subseteq \mathbb{F}(\theta) \subseteq E^u(\theta) \oplus E^c(\theta)\,, \tag{30}$$

which shall be used for theorem E, the other inclusions are similar.

We first prove the second inclusion. Observe that for $\overline{v} = (0, v) \in V(\theta)$ we have that $d\psi_t(\theta)\,\overline{v} = (Y_\theta(t)\,v, *)$. Then, by proposition 1.10, we have that $V(\theta) \cap E^s(\theta) = \{0\}$. For every $\theta \in \Lambda$ there exists a subspace $R(\theta) \subset E^u(\theta) \oplus E^c(\theta)$ and a linear map $L(\theta) : R(\theta) \to E^s(\theta)$ such that $V(\theta) = \operatorname{graph} L(\theta)$.

Let μ be an ergodic invariant measure for ψ_t. Given $\varepsilon > 0$ let $K \subset \Lambda$ be a compact subset such that $\mu(K) > 1 - \varepsilon$ and

(a) $\sup\big\{\, \|L(\theta)\| \mid \theta \in K \,\big\} < +\infty$.

(b) There exists $T > 0$ and $0 < \eta < 1 < \lambda < \eta^{-1}$ such that for $t > T$ and $\theta \in K$

$$\big\|\psi_t|_{E^s(\theta)}\big\| < \eta^t \qquad \text{and} \qquad \big\|d\psi_t^{-1}|_{E^u(\theta)\oplus E^c(\theta)}\big\| < \lambda^t$$

Let $V_{-s}(\theta) := d\psi_s\big(V(\psi_{-s}(\theta))\big)$. Then

$$\begin{aligned} V_{-s}(\theta) &= d\psi_s\big(\operatorname{graph} L(\psi_{-s}(\theta))\big) \\ &= \operatorname{graph}\,\Big[\big((d\psi_s \circ L(\psi_{-s}(\theta)\big) \circ \big(d\psi_s|_{R(\psi_{-s}(\theta)}\big)^{-1}\Big] \end{aligned}$$

For $\theta \in K$ take a sequence $s_n \to +\infty$ such that $\psi_{-s_n}(\theta) \in K$. Since $d\psi_{s_n}\big(R(\psi_{-s_n}(\theta))\big) \subseteq E^u(\theta) \oplus E^c(\theta)$, by properties (a) and (b), we have that

$$\big\|d\psi_{s_n} \circ L(\psi_{-s_n}(\theta)) \circ \big(d\psi_{-s_n}|_{d\psi_{s_n} R(\psi_{-s_n}(\theta))}\big)\big\| \le \mu^{s_n}\,\lambda^{s_n}\,\sup_{\theta\in K}\|L(\theta)\|\,.$$

Since $\mu^{s_n}\lambda^{s_n} \to 0$ when $n \to +\infty$, we have that

$$\mathbb{F}(\theta) = \lim_{n\to+\infty} V_{-s_n}(\theta) = \lim_{n\to+\infty} d\psi_{s_n}\big(R(\psi_{-s_n}(\theta))\big) \subseteq E^u(\theta) \oplus E^c(\theta)\,.$$

Therefore the second inclusion in (30) is satisfied for $\theta \in K$. Since $\mu(K) \ge 1 - \varepsilon$ and ε and μ are arbitrary, this inclusion holds for a set of total measure in Λ.

We prove now the first inclusion. There exist (cf. [1] theorem 3.1.19) a continuous riemannian metric $\langle\,,\,\rangle$ on T^*M and a continuous family of linear isomorphisms $J(\theta) : T_\theta T^*M \hookleftarrow$ such that $J(\theta)^2 = -I$ and the symplectic forms is written as $\omega(x, y) = \langle x, J(\theta)y\rangle$ for all $x, y \in T_\theta T^*M$ and $\theta \in T^*M$. From now we use this metric. Let $\{e_1(\theta), \ldots, e_n(\theta)\}$ be an orthonormal basis of $\mathbb{F}(\theta)$ and define $e_{n+i}(\theta) =$

$Je_i(\theta)$ for $i=1,\dots,n$. Since the subspace $\mathbb{F}(\theta)$ is Lagrangian, the subspace $J\mathbb{F}(\theta) = \operatorname{span}\{e_{n+1}(\theta),\dots,e_{2n}(\theta)\}$ is the orthogonal complement of $\mathbb{F}(\theta)$ with respect to $\langle\,,\,\rangle$. The matrices of ω and $d\psi_t(\theta)$ with respect to the family of basis $\{e_1(\theta),\dots,e_{2n}(\theta)\}$ are

$$J=\begin{bmatrix} & I\\ -I & \end{bmatrix} \qquad \text{and} \qquad d\psi_t(\theta)=\begin{bmatrix} A_t(\theta) & C_t(\theta)\\ & B_t(\theta)\end{bmatrix}$$

Since $d\psi_t$ preserves the symplectic form we have that $(d\psi_t)^*\,J\,(d\psi_t)=J$. Hence $B_t^*(\theta)=A_t(\theta)^{-1}$ and $B_t^*(\theta)\,C_t(\theta)$ is symmetric.

Since $\mathbb{F}(\theta)\subseteq E^u(\theta)\oplus E^c(\theta)$ then

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\left\|B_t^{*\,-1}(\theta)\,w\right\| = \lim_{t\to\pm\infty}\frac{1}{t}\log\left\|A_t(\theta)\,w\right\|\geq 0$$

for all $w\in\mathbb{R}^n$, $\theta\in\Lambda$. This implies that

$$\lim_{t\to\pm\infty}\frac{1}{t}\log\left\|B_t^{-1}(\theta)\,w\right\|\geq 0\,. \tag{31}$$

Suppose that $E^u(\theta)\not\subset\mathbb{F}(\theta)$. Take $v\in E^u(\theta)\setminus\mathbb{F}(\theta)$. Then $v=w+z$ with $z\in\mathbb{F}(\theta)$ and $0\neq w\in J\mathbb{F}(\theta)$. We have that $d\psi_{-t}\,v = u_{-t}+\big(*,B_t^{-1}(\theta)\,w\big)$ with $u_{-t}\in\mathbb{F}(\psi_{-t}(\theta))\subseteq E^u(\psi_{-t}(\theta))\oplus E^c(\psi_{-t}(\theta))$. Since $B_t^{-1}(\theta)\,w\in J\mathbb{F}(\psi_{-t}(\theta))$ and $\mathbb{F}$ and $J\mathbb{F}$ are orthogonal, we have that $\|d\psi_{-t}(v)\|\geq\left\|B_t^{-1}(\theta)\,w\right\|$. Hence from (31) we obtain

$$\lim_{t\to+\infty}\frac{1}{t}\log\left\|d\psi_{-t}(\theta)\,v\right\|\geq 0\,.$$

This contradicts the choice $v\in E^u(\theta)$.

$\square$

Finaly we prove (c). Let $\pi_\theta:\mathbb{F}(\theta)\to H(\theta)$ be the restriction of $d\pi$. Then $\pi_\theta^{-1}(v)=(v,\,\mathbb{U}(\theta)\,v)$ and by corollary 1.8

$$\sup_{\theta\in\Sigma}\left\|\pi_\theta^{-1}\right\|<+\infty\,. \tag{32}$$

Fix $\theta\in\Lambda$, let $Z_\theta(t):V(\theta)\to H(\psi_t(\theta))$ be defined by $Z_\theta(t)\,v=\pi_{\psi_t(\theta)}\;d\psi_t(\theta)\,(v,\mathbb{U}(\theta)v)$. Then

$$d\psi_t(\theta)|_{\mathbb{F}(\theta)}=\pi^{-1}_{\psi_t(\theta)}\circ Z_\theta(t)\circ\pi_\theta$$

By (30) and (32), we have that

$$\begin{aligned}\chi(\theta) &= \lim_{T\to+\infty}\frac{1}{T}\log\left|\det d\psi_t(\theta)|_{\mathbb{F}(\theta)}\right|\\ &= \lim_{T\to+\infty}\frac{1}{T}\log\left|\det Z_\theta(t)\right|\,.\end{aligned}$$

Since the isomorphisms $H(t) = Z_\theta(t)$, $V(t) = \mathbb{U}(\psi_t(\theta))\, Z_\theta(t)$ satisfy the Jacobi equation (6), we have that

$$\dot{Z}_\theta = (H_{pq} + H_{pp}\,\mathbb{U})\, Z_\theta\,.$$

Since $\det Z_\theta(t) \neq 0$ for all $t \in \mathbb{R}$, then $\det Z_\theta(t)$ has constant sign and

$$\frac{d}{dt}\left|\det Z_\theta(t)\right| = \frac{d}{dt}\det Z_\theta(t) = \operatorname{tr}\left(H_{pq} + H_{pp}\,\mathbb{U}\right)\ \left|\det Z_\theta(t)\right|\,,$$

$$\frac{d}{dt}\log\left|\det Z_\theta(t)\right| = \operatorname{tr}\left[H_{pq} + H_{pp}\,\mathbb{U}\right]\,,$$

$$\lim_{T\to+\infty}\frac{1}{T}\log\left|\det d\psi_t(\theta)|_{\mathbb{F}(\theta)}\right| = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \operatorname{tr}\left[H_{pq} + H_{pp}\,\mathbb{U}\right]\,dt\,.$$

□

The exponential map.

The following proposition proves item (c) in theorem 0.1. We present a modern proof that shall be needed in the sequel.

1.15. Proposition. [Hartman][17] *Let $H : T^*M \to \mathbb{R}$ be a convex Hamiltonian. Then*

(a) *A half open segment $\{\,\psi_t(\theta)\,|\,t\in[0,a[\,\}$, $a \in]0,+\infty[\cup\{+\infty\}$ has no conjugate points if and only if there exists a Lagrangian subspace $E \subset T_\theta T^*M$ such that $d\psi_t(E)\cap V(\psi_t(\theta)) = \{0\}$ for all $t \in]0,a[$.*

(b) *A closed segment $\{\,\psi_t(\theta)\,|\,t\in[0,a]\,\}$ has no conjugate points if and only if there exists a Lagrangian subspace $E \subset T_\theta T^*M$ such that $d\psi_t(E)\cap V(\psi_t(\theta)) = \{0\}$ for all $t\in[0,a]$.*

(c) *If a closed segment $\{\,\psi_t(\theta)\,|\,t\in[0,a]\,\}$ has no conjugate points, then there exists $\delta > 0$ such that the segment $\{\,\psi_t(\theta)\,|\,t\in[-\delta,a+\delta]\,\}$ has no cojugate points.*

Proof:

(a) If the half open segment is disconjugate, then the Lagrangian subspace $E = V(\theta)$ satisfies $d\psi_t(V(\theta))\cap V(\psi_t(\theta)) = \{0\}$.

Conversely, suppose that such Lagrangian subspace E is given. Let $E(t) := d\psi_t(E)$. Then the projection $d\pi : E(t) \to H(\psi_t(\theta))$ is an isomorphism. Let $S(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ be the linear isomorphism such that $E(t) = \text{graph}\,(S(t))$ for $t\in]0,a[$. Since $E(t)$ is Lagrangian, then $S(t)$ is symmetric and it satisfies the Ricatti equation (7). Let $Z(t) : E(t) \to H(\psi_t(\theta))$ be given by $Z(t)\,w = d\pi\, d\psi_t(\theta)\,w$. Then

$$\dot{Z} = (H_{pq} + H_{pp}\,S)\,Z \quad , \quad Z(0) = d\pi|_E. \tag{33}$$

Let $w \in E$ and $(h_1(t), v_1(t)) = d\psi_t(w) \in E(t)$, then $h_1(t) = Z(t)\, w$, $v_1(t) = S(t)\, Z(t)\, w$. Let $\xi \in T_\theta T^*M$ and $(h(t), v(t)) = d\psi_t(\xi)$. Suppose that $h(c) = 0$, $c \in]0, a[$. Using these $h_1(t)$, $v_1(t)$, the same arguments as in (9), (10), (11) give that

$$h(t) = Z(t)\, Z(c)^{-1}\, h(c) + Z(t) \int_c^t Z^{-1}(s)\, H_{pp}\, (Z^*(s))^{-1} \big(Z(c)^* v(c)\big)\, ds\,. \tag{34}$$

Since $h(c) = 0$, then

$$\big\langle Z(t)^{-1}\, h(t)\,,\, Z(c)^* v(c)\big\rangle = \int_c^t \big\langle H_{pp}\, (Z(s)^*)^{-1}\, Z(c)^* v(c)\,,\, (Z(s)^*)^{-1}\, Z(c)^* v(c)\big\rangle\, ds\,. \tag{35}$$

In particular $h(t) \neq 0$ for all $t \in]0, a[\setminus \{c\}$. Hence the segment $]0, a[$ does not have conjugate points.

It remains to prove the case $c = 0$. Let $0 < b < a$. We prove that θ is not conjugate to $\psi_b(\theta)$. We already know that $\psi_b(\theta)$ has no conjugate points in $\{\, \psi_t(\theta) \mid 0 < t < a \,\}$. For $0 < d < b$ let $E_d = d\psi_{b-d}(V(\psi_d(\theta))) = \operatorname{graph}(K_d)$, where $K_d : H(\psi_b(\theta)) \to V(\psi_b(\theta))$ is the corresponding solution of the Ricatti equation (7). Let $\varepsilon > 0$ be such that $b < b + \varepsilon < a$ and let $E_{b+\varepsilon} = \psi_{-\varepsilon}(V(\psi_{b+\varepsilon}(\theta))) = \operatorname{graph}(K_{b+\varepsilon})$. By proposition 1.4(b) and remark 1.5, the family K_d is monotone decreasing on d and has the (lower) bound $K_{b+\varepsilon}$. Hence there exists $\mathbb{E} := \lim_{d\to 0^+} E_d = d\psi_b(V(\theta))$ and $\mathbb{E} \cap V(\psi_b(\theta)) = \{0\}$.

(c) If the closed segment $[0, a]$ has no conjugate points, then $d\psi_t(V(\theta)) \cap V(\psi_t(\theta)) = \{0\}$ for $0 < t \le a$. Then $d\psi_t(V(\theta)) \cap V(\psi_t(\theta)) = \{0\}$ for $0 < t \le a + \delta$ for some $\delta > 0$. By item (a), using $E(t) = d\psi_t(V(\theta))$, the segment $[0, a + \delta]$ has no conjugate points. The same argument using $F(t) = d\psi_{a+\delta-t}(V(\psi_{a+\delta}(\theta)))$ shows that there exists $\delta_1 > 0$ such that the segment $[-\delta_1, a + \delta]$ has no conjugate points.

(b) Suppose that the segment $[0, a]$ is disconjugate. By item (c) there exists $\delta > 0$ such that the segment $[-\delta, a + \delta]$ is disconjugate. Now apply item (a) to the segment $[-\frac{\delta}{2}, a + \delta[$. Conversely, if there is a Lagrangian subspace $E \subset T_\theta T^*M$ with $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $0 \le t \le a$, then the argument of equation (35) applies for all $c \in [0, a]$.

□

It may be impossible to find such Lagrangian subspace E of theorem 1.15 (a),(b) satisfying $E \subset T_\theta \Sigma$ as explained in remark 1.17. Compare this with theorem 0.1. *If* there exists a *continuous* invariant Lagrangian bundle $E(\theta)$, one can always suppose that $E(\theta) \subset T_\theta\Sigma$ by taking $(E(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$. *Then* it must satisfy the transversality condition 0.1(a).

1.16. Proposition. *For $\theta \in \Sigma$ define $W(\theta) := (V(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$. Suppose that the orbit of θ has no conjugate points. Then for all $t \neq 0$, $d\psi_t\big(W(\theta)\big) \cap V\big(\psi_t(\theta)\big) = \{0\}$.*

Proof:

Suppose that there exists $b > 0$ such that $(0, w_b) \in d\psi_b(W(\theta)) \cap V(\psi_b(\theta)) \neq \{0\}$. Let $(k_0, w_0) := d\psi_{-b}(0, w_b) \in W(\theta)$. Since the segment $\{\, \psi(\theta) \,|\, t \in [0, +\infty[\,\}$ has no conjugate points, then $k_0 \neq 0$. Write $H_p(t) := H_p(\psi_t(\theta))$, $H_q(t) := H_q(\psi_t(\theta))$. Since $W(\theta) = (V(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta) \rangle$ and $X(\theta) = (H_p(0), -H_q(0))$, then $k_0 = \alpha\, H_p(0)$ for some $\alpha \neq 0$. Multiplying w_b by $\frac{1}{\alpha}$, we can assume that $\alpha = 1$. Let

$$(h(t), v(t)) := d\psi_t(X(\theta)) - (k_0, v_0)) \in d\psi_t(W(\theta)) \subseteq T_{\psi_t(\theta)}\Sigma\,.$$

Then

$$\begin{aligned} h(0) &= H_p(0) - k_0 = 0 \quad , \quad v(0) \neq 0. \\ h(b) &= d\pi\,[X(\psi_t(\theta)) - (0, w_b)] = H_p(b)\,. \end{aligned} \tag{36}$$

Since $(h(0), v(0)) \in T_\theta\Sigma$, then

$$0 = dH(h(0), v(0)) = H_q\, h(0) + H_p\, v(0) = H_p(0) \cdot v(0)\,. \tag{37}$$

Let $\mathbb{E}(t) = \mathbb{E}(\psi_t(\theta))$ be the stable Green bundle of $\psi_{\mathbb{R}}(\theta)$, let $\mathbb{S}(t)$ be from (20) and let $Z(t) := \boldsymbol{h}(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ be from claim 1.3. So that $Z(t)\, w = d\pi\, d\psi_t(w, \mathbb{S}(t)\, w)$ and $Z(t)$ satisfies (33). Using formula (34) for $c = 0$, (with $Z(t) = \boldsymbol{h}(t)$, $Z(0) = I$ and $h(0) = 0$) we obtain that

$$h(t) = Z(t) \int_0^t Z^{-1}(s)\, H_{pp}(s)\, (Z^*(s))^{-1} v(0)\; ds\,.$$

Hence, since $v(0) \neq 0$, we have that

$$\langle Z(b)^{-1} h(b),\, v(0) \rangle = \int_0^b \langle H_{pp}\, (Z^*(s)^{-1} v(0)), (Z^*(s)^{-1} v(0)) \rangle\; ds > 0\,. \tag{38}$$

Since $X(\theta) \in \mathbb{E}(\theta)$, then

$$\begin{aligned} (H_p(b), -H_q(b)) &= X(\psi_b(\theta)) = d\psi_b(X(\theta)) = d\psi_b(H_p(0), \mathbb{S}(0)\, H_p(0)) \\ &= (Z(b)\, H_p(0), \mathbb{S}(b) \cdot Z(b)\, H_p(0))\,. \end{aligned}$$

From the first component we get that $Z(b)^{-1} H_p(b) = H_p(0)$. Using (36) we have that $Z(b)^{-1} h(b) = H_p(0)$. Replacing this equation on (38), we obtain that

$$\langle H_p(0), v(0) \rangle > 0\,.$$

This contradicts equation (37).

$\square$

1.17. Remark.

The same proof applies for the following statement: If $H : T^*M \to \mathbb{R}$ is convex and there is a Lagrangian subspace $E \subset T_\theta\Sigma$ such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for $t \in [0, a]$, then $d\psi_t\big(W(\theta)\big) \cap V\big(\psi_t(\theta)\big) = \{0\}$ for all $t \in [0, a]$.

In particular this hypothesis holds if $\{\,\psi_t(\theta)\,|\,t\geq 0\,\}$ or $\{\,\psi_t(\theta)\,|\,t\leq 0\,\}$ have no conjugate points. Just take the corresponding Green bundles.

The example I.2 does not satisfy the conclusion of proposition 1.16 on $t\in[0,\pi]$, nevertheless the segment $\{\,\psi_t(\theta)\,|\,t\in[0,\pi]\,\}$ has no conjugate points. This implies that the Lagrangian subspace given by proposition 1.15(b) may not satisfy $E\subset T_\theta\Sigma$.

1.18. Corollary. *Let $H:T^*M\to\mathbb{R}$ be convex, $\Sigma=H^{-1}\{e\}$, $q\in\pi(\Sigma)$ and suppose that the orbits $\{\,\psi_t(\theta)\,|\,t\geq 0\,\}$ have no conjugate points for all $\theta\in\pi^{-1}\{q\}\cap\Sigma$. Then the exponential map $\exp_q:T_q^*M\to M$ is an inmersion.*

Proof:

We have that $\exp_q(t\theta)=\pi\circ\psi_t(\theta)$ for $\theta\in\Sigma_q=\pi^{-1}\{q\}\cap\Sigma$. Consider the splitting

$$T_{t\theta}(T_q^*M)\approx T_q^*M=T_\theta\big(\Sigma_q\big)\oplus\langle\theta\rangle=\big(V(\theta)\cap T_\theta\Sigma\big)\oplus\langle\theta\rangle\,.$$

The derivative $d\exp_q(t\theta):T_q^*M\to T_{\pi(\psi_t\,\theta)}M$ is given by

$$d\exp_q(t\theta)|_{V(\theta)\cap T_\theta\Sigma}=d\pi\circ d\psi_t(\theta)|_{V(\theta)\cap T_\theta\Sigma}\,, \tag{39}$$

$$d\exp_q(t\theta)(\theta)=d\pi\big(X(\psi_t(\theta))\big)=d\pi\circ d\psi_t\big(X(\theta)\big)\,. \tag{40}$$

Therefore

$$\begin{aligned} d\exp_q(t\theta)\big(T_q^*M\big)&=d\pi\circ d\psi_t\big(V(\theta)\cap T_\theta\Sigma\oplus\langle X(\theta)\rangle\big)\\ &=d\pi\circ d\psi_t\big(W(\theta)\big)\,. \end{aligned} \tag{41}$$

Since $\dim W(\theta)=n$, then $\dim d\psi_t(W(\theta))=n$. By proposition 1.16 $d\psi_t(W(\theta))\cap V(\psi_t(\theta))=\{0\}$. Hence $d\pi:d\psi_t(W(\theta))\to T_{\pi(\psi_t\,\theta)}M$ is a linear isomorphism. From (41) $d\exp_q(t\theta)$ is a linear isomorphism for all $t>0$, $\theta\in\Sigma_q$.

□

2. The transversal behaviour.

Given $\theta\in\Sigma$, consider $H_p(\theta)=d\pi\,X(\theta)\neq 0$. Let $\boldsymbol{N}(\theta)\subseteq T_\theta\Sigma$ be defined by

$$\boldsymbol{N}(\theta):=\{\,\xi\in T_\theta\Sigma\,|\,\langle\,d\pi\,\xi,\,d\pi\,X(\theta)\,\rangle_{\pi(\theta)}=0\,\}\,. \tag{42}$$

Then $\boldsymbol{N}(\theta)\oplus\langle X(\theta)\rangle=T_\theta\Sigma$. Fix $\theta_0\in\Sigma$ and a smooth coordinate system $(q_1,\dots,q_n,t)$ of $M\times\mathbb{R}$ along the projection $(\pi(\psi_t(\theta_0)),t)$ of the orbit of θ_0 such that

(2.a) $\left.\frac{\partial}{\partial q_1}\right|_{(\pi(\psi_t\theta_0),t)}=d\pi\,X(\psi_t(\theta_0))$.

(2.b) $\frac{\partial}{\partial q_2},\dots,\frac{\partial}{\partial q_n}$ is an orthonormal basis for $d\pi\,\boldsymbol{N}(\psi_t(\theta_0))=\langle\frac{\partial}{\partial q_1}\rangle^\perp$ along $\pi(\psi_t(\theta_0),t)$.

Write $p_i = dq_i|_{T_qM}$, $i = 1, 2, \dots, n$. From (2.a) we have that

$$1 = \tfrac{d}{dt}\, q_1\Big|_{(\pi(\psi_t\theta_0),\, t)} = H_{p_1}(\psi_t(\theta_0)) \neq 0\,.$$

Then the equation $H(q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n) = h$ can be solved locally for p_1:

$$p_1 = -K(Q, P, T; h)\,,$$

where $P = (p_2, \dots, p_n)$, $Q = (q_2, \dots, q_n)$, $T = q_1$. This solution can be extended to a simply connected neighbourhood W of the orbit $\{\, (\psi_t(\theta_0),\, t) \mid t \in \mathbb{R} \,\} \subseteq T^*M \times \mathbb{R}$.

From now on we omit the $\mathbb{R}$-coordinate in $W \subseteq T^*M \times \mathbb{R}$. Let ϕ_T be the reparametrization of the flow ψ_t on W such that it preserves the foliation $q_1 = T =$ constant. In particular $d\phi_T(\theta_0)$ preserves the transversal bundle $\boldsymbol{N}(\phi_T(\theta_0))$ along the orbit of θ_0. Define $\tau(T, \theta)$ by $\phi_T(\theta) = \psi_{\tau(T,\theta)}(\theta)$. Then $\tau(T, \theta_0) = T$ for all $T \in \mathbb{R}$. The following reduction appears in Arnold [2].

2.1. Lemma. *The orbits $\phi_T(\theta) = (T = q_1, Q(T), p_1(T), P(T); \tau(T, \theta)) \in W$ satisfy the equations*

$$\frac{dQ}{dT} = \frac{\partial K}{\partial P} \qquad , \qquad \frac{dP}{dT} = -\frac{\partial K}{\partial Q}\,,$$

where $Q = (q_2, \dots, q_n)$, $P = (p_2, \dots, p_n)$, $T = q_1$ and $K(Q, P; T)$ is defined by $H(T, q_2, \dots, q_n; -K, p_2, \dots, p_n) = h$.

Proof: Consider the canonical 1-form $\Theta = p\, dq$ on TM and the symplectic form $\omega = d\Theta = dp \wedge dq$. On $T\Sigma$ we have that $\omega(X(\theta), \cdot) = -dH = 0$. Let $\frac{d}{dT}\phi_T(\theta) = \big(Y(\phi_T(\theta)),\, \frac{d}{dT}\, \tau(T, \theta)\big)$ be the vectorfield of ϕ_T. Then $Y(\phi_T(\theta))$ is a multiple of the vectorfield $X(\phi_T(\theta))$ of ψ_t, and then $\omega\big(Y(\phi_T(\theta)),\, \cdot\,\big) = 0$ on TW. We have

$$\Theta = p\, dq = P\, dQ - K\, dT,$$

$$\omega|_{T\Sigma} = d\Theta|_{T\Sigma} = \sum_{i=2}^{n} \left[dP_i \wedge dQ_i - K_{Q_i}\, dQ_i \wedge dT - K_{P_i}\, dP_i \wedge dT\right].$$

The matrix of $\omega|_{T\Sigma}$ is given by

$$\begin{bmatrix} 0 & -I & -K_Q^* \\ I & 0 & -K_P^* \\ K_Q & K_P & 0 \end{bmatrix} \begin{array}{l} \}\, Q \\ \}\, P \\ \}\, T = q_1 \end{array}$$

Then the vector $\big(K_P, -K_Q, 1 = \frac{dq_1}{dT}\big) =: Z(\phi_T(\theta))$ satisfies $\omega(Z(\phi_T(\theta)), \cdot) = 0$ on $\Sigma \approx \{\, (T, Q, p_1, P) \mid p_1 = -K(Q, P, T) \,\}$. Since the form ω has maximal rank on Σ, we have that $Z(\phi_T(\theta)) = Y(\phi_T(\theta))$ for $\theta \in W$. This proves the lemma.

$\square$

Since for $\vartheta \in W$, $\Phi_T(\vartheta) = \psi_{\tau(T,\vartheta)}(\vartheta)$, then

$$\begin{aligned} d\phi_T(\vartheta)\cdot\xi &= d\psi_{\tau(T,\vartheta)}\cdot\xi + \Big(\left.\frac{\partial\tau}{\partial\vartheta}\right|_{(T,\vartheta)}\cdot\xi\Big)\left.\frac{d}{dt}\psi_t(\vartheta)\right|_{\tau(T,\vartheta)} \\ &= d\psi_{\tau(T,\vartheta)}\cdot\xi + \alpha(T,\xi)\,X(\phi_T(\vartheta))\,, \end{aligned} \tag{43}$$

where $\alpha(T,\xi) = \frac{\partial\tau}{\partial\vartheta}\big|_{(T,\vartheta)}\cdot\xi$.

Let $\Lambda(\psi_t(\theta_0)) : T_{\psi_t(\theta_0)}\Sigma \to \boldsymbol{N}(\psi_t(\theta_0))$ be the projection along the direction of the vectorfield, i.e.

$$\Lambda\,\xi = \xi + \beta(\xi)\;X(\psi_t(\theta_0)) \text{ with } \beta(\xi)\in\mathbb{R} \text{ such that } \Lambda\,\xi \in \boldsymbol{N}(\psi_t(\theta_0))\,.$$

Through the proof of lemma 2.2 all the quantities will be understood in local coordinates $T = q_1$, $Q = (q_2,\dots,q_n)$, $P = (p_2,\dots,p_n)$ of $W\cap\Sigma$. In particular $T_{\phi_T(\theta_0)}\Sigma = \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ for all $T\in\mathbb{R}$, with coordinates $\left(\frac{\partial}{\partial T},\frac{\partial}{\partial Q},\frac{\partial}{\partial P}\right)$. Since along the orbit of θ_0 we have that

$$\dot q = (\dot T,\dot Q) = (1,\mathbf{0}_{n-1}) = (H_{p_1},H_P) \qquad \text{along } \phi_T(\theta_0) = (T,\mathbf{0}_{n-1},\mathbf{0}_{n-1}),$$

then the subspace $\boldsymbol{N}(\phi_T(\theta_0))$ in (42) is written as $\mathcal{N} := \{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ in coordinates. Define $\mathbb{P} : T_\vartheta\Sigma = \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1} \to \mathcal{N} = \{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ as the projection

$$\mathbb{P}(h_1,H,V) := (0,H,V).$$

Using $\beta = -h_1$ on (43) and $H_p = (1,\mathbf{0})$, the projection Λ is written as

$$\Lambda(h_1,H,V) = (0,H,V + h_1\cdot H_Q)\,,$$

where $H_Q := (H_{q_2},\dots,H_{q_n})$. Define the matrizes

$$\mathbb{H}(T) = \begin{bmatrix} H_{pq} & H_{pP} \\ -H_{Qq} & -H_{QP}\end{bmatrix}_{(2n-1)\times(2n-1)}$$

$$\mathbb{K}(T) = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times(n-1)} \\ K_{Pq} & K_{PP} \\ -K_{Qq} & -K_{QP}\end{bmatrix}_{(2n-1)\times(2n-1)}$$

along the orbit of θ_0.

2.2. Lemma. *Along the orbit $\theta_T := \phi_T(\theta_0)$ of θ_0, we have that*

(a) *The matrix $K_{PP}(\theta_T)$ is positive definite.*
(b) *For $\xi\in\boldsymbol{N}(\theta_T)$, $\mathbb{K}(\theta_T)\,\xi = \Lambda(\theta_T)\,\mathbb{H}(\theta_T)\,\xi$.*
(c) *The operator $\mathbb{K}(\theta_T)|_{\boldsymbol{N}(\theta_T)}$ is uniformly bounded on $T\in\mathbb{R}$.*

Proof:

(b) Write $\theta_T = \phi_T(\theta_0) = \psi_T(\theta_0) = (T, \mathbf{0}, \mathbf{0})$. From lemma 2.1, the Jacobi equation for ϕ_T is written as

$$\tfrac{d}{dT}\left[\mathbb{P}\, d\phi_T\, \xi\right] = \mathbb{K}\left[\mathbb{P}\, d\phi_T\, \xi\right],$$

where $d\phi_T\xi = (h_1(T), H(T), V(T)) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$. for any $\xi \in T_{\theta_0}\Sigma = \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$. Also, from (6),

$$\tfrac{d}{dt}\left[d\psi_t\, \xi\right] = \mathbb{H}\left[d\psi_t\, \xi\right].$$

Since $d\phi_T(\theta_0)$ preserves tha foliation $q_1 =$ constant, then it preserves the subspace $\mathcal{N}$. Therefore

$$\mathbb{P}\, d\phi_T\, \xi = d\phi_T\, \xi = \Lambda\, d\phi_T\, \xi \qquad \text{for } \xi \in \mathcal{N}.$$

In particular, the first coordinate of $d\phi_T\, \xi$ is $h_1(T) \equiv 0$ and also $\frac{d}{dT}h_1 \equiv 0$. Hence if $\xi \in \mathcal{N}$, then

$$\begin{aligned}
\tfrac{d}{dT}\left[\mathbb{P}\, d\phi_T\, \xi\right] &= \tfrac{d}{dT}\left[d\phi_T\, \xi\right] \\
&= \tfrac{d}{dT}\left[\, d\psi_T\, \xi + \alpha(T)\, X(\theta_T)\,\right] \\
&= \mathbb{H}(d\psi_T\, \xi) + \dot{\alpha}(T)\, X(\theta_T) + \alpha(T)\, \tfrac{d}{dT}\, X(\theta_T) \\
&= \mathbb{H}\big[d\phi_T\, \xi - \alpha(T)\, X(\theta_T)\big] + \dot{\alpha}(T)\, X(\theta_T) + \alpha(T)\, \tfrac{d}{dT}\, X(\theta_T),
\end{aligned}$$

where $\alpha(T) = \alpha(T,\xi)$ is from (43).

Since $\frac{d}{dT}X(\theta_T) = \mathbb{H}\, X(\theta_T)$, then

$$\tfrac{d}{dT}\left[\mathbb{P}\, d\phi_T\, \xi\right] = \mathbb{H}\left[d\phi_T\, \xi\right] + \dot{\alpha}(T)\, X(\theta_T).$$

Hence

$$\mathbb{K}\, d\phi_T\, \xi = \mathbb{H}\, \left[d\phi_T\, \xi\right] + \dot{\alpha}(T)\, X(\theta_T) \qquad \text{for } \xi \in \mathcal{N}. \tag{44}$$

Observe that this equation must hold in all the $2n-1$ coordinates of $T_\vartheta\Sigma$. Since $\text{Image}(\mathbb{K}) \subseteq \mathcal{N}$, then $\Lambda\, \mathbb{K} = \mathbb{K}$. So

$$\begin{aligned}
\mathbb{K}(d\phi_T\, \zeta) = \Lambda\, \mathbb{K}(d\phi_T\, \zeta) &= \Lambda\left[\, \mathbb{H}\, d\phi_T\, \zeta + \dot{\alpha}(T,\zeta)\, X(\theta_T)\,\right] \\
&= \Lambda\, \mathbb{H}(d\phi_T\, \zeta) \qquad \text{for } \zeta \in \mathcal{N}.
\end{aligned}$$

Since $d\phi_T : \boldsymbol{N}(\theta_0) = \mathcal{N} \to \boldsymbol{N}(\theta_T) = \mathcal{N}$ is surjective, this proves item (b).

(a) Let $V \in \mathbb{R}^{n-1}$, then

$$V^*\, \mathbb{K}_{PP}\, V = \big[(0, V^*), \mathbf{0}_{n-1}\big]\, \mathbb{K}\, \begin{bmatrix} \mathbf{0}_n \\ V \end{bmatrix}.$$

Since $\xi = (\mathbf{0}_n, V) \in \mathcal{N}$, we can use equation (44) to obtain

$$\begin{aligned}
V^* K_{pp} V &= \big[(0, V^*), \mathbf{0}_{n-1}\big] \left[\mathbb{H} \begin{bmatrix} \mathbf{0}_n \\ V \end{bmatrix} + \dot\alpha(T, d\phi_{-T}\,\xi) \begin{bmatrix} H_p^* \\ -H_Q^* \end{bmatrix}\right] \\
&= 0 + V^* H_{PP} V + \dot\alpha \left((0, V^*) \cdot H_p^*\right) \\
&= V^* H_{PP} V + \dot\alpha \left\langle (1, \mathbf{0}_{n-1}), (0, V^*) \right\rangle \\
&= V^* H_{pp} V > 0\,.
\end{aligned}$$

We now prove (c). Since the operator $\mathbb{H}(\vartheta)$ is uniformly bounded on $\vartheta \in \Sigma$, from (b) it is enough to prove that the projection Λ is bounded on the orbit of θ_0. For, it is enough to see that the angle $\sphericalangle(X(\theta_T), \boldsymbol{N}(\theta_T))$ is uniformly bounded away from 0. Given a vector

$$\xi = (h, v) = \sum_{i=1}^{n} \left(h_i \tfrac{\partial}{\partial q_i} + v_i \tfrac{\partial}{\partial p_i} \right) \in \boldsymbol{N}(\theta_T)\,,$$

written in our coordinates (2.a), (2.b), define the norm $|\xi| := (\sum_{i=1}^n h_i^2 + v_i^2)^{\frac12}$. Suppose that $|\xi| = 1$. Since $\xi \in \boldsymbol{N}(\theta_T)$, then $H_p \cdot h = 0$. Moreover, $H_p = \frac{\partial}{\partial q_1}$ and hence

$$\begin{aligned}
\frac{|X(\theta_T)\cdot\xi|}{|X(\theta_T)|} &= \frac{|H_p \cdot h - H_q \cdot v|}{(|H_p|^2 + |H_q|^2)^{\frac12}} = \frac{|H_q|\;|v|}{\sqrt{1 + |H_q|^2}} \\
&\le 1 \cdot \max_{0 \le x \le A} \frac{x}{\sqrt{1+x^2}} = \frac{A}{\sqrt{1+A^2}} < 1\,,
\end{aligned}$$

where $A := \sup_{T\in\mathbb{R}} |H_q(\theta_T)| < +\infty$ with the norm in our coordinates $\frac{\partial}{\partial p_i}$.

□

2.3. Corollary. *Along the orbit $\theta_T = \phi_T(\theta_0)$, we have*

(a) *The orbit of θ_0 under ϕ_T has no conjugate points.*

(b) *Existence of the Green bundles for $d\phi_s|_{\boldsymbol{N}(\theta_T)}$:*

$$\mathbb{E}^\top(\theta_T) = \lim_{s\to+\infty} d\phi_{-s}\left(V(\theta_{T+s}) \cap \boldsymbol{N}(\theta_{T+s})\right)$$

$$\mathbb{F}^\top(\theta_T) = \lim_{s\to-\infty} d\phi_{-s}\left(V(\theta_{T+s}) \cap \boldsymbol{N}(\theta_{T+s})\right)$$

Moreover, $\mathbb{E}^\top(\theta) = \mathbb{E}(\theta) \cap \boldsymbol{N}(\theta)$ and $\mathbb{F}^\top(\theta) = \mathbb{F}(\theta) \cap \boldsymbol{N}(\theta)$ for all $\theta \in \Sigma$.

(c) *Horizontal growth of iterates of vertical vectors: For all $R > 0$ there exists $S = S(R, \theta_T) > 0$ such that for all $|s| > S(R, \theta_T)$ and all $\xi \in \boldsymbol{N}(\theta_T) \cap V(\theta_T)$ we have that $|d\pi(d\phi_s(\theta_T)\cdot\xi)| > R\,|\xi|$.*

(d) *Define*

$$B^\top(\theta_T) = \left\{\, \xi \in \boldsymbol{N}(\theta_T) \mid \sup_{s\in\mathbb{R}} |d\pi \cdot d\psi_s \cdot \xi| < +\infty \,\right\}.$$

Then $B^\top(\theta_T) \subseteq \mathbb{E}^\top(\theta_T) \cap \mathbb{F}^\top(\theta_T)$.

(e) *Let* $\Psi_s := d\phi_s|_{\boldsymbol{N}} = \Lambda \circ d\psi_s|_{\boldsymbol{N}}$, *then*

$$d\phi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\Psi_s \circ \Lambda(\theta_T) = \Psi_s\,,$$

$$\Psi_{s+t} = \Psi_s \circ \Psi_t\,.$$

Proof:

Let $U(\theta_T) := V(\theta_T) \cap \boldsymbol{N}(\theta_T) = V(\theta_T) \cap T_{\theta_T}\Sigma$. From (43) we have that

$$d\phi_T(\theta_0)\cdot\xi = d\psi_T(\theta_0)\cdot\big[\xi + (\alpha(T,\theta_0)\cdot\xi)\, X(\theta_0)\big]\,.$$

Hence $d\psi_T(U(\theta_0)) \subseteq d\phi_T(W(\theta_0))$, where $W(\theta) := \big(V(\theta)\cap T_\theta\Sigma\big) \oplus \langle X(\theta)\rangle$. From proposition 1.16,

$$U(\theta_T) \cap d\phi_T(U(\theta_0)) \subseteq V(\theta_T) \cap d\psi_T(W(\theta_0)) = \{0\} \qquad \text{for all } T \neq 0\,.$$

This implies item (a).

From lemma 2.2, the first part of item (b) and items (c) and (d) have the same proofs as propositions A, 1.10 and proposition 1.11 respectively.

Since $d\phi_s$ and $d\psi_s$ satisfy (43) and $d\phi_s$ preserves the bundle $\boldsymbol{N}(\theta_T)$, we have that

$$d\phi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\psi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\psi_s \circ \Lambda(\theta_T) = \Psi_s\,.$$

Moreover,

$$\Psi_{s+t} = d\phi_{s+t}|_{\boldsymbol{N}(\theta_T)} = d\phi_s|_{\boldsymbol{N}(\theta_{T+t})} \circ d\phi_t|_{\boldsymbol{N}(\theta_T)} = \Psi_s \circ \Psi_t\,.$$

This proves item (e).

Finally, we have

$$\begin{aligned}\mathbb{E}^\top(\theta_0) &= \lim_{T\to+\infty} \big[d\phi_{-T}\big(V(\theta_T)\cap \boldsymbol{N}(\theta_T)\big)\big]\\ &= \Lambda(\theta_0) \lim_{T\to+\infty}\big[d\psi_{-T}\big(V(\theta_T)\cap T_{\theta_T}\Sigma\big)\big]\\ &\subseteq \Lambda(\theta_0)\cdot\mathbb{E}(\theta_0) \subseteq \mathbb{E}(\theta_0)\,.\end{aligned}$$

Because $X(\theta_0) \in \mathbb{E}(\theta_0)$. Thus $\mathbb{E}^\top(\theta_0) \subseteq \mathbb{E}(\theta_0) \cap \boldsymbol{N}(\theta_0)$. Since $\dim \mathbb{E}^\top(\theta_0) = \dim U = n-1 = \dim(\mathbb{E}(\theta_0)\cap \boldsymbol{N}(\theta_0))$, we have that $\mathbb{E}^\top(\theta_0) = \mathbb{E}(\theta_0)\cap\boldsymbol{N}(\theta_0)$. The equation $\mathbb{F}^\top(\theta_0) = \mathbb{F}(\theta_0)\cap\boldsymbol{N}(\theta_0)$ is proven similarly. This completes the proof of item (b).

□

Proof of proposition B.

Let $\theta_0 \in \Gamma$ and let $\tau > 0$ be the period of θ_0. Suppose that Γ is hyperbolic, then by proposition 1.14, $\mathbb{E}(\theta_0) \subseteq E^s(\theta_0) \oplus \langle X(\theta_0)\rangle$ and $\mathbb{F}(\theta_0) \subseteq E^u(\theta_0)\oplus\langle X(\theta_0)\rangle$. Hence $\mathbb{E}(\theta_0)\cap\mathbb{F}(\theta_0) = \langle X(\theta_0)\rangle$.

Now observe that if we construct the coordinates $(q_1, \dots, q_n)$ in (2.a), (2.b) so that they are periodic with period τ, then $d\phi_\tau(\theta_0) : \boldsymbol{N}(\theta_0) \hookleftarrow$ is the derivative of the Poincaré map of ψ_t defined on the cross-section $\{\, \vartheta \mid q_1(\vartheta) = 0 \,\}$.

Suppose that $\mathbb{E}(\theta_0) \cap \mathbb{F}(\theta_0) = \langle X(\theta_0) \rangle$. By corollary 2.3 (a) and (b) we have that $B^\top(\theta_0) \subseteq \mathbb{E}^\top(\theta_0) \cap \mathbb{F}^\top(\theta_0) = \langle X(\theta_0) \rangle \cap \boldsymbol{N}(\theta_0) = \{0\}$. This implies that Γ is hyperbolic.

$\square$

3. Quasi-hyperbolic actions.

Let B be a compact metric space and $\pi : \boldsymbol{E} \to B$ a vector bundle provided with a continuous norm $|\quad|_p$ on each fibre $\pi^{-1}\{p\}$. Let Ψ be an $\mathbb{R}$-action $\Psi : \mathbb{R} \to \mathrm{Isom}(\boldsymbol{E})$, i.e.

(i) There exists a continuous flow ψ_t on B such that $\pi \circ \Psi_t = \psi_t \circ \pi$.

(ii) $\Psi_t : \boldsymbol{E}(p) \to \boldsymbol{E}(\psi_t(p))$ is a linear isomorphism, where $\boldsymbol{E}(p) := (\pi|_{\boldsymbol{E}})^{-1}\{p\}$, $p \in B$.

(iii) $\Psi_{s+t} = \Psi_s \circ \Psi_t$.

We say that the action Ψ_t is *quasi-hyperbolic* if

$$\sup_{t\in\mathbb{R}} |\Psi_t(\xi)| = +\infty \qquad \text{for all} \qquad \xi \in \boldsymbol{E},\ \xi \neq 0.$$

We say that Ψ is *hyperbolic* if there exists an invariant continuous splitting $\boldsymbol{E}(p) = E^u(p) \oplus E^s(p)$, $p \in B$ and $C > 0$, $\lambda > 0$ such that

(3.a) $\Psi_t\big(E^u(p)\big) = E^u\big(\psi_t(p)\big)$, $\Psi_t\big(E^s(p)\big) = E^s\big(\psi_t(p)\big)$, for all $x \in B$, $t \in \mathbb{R}$.

(3.b) $|\Psi_t(\xi)| \leq C\,\mathrm{e}^{-\lambda t}\ |\xi|$ for all $t > 0$, $\xi \in E^u(p)$, $p \in B$.

(3.c) $|\Psi_{-t}(\xi)| \leq C\,\mathrm{e}^{-\lambda t}\ |\xi|$ for all $t > 0$, $\xi \in E^u(p)$, $p \in B$.

The aim of this section is to prove the following

0.2. Theorem. *If the action Ψ is quasi-hyperbolic and the non-wandering set $\Omega(\psi|_B) = B$, $\psi \circ \pi = \pi \circ \Psi$, then $\Psi|_{\boldsymbol{E}}$ is hyperbolic.*

Applying this theorem to the case in which $B = M$ is a compact manifold, ψ_t is a differentiable flow on M, $\Psi_t = d\psi_t$ and $\boldsymbol{E}$ is a $d\psi_t$-invariant continuous bundle such that $\boldsymbol{E} \oplus \big\langle \frac{d}{dt}\psi_t \big\rangle = TM$, we get

3.1. Theorem. [Freire] [12]

If M is a closed manifold and ψ is a quasi-Anosov flow on M such that $\Omega(\psi) = M$, then ψ is Anosov.

The proof of theorem 0.2 is similar to that of theorem 3.1. We include it here for completeness.

Suppose that the action of Ψ is quasi-hyperbolic. For $p \in B$ define

$$E^s(p) := \left\{ v \in \boldsymbol{E}(p) \;\middle|\; \sup_{t>0} |\Psi_t(p)(v)| < +\infty \right\},$$

$$E^u(p) := \left\{ v \in \boldsymbol{E}(p) \;\middle|\; \sup_{t<0} |\Psi_t(p)(v)| < +\infty \right\}.$$

Observe that by the quasi-hyperbolicity we have that

$$E^s(p) \cap E^u(p) = \{0\} \qquad \text{for all } p \in B.$$

3.2. Lemma. *There exists $\tau > 0$ such that for all $p \in B$,*

$$\left\| \Psi_\tau|_{E^s(p)} \right\| < \tfrac{1}{2} \quad , \quad \left\| \Psi_{-\tau}|_{E^u(p)} \right\| < \tfrac{1}{2}$$

for all $p \in M$.

Proof:

Suppose that the lemma is false for E^s. The proof for E^u is similar. Then there exist sequences $x_n \in B$, $v_n \in E^s(x_n)$, $|v_n| = 1$ such that $|\Psi_n(v_n)| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. We claim that there exists C such that

$$\sup_{t \geq 0} \left\| \Psi_t(p)|_{E^s(p)} \right\| < C < +\infty\,.$$

for all $p \in B$.

Suppose the claim is true. Let $w_n := \Psi_n(v_n)$ and $y_n := \psi_n(x_n)$. We have that $\frac{1}{2} \leq |w_n| \leq C$ for all n and

$$|\Psi_t(w_n)| = |\Psi_{t+n}(v_n)| \leq C \qquad \text{for all } t \geq -n\,.$$

Since B is compact there exists a convergent subsequence of (y_n, w_n). If $y_n \underset{n}{\to} y$ and $w_n \underset{n}{\to} w$, we have that $y \in B$, $w \in \boldsymbol{E}(y)$, $|w| \geq \frac{1}{2}$ and

$$|\Psi_t(w)| \leq C \qquad \text{for all } t \in \mathbb{R}\,.$$

This is a contradiction.

Now we prove the claim. Suppose it is false. Then there exists $x_n \in B$, $t_n \geq 0$, $v_n \in E^s(x_n)$, $|v_n| = 1$, such that

$$\sup_n |\Psi_{t_n}(v_n)| = +\infty\,. \tag{45}$$

Let $s_n > 0$ be such that

$$|\Psi_{s_n}(v_n)| > \tfrac{1}{2} \sup_{s \geq 0} |\Psi_s(v_n)| \geq \tfrac{1}{2}\, |\Psi_{t_n}(v_n)|\,.$$

By (45) we have that $s_n \underset{n}{\to} +\infty$. Let $y_n := \psi_{s_n}(x_n)$ and

$$w_n := \frac{\Psi_{s_n}(v_n)}{|\Psi_{s_n}(v_n)|}\,.$$

Then $|w_n| = 1$ and if $t > -s_n$ we have

$$|\Psi_t(w_n)| = \frac{|\Psi_{t+s_n}(v_n)|}{|\Psi_{s_n}(v_n)|} \le \frac{2\;|\Psi_{t+s_n}(v_n)|}{\sup_{s\ge 0}|\Psi_s(v_n)|} \le 2\,.$$

Since $|w_n| = 1$ and $y_n \in M$, there exists a convergent subsequence $(y_n, w_n) \to (y, w)$. We would have that $y \in B$, $w \in \boldsymbol{E}(y)$, $|w| = 1$ and

$$|\Psi_t(w)| \le 2 \qquad \text{for all } t \in \mathbb{R}\,.$$

This is a contradiction.

□

3.3. Lemma. *There exists $K > 0$ such that for all $x \in B$, $v \in \boldsymbol{E}(x)$*

$$|\Psi_t(v)| \le K\left(\,|v| + |\Psi_s(v)|\,\right) \qquad \textit{for all} \quad 0 \le t \le s.$$

Proof:

Suppose it is false. Then there exist $x_n \in B$, $0 \ne v_n \in \boldsymbol{E}(x)$ and $0 \le t_n \le s_n$, such that

$$|\Psi_{t_n}(v_n)| \ge n\big(\,|v_n| + |\Psi_{s_n}(v_n)|\,\big)\,.$$

Then $t_n \to +\infty$ and $s_n - t_n \to +\infty$ when $n \to +\infty$. We can assume that $|\Psi_{t_n}(v_n)| = \sup\limits_{0\le t\le s_n} |\Psi_t(v_n)|$. Let

$$w_n := \frac{\Psi_{t_n}(v_n)}{|\Psi_{t_n}(v_n)|}\,.$$

For $-t_n < t < s_n - t_n$, we have that

$$|\Psi_t(w_n)| = \frac{|\Psi_{t+t_n}(v_n)|}{|\Psi_{t_n}(v_n)|} \le \frac{1}{|\Psi_{t_n}(v_n)|} \sup_{0\le t\le s_n} |\Psi_t(v_n)| = 1.$$

Taking a subsequence, if necessary, we can assume that $x_n \to x$ and $w_n \to w \in \boldsymbol{E}(x)$. Then $|\Psi_t(w)| \le 1$ for all $t \in \mathbb{R}$, with $|w| = 1$. This is a contradiction.

□

Proof of theorem 0.2:

Standard methods (cf. Hirsch-Pugh-Shub [18]) show that the continuity of the strong stable and unstable bundles is redundant in the definition of a hyperbolic action. So it remains to prove that for all $x \in B$, we have that $E^s(x) \oplus E^u(x) = \boldsymbol{E}(x)$. Given $x \in B$, since $x \in \Omega(\psi|_B)$, there exist $x_n \to x$, $s_n \to +\infty$ such that $\psi_{s_n} x_n \to x$. Let E_n be a subspace of $\boldsymbol{E}(x_n)$ such that

$$\dim E_n = \dim \boldsymbol{E}(x) - \dim E^s(x)\,,$$

$$E^s(x) \oplus \lim_n E_n = \boldsymbol{E}(x)\,.$$

We claim that there exists $C > 0$ such that

$$\big\| \Psi_{-t} \big| \, \Psi_{s_n}(E_n) \big\| \le C \qquad \text{for all } 0 \le t \le s_n.$$

Suppose the claim is true. Taking a subsequence if necessary, we can assume that the limit $\lim_n \Psi_{s_n}(E_n)$ exists. Then

$$\big\| \Psi_{-t} \big| \lim_n \Psi_{s_n}(E_n) \big\| \le C \qquad \text{for all } t > 0.$$

Then $\lim_n \Psi_{s_n}(E_n) \subset E^u(x)$ and $\dim E^u(x) + \dim E^s(x) \ge \dim E_n + \dim E^s(x) = \dim \boldsymbol{E}(x)$. Since $E^s(x) \cap E^u(x) = \{0\}$, this completes the proof.

Now we proof the claim. Suppose it is false, then there exist $v_n \in \Psi_{s_n}(E_n)$, $|v_n| = 1$ and $0 < t_n < s_n$ such that $|\Psi_{-t_n}(v_n)| \ge n$. By lemma 3.3, we have that

$$n \le |\Psi_{-t_n}(v_n)| \le K \left(|\Psi_{-s_n}(v_n)| + |v_n| \right).$$

Hence $|\Psi_{-s_n}(v_n)| \ge \frac{n-K}{K}$. From lemma 3.3 we also have that

$$\frac{|\Psi_t(\Psi_{-s_n}(v_n))|}{|\Psi_{-s_n}(v_n)|} \le K + \frac{K}{|\Psi_{-s_n}(v_n)|} \qquad \text{for } 0 < t < s_n\,.$$

Let $w_n := \dfrac{\Psi_{-s_n}(v_n)}{|\Psi_{-s_n}(v_n)|}$. The estimates above give that

$$|\Psi_t(w_n)| \le K + \frac{K^2}{n-K} \qquad \text{for } 0 < t < s_n\,.$$

If $w_n \to w$, then $|w| = 1$ and $w \in \lim E_n$, thus $w \notin E^s(x)$. But $|\Psi_t(w)| \le K$ for all $t \ge 0$. This is a contradiction.

□

Proof of theorem C.

For $\theta \in \Sigma$, let

$$\boldsymbol{N}(\theta) := \big\{\, \xi \in T_\theta\Sigma \,|\, \langle\, d\pi \cdot \xi \,,\, d\pi\, X(\theta)\, \rangle_{\pi(\theta)} = 0 \,\big\},$$

and let $\Lambda(\theta) : T_\theta\Sigma \to \boldsymbol{N}(\theta)$ be the projection along the direction of $X(\theta)$:

$$\Lambda(\theta) \cdot \xi = \xi + \beta(\xi)\, X(\theta) \qquad \text{such that } \Lambda(\theta) \cdot \xi \in \boldsymbol{N}(\theta)\,.$$

Let $\Psi_s(\theta) = \Lambda(\psi_s(\theta)) \circ d\psi_s|_{\boldsymbol{N}(\theta)}$. By corollary 2.3 (e), Ψ_s defines an $\mathbb{R}$-action on the vector bundle $\pi : \boldsymbol{N} \to \Sigma$.

Suppose that the Green bundles satisfy $\mathbb{E} \cap \mathbb{F} = \langle X \rangle$ on Σ. Then by corollary 2.3(b) and 2.3(d),

$$B^\top(\theta) \subseteq \mathbb{E}^\top(\theta) \cap \mathbb{F}^\top(\theta) = \mathbb{E}(\theta) \cap \mathbb{F}(\theta) \cap \boldsymbol{N}(\theta) = \{0\}\,.$$

Therefore the action Ψ_s is quasi-hyperbolic. Since ψ_s preserves the Liouville measure, which is positive on open sets, then $\Omega(\psi|_\Sigma) = \Sigma$. Hence Ψ_s is a hyperbolic action.

This implies that ϕ_t is Anosov. By the methods in Hirsch-Pugh-Shub [18], we get that ψ_t is Anosov. We sketch the proof below.

For $\theta \in \Sigma$ let $\mathcal{E}^u(\theta) \subseteq \boldsymbol{N}(\theta)$ be the subspace given by item (3.b) in the definition of hyperbolic action. Then $\mathcal{E}^u$ is a continuous subbundle of $T\Sigma$. Define

$$\mathcal{F} := \left\{ L : \mathcal{E}^u \to \mathbb{R} \text{ continuous} \;\middle|\; L(\theta) : \mathcal{E}^u(\theta) \to \mathbb{R} \text{ is linear } \forall \theta \in \Sigma \right\},$$

with $\|L\| = \sup_{\xi \in \mathcal{E}^u \setminus \{0\}} \frac{|L(\xi)|}{|\xi|}$.

To each functional in $\mathcal{F}$ associate a subbundle W_L of $T\Sigma$ by

$$W_L(\theta) := \text{Graph}(L(\theta)) = \left\{ \xi + (L(\theta) \cdot \xi)\, X(\theta) \;\middle|\; \xi \in \mathcal{E}^u(\theta) \right\}.$$

Let $\tau > 0$ be such that $C\,\mathrm{e}^{-\lambda\tau} < \mathrm{e}^{-\lambda} < 1$, where C and λ are from (3.b). Consider the following "graph transformation" $T : \mathcal{F} \to \mathcal{F}$, corresponding to $W_{TL} = d\psi_\tau(W_L)$ and defined by

$$d\psi_\tau\big(\xi + L(\xi)\, X(\theta)\big) = \Psi_\tau(\xi) + \big[T(L)\,(\Psi_\tau(\xi))\big]\, X(\psi_\tau(\theta)), \;\; \theta \in \Sigma, \; \xi \in \mathcal{E}^u(\theta).$$

We claim that T is a contraction. Indeed

$$\begin{aligned} \left| \frac{T(L_1 - L_2)\,(\Psi_\tau \xi)}{\|\Psi_\tau\, \xi\|} \right| \; \|X(\psi_\tau \theta)\| &\leq \frac{\|\, d\psi_\tau[(L_1(\xi) - L_2(\xi))\, X(\theta)]\|}{\mathrm{e}^\lambda \;\; \|\xi\,\|} \\ &\leq \frac{|(L_1 - L_2)(\xi)|}{\mathrm{e}^\lambda \;\; \|\xi\|} \;\; \|X(\psi_\tau \theta)\| \; . \end{aligned}$$

Hence $\|T(L_1) - T(L_2)\| \leq \mathrm{e}^{-\lambda}\; \|L_1 - L_2\|$.

The fixed point L^* of T gives the (continuous) strong unstable bundle of ψ_t. Indeed, if $E^u = W_{L^*}$, then E^u is clearly $d\psi_t$-invariant. Moreover, if $\zeta \in E^u(\theta)$, then $\zeta = \xi + (L^*\xi)\, X(\theta)$ with $\xi = \Lambda\zeta \in \mathcal{E}^u(\theta)$. Since L^* is continuous then there exists $Q_1 > 0$ such that $|\zeta| \leq Q\, |\xi| = Q\, |\Lambda\zeta|$ for all $\zeta \in \mathcal{E}^u$. Since $\boldsymbol{N}(\theta)$ is transversal to $\langle X(\theta) \rangle$ and both are continuous subbundles of $T\Sigma$, (Σ compact), then the angle $\sphericalangle(X(\theta), \boldsymbol{N}(\theta))$ is bounded below and hence there exists $Q_2 > 0$ such that $\frac{1}{Q_2}\,|\Lambda\zeta| = \frac{1}{Q_2}\,|\xi| < |\zeta|$. Then

$$\begin{aligned} |d\psi_t(\zeta)| &= |\; \Psi_t(\xi) + (T(L^*) \cdot \xi)\, X(\psi_t \theta)\; | \\ &\geq \frac{1}{Q_2}\; |\Psi_t(\xi)| \geq \frac{C^{-1}\,\mathrm{e}^{\lambda t}}{Q_2}\; |\xi| \geq \frac{\mathrm{e}^{\lambda t}}{C\, Q_1\, Q_2}\; |\zeta| \; . \end{aligned}$$

Finally,

$$\dim E^u = \dim \text{Graph}(L^*) = \dim \mathcal{E}^u = n - 1\, .$$

The existence of a continuous strong stable subbundle of dimension $n-1$ is proven similarly.

□

4. The Index Form.

Let $L = p\,H_p - H$ be the Lagrangian associated to H. Consider $\frac{\partial L}{\partial v}(x,v) : T_{(x,v)}(T_xM) \approx T_xM \to \mathbb{R}$ as an element $L_v \in T^*M$. The Lagrange transform $\mathcal{F}_L(x,v) = (x, L_v)$ identifies $q = x$, $p = L_v$. Similarly, using $\mathcal{F}_L^{-1} = \mathcal{F}_H$, we get that $v = H_q$ under the same identification. Also

$$H = v\,L_v - L = p\,H_p - L(q, H_p)\,.$$

Hence $H_q = -L_q = -L_x$. With these identifications the Hamiltonian equations (1) become

$$\begin{aligned} \tfrac{d}{dt}\,x = v \qquad &\longleftrightarrow \qquad \tfrac{d}{dt}\,q = H_p\,, \\ \tfrac{d}{dt}\,L_v = L_x \qquad &\longleftrightarrow \qquad \tfrac{d}{dt}\,p = -H_q\,, \end{aligned} \tag{46}$$

Hence, identifying $L_x = H_q \in T_{(p,q)}(T^*M)$, the Euler-Lagrange equation (46) is understood as a first order differential equation on T^*M, where $\frac{d}{dt}L_v$ is the tangent vector to the path $t \mapsto L_v(\gamma(t), \dot\gamma(t))$ in T^*M.

We derive the Jacobi equation in this Lagrangian setting. Let $\gamma(t)$ be a solution of the Euler-Lagrange equation (2). Considering a variation $f(s,t)$ of $\gamma(t) = f(0,t)$ made of solutions $t \mapsto f(s,t)$ of the Euler-Lagrange equation (2) and taking the covariant derivative $\frac{D}{ds}$, we obtain the *Jacobi equation*

$$\tfrac{D}{dt}\,(L_{vx}\,k + L_{vv}\,k') = L_{xx}\,k + L_{xv}\,k'\,, \tag{47}$$

where $k = \frac{\partial f}{\partial s}(0,t)$, $\dot k = \frac{D}{dt}\frac{\partial f}{\partial s}$ and the derivatives of L are evaluated on $\gamma(t) = f(0,t)$. Here we have used that $\frac{D}{ds}\frac{\partial F}{\partial t} = \frac{D}{dt}\frac{\partial F}{\partial s}$ for the variation map $F(s,t) = L_v\big(f(s,t), \frac{\partial f}{\partial s}(s,t)\big) \in T^*M$, where $\frac{D}{ds}$ and $\frac{D}{dt}$ are the covariant derivatives on the riemannian manifold T^*M. The linear operators L_{xx}, L_{xv}, L_{vv} coincide with the corresponding matrices of partial derivatives in local coordinates. The solutions of (47) satisfy

$$D\varphi_t\big(k(0), \dot k(0)\big) = \big(k(t), \dot k(t)\big) \in T_{\gamma(t)}(TM)\,,$$

where φ_t is the Lagrangian flow on TM.

Let Ω_T be the set of continuous piecewise C^2 vectorfields ξ along $\gamma_{[0,T]}$. Define the *index form* on Ω by

$$I(\xi,\eta) = \int_0^T \Big(\dot\xi\,L_{vv}\,\dot\eta + \dot\xi\,L_{vx}\,\eta + \xi\,L_{xv}\,\dot\eta + \xi\,L_{xx}\,\eta\Big)\,dt\,, \tag{48}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s} \in \Omega_T$. For general results on this form see Duistermaat [9].

The following transformation of the index form is taken from Hartman [17] and originally due to Clebsch [7]. Let $\theta \in TM$ and suppose that the orbit of θ, $\varphi_t(\theta)$, $0 \le t \le T$ does not have conjugate points. Let $E \subset T_\theta T^*M$ be a Lagrangian subspace such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $0 \le t \le T$. Such subspace E always exists by proposition 1.15(b). Let $E(t) := d\psi_t(E)$ and let $H(t)$, $V(t)$ be a matrix solution of the Hamiltonian Jacobi equation (6) such that $\det H(t) \neq 0$ and $E(t) = \text{Image}\,(H(t), V(t)) \subset T_{\psi_t(\theta)}(T^*M)$ is a Lagrangian subspace. In particular $H(t)$ satisfies the Lagrangian Jacobi equation (47). We have that

$$\begin{bmatrix} H(t) \\ V(t) \end{bmatrix} = D\mathcal{F}_L\,(\varphi_t(\theta)) \cdot \begin{bmatrix} H(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ L_{vx} & L_{vv} \end{bmatrix} \begin{bmatrix} H \\ H' \end{bmatrix} \tag{49}$$

From (49) and (47), we have that

$$\begin{aligned} V &= L_{vv}\,H' + L_{vx}\,H \\ V' &= L_{xv}\,H' + L_{xx}\,H \end{aligned} \tag{50}$$

Moreover, since $E(t)$ is a Lagrangian subspace, then the Hamiltonian solution of the Ricatti equation (7) $S(t) = V(t)\,H(t)^{-1}$ is symmetric, i.e.

$$H(t)^*\,V(t) = V(t)^*\,H(t). \tag{51}$$

Let $\eta \in \Omega_T$ and define $\zeta \in \Omega_T$ by $\eta(t) = H(t)\,\zeta(t)$. Then the integrand of $I(\eta,\eta)$ in (48) is

$$(L_{vv}\,H\,\zeta' + V\,\zeta)\;(H'\,\zeta + H\,\zeta') + (V'\,\zeta + L_{xv}\,H\,\zeta')\;H\zeta\,.$$

Since $(L_{vv}\,H\zeta')\cdot(H'\zeta) = (H\zeta')\cdot(L_{vv}\,H'\zeta)$ and $(L_{xv}\,H\zeta')\cdot(H\zeta) = (H\zeta')\cdot(L_{vx}\,H\zeta)$, the integrand can be written as

$$L_{vv}\,H\zeta'\cdot H\zeta' + H\zeta'\cdot V\zeta + V\zeta\cdot H'\zeta + V\zeta\cdot H\zeta' + V'\zeta\cdot H\zeta\,.$$

Using (51), we have that $V^*H\zeta'\cdot\zeta = H^*V\zeta'\cdot\zeta = V\zeta'\cdot H\zeta$. hence the integrand in (48) is $L_{vv}\,H\zeta'\cdot H\zeta' + (V\zeta\cdot H\zeta)'$. Since η and ζ are continuous, we have that

$$I(\eta,\eta) = \int_0^T \big(L_{vv}\;H\zeta'\cdot H\zeta'\big)\,dt + H\zeta\cdot V\zeta\big|_0^T \qquad \text{if} \quad \eta = H\zeta \in \Omega_T\,. \tag{52}$$

Let $\mathcal{F}_H(q,p) = (q, H_p)$ be the Legendre transform of the Hamiltonian. Then $\mathcal{F}_L$ and $\mathcal{F}_H$ are inverse maps, in particular

$$(D\mathcal{F}_H)^{-1} = \begin{bmatrix} I & 0 \\ H_{qp} & H_{pp} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ L_{vx} & L_{vv} \end{bmatrix} = D\mathcal{F}_L\,,$$

and hence $L_{vv} = (H_{pp})^{-1}$. We obtain that

$$I(\xi,\eta) = \int_0^T (H\zeta')^*\,(H_{pp})^{-1}\,(H\rho')\,dt + (H\zeta)^*(V\rho)\big|_0^T\,. \tag{53}$$

for $\xi = H\zeta \in \Omega_T$, $\eta = H\rho \in \Omega_T$. This formula can also be written as

$$I(\xi,\eta) = \int_0^T (H\zeta')^* \, (H_{pp})^{-1} \, (H\rho') \, dt + \xi^* \, S \, \eta\big|_0^T, \tag{54}$$

where $S(t) = V(t)\, H(t)^{-1}$ is the corresponding solution to the Ricatti equation (7).

4.1. Corollary. *If $\theta \in T^*M$ and the segment $\{\, \psi_t(\theta) \mid t \in [0,T] \,\}$ has no conjugate points then the index form is positive definite on*

$$\Gamma_T = \left\{ \xi : [0,T] \to TM \,\middle|\, \xi(t) \in T_{\pi\psi_t\theta}M,\ \xi \text{ is piecewise } C^2,\ \xi(0) = 0,\ \xi(T) = 0 \right\}.$$

Proof:

Let $\xi \in \Gamma_T$, $\xi \neq 0$. Write $\xi(t) = H(t)\zeta(t)$. Since $\det H(t) \neq 0$, $\zeta(0) = 0$, $\zeta(T) = 0$ and $\zeta(t) \not\equiv 0$, then $\zeta' \not\equiv 0$. Now use formula (53).

□

We now extend formula (53) to the case in which $H(t)$ may be singular (or $\zeta'(t)$, $\eta'(t)$ may be discontinuous) at a finite set of points. Let $\{t_1, \dots, t_N\}$ be the points in $[0,T]$ such that $\det H(t_i) = 0$. Then

$$I(\xi,\eta) = \int_0^T (h\zeta')^* \, (H_{pp})^{-1} \, (H\rho') \, dt + (H\zeta)^*(V\rho)\big|_0^T - \sum_{i=1}^N (H\zeta)^*(V\rho)\big|_{t_i^-}^{t_i^+}. \tag{55}$$

where $\xi = H\zeta$, $\eta = H\rho$. But now ζ, ρ are piecewise C^2 but may be discontinuous at $t_1, \dots, t_N$. Alternativelly, we can use a sum of formulas (53) or (55) using different Lagrangian subspaces $d\psi_t(E_i)$ and corresponding solutions (H_i, V_i) on disjoint subintervals $]t_i, t_{i+1}[\subset [0,T]$.

Observe that for any convex Hamiltonian on T^*M and any $\theta \in M$ there exists $\varepsilon > 0$ such that the segment $\{\psi_t(\theta) \mid |t| < \varepsilon\}$ has no conjugate points. Indeed, let $Y_\theta(t)\, v = d\pi \, d\psi_t(\theta)\,(0,v)$, where $(0,v) \in V(\theta)$. Then from the Jacobi equation (6) we have that

$$\frac{D}{dt} Y_\theta(t)\bigg|_{t=0} = H_{pp}(\theta).$$

Since $Y_\theta(0) = 0$ and $H_{pp}(\theta)$ is nonsingular, then $Y_\theta(T)$ is an isomorphism for $|t| < \varepsilon$, some $\varepsilon > 0$. In particular $d\psi_t(\theta)\, V(\theta) \cap V(\theta) = \{0\}$ for $|t| < \varepsilon$.

We say that a curve $\gamma(t) \in M$, $t \in [0,T]$ is *minimizing* if it minimizes the action functional

$$\int_0^T L(\delta(t), \delta'(t)) \, dt$$

over all absolutely continuous curves $\delta(t) \in M$, $0 \le t \le T$, such that $\delta(0) = \gamma(0)$ and $\delta(T) = \gamma(t)$.

4.2. Corollary. *If $\theta \in T^*M$ and the segment $\{\, \pi\psi_t(\theta) \,|\, t \in [0, S[\, \}$ is minimizing, then it has no conjugate points.*

Proof:

Suppose it is false. Let $\psi_T(\theta)$, $T < S$ be the first conjugate point in $\{\, \pi\psi_t(\theta) \,|\, t \in [0, S[\, \}$. Then there exists $\xi = (0, v) \in V(\theta)$ such that $d\psi_T(\theta)\,\xi \in V(\psi_T(\theta))$. Let $\eta(t) = d\pi\, d\psi_t(\theta)\,\xi$. Using the limit of (53) on the interval $[0, T[$ we have that $I(\eta, \eta)|_0^T = 0$.

Let $\varepsilon > 0$ be such that (cf. prop. 1.15) the segment $\{\psi_t(\theta) \,|\, T-\varepsilon \le t \le T+\varepsilon\}$ has no conjugate points. Let $\zeta(t) = Y_{\psi_{T+\varepsilon}(\theta)}(t - T - \varepsilon)\, w = d\pi\, d\psi_{t-T-\varepsilon}(\psi_{T-\varepsilon}(\theta))\,(0, w)$, $T - \varepsilon \le t \le T + \varepsilon$, where w is such that $\zeta(T - \varepsilon) = \eta(T - \varepsilon)$. Let $\widehat{\eta} = \eta(t)$ if $T - \varepsilon \le t \le T$ and $\widehat{\eta}(t) = 0$ if $T \le t \le T + \varepsilon$. Then using $H(t) = Y_{\psi_{T+\varepsilon}(\theta)}(t - T - \varepsilon)$ on (53), we have that

$$I(\widehat{\eta}, \widehat{\eta})\Big|_{T-\varepsilon}^{T+\varepsilon} > I(\zeta, \zeta)\Big|_{T-\varepsilon}^{T+\varepsilon}.$$

Extend ζ and $\widehat{\eta}$ by $\zeta(t) = \widehat{\eta}(t) = \eta(t)$ for $0 \le t \le T - \varepsilon$. Then

$$I(\zeta, \zeta)\Big|_0^{T+\varepsilon} < I(\widehat{\eta}, \widehat{\eta})\Big|_0^{T+\varepsilon} = 0.$$

Now consdider the variation $f(s, t) = \exp_{\pi\psi_t(\theta)} s\,\widehat{\eta}$. We have that $f(s, 0) = \pi(\theta)$, $f(s, T + \varepsilon) = \pi(\psi_{T+\varepsilon}(\theta))$, $\frac{\partial f}{\partial s}(0, t) = \widehat{\eta}(t)$ and if

$$A(s) = \int_0^{T+\varepsilon} L\left(f(s, t), \tfrac{\partial f}{\partial s}(s, t)\right)\, dt$$

is the action of $f(s, t)$, then $A'(0) = 0$ because $f(0, t) = \pi\,\psi_t(\theta)$ satisfies the Euler-Lagrange equation and

$$A''(0) = I(\widehat{\eta}, \widehat{\eta}) < 0\,.$$

Therefore the segment $\{\, \pi\,\psi(\theta) \,|\, 0 \le t \le T + \varepsilon \,\}$ is not minimizing.

□

5. Proof of theorem D.

We begin by quoting the following theorem by R. Mañé [23]. Given a (periodic) Lagrangian $L : TM \times S^1 \to \mathbb{R}$ and $\omega \in H^1(M, \mathbb{R})$, let $\mathcal{M}^\omega(L)$ be the set of minimizing measures of $L + \overline{\omega}$, where $\overline{\omega}$ is any 1-form in the class ω.

5.1. Theorem. (Mañé [23])

(a) *For every $\omega \in H^1(M, \mathbb{R})$ there exists a residual subset $\mathcal{O}(\omega) \subset C^\infty(M \times S^1)$ such that $\psi \in \mathcal{O}(\omega)$ implies $\#\mathcal{M}^\omega(L + \psi) = 1$.*

(b) *There exist residual subsets $\mathcal{O} \subset C^\infty(M \times S^1)$ and $\mathcal{H} \subset H^1(M, \mathbb{R})$ such that $\psi \in \mathcal{O}$ and $\omega \in \mathcal{H}$ imply $\#\mathcal{M}^\omega(L + \psi) = 1$.*

Now take an autonomous Lagrangian L and $\omega = 0 \in H^1(M,\mathbb{R})$. Then the item (a) implies the first part of theorem D. Let $\mathcal{O}$ be the residual subset given by this theorem. Let $\mathcal{A}$ be the subset of $\mathcal{O}$ of potentials ψ for which the measure on $\mathcal{M}(L+\psi)$ is supported on a periodic orbit. Let $\mathcal{B} := \mathcal{O} \setminus \mathcal{A}$ and let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ on which the minimizing periodic orbit is hyperbolic. We prove that $\mathcal{A}_1$ is relatively open on $\mathcal{A}$. For, let $\psi \in \mathcal{A}_1$ and

$$\mathcal{M}(L+\psi) = \{\mu_\gamma\}$$

where μ_γ is the invariant probability measure supported on the hyperbolic periodic orbit γ for the flow of $L+\psi$. We claim that if $\phi_n \in \mathcal{A}$, $\phi_n \to \psi$ and $\mathcal{M}(L+\phi_k) = \{\mu_{\eta_k}\}$, then $\eta_n \to \gamma$. Indeed, since L is superlinear, the velocities in the support of the minimizing measures $\mu_k := \mu_{\eta_k}$ are bounded (cf. [26], [22]), and hence there exists a subsequence $\mu_k \to \nu$ converging weakly* to a some invariant measure ν for $L+\psi$. Then if $\nu \neq \mu_\gamma$,

$$\lim_k S_{L+\phi_k}(\mu_k) = S_{L+\psi}(\nu) > S_{L+\psi}(\mu_\gamma)\,. \tag{56}$$

Thus if δ_k is the analytic continuation of the hyperbolic periodic orbit γ to the flow of $L+\phi_k$ in the original energy level $c(L+\psi)$, since $\lim_k S_{L+\psi_k}(\mu_{\delta_k}) = S_{L+\psi}(\mu_\gamma)$, for k large we have that,

$$S_{L+\phi_k}(\mu_{\delta_k}) < S_{L+\phi_k}(\mu_{\eta_k})\,,$$

which contradicts the choice of η_k. Therefore $\nu = \mu_\gamma$. For energy levels h near to $c(L+\psi)$ and potentials ϕ near to ψ, there exist hyperbolic periodic orbits $\gamma_{\phi,h}$ which are the continuation of γ. Now, on a small neighbourhood of a hyperbolic orbit there exists a unique invariant measure supported on it, and it is in fact supported in the periodic orbit. Thus, since $\eta_n \to \gamma$, then η_n is hyperbolic. Hence $\phi_n \in \mathcal{A}_1$ and $\mathcal{A}_1$ contains a neighbourhood of ϕ in $\mathcal{A}$.

Let $\mathcal{U}$ be an open subset of $C^\infty(M,\mathbb{R})$ such that $\mathcal{A}_1 = \mathcal{U} \cap \mathcal{A}$. We shall prove below that $\mathcal{A}_1$ is dense in $\mathcal{A}$. This implies that $\mathcal{A}_1 \cup \mathcal{B}$ is generic. For, let $\mathcal{V} := \text{int}\,(C^\infty(M,\mathbb{R})\setminus\mathcal{U})$, then $\mathcal{U}\cup\mathcal{V}$ is open and dense in $C^\infty(M,\mathbb{R})$. Moreover, $\mathcal{V}\cap\mathcal{A} = \emptyset$ because $\mathcal{A} \subseteq \overline{\mathcal{A}_1} \subseteq \overline{\mathcal{U}}$ and $\mathcal{V}\cap\mathcal{A} \subseteq \mathcal{A}\setminus\overline{\mathcal{U}} = \emptyset$. Since $\mathcal{O} = \mathcal{A}\cup\mathcal{B}$ is generic and

$$\begin{aligned}(\mathcal{U}\cup\mathcal{V})\cap(\mathcal{A}\cup\mathcal{B}) &= (\mathcal{U}\cap\mathcal{A})\cup\big((\mathcal{U}\cup\mathcal{V})\cap\mathcal{B}\big)\\ &\subseteq \mathcal{A}_1\cup\mathcal{B}\,,\end{aligned}$$

then $\mathcal{A}_1\cup\mathcal{B}$ is generic.

Note: The perturbation to achieve hyperbolicity in the case of a periodic orbit and a singularity follow the same spirit. However to prove it in the case of a singular point is much easier because the Jacobi equation (the linear part of the flow) is autonomous. We suggest to read it first.

We have to prove that $\mathcal{A}_1$ is dense in $\mathcal{A}$, i.e. given that a Lagrangian L has a unique minimizing measure supported on a periodic orbit γ, then there exists a perturbation by an arbitrarily small C^∞-potential ϕ, such that the new Lagrangian $L+\phi$ has a unique minimizing measure supported on a hyperbolic periodic orbit.

Fix $\rho\in\mathcal{A}$ and let Γ be the periodic orbit in $\mathcal{M}(L+\rho)$, we can assume that $\rho=0$. By the graph property (cf. [26], [22]), the projection $\pi|_\Gamma:\Gamma\to M$, $\pi(x,v)=x$ is injective. In particular $\pi(\Gamma)\subseteq M$ is a simple closed curve. Choose coordinates $\overline{x}=(x_1,\dots,x_n):U\to S^1\times\mathbb{R}^{n-1}$ on a tubular neighbourhood of $\pi(\Gamma)$ such that $\overline{x}(\Gamma)=S^1\times\{\mathbf{0}\}$ and $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_n}\right\}$ is an orthonormal frame over the points of $\pi(\Gamma)$. In particular $\frac{\partial}{\partial x_1}$ is paralel to $\pi(\Gamma)$. Define $\widetilde{L}=L+\phi$ with $\phi(x_1,\dots,x_n)=\frac{1}{2}\,f(\overline{x})\,\varepsilon\,(x_2^2+x_3^2+\cdots+x_n^2)$, where $f(\overline{x})$ is a C^∞ nonnegative bump function with support in U which is 1 on a small neighbourhood of $\pi(\Gamma)$. Clearly ϕ can be made C^∞ arbitrarily small. On $\pi(\Gamma)$ we have that $\widetilde{L}_{vv}=L_{vv}$, $\widetilde{L}_{vx}=L_{vx}$ and $\widetilde{L}_{xx}=L_{xx}+\varepsilon\left[\begin{smallmatrix}0&\mathbf{0}\\ \mathbf{0}&I\end{smallmatrix}\right]$. Observe that since $\widetilde{L}\geq L$ then $\mathcal{M}(L+\phi)=\{\mu_\Gamma\}$.

Let H and $\widetilde{H}=H-\phi$ be the Hamiltonians associated to L and $\widetilde{L}$ respectively. Let Γ be the corresponding periodic orbit for H and $\widetilde{H}$ on T^*M. We have to prove that the periodic orbit Γ is hyperbolic for the flow of $\widetilde{H}$. Since it is minimizing, by corollary 4.2 the orbit Γ has no conjugate points. By proposition B it is enough to prove that the Green bundles $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{F}}$ satisfy $\widetilde{\mathbb{E}}(\theta)\cap\widetilde{\mathbb{F}}(\theta)=\langle\widetilde{X}(\theta)\rangle$ on a point $\theta\in\Gamma$, where $\widetilde{X}$ is the Hamiltonian vectorfiled of $\widetilde{H}$.

Fix $\theta\in\Gamma$ and let $\widetilde{Z}_T(t)=d\pi\circ d\widetilde{\psi}_t\circ\big[d\pi|_{d\widetilde{\psi}_{-T}(V(\widetilde{\psi}_T(\theta)))}\big]^{-1}$. Let $(\widetilde{Z}_T,\widetilde{V}_T)$ and $\widetilde{K}_T(t)=\widetilde{V}_T(t)\,(\widetilde{Z}_T(t))^{-1}$, $0\leq t<T$, be the corresponding solutions of the Jacobi equation (6) and the Ricatti equation (7) respectively: $\mathrm{graph}(\widetilde{K}_T(t))=d\widetilde{\psi}_t(d\widetilde{\psi}_{-T}(V(\widetilde{\psi}_T\,\theta)))$, $\widetilde{Z}_T(T)=0$, $\widetilde{Z}_T(0)=I$.

Define $\boldsymbol{N}(\theta):=\{\,w\in T_{\pi(\theta)}M\mid\langle w,\dot\gamma\rangle=0\,\}$. Then $\boldsymbol{N}(\theta)$ is the subspace of $T_{\pi(\theta)}M$ generated by the vectors $\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_n}$. Let $v_o\in\boldsymbol{N}(\theta)$, $|v_o|=1$ and let $\xi^T(t):=\widetilde{Z}_T(t)\,v_o$. Denote by $\widetilde{I}_T$ and I_T be the index forms on $[0,T]$ for $\widetilde{L}$ and L respectively. Using the solution $(\widetilde{Z}_T,\,\widetilde{V}_T)$ on formula (53), we obtain that

$$\widetilde{I}_T(\xi^T,\,\xi^T)=-(\widetilde{Z}_T(0)\,v_o)^*\,(\widetilde{V}_T(0)\,v_o)=-v_o{}^*\,\widetilde{K}_T(0)\,v_o\,. \tag{57}$$

Moreover, in the coordinates $(x_1,\dots,x_n;\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ on TU we have that

$$\begin{aligned}\widetilde{I}_T(\xi^T,\,\xi^T)&=\int_0^T\left(\dot\xi^T\,\widetilde{L}_{vv}\,\dot\xi^T+2\,\dot\xi^T\,\widetilde{L}_{xv}\,\xi^T+\xi^T\,\widetilde{L}_{xx}\,\xi^T\right)dt\\&=\int_0^T\left(\dot\xi^T\,L_{vv}\,\dot\xi^T+2\,\dot\xi^T\,L_{xv}\,\xi^T+\xi^T\,L_{xx}\,\xi^T\right)dt+\int_0^T\varepsilon\,\sum_{i=2}^n\left|\xi_i^T\right|^2\,dt\,.\end{aligned} \tag{58}$$

We have that $\widetilde{Z}_T(0) = I$ and for all $t > 0$, $\lim_{t\to\infty} \widetilde{Z}_T(t) = \widetilde{\boldsymbol{h}}(t)$ with $\widetilde{\boldsymbol{h}}(t)$ given on claim 1.3 for $\widetilde{H}$. Writing $\pi_N(\xi) = (\xi_2, \xi_3, \dots \xi_n)$ then $\left|\pi_N \widetilde{\boldsymbol{h}}(0)\, v_0\right| = |v_0| = 1$ because $v_0 \in \mathbf{N}(\theta)$. Hence there exists $\lambda > 0$ and $T_0 > 0$ such that $\left|\pi_N \xi^T(t)\right| = \left|\pi_N \widetilde{Z}_T(t)\, v_0\right| > \frac{1}{2}$ for all $0 \le t \le \lambda$ and $T > T_0$. Therefore

$$\widetilde{I}_T(\xi^T,\, \xi^T) \ge I_T(\xi^T,\, \xi^T) + \frac{\varepsilon\lambda}{4}. \tag{59}$$

Let $(\boldsymbol{h}(t),\, \boldsymbol{v}(t)) = d\psi_t \circ \left(d\pi|_{\mathbb{E}(\theta)}\right)^{-1}$ be the solution of the Jacobi equation for H given by claim 1.3 and let $\mathbb{S}(\psi_t(\theta)) = \boldsymbol{v}(t)\, \boldsymbol{h}(t)^{-1}$ be the corresponding solution of the Ricatti equation, with graph$[\mathbb{S}(\psi_t(\theta)] = \mathbb{E}(\psi_t(\theta))$. Using formula (53), and writing $\xi^T(t) = \boldsymbol{h}(t)\, \zeta(t)$, we have that

$$I_T(\xi^T,\, \xi^T) = \int_0^T (\boldsymbol{h}\, \dot{\zeta})^* \, H_{pp}^{-1}\, (\boldsymbol{h}\, \dot{\zeta})\, dt + 0 - \big(\boldsymbol{h}(0)\, \zeta(0)\big)^* \big(\boldsymbol{v}(0)\, \zeta(0)\big)$$

$$I_T(\xi^T,\, \xi^T) \ge -v_o^*\, \mathbb{S}(\theta)\, v_o. \tag{60}$$

From (57), (59) and (60), we get that

$$v_o^*\, \mathbb{S}(\theta)\, v_o \ge v_o^*\, \widetilde{K}_T\, v_o + \frac{\varepsilon\lambda}{4}.$$

From proposition 1.4, we have that $\lim_{T\to+\infty} \widetilde{K}_T(0) = \widetilde{\mathbb{S}}(\theta)$, where graph$(\widetilde{\mathbb{S}}(\theta)) = \widetilde{\mathbb{E}}(\theta)$, the stable Green bundle for $\widetilde{H}$. Therefore

$$v_o^*\, \mathbb{S}(\theta)\, v_o \ge v_0^*\, \widetilde{\mathbb{S}}(\theta)\, v_o + \frac{\varepsilon\lambda}{4}. \tag{61}$$

Similarly, for the unstable Green bundles we obtain that

$$v_o^*\, \mathbb{U}(\theta)\, v_o + \lambda_2 \le v_o^*\, \widetilde{\mathbb{U}}(\theta)\, v_o \qquad \text{for } v_o \in \boldsymbol{N}(\theta), |v_o| = 1. \tag{62}$$

for some $\lambda_2 > 0$ independent of v_o.

From proposition 1.4 we have that $\mathbb{U}(\theta) \succcurlyeq \mathbb{S}(\theta)$. From (61) and (62) we get that $\widetilde{\mathbb{U}}|_{\boldsymbol{N}} \succ \mathbb{U}|_{\boldsymbol{N}} \succcurlyeq \mathbb{S}|_{\boldsymbol{N}} \succ \widetilde{\mathbb{S}}|_{\boldsymbol{N}}$. Since $\widetilde{\mathbb{E}}(\theta) = \text{graph}(\widetilde{\mathbb{S}}(\theta))$ and $\widetilde{\mathbb{F}}(\theta) = \text{graph}(\widetilde{\mathbb{U}}(\theta))$, we get that $\widetilde{\mathbb{E}}(\theta) \cap \widetilde{\mathbb{F}}(\theta) \subseteq \langle \widetilde{X}(\theta) \rangle$. Then proposition B shows that Γ is a hyperbolic periodic orbit for $L + \phi$.

This proves that $\mathcal{A}_1$ is dense in $\mathcal{A}$. Let $\mathcal{A}_2$ be the subset of $\mathcal{A}_1$ of potentials ψ for which the minimizing hyperbolic periodic orbit Γ has transversal intersections $W^s(\Gamma) \pitchfork W^u(\Gamma)$. Then $\mathcal{A}_2$ is relatively open on $\mathcal{A}_1$. Indeed, if $\phi_n \in \mathcal{A}_1$ and $\phi_n \to \psi \in \mathcal{A}_2$ and $\mathcal{M}(L + \phi_n) = \{\mu_{\gamma_n}\}$, $\mathcal{M}(L + \psi) = \{\mu_\Gamma\}$, we have seen that the γ_n's are continuations of the hyperbolic orbit Γ for nearby flows (on nearby energy levels): $\gamma_n = \Gamma_{\phi_n, h_n}$. Since compact subsets (fundamental domains) of the invariant manifolds $W^s(\Gamma_{\phi,h})$ depend continuously in the C^1 topology on (ϕ, h), then the transversality property $W^s(\Gamma_{\phi,h}) \pitchfork W^u(\Gamma_{\phi,h})$ persists on a neighbourhood of (ψ, h_0). In particular it holds for $\gamma_n = \Gamma_{\phi_n,h_n}$ for (ϕ_n, h_n) sufficiently close to (ψ, h_0).

By the same arguments as above, it is enough to prove that $\mathcal{A}_2$ is dense in $\mathcal{A}$. For it we need the following lemma,

5.2. Lemma. *Let Γ be a hyperbolic periodic orbit without conjugate points of a convex Hamiltonian. Then for all $\theta \in W^s(\Gamma)$ there exists $S = S(\theta) > 0$ such that the segment $\{\, \psi_t(\theta) \mid t \geq S \,\}$ has no conjugate points.*

Proof:

Let $\mathbb{E}$ be the stable Green subspace of Γ. Then $\mathbb{E}(\vartheta) \cap V(\vartheta) = \{0\}$ for all $\vartheta \in \Gamma$. Since the weak stable manifold $W^s(\Gamma)$ is tangent to $\mathbb{E}$; then for all $\theta \in W^s(\Gamma)$ there exists $S = S(\theta) > 0$ such that $E(t) := T_{\psi_t(\theta)} W^s(\Gamma)$ satisfies $E(t) \cap V\big(\psi_t(\theta)\big) = \{0\}$. Moreover, $\dim E(t) = n$ and for all $u \in E(t)$, $\lim_{s \to +\infty} d\psi_s(u) \in \langle X(\theta) \rangle$. Since ψ_t preserves the symplectic form ω, we have that $\omega(u, v) = \lim_{s\to+\infty} \omega(d\psi_s(u), d\psi_s(v)) = 0$ for all $u, v \in E(t)$. Hence $E(t)$ is Lagrangian and $E(t) \cap V\big(\psi_t(\theta)\big) = \{0\}$ for all $t > S$. Then the lemma follows from proposition 1.15.

□

Now we prove that $\mathcal{A}_2$ is dense in $\mathcal{A}$. Let $\rho \in \mathcal{A} \setminus \mathcal{A}_2$ and make the perturbation $\widetilde{L} = (L + \rho) + \phi$ explained above so that $\rho + \phi \in \mathcal{A}_1$. We can assume that $\rho + \phi = 0$ and also that $\rho + \phi \notin \mathcal{A}_2$, otherwise there is nothing to prove. Write $\mathcal{M}(L) = \{\mu_\Gamma\}$ and let $\theta \in W^s(\Gamma) \cap W^u(\Gamma)$ be such that $\dim\big(T_\theta W^s(\Gamma) \cap T_\theta W^u(\Gamma)\big) > 1$. Observe that the α and ω limit sets of θ are Γ. In particular the orbit of θ has no autoacumulation points. Let $S = S(\theta) > 0$ be from lemma 5.2 and $\epsilon > 0$ small. Then there exists $\tau > S(\theta)$ and a neighbourhood $W \subset M$ of $\pi(\psi_t(\theta))$ such that $\{\, t \in \mathbb{R} \mid \pi\psi_t(\theta) \in W \,\} =]\tau, \tau + 3\epsilon[$ and $W \cap \pi(\Gamma) = \emptyset$. Let $U \subset M \times \mathbb{R}$ be a tubular neighbourhood of $\{\, (\pi\psi_t(\theta); t) \mid t > S(\theta) \,\}$. Choose coordinates $\overline{y} = (y_1, \ldots, y_n; t) : U \to \mathbb{R}^n \times \mathbb{R}$ such that $\overline{y}\big(\pi\psi_t(\theta), t\big) = \big(\alpha(t), \mathbf{0}; t\big) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, and $\left\{ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \cdots, \frac{\partial}{\partial y_n} \right\}$ is an orthonormal frame over the points $\pi\psi_t(\theta)$, $t > S(\theta)$. Let $\varphi(y_1, \ldots, y_n) = \frac{1}{2} f(\overline{y})\, \delta\, (y_2^2 + y_3^2 \cdots + y_n^2)$, where $f(\overline{y})$ is a C^∞ bump function with support in W which is 1 on a small neighbourhood of $\big\{\, \pi\psi_t(\theta) \mid t \in [\tau + \epsilon, \tau + 2\epsilon] \,\big\}$. By choosing a small $\delta > 0$ the function φ can be made C^∞ arbitrarily small. Choose δ small enough such that the orbit Γ remains hyperbolic. Then we still have that $\mathcal{M}(L + \phi) = \{\mu_\Gamma\}$ and the orbit of θ is the same for both flows. Write $\widetilde{L} = L + \rho + \phi$, $\widehat{L} = \widetilde{L} + \psi$ and $\widetilde{I}_T$, $\widehat{I}_T$ the corresponding index froms on $\{\, \psi_t(\theta) \mid t \in [\tau, T] \,\}$. Since $\{\, \psi_t(\theta) \mid t > S(\theta) \,\}$ has no conjugate points, the stable Green subspace $\widehat{\mathbb{E}}(\psi_t(\theta))$, $t > S(\theta)$ and the stable solution of the Ricatti equation $\widehat{\mathbb{S}}(\psi_t(\theta))$ exist. The same arguments as in equations (58)–(61) give for $\xi^T = \widehat{Z}_T(t)\, v_o$, $v_o \in \boldsymbol{N}(\psi_\tau(\theta))$, $|v_o| = 1$, $\widehat{Z}_T(t) = d\pi \circ d\widehat{\psi}_{t-T} \circ \big[d\pi|_{d\psi_{-T}(V(\widehat{\psi}_{T+\tau}(\theta)))}\big]^{-1}$, that

$$\widehat{I}_T(\xi^T, \xi^T) = \widetilde{I}_T(\xi^T, \xi^T) + \int_{\tau+\epsilon}^{\tau+2\epsilon} \delta \sum_{i=2}^{n} \left|\xi_i^T\right|^2 \, dt\,,$$

$$v_0^* \, \mathbb{S}\big(\psi_\tau(\theta)\big)\, v_o \geq v_o \, \widehat{\mathbb{S}}\big(\psi_\tau(\theta)\big)\, v_o + \frac{\delta\,\widehat{\lambda}}{4}\,,$$

for some $\widehat{\lambda} > 0$. In particular

$$\widehat{\mathbb{E}}\big(\psi_\tau(\theta)\big) \cap \widetilde{\mathbb{E}}\big(\psi_\tau(\theta)\big) = \big\langle X(\psi_\tau\theta)\big\rangle\,. \tag{63}$$

By the same arguments as in proposition 1.14, we have that $\widehat{\mathbb{E}}\big(\psi_\tau(\theta)\big) = T_{\psi_\tau(\theta)}\big(W^u_{\widetilde{L}}(\psi_\tau\theta)\big)$. Observe that $T_{\psi_\tau(\theta)}\big(W^u_{\widehat{L}}(\psi_\tau\theta)\big) = T_{\psi_\tau(\theta)}\big(W^u_{\widetilde{L}}(\psi_\tau\theta)\big)$ didn't change. Write $T_{\psi_\tau(\theta)}\big(W^u_{\widetilde{L}}(\psi_\tau\theta)\big) = \mathbb{V} \oplus \mathbb{W}$, with $\mathbb{V} \subseteq \widetilde{\mathbb{E}}(\psi_\tau\theta)$ and $\mathbb{W} \cap \widetilde{\mathbb{E}}(\psi_\tau\theta) = \{0\}$. By diminishing δ if necessary we can get (63) and $\widehat{\mathbb{E}}(\psi_\tau\theta) \cap \mathbb{W} = \{0\}$. Thus

$$T_{\widehat{\psi}_\tau(\theta)} W^s_{\widehat{L}}(\Gamma) \cap T_{\widehat{\psi}_\tau(\theta)} W^u_{\widehat{L}}(\Gamma) = \widehat{\mathbb{E}}(\psi_\tau\theta) \cap T_{\psi_\tau(\theta)} W^u(\Gamma) = \big\langle \widehat{X}(\psi_\tau\theta)\big\rangle\,.$$

Finally, by a finite number of these perturbations on a fundamental domain of $W^s(\Gamma)$ one can remove all tangencies of $W^s(\Gamma)$ and $W^u(\Gamma)$ and obtain a potential in $\mathcal{A}_2$ arbitrarily C^∞-near $\rho \in \mathcal{A}$.

□

The case of a singularity.

Suppose that the minimizing measure μ is supported on a singularity $(x_0, 0)$ of the Lagrangian flow. From the Euler-Lagrange equation (2) we get that $L_x(x_0, 0) = 0$. Differentiating the energy function (3) we see that $(x_0, 0)$ is a singularity of the energy level $c(L)$. Moreover, the minimizing property of μ implies that x_0 is a minimum of the function $x \mapsto L_{xx}(x, 0)$. In particular, $L_{xx}(x_0, 0)$ is positive semidefinite in linear coordinates in $T_{x_0}M$.

Choose coordinates $\overline{y} = \{y_1, \dots, y_n\}$ on a neighborhood of x_0 such that $\overline{y}(x_0) = \mathbf{0}$ and $L_{vv}(x_0, 0)$ is the identity on the basis $\big\{ \frac{\partial}{\partial y_1}\big|_{x_0}, \cdots, \frac{\partial}{\partial y_n}\big|_{x_0} \big\}$. Use the coordinates $\big(\{y_1, \dots, y_n\}, \big\{ \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n} \big\}\big)$ on a neighbourhood of x_0. On the orbit $\varphi_t(x_0, 0) \equiv (x_0, 0)$ the matrices L_{xx}, L_{xv}, L_{vv} are constant with respect to the time t. The Jacobi equation (47) at $(x_0, 0)$ becomes

$$\ddot{k} = L_{xx}(x_0, 0)\; k\;,$$

where $L_{vv}(x_0, 0) = I$. Let f be a nonnegative C^∞ function on $\mathbb{R}^n$ with support on $|y| < \varepsilon < 1$ and $f \equiv 1$ on a neighbourhood of $\mathbf{0}$. Let $\phi(y) = \frac{1}{2}\, \delta\, f(y)\, (y_1^2 + \cdots + y_n^2)$. Adjusting $\delta > 0$ we can make ϕ arbitrarily C^∞ small. Let $\widetilde{L} := L + \phi$. Then the atomic measure μ supported on $(x_0, 0)$ is still the unique minimizing measure for $\widetilde{L}$ and the Jacobi equation for the orbit $\widetilde{\varphi}_t(x_0, 0) \equiv (x_0, 0)$ is

$$\ddot{k} = \big(L_{xx}(x_0, 0) + \delta\, I\big)\; \dot{k}\,. \tag{64}$$

Since $L_{xx}(x_0, 0)$ is positive semidefinite, then $A := \big(L_{xx}(x_0, 0) + \delta\, I\big)$ is positive definite. Equation (64) is linear

$$\begin{bmatrix} k' \\ k'' \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \; \begin{bmatrix} k \\ k' \end{bmatrix} \tag{65}$$

with constant coefficients where $k \equiv 0$ is a hyperbolic singularity.

To obtain the transversality property $W^s \pitchfork W^u$ at homoclinic orbits we can use the same arguments as above if we have a replacemnet for lemma 5.2. The argumenmts of lemma 5.2 apply if we show that the stable subspace of $(x_0, 0)$ is transversal to the vertical subspace. For, it is enough to prove that the eigenvectors (x, y) of the (non-singular) matrix in (65) have coordinate $x \neq 0$. These eigenvectors satisfy $y = \lambda x$, $Ax = \lambda^2 x$, $(\lambda \in \mathbb{R})$. Hence if $x = 0$, then $(x, y) = (0, 0)$ is not an eigenvector.

□

6. The Exponential map.

The following lemma was proven for geodesic flows by Freire and Mañé [13] and has intrinsic interest (compare with proposition 1.10).

6.1. Lemma. *Let H be a convex Hamiltonian. Choose a riemannian metric $\|\ \ \|$ on M. If ψ_t has no conjugate points on $\Sigma = H^{-1}\{e\}$, then there exists $B > 0$ such that*

$$\|Y_\theta(t) \cdot v\| > B\; \|v\|$$

for all $\theta \in \Sigma$, $|t| > 1$ and $v \in T_\theta M$. Where $Y_\theta(t) \cdot v = d\pi \circ d\psi_t(\theta) \cdot (0, v)$, $(0, v) \in H(\theta) \oplus V(\theta)$.

We sketch the proof of theorem F. In the case in which $\Theta(X) > 0$ on Σ one can define the vectorfield $Y = \frac{1}{|\Theta(X)|}\, X$. This vectorfield preserves the 1-form Θ on $T\Sigma$. Extending quadratically the Hamiltonian to T^*M by $\mathbb{H}(t\,\theta) = t^2$, where $\theta \in \Sigma$, we obtain a new Hamiltonian $\mathbb{H}$, which is convex and has no conjugate points in Σ. The Hamiltonian vectorfield $\mathbb{Y}$ of $\mathbb{H}$ is called the (convex) *simplectification* of the contact vectorfield Y. The transversal bundle $\boldsymbol{N}(\theta) = \ker \Theta|_{T_\theta \Sigma}$ is invariant under the flow of $\mathbb{Y}$. Let $E_q : T_q\tilde{M} \to \tilde{M}$ be the exponential map for $(\mathbb{Y}, \Sigma)$. The derivative $dE_q(t\theta)$ is given in formulas (39) and (40). The norm of the directional derivative in formula (40) is clearly bounded below. By lemma 6.1 the norm of the derivative $dE_q|_{V(\theta)\cap T_\theta\Sigma}$ in formula (39) is uniformly bounded away from zero. Since $V(\theta)\cap T_\theta\Sigma \subset \boldsymbol{N}(\theta)$, and the angle $\sphericalangle\big(\boldsymbol{N}(\theta), X(\theta)\big)$ is bounded below, we obtain that $\|dE_q\|$ is bounded away from zero. This implies that $E_q : T_q\tilde{M} \to \tilde{M}$ is a covering map and hence a diffeomorphism. In particular, the universal cover $\tilde{M}$ of M is homeomorphic to $\mathbb{R}^n$. By corollary 1.18, the exponential map $\exp_q$ for X is a local diffeomorphism. Once we know that E_q is a diffeomorphism, it is easy to show that $\exp_q$ is bijective and hence a diffeomorphism.

Proof of Lema 6.1:

Since $Y_\theta(t)$ satisfies (8) (with H_{qp} and H_{pp} uniformly bounded on Σ) and by corolary 1.8, $S_\theta(t)$ is uniformly bounded for $|t|>1$, then there exists $D_1>1$ such that

$$\left\|\dot{Y}_\theta(t)\cdot w\right\| \le D_1\, \|Y_\theta(t)\cdot w\| \text{ for all } |t|>1,\ w\in V(\theta),\ \theta\in\Sigma.$$

Let

$$A := (1+D_1)^2 \sup_{\theta\in\Sigma} \left\|H_{pp}(\theta)^{-1}\right\|.$$

Let $H_\theta(t) := Y_{\psi_{-2}(\theta)}(t+2)\circ\big(Y_{\psi_{-2}(\theta)}(2)\big)^{-1}$, then $\det H_\theta(t)\neq 0$ for $t>-2$, $H_\theta(0)=I$ and there exists a linear homomorphism $V_\theta(t) : H(\theta)\to V(\psi_t(\theta))$ such that (H_θ, V_θ) is a matrix solution of the Jacobi equation (6) and Image$(H_\theta(t), V_\theta(t)) = d\psi_{t+2}(V(\psi_{-2}(\theta)))$ is a Lagrangian subspace. Let

$$D_2 := \max_{\substack{-2\le t\le 2\\ \theta\in\Sigma}} \left\|H_\theta(t)^*\,(H_{pp})^{-1}\,H_\theta(t)\right\| < +\infty.$$

Let $\rho:\mathbb{R}\to[0,1]$ be a C^∞ function such that $\rho(t)=0$ for $|t|\ge 1$ and $\rho(0)=1$. Let

$$C := D_2 \int_{-1}^{1} |\rho'(t)|^2\, dt\,.$$

Let

$$0 < B < (A\,C)^{\frac{1}{2}}. \tag{66}$$

Suppose that $\|Y_\theta(T)\,v_o\| < B\,\|v_o\|$ for some $\theta\in\Sigma$, $v_o\in T_{\pi(\theta)}M$, $\|v_o\|=1$ and $|T|>1$. Assume that $T>1$, the case $T<-1$ is proven similarly. Let $J(t)\in H(\psi_t(\theta)) = T_{\pi\psi_t(\theta)}M$ be defined by $J(t)=0$ for $-1\le t\le 0$, $J(t)=Y_\theta(t)\,v_o$ for $0\le t\le T$ and $J(t) = (T+1-t)\,\tau_t\,J(T)$ for $T\le t\le T+1$, where $\tau_t : T_{\pi\psi_T(\theta)}M \to T_{\pi\psi_t(\theta)}M$ is the parallel transport along the path $t\mapsto \pi\psi_t(\theta)$. Then $J(t)$ is continuous and piecewise C^2.

Let $v(t)\in V(\theta)\approx T_{\pi(\theta)}M$ be defined by $J(t) = Y_\theta(t)\,v(t)$. Then $v(t)=0$ for $-1\le t\le 0$ and $v(t)=v_o$ for $0\le t\le T$. Let $\{e_1,\dots,e_n\}$ be a basis of $T_{\pi(\theta)}M$

and write $v(t) = \sum_{i=1}^n v_i(t)\, e_i$, $Y_i(t) = Y(t)\, e_i$. Then the covariant derivative along $\pi\psi_t(\theta)$ of $J(t) = Y(t)v(t)$ is

$$\frac{D}{dt}(Yv) = \frac{D}{dt}\left(\sum_{i=1}^n v_i(t)\, Y_i(t)\right) = \sum_i \left(\frac{DY_i}{dt}\right) v_i + \sum_i \left(\frac{dv_i}{dt}\right) Y_i\,,$$

$$Y\, v' = \frac{D}{dt}(Yv) - \frac{DY}{dt} v\,.$$

For $T \le t \le T+1$, since $J(t) = Y_\theta(t)\, v(t) = (T+1-t)\, \tau_t\, J(T)$, we have that

$$\begin{aligned} \|Y v'\| &\le \left\|\tfrac{D}{dt}(Yv)\right\| + \left\|\tfrac{DY}{dt}\, v\right\| \\ &\le \left\|\tfrac{D}{dt} J(t)\right\| + D_1\, \|J(t)\| \\ &\le \|J(T)\| + D_1\, \|J(T)\| \\ &\le (1+D_1)\, B\,. \end{aligned}$$

Using $Y_\theta(t)$ on formula (55) we have that

$$\begin{aligned} I(J,J) &= \int_T^{T+1} (Yv')^*\, (H_{pp})^{-1}\, (Yv')\, dt \\ &\le (1+D_1)^2\, B^2\, \left\|H_{pp}^{-1}\right\| = A\, B^2. \end{aligned}$$

Let $Z(t) \in H(\psi_t(\theta)) = T_{\pi\psi_t(\theta)} M$ be defined by $Z(t) = \rho(t)\, H_\theta(t)\, v_o$. Then $Z(t) = Y_\theta(t)\, \xi(t)$ for $t \ne 0$, with $\xi(t) = 0$ for $|t| > 1$. Since $d\psi_t(\theta) \cdot V(\theta) =$ Image $(Y_\theta(t)\,,\, S_\theta(t)\, Y_\theta(t))$, we have that $S_\theta(t)\, Y_\theta(t) \to I$ when $t \to 0$. Using $Y_\theta(t)$ on formula (55), we have that the only non zero term is

$$I(Z,J) = -Z(0^+) \cdot v_o + Z(0^-) \cdot 0 = -\, \|v_o\|^2 = -1.$$

Write $Z(t) = H(t)\, \zeta(t)$, $\zeta(t) = \rho(t)\, v_o$. Using $H(t)$ and $V(t)$ on fromula (53), we have that

$$\begin{aligned} I(Z,Z) &= \int_{-1}^1 (H\zeta')^*\, (H_{pp})^{-1}\, (H\zeta')\, dt \\ &= \int_{-1}^1 (\rho'(t))^2\, (H(t)\, v_o)^*\, (H_{pp}^{-1})\, (H(t)\, v_o)\, dt \\ &\le D_2 \int_{-1}^1 |\rho'(t)|^2\, dt = C. \end{aligned}$$

Since by (48) the index form is symmetric, for $\lambda \in \mathbb{R}$ we have that

$$I(Z - \lambda\, J\,,\, Z - \lambda\, J) \le C + 2\,\lambda + \lambda^2\, A\, B^2.$$

Since by (66) $4 - 4\, A\, B^2\, C > 0$, this polynomial is λ has two real roots. Therefore $I(Z-\lambda J\,,\, Z - \lambda J) < 0$ for some value of λ. Hence there must be conjugate points in the orbit segment $\{\, \psi_t(\theta) \,|\, t \in [-1, T+1]\,\}$. $\square$

6.2. Proposition. *Let e be a regular value of H. If $\theta(X) > 0$ on the energy level $\Sigma = H^{-1}\{e\}$ and the flow of X has no conjugate points, then there exists a Hamiltonian $\mathbb{H}$ on T^*M with vectorfield Z such that*

(i) $\mathbb{H}$ *is convex and* $\mathbb{H}^{-1}\{e\} = \Sigma$.
(ii) $Z = \frac{1}{\theta(X)} \cdot X$ *on* Σ.
(iii) *The flow* ζ_t *of* Z *has no conjugate points on* Σ.
(iv) ζ_t *preserves the 1-form* Θ *on* $T\Sigma$.
(v) *If* $N(\theta) := \frac{d}{ds} s\theta|_{s=1}$, *then for* $\theta \in \Sigma$ *we have*

$$d\zeta_t(\theta) \cdot N(\theta) = N(\zeta_t(\theta)) + t\, X(\zeta_t(\theta))\,.$$

6.3. Remark. This proposition holds if $\Theta(X) \neq 0$ on a connected compact foward invariant set $K \subset \Sigma$. In this case we obtain an open connected neighbourhood U of K such that $\inf_{\vartheta \in U} |\Theta(X(\vartheta))| > 0$ and a convex Hamiltonian $\mathbb{H}$ without conjugate points, defined on a neighbourhood of U in T^*M. To apply this on theorem F, we can take $K =$ clousure of $\psi_{\mathbb{R}^+}\big((\pi|_\Sigma)^{-1}\{q\}\big) \subseteq \Sigma$.

Proof:

Recall that $\omega(X, \cdot) = -dH$. Let $\Theta = p\, dq$ be the canonical 1-form on T^*M. Suppose that $\Theta(X) > 0$ on Σ. Let Y be the vectorfield on Σ given by $Y = \frac{1}{\Theta(X)} \cdot X$. Then Y preserves the 1-form Θ on $T\Sigma$ because the Lie derivative

$$\begin{aligned} L_Y \Theta &= d\, i_Y\, \Theta + i_Y\, d\Theta = d[\Theta(Y)] + \omega(Y, \cdot) \\ &= d(1) - \tfrac{1}{\Theta(X)}\, dH = 0 \text{ on } T\Sigma. \end{aligned}$$

Let $\phi_t(q,p)$ be the flow of Y on Σ. Define the simplectification $\widetilde{\phi}_t$ of ϕ_t (cf. Arnold [3], [2]) on the cotangent bundle without its zero section $T^*M \setminus M$ by $\widetilde{\phi}_t(q, \lambda p) = \lambda\, \phi_t(q,p)$, $(q,p) \in \Sigma$. Observe that $\widetilde{\phi}_t$ preserves the canonical 1-form Θ. Indeed, we have that $\pi \circ \phi_t = \pi \circ \widetilde{\phi}_t$ and if $\phi_t(q,p) = (q_t, p_t)$,

$$\begin{aligned} \Theta(q_t, \lambda\, p_t) \cdot (d\widetilde{\phi}_t) &= (\lambda\, p_t) \cdot d\pi\, (d\widetilde{\phi}_t\, w) \\ &= \lambda\, [p_t \cdot (d\pi\, d\phi_t\, w)] \\ &= \lambda\, [p \cdot (d\pi\, w)] = \Theta(q,p) \cdot w\,. \end{aligned}$$

Let $\widetilde{Y}$ be the vectorfield of $\widetilde{\phi}_t$. We have that

$$L_{\widetilde{Y}} \Theta = d[\Theta(\widetilde{Y})] + \omega(\widetilde{Y}, \cdot) = 0\,.$$

Therefore the function $\widetilde{H} = \Theta(\widetilde{Y})$ is a Hamiltonian for $\widetilde{Y}$. But $\widetilde{Y}$ has conjugate points on Σ and $\widetilde{H}$ is not convex.

Let $\mathbb{H} = \frac{1}{2}\widetilde{H}^2$ and let Z be the Hamiltonian vectorfield of $\mathbb{H}$. We have that $Z = \widetilde{H}\cdot\widetilde{Y}$, since $\widetilde{H}|_\Sigma \equiv 1$, Z is also an extension of Y. Let ζ_t be the flow of Z. Observe that

$$\zeta_t(q,sp) = s\,\widetilde{\phi}_{t\widetilde{H}(q,sp)}(q,p) = s\,\widetilde{\phi}_{t\,\frac{1}{2}s^2}(q,p)\,, \qquad \text{for } (q,p)\in\Sigma\,.$$

Taking the derivative $\frac{d}{ds}\big|_{s=1}$, we obtain

$$d\zeta_t(\theta)\cdot N(\theta) = N(\zeta_t(\theta)) + t\,X(\zeta_t(\theta)) \quad \text{ for } \theta\in\Sigma.$$

Write $\zeta(t,\theta) = \psi(s(t,\theta),\theta)$ with $s(t,\theta)>0$, $\theta\in\Sigma$. Then

$$D\,\zeta_t = D\,\psi_s + X(\psi_s(\theta))\,\tfrac{\partial s}{\partial\theta} \qquad \text{on } T\Sigma\,. \tag{67}$$

Let $\mathbb{E}\subset T\Sigma$ be the stable Green bundle for X on Σ. Since $X(\theta)\in\mathbb{E}(\theta)$ and $\mathbb{E}$ is ψ_t-invariant, then equation (67) shows that the flow ζ_t of Z preserves the Lagrangian bundle $\mathbb{E}\subset T\Sigma$, which satisfies $\mathbb{E}(\theta)\cap V(\theta) = \{0\}$ for all $\theta\in\Sigma$. By proposition 1.15, ζ_t has no conjugate points on Σ.

The following lemma completes the proof of proposition 6.2

6.4. Lemma. *The Hamiltonian $\mathbb{H}$ is convex, i.e. $\mathbb{H}_{pp}$ is positive definite.*

Proof:

Let $q\in\Sigma$ and $S(q) := T_q^*M\cap\Sigma$. Since $\mathbb{H}$ is homogeneous of degree 2, then $\mathbb{H}_{pp}$ is homogeneous of degree 0 and it is enough to prove that $\mathbb{H}_{pp}(q,p)$ is positive definite for $p\in S(q)$. Let $p_0\in S(q)$. Let $h: T_{p_0}S(q)\times\mathbb{R}\to T_q^*M$ be $h(x,\lambda) := x+\lambda\,p_0$. Let $G = \mathbb{H}\circ h$ and $K = \Theta(\widetilde{Y}\circ h)$. Then $G = \frac{1}{2}K^2$. Since $D^2h = 0$, then $D^2\mathbb{H}(q_0)$ is positive definite if $D^2G(\mathbf{0},1)$ is positive definite. Since $K(\mathbf{0},\lambda)=\lambda$, we have that

$$G_{\lambda\lambda}(\mathbf{0},1) = \big[K_\lambda^2 + K\cdot K_{\lambda\lambda}\big]\,\big|_{(0,1)} = 1\,.$$

Since in the coordinates (x,λ) the vectors $\frac{\partial}{\partial x_i}\Big|_{(\mathbf{0},1)}$, $i=1,\dots,n-1$ are tangent at p_0 to $S(q)$ and $\Theta(\widetilde{Y})\equiv 1$ on $S(q)$, then $K_x(\mathbf{0},1)=\mathbf{0}$. Hence

$$\begin{aligned} G_{xx}(\mathbf{0},1)(v,v) &= |K_x\cdot v|^2 + K\,K_{xx}(v,v)\\ &= K_{xx}(v,v)\,. \end{aligned}$$

Observe that K is homogeneous of degree 1. Therefore $K_x(\mathbf{0},1) = K_x(\mathbf{0},1)$ for all $\lambda>0$. Hence $K_{x\lambda}(\mathbf{0},1)=0$, so that

$$G_{x\lambda}(\mathbf{0},1)\cdot(v,\mu) = \mu\,\left[K_\lambda\,(K_x\cdot v) + K\,(K_{x\lambda}\cdot v)\right] = 0\,.$$

And then

$$\begin{aligned} D^2G(\mathbf{0},1)(v,\mu)^{(2)} &= \mu^2\,G_{\lambda\lambda} + 2\mu(G_{\lambda x}\cdot v) + G_{xx}(v,v)\\ &= \mu^2 + K_{xx}(\mathbf{0},1)\cdot(v,v)\,. \end{aligned}$$

So it is enough to see that $K_{xx}(\mathbf{0},1)$ is positive definite.

Let N be the unit normal vector to $T_{p_0}S(q)$ such that $p_0\cdot N<0$. Such N exists because $H|_{S(q)}\equiv e$ and $p_0\cdot H_p=\theta(X)\neq 0$. Let $F:T_{p_0}S(q)\times\mathbb{R}\to\mathbb{R}$ be given by $F(x,y)=H(p_0+x+yN)$. Since $\frac{\partial F}{\partial y}\Big|_{(0,0)}<0$, there exists a local C^2 function $y:U\subseteq T_{p_0}S(q)\to\mathbb{R}$ such that $F(x,y(x))=e$. Differentiating this equation with respect to x, we obtain that

$$dy=-\frac{F_x}{F_y}$$

$$d^2y=-\frac{1}{F_y}\,D^2F\cdot(I,dy)^{(2)}.$$

Since the vectors $\frac{\partial}{\partial x_i}\Big|_{(0,0)}$ are tangent to the level $F\equiv e$ at (q,p_0), we have that $F_x(\mathbf{0},1)=\mathbf{0}$. Hence $dy(\mathbf{0},1)=\mathbf{0}$ and

$$d^2y(\mathbf{0})\cdot v=-\frac{1}{F_y}\,D^2F\cdot(v,\mathbf{0})^{(2)}=-\frac{1}{F_y}\,H_{xx}\cdot(v,v)>0\,.$$

Now let $v\in T_{p_0}S(q)$ be a unit vector and $r(s)=|p_0+s\,v|$, $z(s)=y(sv)$, $R(s)=|p_0+sv+z(s)\,N|$. We have that

$$r'(s)=\frac{p_0\cdot v+s}{r(s)}\ \text{hence}\ \frac{1}{2}\,r(s)^2=\frac{1}{2}\ |p_0|^2+(p_0\cdot v)\,s+\frac{1}{2}\,s^2\,.$$

Also

$$R'(s)=\frac{(p_0+sv+z(s)\,N)\cdot(v+z'\,N)}{R(s)}=\frac{p_0\cdot v+s+z'\,(p_0\cdot N)+z\,z'}{R(s)}$$

therefore, since $\alpha\,s^2\leq z(s)\leq\beta s^2$ for some $0<\alpha<\beta$ and $|s|$ small.

$$\begin{aligned}\tfrac{1}{2}R(s)^2&=\tfrac{1}{2}\ |p_0|^2+(p_0\cdot v)s+\tfrac{1}{2}\,s^2+z\,(p_0\cdot N)+\tfrac{1}{2}\,z^2\\&=\tfrac{1}{2}\,r(s)^2+z(s)\,(p_0\cdot N)+\tfrac{1}{2}\,z(s)^2\\&\leq\tfrac{1}{2}\,r(s)^2-\tfrac{1}{2}\,\gamma s^2\,,\end{aligned}$$

for some $\gamma>0$ and $|s|$ small. Then for some $\delta>0$, we have that

$$\frac{r(s)}{R(s)}\geq 1+\delta\,s^2$$

$$K_{xx}\cdot(v,v)=\frac{d^2}{ds^2}\,K(sv,1)\big|_{s=0}=\frac{d^2}{ds^2}\left(\frac{r(s)}{R(s)}\right)\bigg|_{s=0}\geq\delta>0\,.\quad\square$$

Proof of Theorem F.

Trough the proof of theorem F we work with the lifted Hamiltonian to the cotangent bundle of the universal cover $\tilde{M}$ of M. Let ζ_t be the flow of the Hamiltonian $\mathbb{H}$ of proposition 6.2. Let $q \in \tilde{M}$ and let $E_q : T_q\tilde{M} \to \tilde{M}$ be the exponential map associated to the flow ζ_t and the energy level Σ. We first prove that E_q is a diffeomorphism. The derivative $dE_q(t\theta)$ is non-singular by corollary 1.18 and is given by formulas (39) and (40). We claim that the norm $\|dE_q(t\theta)\|$ is uniformly bounded below. By lemma 6.1 the norm $\left\|dE_q(t\theta)|_{V(\theta)\cap T_q\Sigma}\right\|$ is uniformly bounded below for $t > \epsilon$. From equation (40) $\left\|dE_q(t\theta)|_{\langle\theta\rangle}\right\|$ is also uniformly bounded below. It remains to see that the angle $\sphericalangle\big[dE_q(t\theta)(V(\theta)\cap T_q\Sigma)\,,\, d\pi(X(\zeta_t\,\theta))\big]$ is uniformly bounded below.

Since ζ_t preserves the 1-form Θ, the bundle $\mathbf{K}(\theta) = \ker\,\Theta|_{T_\theta\Sigma}$ is invariant under $d\zeta_t$. Since $V(\theta)\cap T_q\Sigma \subset \mathbf{K}(\theta)$, then

$$dE_q(t\theta)\,\big(V(\theta)\cap T_q\Sigma\big) \subseteq d\pi\, d\zeta_t(\theta)\,\big(\mathbf{K}(\theta)\big) \subseteq d\pi\big(\mathbf{K}(\zeta_t(\theta))\big)\,. \tag{68}$$

For $\theta = (q,p) \in \Sigma$ write $H_p(q,p) = \alpha\, p + k$, with $\langle k, p\rangle = 0$. Since $\Theta(X) = p\cdot H_p \neq 0$ on Σ there exists $0 \le \lambda < 1$ such that $|\,k| \le \lambda\,|H_p|$ on Σ. Let $h \in d\pi\big(\mathbf{K}(\zeta_t\,\theta)\big)$ with $|h| = 1$, i.e. $\langle p, h\rangle = 0$, $|h| = 1$. Then

$$\langle h, H_p\rangle = \alpha\,\langle p, h\rangle + \langle k, h\rangle = \langle k, h\rangle\,.$$

$$|\langle h, H_p\rangle| \le 1\cdot\lambda\;|H_p|\;.$$

Hence $\cos\sphericalangle(h, H_p) \le \lambda < 1$. Therefore the angle $\sphericalangle\big[d\pi(\mathbf{K}(\theta)), d\pi(X(\theta))\big]$ is uniformly bounded below on $\theta \in \Sigma$. By (68) $\sphericalangle\big[dE_q(t\theta)\big(V(\theta)\cap T_\theta\Sigma\big), d\pi\big(X(\zeta_t\,\theta)\big)\big] > \sphericalangle\big[d\pi\big(\mathbf{K}(\zeta_t\,\theta)\big), d\pi(X(\zeta_t\,\theta))\big]$ is uniformly bounded below. This proves that $\|dE_q(t\theta)\|$ is uniformly bounded below on $t \in \mathbb{R}^+$, $\theta \in \Sigma$. Hence the map $E_q : T_q\tilde{M} \to \tilde{M}$ is a differentiable covering map. Since $\tilde{M}$ is simply connected, it must be a diffeomorphism.

Now let $\exp_q : T_q\tilde{M} \to \tilde{M}$ be the exponential map of (ψ_t, Σ). Since ζ_t is a reparametrization of ψ_t for $t > 0$, $\theta \in \Sigma$ there exists $\tau(t,\theta) > 0$ such that $\psi_t(\theta) = \zeta_{\tau(t,\theta)}(\theta)$. By corollary 1.18 $\exp_q$ is a local diffeomorphism. It remains to prove that it is bijective. Since $|\Theta(X)| = \frac{|X|}{|Y|}$ is bounded then $t \mapsto \tau(t,\theta)$ is a (bijective) homeomorphism of $\mathbb{R}^+$. This implies that the map $t\,\theta \mapsto \tau(t,\theta)\,\theta$, with $\theta \in T_q^*M\cap\Sigma$, is a homeomorphism of T_q^*M. Since E_q is bijective and $\exp_q(t\theta) = E_q\big(\tau(t,\theta)\,\theta\big)$, then $\exp_q$ is bijective.

□

APENDIX I.

I. Examples.

The following formalism for twisted Hamiltonians on surfaces was derived by G. Paternain and M. Paternain in [30]. Let M be a closed orientable riemannian surface with riemannian metric $\langle\ ,\ \rangle_x$. Let $K : TTM \to TM$ be the connection map $K\xi = \nabla_{\dot{x}} v$, where $\xi = \frac{d}{dt}\big(x(t), v(t)\big)$. Let $\pi : TM \to M$ be the canonical projection. Let ω_0 be the symplectic form in TM obtained by pulling back the canonical symplectic form via the Legendre transform associated to the riemannian metric, i.e.

$$\omega_0(\xi,\zeta) = \langle d\pi\,\xi\,,\,K\zeta\rangle - \langle d\pi\,\zeta\,,\,K\xi\rangle\,.$$

Denote by Ω the area 2-form on M. Given a smooth function $F : M \to \mathbb{R}$, define a new symplectic form ω_F on TM by

$$\omega_F = \omega_0 + (F\circ\pi)\,(\pi^*\Omega)\,.$$

This is called a *twisted symplectic structure* on TM (cf. [30]). Let $H : TM \to \mathbb{R}$ be the Hamiltonian

$$H(x,v) = \frac{1}{2}\langle v, v\rangle_x\,.$$

Consider the Hamiltonian vectorfield X_F corresponding to (H,ω_F), i.e.

$$\omega_F\big(X_F(\theta),\cdot\big) = dH\,. \tag{69}$$

Define $Y : TM \to TM$ as the bundle map such that

$$\Omega_x(u,v) = \langle Y(u), v\rangle_x\,. \tag{70}$$

Then $Y(u) = iu$ is the rotation of angle $+\frac{\pi}{2}$. From (69) and (70) we have that

$$dH_\theta(\xi) = \omega_0\big(X_F(\theta),\xi\big) + F\big(\pi(\theta)\big)\,\big\langle Y\big(d\pi\,X_F(\theta)\big), d\pi\xi\big\rangle_{\pi(\theta)}\,. \tag{71}$$

Hence if $\xi = (\xi_1,\xi_2) = (d\pi\,\xi, K\xi) \in H(\theta)\oplus V(\theta)$ and $X_F = (X_1,X_2) \in H(\theta)\oplus V(\theta)$, equation (71) becomes

$$\langle\theta,\xi_2\rangle = \langle X_1,\xi_2\rangle - \langle X_2,\xi_1\rangle + F\,\langle Y(X_1),\xi_1\rangle\,,$$

$$X_F(\theta) = \big(\theta, F\,Y(\theta)\big)\,. \tag{72}$$

In particular, the orbits of the Hamiltonian flow are of the form $\big(\gamma(t),\dot\gamma(t)\big)$ for $\gamma : \mathbb{R} \to M$, with

$$\frac{D}{dt}\,\dot\gamma = F(\gamma)\;Y(\dot\gamma)\,. \tag{73}$$

If $\int_M F\,\Omega = 0$ this flow can be seen as a Lagrangian flow as follows. The condition $\int_M F\,\Omega = 0$ implies that the cohomology class of $F\,\Omega$ is zero. Hence there exists a 1-form η on M such that $d\eta = F\,\Omega$. Consider the Lagrangian

$$L(x,v) = \frac{1}{2}\langle v, v\rangle_x + \eta_x(v)\,.$$

The corresponding Euler-Lagrange equation is

$$\frac{D}{dt}\,\langle \dot\gamma, \cdot\,\rangle = d\eta_x(\dot\gamma) = \langle F\,Y(\dot\gamma), \cdot\,\rangle\,.$$

which gives the same differential equation (73). This flow corresponds to a magnetic field with Lorentz force $F(x)\,Y(v)$.

Observe that the theory developed in the previous sections also applies to the case in which $\int_M F\,\Omega \neq 0$. Indeed, let $q : U \subseteq M \to \mathbb{R}^2$ be a local chart defined on a simply connected domain $U \subset M$. Then the 2-form $F\,\Omega$ is exact on U and there exists a 1-form $\eta|_U$ such that $d\eta = F\,\Omega$ on U. The Hamiltonian flow on $TU \subset TM$ is the Lagrangian flow of $L(x,v) = \frac{1}{2}\langle v,v\rangle_x + \eta_x(v)$, $x \in U$. Let (q,p) be the natural chart on $T^*U \subseteq T^*M$ corresponding to the chart q and let $\mathcal{F}^{-1}$ be the inverse of the Legendre transform corresponding to L. Let $(q,w) = \mathcal{F}^{-1}\circ(q,p)$. Since $\mathcal{F}^*(d\Theta) = \omega_F$, then (q,w) is a symplectic chart for ω_F and sends the vertical bundle of $T_{q(U)}\mathbb{R}^n$ to the vertical bundle of $T_U M$.

We derive the Jacobi equation. Let $(c(t), \dot c(t))$ be an orbit of the flow. Consider a variation $\gamma_s(t) = f(s,t)$, where $f(0,t) = c(t)$ and the paths $t \mapsto f(s,t)$ are solutions of (73):

$$\frac{D}{dt}\,\dot\gamma_s = F\,Y(\dot\gamma_s)\,. \tag{74}$$

Denote the variational field of f by $J(t) = \frac{\partial f}{\partial s}(0,t)$. Using the formula

$$\frac{D}{ds}\,\frac{D}{dt}\,\frac{\partial f}{\partial t} = \frac{D}{dt}\,\frac{D}{ds}\,\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\,.$$

and taking the covariant derivative of equation (74) in the direction $J(t)$, we obtain

$$\ddot J + R(\dot c, J)\,\dot c = \frac{D}{ds}\big[F\,Y(\dot\gamma_s)\big]\,.$$

Since $Y(v)$ is linear on the fibers of TM and $\dot J = \frac{D}{ds}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{\partial f}{\partial s}$, we have

$$\begin{aligned} \tfrac{D}{ds}\,FY(\dot\gamma_s) &= \big[\nabla_J(F\,Y)\big](\dot\gamma_s) + F\,Y(\dot J)\,, \\ &= (\nabla_J F)\,Y(\dot c) + F\,Y(\dot J)\,. \end{aligned}$$

Hence we obtain the Jacobi equation

$$\ddot J + R(\dot c, J)\,\dot c - (\nabla_J F)\,Y(\dot c) - F\,Y(\dot J) = 0\,. \tag{75}$$

Consider the orthonormal basis $\{\, \dot c,\ Y(\dot c) = i\,\dot c\,\}$ of $T_{c(t)}M$. We have that

$$\begin{aligned}
\tfrac{D}{dt}\,\dot c &= F\,Y(\dot c)\,,\\
\tfrac{D}{dt}\,Y(\dot c) &= (\nabla_{\dot c} Y)(\dot c) + Y\left(\tfrac{D}{dt}\,\dot c\right)\\
&= -F\,\dot c\,.
\end{aligned}$$

Write

$$J = x\,\dot c + y\,Y(\dot c)\,.$$

Then

$$\begin{aligned}
\dot J &= (\dot x - F\,y)\,\dot c + (\dot y + F\,x)\,Y(\dot c)\,,\\
\ddot J &= (\ddot x - 2\,F\,\dot y - \dot F\,y - F^2\,x)\,\dot c + (\ddot y + 2\,F\,\dot x + \dot F\,x - F^2\,y)\,Y(\dot c)\,,\\
F\,Y(\dot J) &= -F\,(\dot y + F\,x)\,\dot c + F\,(\dot x - F\,y)\,Y(\dot c)\,.
\end{aligned}$$

Replacing these formulas on equation (75) we obtain

$$\ddot x - (F\,y)' = 0\,, \tag{76}$$

$$\ddot y + K\,y + F\dot x = y\,(\nabla_{Y(\dot c)}F)\,, \tag{77}$$

where $K = \big\langle R\big(\dot c, Y(\dot c)\big)\,\dot c\,,\ Y(\dot c)\big\rangle$ is the sectional curvature along $c(t)$. Fix the energy level $\Sigma = H^{-1}\{\frac12\} = SM$. If $(J,\dot J) \in T_{\dot c(t)}\Sigma \subset H(\theta)\oplus V(\theta)$, we have that $dH(J,\dot J) = \langle \dot c, \dot J\rangle = 0$. Hence

$$\dot x - F\,y = 0\,. \tag{78}$$

This implies equation (76), and from (77) and (78), we get

$$\ddot y + \big(K + F^2 - \nabla_{Y(\dot c)}F\big)\,y = 0\,. \tag{79}$$

Hence the Jacobi equations on $T_{\dot c(t)}\Sigma$ are given by (78) and (79).

Example 1. *A convex Hamiltonian without conjugate points and no continuous invariant transversal bundle.*

Take $K \equiv -1$, $F \equiv 1$. Then equations (78) and (79) become

$$\begin{aligned}
\dot x &= y\,,\\
\ddot y &= 0\,.
\end{aligned} \tag{80}$$

The solutions have the form

$$\begin{aligned}
x(t) &= \tfrac12\,a\,t^2 + b\,t + c\,,\\
y(t) &= a\,t + b\,,\\
\dot y(t) &= a\,.
\end{aligned} \tag{81}$$

The derivative of the Hamiltonian flow ψ_t on $\big(c(t), \dot c(t)\big)$ is given by

$$\begin{aligned} d\psi_t\big(J(0), \dot J(0)\big) &= \big(J(t), \dot J(t)\big) \in H(\psi_t\theta) \oplus V(\psi_t\theta) \\ &= \big(x\,\dot c + y\,Y(\dot c)\,;\, 0\,\dot c + (\dot y + x)\,Y(\dot c)\big) \\ &= \big(x(t)\,\dot c + y(t)\,Y(\dot c)\,;\, z(t)\,Y(\dot c)\big)\,, \end{aligned} \tag{82}$$

where $z(t) = \dot y(t) + x(t)$ and $\theta = \big(c(0), \dot c(0)\big)$.

Every orbit $(c, \dot c)$ has no conjugate points because the only solution of (81) with $x(0) = y(0) = 0$ and $y(T) = 0 = x(T)$, $T \neq 0$ is $x(t) = y(t) \equiv 0$. The Green bundles coincide and are generated by $X(\theta) = (\dot c; -Y(\dot c))$ and $(Y(\dot c); \mathbf{0})$. The last vector can be found by solving $x(0) = 0$, $y(0) = 1$, $x(T) = y(T) = 0$ and letting $T \to \pm\infty$.

Now suppose that there exists an invariant continuous subbundle $\boldsymbol{N}(\theta) \subseteq T_\theta\Sigma$ which is transversal to the vectorfield $X(\theta)$. Let $\theta \in \Sigma$ be a recurrent point and let $t_n \underset{n}{\to} +\infty$ be such that $\psi_{t_n}(\theta) \underset{n}{\to} \theta$. Let $\xi \in \boldsymbol{N}(\theta) \setminus \{0\}$. If $d\psi_t(\xi) = \big(x(t), y(t); z(t)\big) = \big(J(t); \dot J(t)\big)$, then

$$\begin{aligned} \lim_{n \to \pm\infty} \frac{1}{|x(t_n)|}\, d\psi_{t_n}(\xi) &= \lim_n \frac{1}{x(t_n)}\, \big(x(t_n), y(t_n); z(t_n)\big) \\ &= (1, 0; 1) = X(\theta)\,. \end{aligned} \tag{83}$$

By the continuity of $\boldsymbol{N}$ we have

$$X(\theta) = \lim_n \frac{1}{x(t_n)} d\psi_t(\xi) \in \boldsymbol{N}(\theta)\,.$$

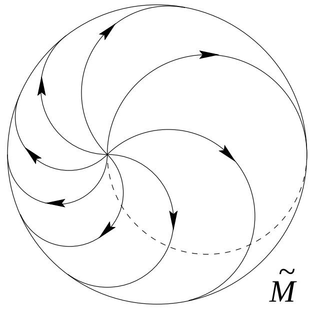
This contradicts the transversality hypothesis.

The same argument as in equation (83) shows that for every $\theta \in \Sigma$, the angle $\sphericalangle\Big(d\psi_t\big(V(\theta) \cap T_\theta\Sigma\big), X(\psi_t\theta)\Big) \underset{t\to+\infty}{\longrightarrow} 0$.

In this case the trajectories of the Hamiltonian flow are the unit tangent vectors of the horospheres of the riemannian metric, i.e.

$$\psi_t(\theta) = -Y\big(h_t(Y(\theta))\big)\,, \tag{84}$$

where h_t is the horocycle flow. In particular, the exponential map $\exp_q : T_q\tilde M \to \tilde M$ is a diffeomorphism for all $q \in M$. This flow has no periodic orbits because the horocycles $W^{uu} \subset \Sigma = T^1\tilde M/_{\pi_1(M)}$ are homeomorphic to $\mathbb{R}$.



EXAMPLE I.1

We prove (84). Let $\theta \in T_q\tilde{M}$, $\|\theta\| = 1$. Let $\phi_t : T\tilde{M} \hookleftarrow$ be the geodesic flow of $\tilde{M}$. Consider a sphere

$$S_T(\theta) = \big\{\, q \in \tilde{M} \mid d_{\tilde{M}}\big(q, \pi\phi_T(\theta)\big) = T \,\big\}$$

and a parametrization by arc length $\gamma_T(s)$ of $S_T(\theta)$ with $\gamma_T(0) = \pi(\theta)$. Then $\frac{D}{ds}\dot{\gamma}_T(s)$ is orthogonal to $\dot{\gamma}_T(s)$. Moreover, it has constant norm, because it is the image of $\frac{D}{ds}\dot{\gamma}_T(0)$ by the isometry given by a rotation about the center $\pi\phi_T(\theta)$ of $S_T(\theta)$. Therefore γ_T satisfies

$$\frac{D}{ds}\,\dot{\gamma}_T = \lambda_T\, Y(\dot{\gamma}_T)$$

for some constant $\lambda_T > 0$. Passing to the limit when $T \to +\infty$ we obtain that the reparamerizations by arc length γ of the horospheres satisfy

$$\tfrac{D}{ds}\,\dot{\gamma} = \mu\, Y(\dot{\gamma})\,, \tag{85}$$

with $\mu = \lim_{T\to+\infty} \lambda_T$, (in fact $T \to \lambda_T$ is strictly decreasing). We claim that $\mu = 1$. Indeed, the solutions of (85) of are closed circles if μ is big enough. The infimum μ_0 of such μ's corresponds to the horospheres . For $\mu > \mu_0$ the flow has conjugate points corresponding to the period (length) of these closed circles (see example 3). From the Jacobi equation (79) with $F = \mu$, we see that $\Sigma = H^{-1}\{\frac{1}{2}\}$ has conjugate points for $\mu > 1$ and the flow is Anosov for $\mu < 1$. Hence $\mu_0 = 1$.

Indeed, for $\mu < 1$ the solution of (78) and (79) are $y = A\,e^{\lambda t} + B\,e^{-\lambda t}$, $x = \mu(\frac{A}{\lambda}\,e^{\lambda t} - \frac{B}{\lambda}\,e^{-\lambda t} + C)$, with $\lambda = \sqrt{1-\mu^2}$. Writing $\xi = \big(x(0), y(0); \dot{x}(0), \dot{y}(0)\big)$ we have that $d\psi_t(\xi) = \big(x(t), y(t); \dot{x}(t), \dot{y}(t)\big)$. For $A \neq 0$ the function $t \mapsto \|d\psi_t(\xi)\|$ has exponential growth and for $B \neq 0$, $t \mapsto \|d\psi_{-t}(\xi)\|$ has exponential growth. As in section §3, this implies that the flow is Anosov for $\mu < 1$. For $\mu > 1$, the Jacobi equations (78), (79) exhibit conjugate points, by the same arguments as for equations (87) and (88).

Example 2. *A periodic orbit without conjugate points and no transversal invariant subspace.*

We construct a convex Lagrangian which has the Jacobi equations (80) on a periodic orbit. Let $\theta \in \Sigma$ be such a periodic point with period $\tau > 0$. Changing the coordinates $(x, y; z)$ on $T_\theta\Sigma$ of equation (82) to $(x, y; \dot{y}) = (x, y; z - x)$, the matrix of the derivative $d\psi_\tau(\theta)$ is

$$d\psi_\tau(\theta) = \begin{array}{cc} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{bmatrix} & \begin{array}{l} \}\, x \\ \}\, y \\ \}\, \dot{y} \end{array} \end{array} \qquad \approx \qquad \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \quad \text{in Jordan form.}$$

Let $\{e_1, e_2, e_3\}$ be the basis of $T_\theta\Sigma$ in which $d\psi_\tau(\theta)$ is written in canonical form. The transformations $d\psi_\tau$ and

$$d\psi_\tau - I = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

have the same invariant subspaces. The vectorfield corresponds to the unique eigenspace $\langle e_1 \rangle$ and the only invariant subspaces are $\langle e_1 \rangle = \langle X(\theta) \rangle$, $\langle e_1, e_2 \rangle = \mathbb{E}(\theta) = \mathbb{F}(\theta)$ and $\langle e_1, e_2, e_3 \rangle = T_\theta\Sigma$. None of them is transversal to $X(\theta) = e_1$.

Now we make the construction. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/_{5\mathbb{Z}^2}$ with the flat metric and polar coordinates (r, θ) centered at $(0, 0) \in \mathbb{R}^2/_{5\mathbb{Z}^2}$ defined on $r \leq 2$. Let $F : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞ function such that $F(r, \theta) = F(r)$, $F(r = 1) = 1$, $\frac{\partial F}{\partial r}\big|_{r=1} = -1$, and $\int_{\mathbb{T}^2} F\, \Omega = 0$. Let η be a 1-form in $\mathbb{T}^2$ such that $d\eta = F\,\Omega$. Define $L(x, v) = \frac{1}{2}\langle v, v \rangle_x + \eta_x(v)$. At $r = 1$, the Euler-Lagrange equation (73) is

$$\tfrac{D}{dt}\,\dot{\gamma} = F(\gamma)\, Y(\dot{\gamma}) = Y(\dot{\gamma}) \qquad \text{at } r = 1\,.$$

Hence $r = 1$, $\dot{\theta} = -1$ is a periodic solution. Moreover, at $r = 1$ we have that $F = 1$, $\nabla_{Y(\dot{c})}F = -\frac{\partial F}{\partial r} = +1$. Hence the Jacobi equations (78), (79) are the same as (80).

Example 3. *A segment without conjugate points with crossing solutions.*

Let $W = \left\{\, (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leq 4 \,\right\}$ and consider the Lagrangian

$$L(x, v) = \frac{1}{2}\,\langle v, v \rangle + \eta_x(v)\,,$$

where $\langle\ ,\ \rangle$ is the euclidean metric and $d\eta = -\Omega$. This Lagrangian corresponds to $F|_W = -1$ in the formalism above and can be embedded into a convex Lagrangian on a closed surface. The Euler-Lagrange equation (73) is

$$\dot{v} = -i\, v\,. \tag{86}$$

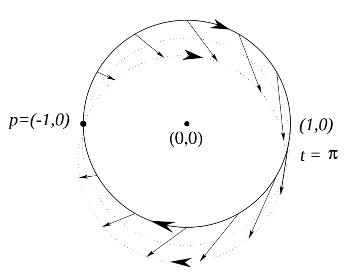
The solutions of (86) on the energy level $|v| = 1$, are all the circles contained in W with constant angular velocity $\dot\theta = -1$. Fix $\mathcal{C}$: $R(t) = 1$, $\theta(t) = \pi - t$ in polar coordinates in W with center $(0,0)$. Let $\vartheta = (0,1)_{(-1,0)}$ be the initial vector of this solution. Then the circles which are rotations of $\mathcal{C}$ fixing the initial point $p = (-1,0)$ are also solutions of (86) (see figure I.3) with the same energy $E = \frac{1}{2}\,|v| = \frac{1}{2}$. The solution $\mathcal{C}$ has conjugate points, but the conjugate points appear at $t = 2\pi$. In particular the segment $\left\{\,\psi_t(\vartheta)\,\middle|\,0 \le t \le \frac{3\pi}{2}\,\right\}$ has no conjugate points. But the intersections with other solutions starting at $p = (-1,0)$ appear at $t < \frac{3}{2}\pi$ and at $t = \pi$ there is a vector $\xi \in V(\vartheta) \cap T_\vartheta\Sigma$ such that $d\pi\,(d\psi_t\,\xi) = d\pi\,X(\psi_t\vartheta) = (0,-1)$. Proposition 1.16 says that these phenomena can not occur if all the positive (negative) orbit of ϑ has no conjugate points. The claims stated above can be proved using the Jacobi equations (78) and (79):

$$\dot x + y = 0\,, \qquad \ddot y + y = 0\,, \tag{87}$$

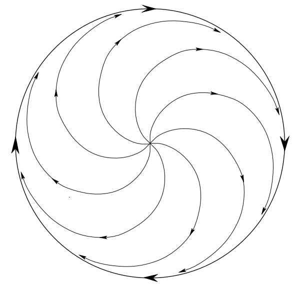
whose solutions are

$$\begin{aligned} x(t) &= a\,\cos t + b\,\sin t + c\,,\\ y(t) &= a\,\sin t - b\,\cos t\,. \end{aligned} \tag{88}$$

Then if $x(0) = y(0) = 0$ and $\dot y(0) \ne 0$ then $\big(x(t), y(t)\big) \ne 0$ for all $0 < t < 2\pi$. Moreover, if $a = -1$, $b = 0$, $c = +1$, then $x(0) = y(0) = 0$, $y(\pi) = 0$, $x(\pi) = +2$.



EXAMPLE I.3



EXAMPLE I.4

Example 4. *A non surjective exponential map without conjugate points.*

Consider the flat torus $\mathbb{T}^2 = \mathbb{R}^2/_{6\mathbb{Z}} = [-3,3]^2/_{6\mathbb{Z}}$ and polar coordinates (r,θ) on $\left\{\,(x,y)\in\mathbb{T}^2\,\middle|\,x^2+y^2<3\,\right\}$ centered at $\mathbf{0} = (0,0)$. Define the Lagrangian

$$L(x,v) = \tfrac{1}{4}\,\langle v,v\rangle_x + \eta_x(v) + \phi(x)\,,$$

where ϕ is a C^∞ function on $\mathbb{T}^2$ such that $\phi(r,\theta) = \frac{1}{4}r^2 + \cos\pi r$ on $r \le 2$ and $\phi(r,\theta) \ge 1$ outside $r \le 2$; and $\eta(r,\theta) = -\frac{1}{2}\,r^2\,f(r)\,d\theta$ with $f(r)$ a C^∞ function with support on $r \le \frac{5}{2}$ such that $f(r) = 1$ on $r \le 2$ and $f(r) \le 1$.

Observe that the path $\mathcal{C} : r(t) \equiv 1$, $\theta(t) = t$ is *static* (cf. Mañé [25]). Indeed, the Lagrangian

$$L(r, \theta; \dot{r}, \dot{\theta}) = \tfrac{1}{4}\,(\dot{r}^2 + r^2\,\dot{\theta}^2) - \tfrac{1}{2}\,r^2\,\dot{\theta} + \tfrac{1}{4}\,r^2 + \cos \pi r \qquad \text{on } r \leq 2\,,$$

is minimized on $\dot{r} = 0$, $\dot{\theta} = 1$, $r = 1$, with $L = -1$; and on $r \geq 2$, we have that $\phi(r) \geq 1$ and $f \leq 1$, so that

$$\tfrac{1}{4}\,r^2\,\dot{\theta}^2 + \eta(v) \geq r^2\,\left(\tfrac{1}{4}\,\dot{\theta}^2 - \tfrac{1}{2}\,\dot{\theta}\right) \geq -\tfrac{1}{4}\,r^2 \geq -1$$

Hence $L \geq \phi(r) \geq 1$ on $r \geq 2$.

By theorem VII in Mañé [25] (see also [8]), there exists a semistatic (hence foward minimizing) solution $\gamma(t)$ such that $\gamma(0) = (0, 0)$ and the ω-limit of $(\gamma, \dot{\gamma})$ is the (hyperbolic) closed orbit $\mathcal{C}$. By the rotational symmetry of the Lagrangian, for every 1-dimensional subspace $\mathcal{L} \subset T_{(0,0)}\mathbb{T}^2$ we obtain a semistatic curve γ, which is tangent to $\mathcal{L}$, with $\gamma_{\mathcal{L}}(0) = (0, 0)$ and ω-limit$(\gamma_{\mathcal{L}}, \dot{\gamma}_{\mathcal{L}}) = \mathcal{C}$. Since the semistatic solutions are minimizing, none of them $\gamma_{\mathcal{L}}|_{[0,+\infty[}$ have conjugate points. Moreover, by theorem X in Mañé [25] (also [8]), $\mathcal{C}$ and all the $(\gamma_{\mathcal{L}}, \dot{\gamma}_{\mathcal{L}})$'s are in the same energy level $E = c(L)$. By the graph property (theorem VII, Mañé [25] and also [8]), the $\gamma_{\mathcal{L}}|_{]0,+\infty[}$'s don't intersect $\pi(\mathcal{C})$ neither intersect each other.

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