THE SPACING DISTRIBUTIONS BETWEEN ZEROS OF ZETA FUNCTIONS

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0. Introduction

In a remarkable numerical experiment, Odlyzko [Od] has found that the local spacing distribution between the zeros of the Riemann Zeta function is modelled by the eigenvalue distributions coming from random matrix theory. In particular by the "GUE" (Gaussian Unitary Ensemble) model [Gau]. His experiment was inspired by the paper of Montgomery [Mon] who determined the pair correlation distribution for the zeros (in a restricted range). We will refer to the above phenomenon as the Montgomery–Odlyzko Law. Rudnick and Sarnak [Ru–Sa] have determined the $n \geq 2$ correlations for the zeros of the zeta function, as well as for more general automorphic L–functions (again only in restricted ranges). These are in perfect agreement with GUE predictions. It appears that the Montgomery–Odlyzko Law is a universal feature for such L–functions. However, a complete proof of this law is well beyond the range of existing techniques. If one believes that the above phenomenon is a manifestation of the spectral nature of the zeros, then it is natural to ask if there is such a law for the zeta and L–functions associated to curves and exponential sums over finite fields. For, in these cases, their zeros may be realized as eigenvalues of Frobenius on cohomology groups (see Section 7). One of the goals of this paper is the formulation and proof of an analogue of the Montgomery–Odlyzko Law for these zeta and L–functions.

A major part of this work is devoted to an analysis of the distribution of the local spacings between the eigenvalues of matrices in the classical groups. Let $G(N)$ denote one of the compact groups $U(N)$, $SU(N)$, $O(N)$, $SO(N)$, $USp(N)$, i.e. the unitary, orthogonal and symplectic groups (Weyl) [We]) realized in their standard representations as $N \times N$ unitary matrices (N being even in the symplectic case). Denote by A a typical element of $G(N)$ and by dA the unit normalized Haar measure on $G(N)$. The eigenvalues of A are all on the unit circle and may be ordered cyclically counterclockwise

$$z_1 \to z_2 \ldots\ldots \to z_N \to z_1 \,. \tag{0.1}$$

Set $z_j = z_{j \bmod N}$ for $j \in \mathbb{Z}$. We are interested in the arc length spacings, $\overline{z_j z_{j+1}}$, between z_j and z_{j+1}, or more generally between z_j and z_{j+a} where $a \geq 1$ is an integer. Define for $j \in \mathbb{Z}$

$$\Delta_{j,a} = \frac{N}{2\pi} \quad \overline{z_{j+a} z_j} \,. \tag{0.2}$$

The mean value of $\Delta_{j,a}$ satisfies

$$\frac{1}{N} \quad \sum_{j=1}^{N} \quad \Delta_{j,a} = a \,. \tag{0.3}$$

The a^{th} consecutive spacing distribution is described by the probability measure on $\mathbb{R}_{\geq 0}$;

$$\mu_a(A) = \frac{1}{N} \quad \sum_{j=1}^{N} \delta_{\Delta_{j,a}} \,. \tag{0.4}$$

Here δ_ζ is the point mass at $\zeta \in \mathbb{R}$. In view of (0.3), $\mu_a(A)$ has mean a. Note that $\mu_a(A)(I)$ gives the proportion of $\Delta_{j,a}(A)$'s lying in I, for any interval I. For any function f on $G(N)$ its expectation over $G(N)$, denoted $E(f(A), G(N))$ or simply $E(f)$, is by definition

$$E(f) = \int_{G(N)} f(A) dA \,.$$

Our first result asserts that the expectations of $\mu_a(A)$ have universal limits (that is independent of the particular family $G(N)$).

THEOREM 0.1. *For each $a \geq 1$ there is a measure μ_a on $\mathbb{R}_{\geq 0}$ of mean a, such that*

$$\lim_{N \to \infty} \quad E\Big(\mu_a(A),\, G(N)\Big) = \mu_a \,.$$

The proof of this theorem which goes via n–level correlations, provides an explicit formula for the density of μ_a (see Proposition 3, section 5) which can be expressed in terms of a Fredholm determinant:

$$d\mu_a = \left(\frac{d^2}{ds^2} \sum_{j=0}^{a-1} \frac{a-j}{j!} \left(\frac{\partial}{\partial T} \right)^j \det(I + TK_s))_{T=-1} \right) ds \tag{0.5}$$

where K_s is the trace class operator on $L^2[-s/2, s/2]$ whose kernel is

$$K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{0.6}$$

(0.6) allows us to identify μ_a with the conditional spacing probabilities computed by Gaudin for GUE [Gau] (the case of Theorem 0.1 with $G(N) = U(N)$ is, essentially due to Gaudin). Incidentally μ_1 is quite well approximated by "Wigner's surmise" [Wi],

$$\frac{32}{\pi^2} s^2 e^{-4s^2/\pi} \qquad \text{(see figure 1)} . \tag{0.7}$$

The identification (0.5) is important to our analysis, for with it we establish an important technical estimate for the tail of μ_1:

$$\mu_1[s,\infty) \leq \frac{4}{3}\, e^{-s^2/8}, \qquad \text{for } s \geq 0 . \tag{0.8}$$

In passing, we note that a complete asymptotic expansion as $s \to \infty$ of the Fredholm determinant appearing in (0.5) has been recently developed in [T–W] and [D–]. As was noted by Gaudin [Gau], (0.5) may be used to compute μ_a numerically (see also section 8). In figures 1, 2, 4, 5, 6, and 7 the densities of μ_1 and μ_2 are drawn together with the empirical spacing distribution for the zeros of the Riemann Zeta function as computed by Odlyzko.

For our applications, it is not only the average of $\mu_a(A)$ that is of interest, but also the individual behavior of $\mu_a(A)$ for almost all A (N large). That is we seek a law of large numbers. To this end let $D(\nu,\mu)$ denote the Kolmogorov–Smirnov discrepancy between two measures ν, μ on $\mathbb{R}$, that is

$$D(\nu,\mu) = \sup_{-\infty<x<\infty} \{|\mu(-\infty,x] - \nu(-\infty,x]|\} . \tag{0.9}$$

We are interested in $D(\mu_a(A),\mu_a)$. D measures how well $\mu_a(A)$ approximates μ_a. Clearly, $0 \leq D \leq 1$ and tends to zero iff the $\Delta_{j,a}(A)$'s become equidistributed w.r.t. μ_a, as $N \to \infty$. The following yields the behavior of $\mu_a(A)$ for almost all A.

THEOREM 0.2. *For $\epsilon > 0$ and $a \geq 1$ there is an explicit $N_0 = N_0(\epsilon, a)$ such that for $N \geq N_0$,*

$$E(D(\mu_a(A),\mu_a),\ G(N)) \leq N^{-\frac{1}{6}+\epsilon} .$$

An immediate consequence of this is that for N large ($\to \infty$) almost all A's have their eigenvalue spacings follow the Gaudin GUE distributions. For example, if $\alpha + \beta < 1/6$ and N is sufficiently large, then Haar $\{A \in G(N) | D(\mu_a(A),\mu_a) \geq N^{-\alpha}\} \leq N^{-\beta}$.

REMARK. The exponent 1/6 in Theorem 0.2 is certainly not sharp. In fact S. Miller has carried out some Monte–Carlo simulations with random elements of $G(N)$ with N of size about 500. He finds that as N gets large, $N^{1/2}D(\mu_1(A), \mu_1)$ $(A \in G(N))$ has a universal limiting law which is depicted in Figure [10] (where it is compared with the usual Kolmogorov law for the Kolmogorov statistic). The same statistic computed with consecutive blocks of sizes about 200 of zeros of the Riemann zeta function at height 10^{20} (courtesy of Odlyzko [Od]) gives the same law–see Figure [11]. Recently, Soshnikov [Sos] has made a first step towards establishing that $N^{1/2}D(\mu_1(A), \mu_1)$ has a limiting distribution as $N \to \infty$.

We turn to the applications mentioned at the outset. Let $\mathbb{F}_q$ be the field with q elements. The extensions $\mathbb{F}_{q^m}$, $m \geq 1$, of $\mathbb{F}_q$ exhaust the algebraic extensions of $\mathbb{F}_q$. Let C be a smooth projective curve defined over $\mathbb{F}_q$. For example C may be a plane curve

$$C: \qquad f(x_1, x_2, x_3) = 0 \tag{0.13}$$

where f is a nonsingular homogeneous polynomial with coefficients in $\mathbb{F}_q$. In analogy with the Riemann Zeta Function, Artin [Ar], introduced a zeta function $\zeta(C/\mathbb{F}_q, T)$ which may be defined by

$$\zeta(C/\mathbb{F}_q, T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m T^m}{m}\right) \tag{0.14}$$

where $N_m = \#C(\mathbb{F}_{q^m})$, that is the number of points of C with coordinates in $\mathbb{F}_{q^m}$ (see section 6, (11)), for an equivalent definition which makes the analogy with the Riemann zeta function more transparent.) It follows from Riemann–Roch Theorem on C (see [Sch]) that this zeta function is rational and is of the form

$$\zeta(C/\mathbb{F}_q, T) = \frac{P(C/\mathbb{F}_q, T)}{(1-qT)(1-T)} \tag{0.15}$$

where P is an integral, self reciprocal* polynomial of degree $2N, N$ being the genus of C. The Riemann hypothesis for $\zeta(C/\mathbb{F}_q, T)$ lies deeper and asserts that the zeros of $P(C/\mathbb{F}_q, T)$ all lie on the circle $|T| = 1/\sqrt{q}$. This was proven by Weil [WEI]. So we may write the zeros ξ_j of P as

$$\xi_j = q^{-1/2} z_j \quad , \quad j = 1, \ldots, 2N \tag{0.16}$$

with $|z_j| = 1$.

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*$a_0 + a_1 T + \cdots + a_\nu T^\nu$ is self reciprocal if $a_j = a_{\nu - j, j=1,\ldots,\nu}$.

Proceeding now as in (0.1), (0.2), (0.3), (0.4), we get spacing measures $\mu_a(C/\mathbb{F}_q)$ for any $a \in \mathbb{N}$. In order to obtain a limiting behavior for the individual spacing measures $\mu_a(C/\mathbb{F}_q)$ We must let the genus N of C, go to infinity. This alone will not suffice to get any reasonable behavior because one can easily show, using the equidistribution result (0.19) below, that we can obtain any probability measure with mean at most equal to a as a limit of $\mu_a(C/\mathbb{F}_q)$'s.

Given this, one natural question to ask is what does $\mu(C/\mathbb{F}_q)$ look like for the typical curve of large genus? To answer this, let $\mathcal{M}_N(\mathbb{F}_q)$ denote the set of isomorphism classes of curves of genus N defined over $\mathbb{F}_q$. This is a finite set whose cardinality goes to infinity if either of N or q go to infinity. Our main result is the following:

THEOREM 0.3. *Given $a \in \mathbb{N}$ we have*

$$\lim_{N\to\infty} \lim_{q\to\infty} \frac{1}{|\mathcal{M}_N(\mathbb{F}_q)|} \sum_{C\in\mathcal{M}_N(\mathbb{F}_q)} D(\mu_a(C/\mathbb{F}_q), \mu_a) = 0 \, .$$

As a consequence, we see that the typical curve over a large field and of large genus will have the local spacing distributions of the zeros of its zeta function, follow the universal μ_a's of GUE. That is the Montgomery–Odlyzko law is valid for the typical such Artin Zeta function.

The connection between Theorems 0.2 and 0.3 comes through monodromy (see Section 7). The monodromy group for a family of curves related to $\mathcal{M}_N(\mathbb{F}_q)$, is the full symplectic group $Sp(2N)$ (see Section 7). It allows one to associate to each curve $C/\mathbb{F}_q$ a Frobenius conjugacy class $\theta(C/\mathbb{F}_q)$ in $Sp(2N)$. In this case this is the conjugacy class in $Sp(2N,\mathbb{C})$ whose characteristic polynomial is $P(C/\mathbb{F}_q, q^{-1/2}T)$. That the latter is the characteristic polynomial of a symplectic matrix, is a consequence of P being self reciprocal. The Riemann Hypothesis for $\zeta(C/\mathbb{F}_q, T)$ ensures that this is a conjugacy class meets $USp(2N)$. In Section 7 we explain how to use Deligne's main Theorem of "Weil II" [DE] to establish the following Chebotarev equidistribution theorem:

Let f be a continuous class function on $USp(2N)$, then

$$\lim_{|\mathbb{F}_q|\to\infty} \frac{\sum\limits_{C\in\mathcal{M}_N(\mathbb{F}_q)} |\text{aut } (C/\mathbb{F}_q)|^{-1} f(\theta(C/\mathbb{F}_q))}{\sum\limits_{C\in\mathcal{M}_N(\mathbb{F}_q)} |\text{aut } (C/\mathbb{F}_q)|^{-1}} = \int\limits_{USp(2N)} f(A)dA \qquad (0.17)$$

where aut $(C/\mathbb{F}_q)$ is the number of automorphisms of the curve C over $\mathbb{F}_q$.

Theorem (0.3) follows by combining (0.17) with Theorem (0.2), applied with

$$f(A) = D(\mu_a(A), \mu_a)\ . \tag{0.20}$$

The other application concerns similar questions for L–functions arising from exponential sums. We consider the case of Kloosterman sums. For $n \geq 2$, q as above, $t \in \mathbb{F}_q^*$ and ψ a nontrivial additive character of $\mathbb{F}_q$, define

$$K\ell_\psi^{(n)}(t,q) = \sum_{\substack{x_1 x_2 \cdots x_n = t \\ x_j \in \mathbb{F}_q^*}} \psi(x_1 + x_2 + \cdots + x_n)\ . \tag{0.19}$$

To this sum we associate the $L-$ function

$$L(\psi,t,q,T) = \exp\left((-1)^n \sum_{m=1}^{\infty} \frac{KL_{\psi_0 TR}(t,q^m)T^m}{m}\right) \tag{0.20}$$

where

$$K\ell_{\psi_0 TR}(t,q^m) = \sum_{\substack{x_1 x_2 \cdots x_n = t \\ x_j \in \mathbb{F}_{q^m}^*}} \psi(TR_{\mathbb{F}_q^m/\mathbb{F}_q}(x_1 + \cdots + x_n)) \tag{0.21}$$

$L(\psi,t,q,T)$ is a polynomial of degree n (in the variable T) [ ] and Deligne [Del] has proven that it satisfies the Riemann Hypothesis: that is all its zeros are of absolute value $q^{-(n-1)/2}$. After normalizing the zeros as in (0.1) to (0.4) we define the measures $\mu_a(K\ell\psi^{(n)}(t,q))$. These measure the local spacings between the zeros of such an L–function, $L(\psi,t,q,T)$.

THEOREM 0.4. *For any $a \in \mathbb{N}$,*

$$\lim_{n\to\infty} \lim_{q\to\infty} \frac{1}{q-1} \sum_{t\in\mathbb{F}_q^*} D(\mu_a(K\ell_\psi^{(n)}(t,q)), \mu_a) = 0\ .$$

So again we see that for q and n large, the typical Kloosterman $L-$ function satisfies the Montgomery–Odlyzko Law. This time the analysis leading to Theorem 0.4 involves all the families of groups in Theorem 0.2. The T–polynomial $L(\psi,t,q,Tq^{-(n-1)/2})$ is naturally the characteristic polynomial of the Frobenius conjugacy class $\theta(K\ell_n(t,q))$ in the geometric monodromy group G_{geom}, of the corresponding Kloosterman sheaf [Ka]. These have been determined in [Ka]. For $n \geq 2$ even, $G_{\text{geom}} = Sp(n)$ (so that the

Frobenius is in $G_n = USp(n)$). For $n \geq 3$ odd and $q = p$ an odd prime $G_{\text{geom}} = SL(n)$ (so that the Frobenius is in $G_n = SU(n)$). Finally if $q = 2$ and n is odd ($n > 7$), $G_{\text{geom}} = SO(n)$. The equidistribution Theorem established in Katz [Ka] asserts that if f is a continuous class function on G_n as above, then

$$\lim_{q\to\infty} \frac{1}{q-1} \sum_{t\in\mathbb{F}_q^*} f(\theta(K\ell_n(t,q))) = \int_{G_n} f(A)dA\,. \tag{0.22}$$

Again applying (0.24) with f as in (0.18), together with Theorem 0.2, yields Theorem 0.4.

We next outline the contents of this paper. The first five sections are concerned with the proofs of Theorems 0.1 and 0.2. In more detail, Section 1 deals with some elementary reductions. The spacing distributions $\mu_a(A)$ as defined in (0.4) are not the most convenient to deal with analytically. We introduce close relatives of these which we call the naive spacings, $\mu_a(A,\ \text{naive})$. The analogues of Theorems 0.1 and 0.2 for the latter are then established in the later sections. With section 2, the proof begin in earnest. To compute and estimate expectations and variances of class functions on these classical groups, we make use of the explicit Weyl integration formulae. These are recalled in Section 2. Using the method of orthogonal polynomials, which was introduced into calculations of this type by Gaudin [Gau] and Mehta [Meh], we give explicit formulae in terms of determinants formed out of projection kernels (which turn out to be simple trigonometric functions) for the Haar measure when any number of the variables in the Weyl formula are integrated out. These are summarized in (2.25) and serve as a basis for all further calculations. Section 3 is devoted to the analysis of certain random variables Z on $G(N)$, called n–level correlation functions. These measure the distribution in $\mathbb{R}^n$ of subsets of size n, of the N eigenvalues of an $A \epsilon G(N)$. The key result of this Section is Theorem 2 which describes the behavior as $N \to \infty$ of the expectation and variance of Z. Part 1 of Theorem 2 (Section 3) shows that the limit of the expectations of the Z's does not depend on the particular family $G(N)$. This is responsible for the universality in Theorem 0.1. Once we have control over the Z's we attack the measures $\mu_a(A,\ \text{naive})$ using combinatorial inclusion/exclusion. This is carried out in Section 4. Corollary 3 of Section 4, gives the precise relation between the Z's and the μ_a's while Lemma 4 gives suitable inequalities for a truncation of this relation. The proofs of Theorem 0.1 and 0.2 are then carried out in Section 4. However, along the way sharp estimates on the size of the tails of the measures μ_a (univ) are needed (see 4.41). The derivation of these estimates is carried out in Section 5. This is achieved by expressing μ_a (univ) in terms of Fredholm determinants, see (5.18). (This also allows us to identify μ_a (univ) with Gaudin's "GUE" spacing measures [Gau]. These Fredholm determinants are scaling limits of finite Fredholm determinants associated with the families $G(N)$. In particular, there is a remarkable relation between the determinants associated with $U(2N-1))$, $SO(2N)$ and $USp(2N-2)$ (see Corollary 6 of Section 5). With this relation and a direct estimation of the determinant, which is possible for the $SO(2N)$ case (see Proposition 7 of section 5) we eventually obtain the sharp bounds for the tails of the μ_a (univ), completing the proof of Theorems 0.1 and 0.2. In the supplement to Section 5, we also investigate some nonuniversal statistics. For $j \geq 1$, we consider the distribution of the distance along the unit circle to the point $z = 1$, of the j^{th} eigenvalue of A, as A varies over $G(N)$. Renormalizing (see 5.2) this distance and letting N go to infinity, we obtain certain limiting probability measures on $\mathbb{R}_{\geq 0}$, denoted ν_j and $\nu_{\pm,j}$. The ν_j being the $U(N)$ limit while $\nu_{\pm,j}$ are the $SO(2N)$ and $USp(N)$ limits, respectively. These measures can be expressed in terms of Fredholm determinants (see Corollary 2) which allow us to compute their densities numerically – see Figures 1, 2, and 3. Being sensitive to the particular family $G(N)$, these allow for some interesting applications which we describe below. Their means are not equal to j so we denote the rescaling of these which have mean j by $\widetilde{\nu_j}$ and $\widetilde{\nu_{\pm,j}}$.

Sections 6 and 7 are devoted to the proof of Theorem 0.3 and, in particular, of (0.17). In Section 6, we discuss some Theorems concerning the equidistribution of Frobenius under an ℓ–adic representation ρ of the fundamental group of a scheme over a finite field k. These results are essentially due to Deligne [Del]. The key ingredients in the proof are the Lefshetz Trace Formula (section 6, (3)) and Deligne's main result in "Weil II" (see section 6, after (3)) which concerns the size of the eigenvalues of Frobenius acting on cohomology. A combination of these, leads after some analysis, to an estimation of "Weyl sums" measuring the equidistribution of the Frobenius conjugacy classes in the geometric monodromy group G_{geom} (section 6, top of page 4) associated with ρ. A uniform version of this (which allows us to vary the field k and, in particular, its characteristic) is given in Theorem 6.7. In Section 7, we apply the above abstract theorem to specific geometric examples. We begin with families of hyperelliptic curves over $k = \mathbb{F}_q$. These give rise to a monodromy representation ρ of π_1 of the parameter space (whose fibres are the hyperelliptic curves of a given genus g). We describe in some detail the computation of G_{geom} (for such a ρ), which turns out to be $Sp(2g)$. For certain one parameter families of such hyperelliptics, we obtain entirely effective bounds for the Weyl equidistribution sums (see section 7, $(3')$). Now (0.17) is concerned with the universal family $\mathcal{M}_g(\mathbb{F}_q)$ of curves of genus g over $\mathbb{F}_q$. Since this family is not representable, we make use of the tri–canonical structure and the space $\mathcal{M}_{g,3K}$ to set up the appropriate monodromy representation ρ. This leads naturally to counting curves weighted by the inverse of the order of their automorphism groups, as is done in (0.17). Since most curves of genus $g \geq 3$ over $\mathbb{F}_q$ (q large)have no automorphisms (0.17) leads to Theorem 0.3. The precise connection between the zeta function $\zeta(C/\mathbb{F}_q, T)$ and the Frobenius class under ρ is described in interlude page 5 of section 6. We note that in our analysis we have not determined explicitly the bounds on the equidistribution sums leading to (0.17).

We end the introduction with some further results along these lines which use Theorem 0.2, and the analysis in Section 5. We also put forth some conjectures which suggest themselves from this work. Theorem 0.3 tells us what happens to most curves of large genus and over a large $\mathbb{F}_q$. One may ask what happens if we choose a curve C of large genus g defined over Q, and ask the same question about the zeros of $\zeta(C/\mathbb{F}_p, T)$ for almost all primes p.

To answer this, we need to assume some standard conjectures. Let C/Q be such a curve. For ℓ a prime we obtain via the action of $GAL(\overline{Q}/Q)$ on $H^1(C, Q_\ell)$ (or equivalently on the Tate module $T_\ell(J)$ of the Jacobian J of C) a $2g$ dimensional ℓ–adic representation ρ of $GAL(\overline{Q}/Q)$. The image $\rho(GAL(\overline{Q}/Q))$ lies in $G = SpS(2g, Q_\ell)$ the group of symplectic similitudes with ℓ–adic coefficients. If this image is open in G, we will say C is general. Explicit examples of such C's of large genus are given in Appendix . In this case, it is conjectured (this being the generalization of the "Sato–Tate" conjectures [Sa], [Ta]) that the unitarized Frobenius conjugacy classes $\theta(C/\mathbb{F}_p)$ become equidistributed in $USp(2g)$ as $p \to \infty$. Precisely this means that for any continuous class function f on $USp(2g)$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p<x}{}' f(\theta(C/\mathbb{F}_p)) = \int_{USp(2g)} f(A)dA \tag{0.23}$$

Here $'$ denotes omit the primes p for which C has bad reduction $\bmod p$ and $\pi(x)$ is the number of primes less than x. Combining (0.25) and Theorem 0.2, we obtain

THEOREM 0.5. *Assume the Sato–Tate conjectures described above. Let C_N be a sequence of curves of genus g_N and which are of general type. If $g_N \to \infty$ as $N \to \infty$, then for any $a \in \mathbb{N}$*

$$\lim_{N\to\infty} \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x}{}' D(\mu_a(C_N/\mathbb{F}_p), \mu_a(univ)) = 0$$

Hence, for such a C_N of large genus, $\zeta(C_N/\mathbb{F}_p, T)$ satisfies GUE statistics (for the spacings of its zeros) for most primes p.

In the above it is crucial that some assumption like C_N being of general type be made. To see this consider the example of the sequence of modular curves $C_N = X_0(N)$ (see [ ] for definitions). We will assume for simplicity that N is prime. As a complex curve, $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ carries a basis $f_1, f_2, \ldots, f_g$ ($g =$ genus(C_N)) of Hecke eigenforms of weight 2. Since N is prime, these are all new forms. By Eichler–Shumura [EI], [SH] and Igusa [Ig] we have that for $p \dagger N$ the eigenvalues of $\theta(X_0(N)/\mathbb{F}_p)$ are of the form $(\theta_1(p), \theta_2(p), \ldots, \theta_g(p)) \in [0.\pi]^g$ where $\lambda_j(p) = 2\sqrt{p}\cos\theta_j(p)$ is the eigenvalue of the Hecke operator T_p on f_j. The image of the ℓ–adic representation ρ of $GAL(\bar{Q}/Q)$ on $H^1(X_0(N), \bar{Q}_L)$ is contained in the algebraic group G, where G is the subgroup of $x\epsilon(GL(2))^g$ with $x = (x_1, \ldots, x_g)$ and $\det(x_j) = \det(x_K), j \neq k$. Here ℓ is any prime different from N. It follows from results of Ribet [Ri] that the image of ρ is Zariski dense in G. Correspondingly, the "Sato–Tate" conjecture for $X_0(N), N$ prime, asserts that for any continuous f on $[0,\pi]^g$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{\substack{p\le x\\ p\dagger N}} f(\theta_1(p), \ldots, \theta_g(p)) = \int\limits_{[0,\pi]^g} f(\theta)\frac{2\sin^2\theta_1}{\pi}d\theta_1 \ldots \frac{2\sin^2\theta_g}{\pi}d\theta_g\,. \qquad (0.24)$$

It is a straight forward matter to prove the analogue of Theorem 0.2 for the product measure $\prod_{j=1}^{g} \frac{1}{\pi}\sin^2\theta_j d\theta_j$, when $g \to \infty$ and hence to develop analogues results about the spacing distributions between the zeros of the zeta functions of $X_0(N)/\mathbb{F}_p$. Doing so in this way will lead to a complicated answer (i.e. for the spacing measures) simply because the asymptotic density of the zeros of $X_0(N)/\mathbb{F}_p$ is not uniform on $[0,\pi]$. Indeed according to (0.24)

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \frac{\#\{j|\theta_j(p) \in [a,b]\}}{g} = \int_a^b \frac{2}{\pi}\sin^2\theta\, d\theta \qquad (0.25)$$

That is, the density is $\frac{2\sin^2\theta}{\pi}$ which is not uniform. [In the $U(N), USp(2N), SO(2N)$ and $SO(2N+1)$ cases the densities are (see (2.55)) $\frac{d\theta}{2\pi}$ on $[0,2\pi)$, $\left(\frac{1}{\pi} + \frac{1}{2N\pi} - \frac{\sin(2N+1)\theta}{(\sin\theta)2N\pi}\right) d\theta$ on $[0,\pi]$, $\left(\frac{1}{\pi} - \frac{1}{2N\pi} + \frac{\sin(2N-1)\theta}{2\pi N\sin\theta}\right) d\theta$ on $[0,\pi]$ and $\left(\frac{1}{\pi} - \frac{\sin N\theta}{2\pi N\sin\theta}\right) d\theta$ on $[0,\pi]$, respectively. For the purpose of determining the spacing distributions as $N\to\infty$, it turns out that these approach the uniform density fast enough as Theorems 0.1 and 0.2 indicate. For this reason, this issue never came up in say Theorem 0.3.]

To get around this variable density problem in (0.25) one can proceed in two ways. One, is to localize, i.e. to consider the spacings between the zeros of $\zeta(X_0(N)/\mathbb{F}_p, T)$ which are near a fixed $0 < \theta_0 < \pi$ (exactly how near introduces a new parameter), the second is to make a monotone change of parameter which renders the density uniform. We choose the second approach, which we call "straightening the angles". Set

$$t = \frac{\theta}{\pi} - \frac{\sin 2\theta}{2\pi} \qquad 0 < \theta \leq \pi \ . \tag{0.26}$$

Then $0 \leq t \leq 1$ and in the parameter t the zeros $0 \leq t_1 \leq t_2 \ldots \leq t_g \leq 1$ have a constant density as we vary over all p. In this parameter (0.24) asserts that in the p limit, $t_1, t_2, \ldots, t_g$ are *uniformly* distributed *independent* random variables on $[0,1]$. The rescaled spacing distributions for these as $g \to \infty$, that is the analogues of Theorems 0.1 and (0.2), are well known [P]. A similar result holds for these except that μ_a is replaced by μ_a (random) where

$$\mu_a \text{ (random)} = \frac{s^{a-1}}{(a-1)!} e^{-s} ds \tag{0.27}$$

(random here indicates the spacings between random numbers i.e. identically independently distributed uniform variables). The shapes of the measures μ_a (random) and μ_a are quite different. For example, the density of μ_1 (classical) vanishes to second order at $s = 0$ (see Figure 1) reflecting the fact that consecutive spacings are rarely very small (this is referred to as level repulsion), while for μ_1 (random) $= e^{-s} ds$ small consecutive spacings are the mode.

Applying the above discussion to $X_0(N)$ we get the following: Let $\tilde{\mu}_a(X_0(N)/\mathbb{F}_q)$ be the straightened spacing measure for the zeros of $\zeta(X_0(N)/\mathbb{F}_p, T)$ (i.e. the measure defined as before, but using the parameter t for the zeros), then we have

THEOREM 0.6. *Let N be prime and assume the general Sato–Tate conjectures. For any $a \in \mathbb{N}$, we have*

$$\lim_{N\to\infty} \lim_{x\to\infty} \sum_{\substack{p \leq x \\ p \nmid N}} D(\tilde{\mu}_a(X_0(N)/\mathbb{F}_p), \mu_a \ (random)) = 0 \ .$$

This says that the spacings between zeros of $\zeta(X_0(N)/\mathbb{F}_p, T)$ behave like random numbers (or what is sometimes referred to as "Poissonian") as $N \to \infty$, for almost all p. It does not seem unreasonable to expect that the last holds in the stronger form without averaging over p. To formulate this, we need to straighten a little differently. The reason is that for fixed p, the density of zeros of $\zeta(X_0(N)/\mathbb{F}_p, T)$ approaches the p–adic Plancherel density $d\mu_p(\theta)$ on $[0, \pi]$ as $N \to \infty$, see Serre [SE] (note that $d\mu_p \to \frac{2}{\pi} \sin^2 \theta d\theta$ as $p \to \infty$). Set $s(\theta) = \int_0^\theta d\mu_p$ and let $\tilde{\tilde{\mu}}_a(X_0(N)/\mathbb{F}_p)$ be the corresponding spacing distribution between the zeros in the s–parameter.

CONJECTURE 0.7. Fix p and a. Then

$$\lim_{\substack{N\to\infty \\ p\nmid N}} D(\tilde{\tilde{\mu}}_a(X_0(N)/\mathbb{F}_p), \mu_a\,(\text{random})) = 0\,.$$

In view of the interpretation of the zeros of $\zeta(X_0(N)/\mathbb{F}_p, T)$ in terms of the eigenvalues of a $p+1$ regular graph on $g(X_0(N))$ vertices (see for example Mestre [Mes]) this conjecture can be tested numerically. The conjecture asserts that the spacings between the eigenvalues of the Hecke operator T_p on weight 2 modular forms for $X_0(N)$ is Poissonian in the limit $N \to \infty$. One may conjecture that the same is true for the eigenvalues of the Laplacian Δ (i.e. "T_p" for p the infinite place). Interestingly, this Poissonian phenomenon for $X(1)$ and Δ was discovered by physicists in numerical experiments [SC].

An analogue the individual type behavior in Conjecture 0.7, but for Kloosterman sums also seems quite plausible though in this case we know of no feasible way of testing the following conjecture:

CONJECTURE 0.8. Fix $\mathbb{F}_q$ and $t \in \mathbb{F}_q^\star$ and ψ an nontrivial additive character of $\mathbb{F}_q$. Then for $a \in \mathbb{N}$

$$\lim_{n\to\infty} D(\mu_a(K\ell_\psi^{(n)}(t,q)),\ \mu_a(\text{univ})) = 0\,.$$

With the exception of the discussion in the supplement to Section 5, the above has been concerned with universal features – i.e. ones not sensitive to the particular family $G(N)$. The distribution of eigenvalue closest to 1, is sensitive to the particular family. This can be pursued arithmetically in a number of ways, including the following. For $\mathbb{F}_q$ as above, and $n \geq 1$, let $H_n(\mathbb{F}_q) = \{D | D$ is a square free monic polynomial of degree n in $\mathbb{F}_q[t]\}$. The quadratic extensions $\mathbb{F}_q(t)(\sqrt{D})$ of $\mathbb{F}_q(t)$ are precisely the function fields whose zeta functions were considered by Artin in his thesis. For $j \geq 1$, let $\lambda_j(q,D)$ be $\frac{n}{2\pi}$ times the distance along the unit circle from the j^{th} zero of $\zeta(\mathbb{F}_q(t)(\sqrt{D}), T)$ to 1. This zeta function is the same as $\zeta(C_D/\mathbb{F}_q, T)$ where C_D is the curve $y^2 = D(x)$ over $\mathbb{F}_q$. Using the equidistribution techniques of Chapter 3, (see also the preprint of Yu [Yu] who investigates the distribution of the class numbers of $\mathbb{F}_q(t)(\sqrt{D})$) together with the scaling limits derived in Section 5, we obtain:

THEOREM 0.9. *Fix $j \geq 1$ and let $f \in C_0(\mathbb{R}_{\geq 0})$. Then*

$$\lim_{n\to\infty} \lim_{q\to\infty} \frac{1}{|H_n(\mathbb{F}_q)|} \sum_{D\in H_n(\mathbb{F}_q)} f(\lambda_j(q,D)) = \int_0^\infty f(t)\,d\nu_{-,j}(t)$$

where $\nu_{-,j}$ is the measure in Proposition 4, Supplement to Section 5.

The point is that as is shown in Section 7, the monodromy associated to the family of curves $H_n(\mathbb{F}_q)$ is the full symplectic group. According to Section 5 (supplement) this leads to the measures $\nu_{-,j}$ which are associated with "$USp(\infty)$" and characteristically so (i.e. the other classical groups give the entirely different measures ν_j and $\nu_{+,j}$).

Again one can optimistically conjecture that Theorem 0.9 should hold in the stronger form without the q–limit.

CONJECTURE 0.10. Fix q and j. For any $f \in C_0(\mathbb{R}_{\geq 0})$ we have

$$\lim_{n\to\infty} \frac{1}{|H_n(\mathbb{F}_q)|} \sum_{D\in H_n(\mathbb{F}_q)} f(\lambda_j(q,D)) = \int_0^\infty f(t)d\nu_{-,j}(t) \ .$$

In this form Conjecture 0.10 asserts that if we fix the function field $K = \mathbb{F}_q(t)$, and consider the distribution of the j^{th} zero from 1 of the zeta functions of quadratic extensions of K, then these follow the distributions associated with the underlying monodromy $USp(\infty)$. While we don't know how to test Conjecture 0.10, even numerically, one can test its analogue over the rational number field. So we return to the Riemann zeta function and Dirichlet L–functions. The universal "GUE" feature common to all of these when averaging over all zeros of a given such L–function, was our starting point. However, we have now identified in the function field setting a nonuniversal feature which betrays further structure underlying the zeros, such as the symplectic structure. The analogue of Conjecture 0.10 for $K = Q$ is to consider the distribution of the j^{th} closest zero to $s = 1/2$ of $\zeta_L(s)$, where L/Q is quadratic extension, as $D_L = \text{disc } (L) \to \infty$. Rubinstein [Rub] has recently developed a method to compute the j^{th} such zero for D_L of order 10^9. He has gathered data on the distribution (normalized) of the lowest and second to lowest such zeros on varying D_L (through primes). His results are displayed in Figures 7, 8 and 9. The fit with $\tilde{\nu}_{-,1}$ and $\tilde{\nu}_{-,2}$ is excellent (to the extent that one expects convergence at a rate of $0(1/\log D_L)$ – compare with Odlyzko [Od] where t plays the role of D_L). Further evidence of the $Sp(\infty)$ influence can be given by considering the density of zeros of $\zeta_L(s)$ near $s = 1/2$. In the function field setting, this density is dictated by the symplectic measure $\mu_{1,N}(x_1)$ in (2.46). As $N \to \infty$ this density converges to

$$\left(1 - \frac{\sin 2\pi x}{2\pi x}\right) dx \ . \tag{0.28}$$

On the other hand, using the analytic methods for studying correlations of the zeros of the Riemann zeta function mentioned in the first paragraph, Ozluk and Shnider [Sh] have shown that, if one assumes the Riemann-Hypothesis for $\zeta_L(s)$, then for a restricted class of test functions; viz $f \in \mathcal{S}(\mathbb{R})$ whose Fourier transforms are supported in $(-2/3, 2/3)$, we have

$$\lim_{x\to\infty} \frac{1}{\#\{L|D_L < x\}} \sum_{D_L<x} \Big(\sum_{\zeta_L(\frac{1}{2}+i\gamma)=0} f\big(\frac{\log D_L}{2\pi}\gamma\big)\Big) = \int_{-\infty}^{\infty} \big(1 - \frac{\sin 2\pi x}{2\pi x}\big) f(x) dx \ . \tag{0.29}$$

So (0.29) and (0.28) are in perfect agreement. Note that the scaling of the imaginary parts by $(\log D_L)/2\pi$ is the natural one here.

With this numerical and theoretical evidence, we are led to

Conjecture 0.11. Let the j^{th} lowest zero of $\zeta_L(s)$ be denoted by $\rho = 1/2 + i\gamma_L^{(j)}$. Then for any $f \in C_0(\mathbb{R}_{\geq 0})$

$$\lim_{x\to\infty} \frac{1}{\#\{L|D_L < x\}} \sum_{D_L<x} f\left(\frac{\gamma_L^{(j)} \log D_L}{2\pi}\right) = \int_0^{\infty} f(t) d\tilde{\nu}_{-,j}(t) \ .$$

The density of the measure $\tilde{\nu}_{-,1}$ (unlike $\tilde{\nu}_1$ or $\tilde{\nu}_{+,1}$) corresponding to $USp(\infty)$ has the remarkable property that it vanishes to second order at $s = 0$. That is the zeros are rarely close to 1/2 (or they repel the point 1/2). At a crude level this was already observed by Hazelgrove [Haz] who was the first to experiment numerically with zeros of $\zeta_L(s)$. What Conjecture 0.11 offers is a precise version of this phenomenon together with a theoretical explanation. That $Sp(\infty)$ appears to be gluing these $\zeta_L(s)$ together as it does in the function field case, hints at the existence in the global case of a remarkable structure yet to be discovered.

1. Measures attached to spacings of eigenvalues

Let $N \geq 1$ be an integer. Given an element A in the unitary group $U(N)$, all of its eigenvalues lie on the unit circle, so there is a unique increasing sequence of angles in $[0, 2\pi]$

$$0 \leq \phi_1 \leq \phi_2 \ldots \leq \phi_N < 2\pi \tag{1.1}$$

such that

$$\det\,(T - A) = \prod_{j=1}^{N} (T - e^{i\phi_j}) \ . \tag{1.2}$$

The N non-negative real numbers

$$\left.\begin{array}{ll} & \phi_{j+1}-\phi_j, \quad j=1,\ldots,N-1 \\ \text{and} & \\ & 2\pi+\phi_1-\phi_N \end{array}\right\} \tag{1.3}$$

are called the "literal" spacings between the adjacent eigenvalues of A. Their sum is 2π, so their mean is $2\pi/N$. By the normalized spacings between adjacent eigenvalues of A, we mean the N non–negative numbers $\Delta_1,\ldots,\Delta_N$ defined by

$$\Delta_j = \frac{N}{2\pi}(\phi_{j+1}-\phi_j), j=1,\ldots,N-1$$

and

$$\Delta_N = \frac{N}{2\pi}(2\pi+\phi_1-\phi_N)\ . \tag{1.4}$$

Clearly their mean is 1.

More generally if $a\geq 1$ is an integer, we can define the "$a-th$" spacings. Firstly, we prolong the sequence $j\to\phi_j$ to all integers j by requiring that $\phi_{j+N}=\phi_j+2\pi$. For $j=1,\ldots,N$ define

$$\Delta_{j,a} = \frac{N}{2\pi}(\phi_{j+a}-\phi_j). \tag{1.5}$$

The mean of these nonnegative numbers is clearly equal to a.

Corresponding to these spacings, we have the measures defined in the introduction:

$$\mu_a(A) = \frac{1}{N}\sum_{j=1}^{N}\delta_{\Delta_{j,a}}\ . \tag{1.6}$$

That is to say, for any continuous f on $\mathbb{R}$

$$\int_{\mathbb{R}} f\,d\mu_a(A) = \frac{1}{N}\sum_{j=1}^{N} f(\Delta_{j,a})\ . \tag{1.7}$$

Note that in the above discussion we chose angles $\phi_j, j=1,\ldots,N$ in a particular fundamental domain, viz $[0,2\pi)$. Suppose that instead we had fixed a real number α and had chosen the angles in $[\alpha,\alpha+2\pi)$. The corresponding spacing vectors $\Delta_{j,a}$ for A computed using $[\alpha,\alpha+2\pi)$ would be some cyclic permutation of those computed using $[0,2\pi)$. Hence, the measure $\mu_a(A)$ would be the same.

We begin with some easily proven elementary facts.

LEMMA 1.

(1) *On $U(N)$, each of the angles ϕ_j (computed using $[0, 2\pi)$) is a (Borel) measurable function. Each is continuous on the open set $U(N)[1/\det(I-A)]$ of $U(N)$, where 1 is not an eigenvalue. More generally, for each x, the angles computed using $[\alpha, \alpha+2\pi)$ are measurable and continuous on the open set $U(N)[1/\det(e^{i\alpha}-A)]$.*
(2) *For any $a \geq 1$ and α, the N normalized spacings $\Delta_{j,a}(A)$ computed using $[\alpha, \alpha+2\pi)$ are measurable and are continuous on $U(N)[1/\det(e^{i\alpha}-A)]$.*
(3) *If f is a continuous function on $\mathbb{R}$*

$$A \longrightarrow \int_{\mathbb{R}} f d\mu_a(A)$$

is continuous on $U(N)$.

PROOF. (2) is immediate from (1). For (3) we observe that, if we compute angles using $[\alpha, \alpha+2\pi)$ each of $f(\Delta_{j,a}(A))$ is continuous on $U(N)[1/\det(e^{i\alpha}-A)]$ and so $A \to \int f d\mu_a(A)$ is continuous on $U(N)[1/\det(e^{i\alpha}-(A)]$. Since these open sets cover $U(N)$ as we vary α, the result follows. The proof of (1) is given in Appendix 1.

For ν a measure (positive) on $\mathbb{R}$ its cummulative distribution function CDF_ν is the function on $\mathbb{R}$ defined by

$$CDF_\nu(x) = \nu((-\infty, x]) \ . \tag{1.8}$$

Recall that we defined the discrepancy $D(\nu, \mu)$ between two such measures to

$$D(\nu, \mu) = \sup_{-\infty<x<\infty} |CDF_\nu(x) - CDF_\nu(x)| \ . \tag{1.9}$$

For probability measures (or measures of mass at most 1), $0 \leq D \leq 1$. The following Lemma will be very useful later on when we apply various equidistribution theorems.

LEMMA 2. *Let ν be a probability measure on $\mathbb{R}$ with a continuous CDF. Then the function*

$$A \mapsto D(\mu_a(A), \nu)$$

from $U(N)$ to $[0,1]$ is continuous.

PROOF. As before, it suffices to show that this function is continuous on each open set $U(N)[1/\det(e^{i\alpha}-A)]$. On this open set, the spacings $\Delta_{j,a}(A)$ are continuous. So the Lemma follows from:

LEMMA 3. *Let ν be a probability measure on $\mathbb{R}$ with a continuous CDF. Fix N and for each $P = (x_1, \ldots, x_N) \in \mathbb{R}^N$ let $\mu(P)$ be the measure (on $\mathbb{R}$)*

$$\mu(P) = \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j} \ .$$

The function $P \to D(\mu(P), \nu)$ is a continuous function on $\mathbb{R}^N \to [0,1]$.

PROOF. see Appendix 2.

Theorems 0.1 and 0.2 are concerned with the measures $\mu_a(A)$ for A in various subgroups of $U(N)$. Let K be a compact group and

$$\rho : K \to U(N) \tag{1.10}$$

a continuous representation. In this way, we have the measures $\mu_a(\rho(A))$. We will denote this measure by $\mu_a(A, K, \rho)$ if we wish to emphsize K. The Haar measure (total mass 1) on K will be denoted by Haar_K or $d_K(A)$ (or simply dA) when it appears in an integral. The average, or expected value, is defined in the obvious way:

$$\begin{aligned} E(\mu_a(A, K, \rho), \text{Haar}_K(A)) &= \mu_a(K, \rho) \\ &= \frac{1}{N}\sum_{j=1}^{N} (\Delta_{j,a} \circ \rho)_*(\text{Haar}_K) \ . \end{aligned} \tag{1.11}$$

Put another way, for any bounded continuous function f on $\mathbb{R}$

$$\int_{\mathbb{R}} f d\mu_a(K, \rho) = \int_K \left(\int_{\mathbb{R}} f d\mu_a(A, K, \rho \right) d_K A \ . \tag{1.12}$$

We may also write

$$\begin{aligned} \mu_a(K, \rho) &= E(\mu_a(A, K, \rho),\ \text{Haar}_K) \\ &= \int_K \mu_a(A, K, \rho) dA \ . \end{aligned} \tag{1.13}$$

It is clear tht $\mu_a(K, \rho)$ is a probability measure. If the representation ρ is understood, we will omit it from the notation. In what follows, we are primarily interested in the case when K is one of the compact classical groups $G(N)$ in their standard representations. That is $G(N)$ is one of:

(i) $U(N)$ the group of $N \times N$ unitary matrices and its subgroup $SU(N)$ consisting of those of determinant equal to 1.
(ii) $O(N)$, the group of $N \times N$ real orthogonal matrices and its subgroup $SO(N)$.
(iii) $USP(N)$, (N even) the compact form of $SP(N)$. That is the set of $N \times N$ complex matrices A which are

(a) unitary *i.e.* ${}^*AA = I$

(b) symplectic *i.e.* ${}^tAJA = J$

$$\text{where} \qquad J = \begin{bmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{bmatrix}.$$

Some cases of Theorems 0.1 and 0.2 for the various $G(N)$'s are simply related to each other. To see this, we make the following observation.

LEMMA 4. *Let H and K be compact groups, and $\Pi : H \to K$ a continuous surjective group homomorphism. Then Π_* $Haar_H = Haar_K$.*

PROOF. Since Π is surjective and Haar_H is translation invariant on H its direct image Π_* Haar_H is translation invariant on K. It is also of mass 1 and so the result follows from the uniqueness of Haar measure.

COROLLARY 5. *Let H be a compact group, M a compact normal subgroup of H, and K a closed subgroup of H, such that $H = MK$. Then for any bounded function*[****] *f on H which is M invariant, we have*

$$\int_H f(A) d_H(A) = \int_K f(A) d_K(A) \, .$$

PROOF. Consider the quotient $H/N \simeq K/N \cap K$. ¿From Lemma 4 we have

$$\begin{aligned} \int_H f(A) d_H(A) &= \int_{H/M} f(A) d_{H/N}(A) \\ &= \int_{K/K\cap M} f(A) d_{K/K\cap M}(A) \\ &= \int_K f(A) d_K(A) \, . \end{aligned}$$

We apply these remarks as follows. The various spacing measures attached to an element A of $U(N)$ are the same for A and for λA, for any scalar $\lambda \in S^1$. So for each $a \geq 1$, the expected values of $\mu_a(A, U(N))$ and of $\mu_a(A, SU(N))$ coincide (take M to be the group of scalars in $H = U(N)$ and K the subgroup $SU(N)$). Similarly, the expected values of these measures for $O(2N+1)$ and $SO(N+1)$ will coincide (take $M = \{1, -1\}$,

[****]Here and everywhere else all functions will be Borel measurable.

$H = O(2N+1)$ and $K = SO(2N+1)$). Thus, the $SU(N)$ case of Theorem 0.1 is equivalent to the $U(N)$ case of it and the $SO(2N+1)$ case to the $O(2N+1)$ case. Similarly, the discrepancy integrals in Theorem 0.2 coincide for $U(N)$ and $SU(N)$, and for $O(2N+1)$ and $SO(2N+1)$. So Theorem 0.2 for $SU(N)$ and $SO(2N+1)$ follow from the cases $U(N)$ and $O(2N+1)$, and visa–versa.

We may also apply the above to give slight extensions of Theorems 0.1 and 0.2, that is to groups $H(N)$ which are close to $G(N)$.

LEMMA 6.

(a) $G(N) = SU(N) \subset H(N) \subset$ *Normalizer of* $G(N)$ *in* $U(N)$
(b) $G(N) = SO(2N+1) \subset H(N) \subset$ *Normalizer of* $G(N)$ *in* $U(2N+1)$
(c) $G(N) = USP(2N) \subset H(N) \subset$ *Normalizer of* $G(N)$ *in* $U(2N)$
(d) $G(N) = SO(2N) \subset H(N) \subset$ *Normalizer of* $G(N)$ *in* $U(2N)$

Let f be a bounded function on the ambient group (i.e. $U(N)$ in case (a), etc.) which is invariant by the subgroup S^1 of unitary scalars. In case (a), (b), and (c), we have

$$\int_{H(N)} f(A)dA = \int_{G(N)} f(A)dA\,.$$

In case (d), we have either

$$\int_{H(N)} f(A)dA = \int_{SO(2N)} f(A)dA$$

or

$$\int_{H(N)} f(A)dA = \int_{O(2N)} f(A)dA$$

depending on whether or not every element of $H(N)$, acting by conjugation on $SO(2N)$, induces an inner automorphism of $SO(2N)$.

PROOF. In cases (a), (b), and (c) we claim that the normalizer of $G(N)$ in the ambient unitary group is $S^1G(N)$, while in case (d) we claim it is $S^1O(2N)$. It is clear that the named group normalizes $G(N)$, what must be shown is that the normalizer is no bigger. In other words, we must show that $H(N) \subset S^1G(N)$ in cases (a), (b), (c), while $H(N) \subset S^1O(2N)$ in case (d).

In the case (a), $S^1G(N) = U(N)$ so there is nothing to prove. In cases (b) and (c), we use the fact that the Dynkin diagram has no nontrivial automorphisms, so every automorphism of $G(N)$ is inner. So any $h \in H(N)$, acting by conjugation on $G(N)$ induces conjugation by some element $g \in G(N)$. Then $h^{-1}g$ commutes with $G(N)$ in its standard representation. As this representation is irreducible, $h^{-1}g$ must be scalar, and this scalar being unitary, lies in S^1. Thus $H(N) \subset S^1G(N)$.

Case (d) requires some additional attention. If $N = 1$, one can argue by inspection, and is left to the reader. If $N \geq 2$, $N \neq 4$ there is one nontrivial automorphism of the Dynkin diagram and it is induced by conjugation by any element of $O_-(2N)$. So any h in $H(N)$, induces conjugation by some element $g \in O(2N)$. Repeating the irreducibility argument given above in the (b) and (c) cases, we get $H(N) \subset S^1O(2N)$.

It remains to examine case (d) with $N = 4$. Here the Dynkin diagram has three extreme points, and its automorphism group is $\sum_3$ acting by permutations of these three points. Think of these three points as (the highest weights of) the three 8–dimensional irreducible representations of Lie($SO(8)$) (namely the "standard" one ρ_{std} and the two spin representations). The action of any h in $H(N)$ preserves the isomorphism class ρ_{std} indeed for A in $SO(2N) \subset U(2N)$ and h in $U(2N)$ which normalizes $SO(2N)$,

$$\rho_{std}(hAh^{-1}) = hAh^{-1} = h\rho_{std}(A)h^{-1} .$$

Therefore conjugaion by h is either inner, or interchanges the two spin representations. In the first case, we have h in $S^1G(N)$. In the latter case, use the fact that any γ_- in $O_-(2N)$ also interchanges the two spin representations. Then $h\gamma_-$ induces an inner automorphism and hence $h\gamma_-$ is in $S^1G(N)$, whence $h \in S^1O(2N)$, as required.

In cases (a), (b), and (c) the inclusions

$$G(N) \subset H(N) = S^1G(N)$$

make clear that we have

$$S^1H(N) = S^1G(N) .$$

¿From this, Lemma 4 and its corrollary, we infer that

$$\int_{H(N)} f(A)dA = \int_{S^1H(N)} f(A)dA = \int_{S^1G(N)} f(A)dA = \int_{G(N)} f(A)dA$$

as required.

In case (d) we must distinguish two cases. If $H(N)$ acts on $SO(2N)$ through inner automorphisms, we have the inclusion

$$G(N) \subset H(N) = S^1G(N)$$

and we argue as above. If some h in $H(N)$ induces a non–inner automorphism, then $H(N)$ contains an element of $S^1O_-(2N)$, in which case the inclusion $G(N) \subset H(N) = S^1O(2N)$ forces the equality

$$S^1H(N) = S^1O(2N)\ .$$

Applying the same arguments to $H(N) \subset S^1H(N)$ and $O(2N) \subset S^1O(2N)$ yields the result in these cases.

If we apply Lemma 6 to the discrepancy function we obtain

$$\int_{H(N)} D(\mu_a(A), \mu_a)dA = \int_{G(N)} D(\mu_a(A), \mu_a)dA \tag{1.14}$$

for case (a) (b) (c) while

$$\int_{H(N)} D(\mu_a(A), \mu_a)dA = \begin{cases} \displaystyle\int_{SO(2N} D(\mu_a(A), \mu_a)dA \\ \displaystyle\int_{O(2N)} D(\mu_a(A), \mu_a)dA \end{cases} \tag{1.15}$$

depending on whether or not $H(N)$ acts by inner automorphism on $SO(2N)$ in case (d). In particular, we see that Theorem 0.2 for $G(N)$ implies the same theorem with $G(N)$ replaced by $H(N)$.

We end this section with some further technical remarks which will fascilitate the analysis in the proof of Theorems 0.1 and 0.2. The eigenvalues of any $A \in G(N)$ are on the unit circle and this makes it difficult to carry out explicit calculations. For this reason we define, case by case, for each $N > a$, measures $\mu_a(A, G(N), \text{naive})$ which approximate $\mu_a(A)$ and which are well adapted to calculation because they involve points on a line.

(I) The $U(N)$ case: Attached to $A \in U(N)$ are its N angles

$$0 \le \phi \le \phi_2 \cdots \le \phi_N < 2\pi\ .$$

We use only the first $N-a$ of the normalized spacing measures $\Delta_{j,a}(A)$, that is precisely those which don't require wrapping around the unit circle. Define

$$\mu_a(A, U(N), \text{naive}) = \frac{1}{N} \sum_{j=1}^{N-a} \delta_{\Delta_{j,a}}\ . \tag{1.16}$$

This measure has total mass $\frac{N-a}{N}$.

(II) The $USP(2N)$ case: The eigenvalues of A in $USP(2N)$ occur in N complex conjugate pairs which we may write uniquely as $e^{\pm\phi_j}$ with N angles

$$0 \leq \phi_1 \leq \phi_2 \ldots \leq \phi_N \leq \pi \ . \tag{1.17}$$

Each of these angles is a continuous function on $USP(2N)$. For $1 \leq i \leq N-a$, we define the a–spacings as before and set

$$\mu_a(A, USP(2N), \text{naive}) = \frac{1}{N} \sum_{j=1}^{N-a} \delta_{\Delta_{j,a}} \tag{1.18}$$

which has mass $(N-a)/N$.

(III) The $SO(2N+1)$ case: For this case, 1 is an eigenvalue of every A. The remaining eigenvalues of A occur in N complex conjugate pairs, which we may write uniquely as $e^{\pm i\phi_j}$ with N angles in $[0, \pi]$,

$$0 \leq \phi_1 \leq \phi_2 \ldots \leq \phi_N \leq \pi \ . \tag{1.19}$$

Again, each of these angles ϕ_j are continuous on $SO(2N+1)$. With the corresponding spacings $\Delta_{j,a}$, $1 \leq j \leq N-a$, we set

$$\mu_a(A, SO(2N+1)\, , \text{ naive}) = \frac{1}{N} \sum_{j=1}^{N-a} \delta_{\Delta_{j,a}} \tag{1.20}$$

which again has mass $(N-a)/N$.

(IV) The $SO(2N)$ case: The eigenvalues of an A in $SO(2N)$ occur in N complex conjugate pairs, which we may write uniquely as $e^{\pm\phi_j}$ with N angles in $[0, \pi]$ and the ϕ_j's as in (1.19). The measures $\mu_a(A, SO(2N)\, , \text{ naive})$ are then defined as in (1.20) with the corresponding $\Delta_{j,a}$'s.

(V) The $O_-(2N+2)$ case: Here $O_-(2N)$ denotes the set of elements in $O(2N+2)$ with determinant -1. $O_-(2N+2)$ is the principal homogeneous space under $SO(2N+2)$ and $O(2n+2)$ is the disjoint union $SO(2N+2) \cup O_-(2N+2)$. We will denote by $\text{Haar}_{O_-(2N+2)}$ the restriction to $O_-(2N+2)$ of Haar measure on $O(2N+2)$, but normalized so that $O_-(2N+2)$ has mass 1.

Any element A in $O_-(2N+2)$ has both 1 and -1 as eigenvalues. The remaining eigenvalues of A occur in N complex conjugate pairs, which we may write uniquely as $e^{\pm\phi_j}$ with N angles in $[0,\pi]$ as in (1.20). We form the corresponding measures $\mu_a(A, O(2N+2)\,,\ \text{naive})$ as in (1.20).

This concludes our case by case definition of the naive spacing measures. In view of the comments following Lemma 4, these cases will suffice for the purpose of understanding the general $G(N)$. ¿From the definitions of the various spacing measures it***** follows that for $G(N) = U(N),\ SO(2N+1),\ USP(2N),\ SO(2N)$ and $O_-(2N+2)$ and $N > a$

$$D(\mu_a(A, G(N)),\ \mu_a(A, G(N),\ \text{naive})) \leq \frac{a+1}{N}\ . \tag{1.21}$$

Hence $E(\mu_a(A, G(N))$ and $E(\mu_a(A, G(N),\ \text{naive})$ will have the same limits, if they have limits at all. In particular, the above implies that Theorem 0.1 will follow from:

THEOREM 0.1 (BIS). *For $G(N)$ one of $U(N),\ SO(N),\ USP(2N)$ or $0_-(2N)$ and $a \geq 1$*

$$\lim_{N\to\infty} E(\mu_a(A, G(N),\ \textit{naive})) = \mu_a\ .$$

Similarly the bound (1.21), together with the triangle inequality, implies that Theorem 0.2 follows from:

THOREM 0.2 (BIS). *Given $A > 0$ and $a \geq 1$, there is an effectively computable $N = N(\epsilon, a)$ such that for $G(N)$ one of $U(N)$, $SO(N)$, $USP(2N)$, $O_-(2N)$ and $N \geq N(\epsilon, a)$*

$$\int_{G(N)} D(\mu_a(A, G(N),\ \textit{naive}),\ \mu_a) dA \leq N^{-\frac{1}{6}+\epsilon}\ .$$

Note that the case $O(2N)$ follows from each of $SO(2N)$ and $O_-(2N)$. In the next sections, we will establish Theorems 0.1 and 0.2 bis.

2. Haar Measure

In each of the cases (I) to (V) at the end of section 1, we described the basic angles $\phi_1, \ldots, \phi_N$ corresponding to $A \in G(N)$. We denote by $X(A) \in \mathbb{R}^N$ this vector of eigenvalue angles for $A \in G(N)$. Thus, for $A \in U(N)$, $X(A) \in [0, 2\pi)^N$, while in all

*****and noting that one obtains the same spacing measures $\mu_a(A\,,\ \text{naive})$ on reversing the order of the points.

the other cases, $X(A)$ lies in $[0,\pi]^N$. The Weyl integration formula gives the measure on these ϕ_j's induced by Haar measure on $G(N)$. The version of the formula which we need, especially in the non–$U(N)$ case, is precisely the one given in Weyl ["The classical groups" pp 197 (7.4B), 218 (7.8B), 224 (9.7), and 226 (9.15)]. In all but the $O_-(2N+2)$ case, this formula is, to the modern reader, a straight forward deciphering of the "intrinsic" one given in Bourbaki [Lie IX, §6 N 2, Cor.1]. The $O_-(2N+2)$ formula, which seems to have been all but forgotten in modern treatments, is discussed in the appendix to this chapter, cf also [Deligne letter].

(I) the $U(N)$ case: An element A in $U(N)$ is determined up to conjugacy by its vector of angles $X(A)$. Bounded central functions on $U(N)$ are in one–one correspondence with bounded functions $\tilde{g}$ on $[0,2\pi)^N$ which are $\sum_N$ invariant, via $g(A)=\tilde{g}(x(A))$. We denote by $V_{U(N)}$ the measure on $[0,2\pi)^N$ (with coordinates $x_j, j=1,\dots,N$) given by

$$dV_{U(N)}=\frac{1}{N!}(\prod_{j<k}|e^{ix_j}-e^{ix_k}|^2)\prod_{\ell}dx_\ell\ . \tag{2.1}$$

The Weyl integration formula asserts that for g a central function on $U(N)$, corresponding to a $\sum_N$–invariant $\tilde{g}$ on $[0,2\pi)^N$ we have

$$\int_{U(N)}g(A)dA=\int_{[0,2\pi)^N}\tilde{g}(x)dV_{U(N)}(x)\ . \tag{2.2}$$

(II) The $USP(2N)$ case: An element A in $USP(2N)$ is determined up to conjugacy by its vector of angles $X(A)$. Central functions g on $UP(2N)$ are in one-one correspondence with functions $\tilde{g}$ on $[0,\pi)^N$ which are $\sum_N$–invariant, via $g(A)=\tilde{g}(X(A))$. Denote by $V_{USP(2N)}$ the measure on $[0,\pi)^N$ (with coordinates $x_j, j=1,\dots,N$) given by

$$dV_{USP(2N)}=\frac{1}{N!}(\prod_{j<k}(2\cos x_j-2\cos x_k)^2)\cdot\prod_{\ell}(\frac{2}{\pi}\sin^2 x_\ell)dx_\ell\ . \tag{2.3}$$

The Weyl integration formula asserts that for g and $\tilde{g}$ as above

$$\int_{USP(2N)}g(A)dA=\int_{[0,\pi]^N}\tilde{g}(x)dV_{USP(2N)}(x)\ . \tag{2.4}$$

(III) The $SO(2N+1)$ case: An element A in $SO(2N+1)$ is determined up to conjugacy by its vector of angles $X(A)$. Central functions g on $SO(2N)$ correspond to function

$\tilde{g}$ on $[0,\pi]^N$ which are $\sum_N$–invariant, via $g(A) = \tilde{g}(X(A))$. Denote by $V_{SO(N+1)}$ the measure on $[0,\pi]^N$ (with coordinates $x_j, j = 1,\ldots,N$) given by

$$dV_{SO(2N+1)} = \frac{1}{N!}(\prod_{j<K}((1+2\cos x_j) - (1+2\cos x_k))^2 \cdot \prod_\ell \frac{1}{\pi}\sin^2(x_\ell/2)dx_\ell\ . \tag{2.5}$$

The Weyl integration formula asserts that for such g and $\tilde{g}$

$$\int_{SO(2N+1)} g(A)dA = \int_{[0,\pi]^N} \tilde{g}(x)dV_{SO(2N+1)}(x)\ . \tag{2.6}$$

(IV) The $SO(2N)$ case: An element A in $SO(2N+1)$ is determined up to conjugation by elements in the ambient group $O(2N)$ by $X(A)$. Central functions g on $O(2N)$ correspond to functions $\tilde{g}$ on $[0,\pi]^N$ which are $\sum_N$–invariant, via $g(A) = \tilde{g}(X(A))$. Denote by $V_{SO(2N)}$ the measure on $[0,\pi]^N$ given by

$$dV_{SO(2N)} = \frac{2}{N!}\prod_{j<k}(2\cos x_j - 2\cos x_k)^2 \prod_\ell \frac{dx_\ell}{2\pi}\ . \tag{2.7}$$

Moreover, we have the formula for such g and $\tilde{g}$

$$\int_{SO(2N)} g(A)dA = \int_{[0,\pi]^N} \tilde{g}(x)dV_{SO(2N)}(x)\ . \tag{2.8}$$

(V) The $O_-(2N+2)$ case: An element $A \in O_-(2N+2)$ is determined up to $O(2N+2)$ conjugation by its vector of angles $X(A)$. Central functions g on $O_-(2N+2)$ (i.e. $O(2N+2)$ central) correspond to functions $\tilde{g}$ on $[0,\pi]^N$ which are $\sum_N$–invariant, via $g(A) = \tilde{g}(X(A))$. We denote by $V_{O_-(2N+2)}$ the measure on $[0,\pi]^N$ given by

$$dV_{O_-(2N+2)} = \frac{1}{N!}\prod_{j<k}(2\cos x_j - 2\cos x_k)^2 \cdot \prod_\ell \frac{2}{\pi}\sin^2 x_\ell dx_\ell\ . \tag{2.9}$$

The Weyl integration formula asserts that for g an $O(2N+2)$–central function on $O_-(2N+2)$, and $\tilde{g}$ as above

$$\int_{O_-(2N+2)} g(A)dA = \int_{[0,\pi]^N} \tilde{g}(x)dV_{O_-(2N+2)}(x)\ . \tag{2.10}$$

Remark.

(1) In all cases the measure $V_{G(N)}$ is visibly $\sum_N$–invariant. So for any function f on $[0,\sigma\pi]^N$, σ being 2 in the $U(N)$ case and 1 in the other cases, both f and its $\sum_N$–symmetrization have the same integral against $dV_{G(N)}$.

(2) In the case of either $SO(2N)$ or $O_-(2N)$, the normalized Haar measure is, by its uniqueness, invariant under $O(2N)$–conjugation. Therefore, any function f on $SO(2N)$ or $O_-(2N)$ has the same integral against Haar measure as its $O(2N)$–centralization, that is the function $x \mapsto \int_{O(2N} f(gxg^{-1})dg$.

We will next give another expression for the measures $V_{G(N)}$. This version generalizes and allows one to integrate out the x_j's one at a time, and this in turn, allows us to calculate various probabilities. The method is due to Gaudin and Mehta [GAU], [MEA] and is based on orthogonal polynomials and related projection operators. The formulae also show how the general $V_{G(N)}$ is built up from the $N = 1$ case.

We first recall the Vandermonde determinant. Given an integer $N \leq 1$ and N elements $f_1, \ldots, f_N$ in a (cumutative) ring R, the Vandermonde determinant, $\mathrm{Van}(f_1, \ldots, f_N)$ is the $N \times N$ determinant whose (i, j) entry is f_i^{j-1} $1 \leq i,\ j \leq N$. One has

$$\mathrm{Van}(f_1, \ldots,\ f_N) = \prod_{i<j} (f_j - f_i)\ . \tag{2.11}$$

Let T be a topological space, μ a positive Borel measure on T with finite mass. Let f be a bounded $\mathbb{C}$–valued function on T satisfying

$$\int_T f(t) d\mu(t) = 0, \quad \int_T |f(t)|^2 d\mu(t) = 1\ . \tag{2.12}$$

Suppose that for each $n \geq 1$, there is a monic polynomial $P_n(X) \in \mathbb{Z}[X]$ of degree n, such that the sequence $\phi_n,\ n \geq 0$ on T, defined by

$$\left.\begin{aligned} \phi_0 &= 1/\sqrt{\mu(T)} \\ \phi_n(t) &= P_n(f(t)) \qquad \text{for } n \geq 1 \end{aligned}\right\} \tag{2.13}$$

is an orthonormal sequence, i. e.

$$\int_T \phi_j(t)\overline{\phi_k(t)} d\mu(t) = \delta_{j,k}\ . \tag{2.14}$$

For $N \geq 1$ let T^N be the N–fold product of T with itself N–times and let $Pr_j \colon T^N \to T\ ;\ t \mapsto t_j$ be the j^{th} projection, $1 \leq j \leq N$. Denote by $K_N(x, y)$ the kernel on T^2

$$K_N(x, y) = \sum_{n=0}^{N-1} \phi_n(x)\phi_n(y) \tag{2.15}$$

K_N is the kernel for projection on span $\{\phi_0, \ldots, \phi_{N-1}\}$. For $n \geq 1$ define the function $D_{n,N}$ on T^n by

$$D_{n,N}(t_1, \ldots, t_n) = \det_{n\times n}\left(K_N(t_j,t_k)_{\substack{j=1,\ldots,n\\ k=1,\ldots,n}} \right) \quad . \tag{2.16}$$

Also set $D_{0,N} \equiv 1$ (as a function on T^0).

LEMMA 1. *With the above definitions, we have the following identities:*

(1)
$$\frac{1}{\sqrt{\mu(T)}}\, Van(f(t_1), \ldots, f(t_N)) = \det_{N\times N}\left((\phi_{i-1}(t_j))_{\substack{i=1,\ldots,N\\ j=1,\ldots,N}} \right)$$

(2)
$$\frac{1}{\mu(T)}|\, Van(f(t_1), \ldots, f(t_N))|^2 = D_{N,N}(t_1, \ldots, t_N)$$

(3) *For* $1 \leq n \leq N$

$$\int_T D_{n,N}(t_1, \ldots, t_n) d\mu(t_n) = (N+1-n) D_{n-1,N}(t_1, \ldots, t_{n-1})$$

(4) *For* $1 \leq n \leq N$*, the function* $D_{n,N}$ *on* T^n *is real, nonnegative and* $\sum_n$ *invariant. For* $n > N$*,* $D_{n,N} \equiv 0$.

(5) *For* $n \geq 1$*, let* F *be a symmetric function on* T^n*. Define* $F_{n,N}$ *on* T^N *by*

$$F_{n,N}(t_1, \ldots, t_N) = \sum_{1\leq i_1 < i_2 \cdots < i_n \leq N} F(t_{i_1}, \ldots, t_{i_n}) \, . \tag{2.17}$$

So $F_{n,N}$ *is symmetric on* T^N*. Let* $\mu_{n,N}$ *be the measure on* T^n *defined by*

$$d\mu_{n,N}(t_1, \ldots, t_n) = \frac{1}{n!} D_{n,N}(t_1, \ldots, t_n) d\mu(t_1) \cdots d\mu(t_n) \tag{2.18}$$

then we have the integration formula

$$\int_{T^N} F_{n,N}(t_1, \ldots, t_N) d\mu_{N,N}(t_1, \ldots, t_N) = \int_{T^n} F(t_1, \ldots, t_n) d\mu_{n,N}(t_1, \ldots, t_n) \, .$$

PROOF.

(1) For $n \geq 1$, $\phi_n = P_n(f) = f^n +$ lower terms. This means that we can pass from $\det_{N\times N(\phi_{i-1}(t_j))}$ to the $N \times N$ determinant whose i^{th} row for $i \geq 2$ is $f(t_j)^{i-1}$

and whose first row is $\phi_0(t_j) = \frac{1}{\sqrt{\mu(T)}}(1, \ldots, 1)$, by elementary row operations. Since the determinant does not change, we get the first assertion.

(2) Taking the square absolute value of (1) we get

$$
\begin{aligned}
\frac{1}{\mu(T)} |\mathrm{Van}(f(t_1), \ldots, f(t_N))|^2 &= |\det{}_{N\times N}(\phi_{i-1}(t_j))|^2 \\
&= \det{}_{N\times N}(\phi_{i-1}(t_j)) \times \det{}_{N\times N}(\overline{\phi_{i-1}(t_j)}) \\
&= \det{}_{N\times N}(\phi_{j-1}(t_i)) \times \det{}_{N\times N}(\overline{\phi_{i-1}(t_j)}) \\
&= \det{}_{N\times N}((i,j) \mapsto \textstyle\sum_{k=1}^{N} \phi_{k-1}(t_i)\overline{\phi_{k-1}}(t_j)) \\
&= D_{N,N}(t_1, \ldots, t_N)\ .
\end{aligned}
$$

(3) From the orthonormality of the ϕ_n's and the definition of K_N we have immediately

$$\int_T K_N(t,t)d\mu(t) = N \tag{2.19}$$

and

$$\int_T K_N(x,t)K_N(t,y)d\mu(t) = K_N(x,y)\ . \tag{2.20}$$

The first of these is the $n = 1$ case of the assertion. For $n > 1$, we expand $D_{n,N}$ by its n^{th} column:

$$D_{n,N}(t_1, \ldots, t_n) = \sum_{k=1}^{n} (-1)^{k+n} K_N(t_k, t_n)\ \mathrm{Cofactor}_{k,n}\ .$$

The term with $k = n$ is $K_N(t_n, t_n)D_{n-1,N}$ which integrates to give $N \times D_{n-1,N}(t_1, \ldots, t_{n-1})$. It remains to see that for each of the $n-1$ values of k from 1 to $n-1$, the term $(-1)^{k+n} K_N(t_k, t_n) \times$ $\mathrm{Cofactor}_{k,n}$ integrates to give $-D_{n_1,N}(t_1, \ldots, t_{n-1})$. For each such k, we expand $\mathrm{Cofactor}_{k,n}$ by its n^{th} row

$$\mathrm{Cofactor}_{k,n} = \sum_{\ell=1}^{n-1} (-1)^{n-1+\ell} K_N(t_n, t_\ell) \times \mathrm{Cofactor}_{(n,k),(\ell,n)}$$

where

$$\mathrm{Cofactor}_{(n,k),(\ell,n)} = \text{ the } (n,\ell) - \text{cofactor of } \mathrm{Cofactor}_{k,n}$$

is the $(n-2) \times (n-2)$ matrix obtained by removing the indicated rows and columns. So we obtain

$$
\begin{aligned}
(-1)^{k+n} K_N(t_k, t_n) \times \mathrm{Cofactor}_{k,n} = \sum_{\ell=1}^{n-1} (-1)^{k-1+\ell} K_N(t_n, t_\ell) K_N(t_k, t_n) \cdot \\
\mathrm{Cofactor}_{(n,k),(\ell,n)}\ .
\end{aligned}
$$

The term $\text{Cofactor}_{(n,k),(l,n)}$ is just the (k,ℓ) cofactor of $D_{n-1,N}$ (itself the (n,n)–cofactor of $D_{n,N}$), and $K_N(t_n,t_\ell)K_N(t_k,t_n)$ integrates to give $K_N(t_k,t_\ell)$. So after integration we get

$$\sum_{\ell=1}^{n-1}(-1)^{k-1+\ell}K_N(t_k,t_\ell)\times(\text{the } (k,\ell) \text{ cofactor of } D_{n-1,N})$$

which is precisely (-1) times (the expansion by minors along the k^{th} row) of $D_{n-1,N}$.

(4) In view of the integration formula (3), it suffices to treat the case $n\le N$, to show that $D_{N,N}$ is real, symmetric and nonnegative in its N–variables. This is obvious from (2) since the Vandermonde determinant transforms under $\sum_N$ by the sign character, and hence its square absolute value is symmetric as well as real and nonnegative.

To show that $D_{n,N}$ vanishes for $n>N$, think of $D_{n,N}$ as the $n\times n$ determinant made from $K_N(x,y)$. Introduce functions ψ_i on T for $i=0,\dots,n-1$ by defining

$$\psi_i=\phi_i \quad \text{for} \quad 0\le i\le N-1$$

and

$$\psi_i\equiv 0 \quad \text{for} \quad N\le i\le n-1\ .$$

Then it is trivially the case that

$$K_N(x,y)=\sum_{i=0}^{n-1}\psi(x)\overline{\psi(y)}$$

and hence (as in the proof of (2)) that

$$\det_{n\times n}(K_N(x_i,x_j))=|\det_{n\times n}(\psi_{i-1}(x_j))|^2\ .$$

In as much as the last row $(n>N)$ is zero, the determinant is zero.

(5) The $\sum_n$–invariance of $\mu_{n,N}$ is obvious from (4). If $n>N$, both $F_{n,N}$ and $\mu_{n,N}$ vanish identically, so the integration formula is true, but nugatory. For $n=N$, there is nothing to prove. So we take $1\le n<N$. Consider the integral

$$\int_{T^N}F_{n,N}(t_1,\dots,t_N)D_{N,N}(t_1,\dots,t_N)d\mu(t_1)\cdots d\mu t_N=$$
$$\sum_{1\le i_1<i_2\cdots<i_n\le N}\int_{T^N}F(t_{i_1},,\dots,t_{i_n})D_{N,N}(t_1,\dots,t_N)d\mu(t_1)\cdots d\mu(t_N)\ .$$

By symmetry, each summand is equal to

$$\int_{T^N} F(t_1,\dots,t_n)D_{N,N}(t_1,\dots,t_N)d\mu(t_1)\cdots d\mu(t_N)\ .$$

Using (3) to successively integrate out the variables $t_N, t_{N-1}, \dots, t_{n+1}$, we get

$$\int_{T^N} F(t_1,\dots,t_n)D_{N,N}(t_1,\dots,t_N)d\mu(t_1)\cdots d\mu(t_N) =$$
$$1.2.\cdots(N-n)\int_{T^N} F(t_1,\dots,t_n)D_{n,N}(t_1,\dots,t_n)d\mu(t_1)\cdots d\mu(t_n)\ .$$

Since there are $\binom{N}{n}$ summands we get

$$\frac{1}{N!}\int_{T^N} F_{n,N}(t_1,\dots,t_N)D_{N,N}(t_1,\dots,t_N)d\mu(t_1)\cdots d\mu(t_N) =$$
$$\frac{1}{N!}\binom{N}{n}\int_{T^n} F(T_1,\dots,t_n)D_{n,N}(t_1,\dots,t_n)d\mu(t_1)\cdots d\mu(t_n)$$

as required. This completes the proof of Lemma 1.

We now apply this Lemma to rewrite the Weyl measure $V_{G(N)}$ as being the measure $\mu_{N,N}$ on T^N for a suitable situation of the type (T,μ,f) considered in the Lemma. We proceed case by case according to the cases at the beginning of this section. Note that since $V_{O_-(2N+2)} = V_{USP(2N)}$ we do not discuss the $O_-(2N+2)$ case separately.

(I) The $U(N)$ case: The group $G(1) = U(1)$ is abelian, so $U(1)$ is its own space of conjugacy classes. We take for T this space of conjugacy classes viewed as being $[0,2\pi)$ endowed with Haar measure $d\mu = \frac{dx}{2\pi}$. For f, we take the function $f(x) = e^{ix}$, the character of the standard representation of $U(1)$. The powers f^n, $n\in\mathbb{Z}$, are orthonormal, so we may take $P_n(X) = X^n$. Since μ has total mass 1, we have $\phi_n = f^n$ for $n\ge 0$. For $N\ge 1$, $K_N(x,y)$ is

$$K_N(x,y) = \sum_{n=0}^{N-1} e^{in(x-y)}\ . \tag{2.21}$$

The measure $V_{U(N)}$ on $T^N = [0,2\pi)^N$ is equal to

$$\begin{aligned} dV_{U(N)} &= \frac{1}{N!}\prod_{j<k}|e^{ix_j}-e^{ix_k}|^2\ \prod_\ell \frac{dx_\ell}{2\pi}\\ &= \frac{1}{N!}\left|\mathrm{Van}(f(x_1),\dots,f(x_N))\right|^2\prod_\ell\frac{dx_\ell}{2\pi} \qquad (2.22)\\ &= \frac{1}{N!}\det{}_{N\times N}(K_N(x_j,x_k))\prod_\ell\frac{dx_\ell}{2\pi}\ . \end{aligned}$$

This is the measure $\mu_{N,N}$ attached to the data, $T = [0, 2\pi)$, $d\mu = \frac{dx}{2\pi}$, $f(x) = e^{ix}$. The measure $\mu_{n,N}$ attached to this data is

$$d\mu_{n,N}(x_1, \ldots, x_n) = \frac{1}{n!} \ \det{}_{n\times n}(K_N(x_j, x_k)) \cdot \prod_{\ell} dx_\ell/2\pi \ . \tag{2.23}$$

(II) The USP(2N) case: The group $G(1) = USP(2) = SU(2)$ has as its space of conjugacy classes the space $T = [0, \pi]$ and its corresponding Weyl measure is $\frac{2}{\pi}\sin^2 x dx = d\mu$.

For f we take the function

$$f(x) = 2\cos x = \frac{\sin 2x}{\sin x} \tag{2.24}$$

which is the character of the standard 3–dimensional representation of $SU(2)$. The group $SU(2)$ has a single irreducible representation of each degree $n \geq 1$ (namely Sym^{n-1} of the standard representation), whose character is

$$\frac{\sin nx}{\sin x} = \delta_{n,\mathrm{odd}} + \sum_{\substack{k=1 \\ k\equiv(n-1) \pmod 2}}^{n-1} (e^{ikx} + e^{-ikx}) \ . \tag{2.25}$$

¿From this it is clear that $\frac{\sin nx}{\sin x}$ is a $\mathbb{Z}$–monic polynomial of degree $n-1$ in $2\cos x = e^{ix} + e^{-ix}$. By the Peter Weyl Theorem (or elementarily) we know that the functions $\frac{\sin nx}{\sin x}$ are orthonormal on $[0, \pi]$ for the measure $d\mu$. So we have

$$\phi_n(x) = \frac{\sin(n+1)x}{\sin x} \ , \quad \text{for } n \geq 0 \ . \tag{2.26}$$

For $N \geq 1$, the function $K_N(x, y)$ is thus

$$K_N(x, y) = \frac{1}{\sin x \sin y} \sum_{n=1}^{N} \sin(nx)\sin(ny) \ . \tag{2.27}$$

The measure $V_{USP(2N)}$ on $T^N = [0, \pi]^N$ is equal to

$$\begin{aligned} \frac{1}{N!}\prod_{i<j}(2\cos x_i - 2\cos x_j)^2 \prod_{\ell} \frac{2}{\pi}(\sin^2 x_\ell)dx_\ell &= \frac{1}{N!} \ |\mathrm{Van}(f(x_1), \ldots, f(x_n))|^2 \prod_{\ell} \frac{2}{\pi}\sin^2 x_\ell dx_\ell \\ &= \frac{1}{N!} \ \det{}_{N\times N}(K_N(x_j, x_k)) \prod_{\ell} \frac{2}{\pi}\sin^2 x_\ell dx_\ell \ . \end{aligned}$$

The last is the measure $\mu_{N,N}$ attached to the data

$$T = [0, \pi],\ d\mu = \frac{2}{\pi} \sin^2 x dx,\ f(x) = 2\cos x\ . \tag{2.28}$$

Thus, the measure $\mu_{n,N}$ attached to this data is

$$d\mu_{n,N}(x_1, \dots, x_n) = \frac{1}{n!} \det_{n\times n}(K_N(x_j, x_\ell)) \cdot \prod_\ell \frac{2}{\pi} \sin^2 x_\ell dx_\ell\ . \tag{2.29}$$

(III) The $SO(2N+1)$ case: The group $G(1) = SO(3)$ has $T = [0, \pi]$ for its conjugacy classes and the Weyl measure on T is

$$d\mu = \frac{2}{\pi} \sin^2 \frac{x}{2} dx\ . \tag{2.30}$$

We take for f the function

$$f(x) = 1 + 2\cos x = (\sin \frac{3x}{2}) / \sin(\frac{x}{2}) \tag{2.31}$$

which is the character of the standard 3–dimensional representation of $SO(3)$. The group $SO(3)$ has a single irreducible representation of each odd degree $2n+1$ (namely sym^{2n} of the standard representation of $SU(2)$, viewed as a representation of $SU(2)/(\pm 1) = SO(3)$), whose character is

$$\frac{\sin((2n+1)x/2)}{\sin(x/2)} = 1 + \sum_{k=1}^{n} (e^{ikx} + e^{-ikx})\ . \tag{2.32}$$

This formula makes it clear that $\sin((2n+1)x/2)/\sin(x/2)$ is a monic $\mathbb{Z}$–polynomial of degree n in the quantity in equation (2.31). By the Peter Weyl Theorem (the part concerning orthonormality of characters of irreducibles) or by inspection, we know that the functions in (2.32) are orthonormal w.r.t. the measure in (2.30). So we have

$$\phi_n(x) = \frac{\sin((2n+1)x/2)}{\sin x/2}\ , \quad n \geq 0\ . \tag{2.33}$$

For $N \geq 1$,

$$K_N(x, y) = \frac{1}{(\sin x/2)(\sin y/2)} \sum_{n=0}^{N-1} \sin((2n+1)x/2) \sin((2n+1)y/2) \tag{2.34}$$

$dV_{SO(2N+1)}$ is equal to

$$\frac{1}{N!}\prod_{j<k}((1+2\cos x_j)-(1+2\cos x_k))^2\prod_m \frac{2}{\pi}\sin^2(\frac{x_m}{2})dx_m =$$
$$\frac{1}{N!}\det{}_{N\times N}(K_N(x_j,x_k))\prod_m \frac{2}{\pi}\sin^2(x_m/2)dx_m \; . \tag{2.35}$$

The last is the measure attached to the data (2.30), (2.31) and $T=[0,\pi]$. Hence the associated measure $\mu_{n,N}$ is

$$d\mu_{n,N}(x_1,\dots,x_n)=\frac{1}{n!}\det{}_{n\times n}(K_N(x_j,x_k))\prod_m \frac{2}{\pi}\sin^2(x_m/2)dx_m \; . \tag{2.36}$$

(IV) The $SO(2N)$ case: The group $G(1)=SO(2)$ is the same as the abelian group $U(1)$. The conjugation action of $O(2)/SO(2)=\pm 1$ on $SO(2)$ is inversion. The quotient of $SO(2)$ by $0(2)$–conjugation is the space $T=[0,\pi]$. We take for μ the measure

$$d\mu=\frac{dx}{2\pi} \tag{2.37}$$

on T, which has total mass $1/2$, which is half the direct image of Haar measure from $SO(2)$. For f we take

$$f(x)=2\cos x=e^{ix}+e^{ix} \tag{2.38}$$

which is the character of the standard two–dimensional representation of $SO(2)$. For $n\geq 1$ define

$$f_n(x)=2\cos nx=e^{inx}+^{-inx} \quad . \tag{2.39}$$

Each f_n is the character of the represention V_n of $SO(2)$ which is the sum of two inequivalent irreducible representations, and for $n\neq m$ the representations V_n and V_m have no common constituent. Intrinsically, the V_n for $n\geq 1$ are the restrictions to $SO(2)$ of the nontrivial irreducible representations of $O(2)$. Plainly f_n is a $\mathbb{Z}$–monic polynomial of degree n in $f(x)=e^{ix}+e^{ix}$. As before the functions $\{2\cos nx\}_{n\geq 1}$ are orthonormial on $[0,\pi]$ for $d\mu$ (this was the reason for the nomalization of μ). So

$$\left.\begin{array}{ll} & \phi_n = 2\cos(nx) \quad , \; n\geq 1 \\ \text{and} & \\ & \phi_0=\dfrac{1}{\sqrt{2}} \end{array}\right\} . \tag{2.40}$$

For $N \geq 1$, $K_N(x,y)$ is given by

$$K_N(x,y) = 2 + 4\sum_{n=1}^{N-1} \cos(nx)\cos(ny) \tag{2.41}$$

$V_{SO(2N)}$ on T^N is equal to

$$\begin{aligned} \frac{2}{N!}\prod_{j<k}(2\cos x_j - 2\cos x_k)^2 \prod_m \frac{dx_m}{2\pi} &= \frac{2}{N!}|\mathrm{Van}(f(x_1),\ldots,f(x_N))|^2 \prod_m \frac{dx_m}{2\pi} \\ &= \frac{1}{N!}\det{}_{N\times N}(K_N(x_j,x_k))\prod_m \frac{dx_m}{2\pi}\,. \end{aligned} \tag{2.42}$$

This then is the $\mu_{N,N}$ measure data attached to (35), (36). The $\mu_{n,N}$ measure is therefore

$$d\mu_{n,N}(x_1,\ldots,x_n) = \frac{1}{N!}\det{}_{n\times n}(K_N(x_j,x_k))\prod_m \frac{dx_m}{2\pi}\,. \tag{2.43}$$

For analytic purposes it will be convenient to rewrite the measures $\mu_{n,N}$ in terms kernels $L_N(x,y)$ which absorb the weights $\prod\limits_m \omega(x_n)dx_m$ in the above formula. We do so as follows.

$U(N)$: Here we don't do anything so set

$$L_N(x,y) = K_N(x,y) \tag{2.44}$$

$USP(2N)$ (and $0_-(2N+2)$): Define

$$\begin{aligned} L_N(x,y) &= 2\sin x \sin y K_N(x,y) \\ &= \sum_{n=1}^{N} 2\sin(nx)\sin(ny) \end{aligned} \tag{2.45}$$

Thus $\mu_{n,N}$ on $[0,\pi]^n$ is

$$d\mu_{n,N}(x_1,\cdots,x_n) = \frac{1}{n!}\det{}_{n\times n}(L_N(x_j,x_k))\prod_m \frac{dx_m}{\pi} \tag{2.46}$$

$SO(2N+1)$: Define

$$\begin{aligned} L_N(x,y) &= 2\sin(x/2)\sin(y/2)K_N(x,y) \\ &= \sum_{n=0}^{N-1} 2\sin((2n+1)x/2)\sin((2n+1)y/2) \end{aligned} \tag{2.47}$$

$$d\mu_{n,N}(x_1, \ldots, x_n) = \frac{1}{n!} \det_{n\times n}(L_N(x_j, x_k)) \prod_m \frac{dx_m}{\pi} \tag{2.48}$$

$SO(2N)$: Define

$$\begin{aligned} L_N(n, y) &= \frac{1}{2} K_N(x, y) \\ &= 1 + \sum_{n=1}^{N-1} 2\cos(nx)\cos(ny) \end{aligned} \tag{2.49}$$

so

$$\mu_{n,N}(x_1, \ldots, x_n) = \frac{1}{n!} \det_{n\times n}(L_N(x_j, x_k)) \prod_m \frac{dx_m}{\pi} \ . \tag{2.50}$$

Finally the series definning $L_N(x, y)$ may be added and expressed in terms of the function

$$S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)} \tag{2.51}$$

We get the following:
$U(N)$:

$$\det_{n\times n}(L_N(x_j, x_k)) = \det_{n\times n}(S_N(x_j - x_k)) \tag{2.52}$$

Indeed

$$L_N(x, y) = \sum_{n=0}^{N-1} e^{in(x-y)}$$

which is the value at $z = x - y$ of

$$e^{i(N-1)z/2} S_N(z) \ .$$

Therefore, the matrix made from $L_N(x_j, x_k)$ is conjugate to that made from $S_N(x_j - x_k)$ by the diagonal matrix with j^{th} entry $e^{i(N-1)x_j/2}$.
$USP(2N)$: One checks that

$$L_N(x, y) = \frac{1}{2} S_{2N+1}(x - y) - \frac{1}{2} S_{2N+1}(x + y) \tag{2.53}$$

$SO(2N + 1)$: One checks that

$$L_N(x, y) = \frac{1}{2}(S_{2N}(x - y) - S_{2N}(x + y)) \tag{2.54}$$

$SO(2N)$:

$$L_N(x, y) = \frac{1}{2}(S_{2N-1}(x - y) + S_{2N-1}(x + y))$$

We may summarize the formulae of this section with the following table. Let λ, ρ, τ and ϵ denote quantities below.

Table 1

$G(N)$	λ	σ	ρ	τ	ϵ
$U(N)$	0	2	1	0	0
$USP(2N)$	0	1	2	1	-1
$SO(2N+1)$	1/2	1	2	0	-1
$SO(2N)$	0	1	2	-1	1
$O_{-}(2N+2)$	1	1	2	1	-1

The measure $\mu_{n,N}$ on $[0, \sigma\pi]^n$ is

$$\frac{1}{n!} \det_{n \times n}(L_N(x_j, x_k)) \prod_m \frac{dx_m}{\sigma\pi} \tag{2.55}$$

with

$$L_N(x,y) = \frac{\sigma}{2}[S_{\rho N+\tau}(x-y) + \epsilon S_{\rho N+\tau}(x+y)] \ . \tag{2.56}$$

For later, we define

$$L_{N,-}(x,y) = \frac{\sigma}{2} S_{\rho N+\tau}(x-y) = \frac{1}{\rho} S_{\rho N+\tau}(x-y) \tag{2.57}$$

and

$$L_{N,+}(x,y) = \epsilon \frac{\sigma}{2} S_{\rho N+\tau}(x+y) \ . \tag{2.58}$$

These evaluations complete section 2.

3. Correlation Functions

In this section we introduce certain random variables Z on $(G(N), dA)$, called correlations. These measure the distribution in $\mathbb{R}^n$, $n \geq 2$, of subsets of the eigenvalue angles $X(A)$, of size n. We develop suitable estimates for the expectations and variances of these variables as $N \to \infty$. This will allow us in the next section to tackle the local consecutive spacing distributions. The key in the analysis that follows are the formulae of the last section.

We begin by introducing a class of test functions which we will use to measure the joint distributions. Let $\mathcal{T}(n)$ denote the space of functions F on $\mathbb{R}^n$ satisfying

(1) F is $\sum_n$– invariant.
(2) F is invariant under additive translation by the diagonal vector

$$\Delta_n(t) = (t, t, \ldots, t) \epsilon \mathbb{R}^n \ , \quad t \epsilon \mathbb{R} \ .$$

Condition (2) asserts that F depends on only the differences of the coordinates which is what the spacing distributions are concerned with. We denote by $\mathcal{T}_0(n)$ the space of F's as above which also satisfy:

(3) F has compact support modulo the diagonal – that is there is an $\alpha \geq 0$ such that $F(x) = 0$ whenever

$$\max_{j,k} |x_j - x_k| > \alpha \; . \tag{3.1}$$

(If F satisfies this last condition with some α, we write $\operatorname{supp}(F) \leq \alpha$). Condition (3) ensures that F measures the local spacings between coordinates.

Now given $A \epsilon G(N)$ we have its vector of eigenvalue angles $X(A)$ lying in $[0, \sigma\pi]^N$. For $F \epsilon \mathcal{T}_0(n)$ we define the random variable Z:

$$\begin{aligned} Z[n, F, G(N)](A) &= \frac{1}{N+\lambda} \sum_{\substack{S \subset X(A) \\ |S|=n}} F(\frac{(N+\lambda)}{\sigma\pi} S) \\ &= \frac{1}{N+\lambda} \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq N} F(\frac{(N+\lambda)}{\sigma\pi} (x_{j_1}(A), \ldots, x_{j_n}(A))) \end{aligned} \tag{3.2}$$

(The quantities σ, λ, etc. are those defined in Table 1 of section 2.) Note that the scale of $\frac{N}{\sigma\pi} X(A)$ normalizes these numbers on the line so that the mean spacing is 1. We also note that $Z[n, F, G(N)](A)$ is actually continuous on $G(N)$, if $G(N) \neq U(N)$. When $G(N) = U(N)$, we have artificially chosen the origin to be the point 1 on the unit circle. So in this case Z is bounded and corresponds to the naive spacing measures of section 1. Lemma 1 of section 2, and in particular part (IV) thereof, implies the following basic relation.

Proposition 1.

$$E(Z[n, F, G(N)]) = \frac{1}{N+\lambda} \int_{[0,\sigma\pi]^n} F((N+\lambda)x/\sigma\pi) d\mu_{n,N}(x) \; .$$

This relation allows us to investigate $E(Z[n, F, G(N)])$ as $N \to \infty$. The renormalized limiting density of the densities of $\mu_{n,N}$, is the density on $\mathbb{R}^n$

$$W_n(x_1, \ldots, x_n) = \det_{n \times n} \left(\frac{\sin \pi(x_j - x_k)}{\pi(x_j - x_k)} \right) \tag{3.3}$$

For $F \epsilon \mathcal{T}_0(n)$ set

$$\begin{aligned} E(n, F, \text{univ}) &= \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0, z_2, \ldots, z_n) W_n(0, z_2, \ldots, z_n) dz_2 \cdots dz_n \\ &= \frac{1}{(n-1)!} \int\limits_{[0,\alpha]^{n-1}} F(0, z_2, \ldots, z_n) W_n(0, z_2, \ldots, z_n) dz_2 \cdots dz_n \end{aligned} \tag{3.4}$$

Here we are assuming that supp$F \le \alpha$ and the notation $\int_{\text{order}}$ means the integral is over the set $z_2 \le z_3 \cdots \le z_n$. Since F and W are $\sum_n$–invariant, it follows that the two integrals in (3.4) must coincide.

The main result of this section is

THEOREM 2.

$$|E(Z[n, F, G(N)]) - E(n, F, \mathit{univ})| \le \|F\|_\infty (8\alpha)^{n-1}((\pi\alpha)^2 + \alpha + 1 + 10 \log N)/N \tag{1}$$

$$|E(Z[n, F, G(N)])| \le \|F\|_\infty 2(2\alpha)^{n-1}/(n-1)! \tag{2}$$

$$\mathit{Var}(Z[n, F, G(N)]) \le \frac{(3(8\alpha)^{n-1} + 65(8\alpha)^{2n-2})\|F\|_\infty}{N} . \tag{3}$$

The variance is the quantity $E((Z - E(Z))^2) = E(Z^2) - (E(Z))^2)$. Part (3) of the above theorem shows that Z concentrates about its mean with $O(1/\sqrt{N})$ deviation.

The rest of this section is devoted to the proof of Theorem 2. We begin with some estimates for $L_N(x, y)$ and the related determinants.

LEMMA 3. *For* $x, y \epsilon \mathbb{R}$,

$$|L_N(x, y)| \le \frac{2N}{\sigma},\ |L_{N,-}(x, y)| \le N + 1,\ \textit{and}\ |L_{N,+}(x, y)| \le N + 1$$

PROOF. Obvious from the series representations (42), (48) of section 2 and the definitions of $L_{N,-}(x, y)$ and $L_{N,+}(x, y)$.

LEMMA 4. $L_N(x, y)$ *is periodic in each variable with period* 4π *and*

$$\int_0^{4\pi} \int_0^{4\pi} |L_N(x, y)|^2 dx dy = 4\pi N$$

PROOF. Clear from (42), $\cdots$ (48) and Plancherel.

LEMMA 5. *For* $1 \le n \le N$ *and* $x \epsilon \mathbb{R}^n$

$$|\det{}_{n\times n}(L_N(x_i, x_j))| \le (\frac{2N}{\sigma})^n$$

and

$$|\det{}_{n\times n}(L_{N,-}(x_i, x_j))| \le (N + \frac{\tau}{2})^n$$

PROOF. Interpret $L_N(x_i, x_j)$ as the standard Hermitian inner product $\langle v(i), v(j)\rangle$ of vectors in $\mathbb{C}^N$, where $v(j)$ in $\mathbb{C}^N$ is the vector

$$\begin{aligned}
(1, e^{ix_j}, e^{2ix_j}, \dots, e^{i(N-1)x_j}) &\qquad \text{for } U(N),\\
\sqrt{2}(\sin x_j, \sin 2x_j, \dots, \sin Nx_j) &\qquad \text{for } USP(2N),\\
\sqrt{2}(\sin x_j/2, \sin 3x_j/2, \dots, \sin(2N-1)x_j/2)) &\qquad \text{for } SO(2N+1),\\
(1, \sqrt{2}\cos x_j, \sqrt{2}\cos(2x_j), \dots, \sqrt{2}\cos((N-1)x_j)) &\qquad \text{for } SO(2N)\ .
\end{aligned}$$

Now each $v(j)$ satisfies

$$\|v(j)\|^2 \le \frac{2N}{\sigma}$$

so our assertion follows from the following (presumably) well known inequality.

LEMMA 6. *Given $n \ge 1$ vectors $v(1), \dots, v(n)$ in a Hilbert space, we have*

$$|\det_{n\times n}(\langle v(i), v(j)\rangle)| \le \prod_j \|v(j)\|^2$$

PROOF. The truth of the asserted inequality is invariant under scaling the vectors $v(i)$ by positive real numbers β_i. This allows us to reduce to the case where all $v(i)$'s of length 1 (if any has $\|v(i)\| = 0$, the result is trivially true). In this case we must prove that

$$|\det_{n\times n}((v(i), v(j)))| \le 1$$

If the vectors are linearly dependent, then the determinant clearly vanishes and the result is true. So we may assume that the vectors are linearly independent. Then the matrix $A = (\langle v(i), v(j)\rangle)$ is the matrix of a positive definite Hermitian form (namely the Hilbert space inner product) on the n–dimensional space spanned by the $v(i)$, expressed in that basis. Therefore, the n–eigenvalues $\lambda_1, \dots, \lambda_n$ of A are real and positive. So $|\det A| = \prod_{j=1}^{n} \lambda_j$. By the geometric arithmetic mean inequality, we have

$$\begin{aligned}
|\det_{n\times n}(\langle v(i), v(j)\rangle)| &= (\det A)^{1/n}\\
&= \left(\prod_{j=1}^{n} \lambda_j\right)^{1/n} \le \frac{1}{n}\sum_{j=1}^{n}\lambda_j\\
&= \frac{1}{n}\,\text{trace}\,(A) = \frac{1}{n}\sum_{j=1}^{n}\langle v(j), v((j)\rangle = 1
\end{aligned}$$

This complete the proof of Lemma 6, and hence Lemma 5.

With these Lemma's we can now give the proof of part (2) of Theorem 1. Clearly we may assume that $n \leq N$, and that $\|F\|_\infty = 1$. ¿From Proposition 1

$$
\begin{aligned}
|E(Z[n,F,G(N)])| &\leq \frac{1}{N+\lambda}\int_{[0,\sigma\pi]^n} |F((N+\lambda)x/\sigma\pi)|d\mu_{n,N}(x) \\
&\leq \frac{1}{n!}\left(\frac{2N}{\sigma}\right)^n \frac{1}{N+\lambda}\int_{[0,\sigma\pi]^n} |F((N+\lambda)x/\sigma\pi)| \prod_j \frac{dx_j}{\sigma\pi} \\
&\qquad\qquad \text{by Lemma 3} \\
&\leq \frac{1}{n!}\left(\frac{2N}{\sigma}\right)^n \frac{1}{N+\lambda}\int_{[0,1]^n} |F((N+\lambda)x)| \prod_j dx_j \\
&\leq \frac{1}{n!}\left(\frac{2N}{\sigma}\right)^n \frac{1}{(N+\lambda)}\mathrm{Vol}(\Delta(n,\frac{\alpha}{N+\lambda})\cap[0,1]^n)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(n,\alpha) &= \{x\epsilon\mathbb{R}^n \,|\, \max_{i,j}|x_i-x_j| \leq \alpha\} \\
&\leq \frac{1}{n!}\left(\frac{2}{\sigma}\right)^n \mathrm{Vol}(\Delta(n,\frac{\alpha}{N})\cap[0,1])^n
\end{aligned}
$$

Clearly part (2) will follow from this and the following Lemma.

LEMMA 7. *: For $n \geq 2$, $\alpha \geq 0$, we have*

$$\frac{1}{n!}\,\mathit{Vol}(\Delta(n,\alpha)\cap[0,1]^n) \leq \frac{\alpha^{n-1}}{(n-1)!}\,.$$

PROOF.
The region $\Delta(n,\alpha)\cap[0,1]^n$ is stable by the symmetric group so that the ℓ.h.s. of the inequality to be proven is the volume of the region

$$
\begin{aligned}
0 \leq x_1 \leq x_2 \cdots \leq x_n \leq 1 \\
x_n - x_1 \leq \alpha
\end{aligned}
$$

This region is contained in

$$0 \leq x_1 \leq 1, \qquad 0 \leq x_1 \leq x_2 \cdots \leq x_n, \qquad x_n - x_1 \leq \alpha\,.$$

By the unimodular change of coordinates

$$
\begin{aligned}
&y_0 = x_1 \\
&y_i = x_{i+1} - x_1 \qquad \text{for } i = 1, \ldots, n-1\,,
\end{aligned}
$$

the last region is the product

$$[0,1] \times \{y\epsilon\mathbb{R}^{n-1}\,; \qquad 0 \leq y_1 \leq y_2 \cdots \leq y_{n-1} \leq \alpha\}$$

whose volume in $\mathbb{R}^n$ is that of

$$\{y\epsilon\mathbb{R}^{n-1}\,; \qquad 0 \leq y_1 \cdots \leq y_{n-1} \leq \alpha\}\ .$$

This in turn is a fundamental domain for the action of $\sum_{n-1}$ on the α–cube $[0,\alpha]^{n-1}$. Thus it has volume $\alpha^{n-1}/(n-1)!$. This completes the proof of Lemma 7 and hence of part (2) of Theorem 2.

Proof of Part 1 of Theorem 2: What underlies this analysis is the elementary limit

$$\lim_{N\to\infty} \frac{1}{N} S_N(2\pi x/N) = \frac{\sin \pi x}{\pi x}\ . \tag{3.5}$$

¿From this, Lemma 5 for the case of $U(N)$ and definition (3), we have

LEMMA 8.

$$|W_n(x_1, \dots, x_n)| \leq 1$$

The reader may verify for himself the following elementary inequalities:

For $-1 \leq x \leq 1$,

$$\left|\frac{\sin x}{x} - 1\right| \leq \frac{x^2}{5}\ . \tag{3.6}$$

For $-M \leq x \leq M \quad , \ M > 0$

$$\left|1 - \frac{x}{M\sin(x/M)}\right| \leq \frac{x^2}{4M^2}\ . \tag{3.7}$$

For $-M \leq x \leq M \quad , \ M > 0$

$$\left|\frac{\sin x}{x} - \frac{\sin x}{M\sin(x/M)}\right| \leq \frac{x^2}{4M^2}\ . \tag{3.8}$$

For $-M \leq x \leq M\,,\ M > 0\,, \quad \delta\epsilon\mathbb{R}$

$$\left|\frac{\sin x}{x} - \frac{\sin(1+\delta)x}{M\sin(x/M)}\right| \leq |\delta| + \frac{(1+|\delta|)x^2}{4M^2}\ . \tag{3.9}$$

The above allow us to deduce

LEMMA 9. *For the $G(N)$'s in Theorem 2:*

(1) *If* $|\pi x| \le N$

$$\left|\frac{\sin \pi x}{\pi x} - \frac{\sigma}{2(N+\lambda)} S_{\rho N+\tau}\left(\frac{\sigma \pi x}{N+\lambda}\right)\right| \le \frac{1}{2N} + \left(1+\frac{1}{2N}\right)\frac{\pi^2 x^2}{4N^2}$$

(2) *For any* x

$$\left|\frac{\sin \pi x}{\pi x} - \frac{\sigma}{2(N+\lambda)} S_{\rho N+\tau}\left(\frac{\sigma \pi x}{N+\lambda}\right)\right| \le \frac{1}{2N} + \frac{2\pi^2 x^2}{N^2} .$$

PROOF. According to the definition of S we have

$$\frac{\sigma}{2(N+\lambda)} S_{\rho N+\tau}\left(\frac{\sigma \pi x}{N+\lambda}\right) = \frac{\sin((\rho N+\tau)\pi x/\rho(N+\lambda))}{\rho(N+\lambda)\sin(\pi x/\rho(N+\lambda))} .$$

So (1) follows from (9) with

$$\begin{aligned} \delta &= (\rho N+\tau)/\rho(N+\lambda) - 1 = \frac{\tau - \rho\lambda}{\rho(N+\lambda)} \\ M &= \rho N + \tau \end{aligned}$$

and noting that $|\delta| \le 1/2N$.

If $|\pi x/N| > 1$, the second assertion is immediate since the terms being subtracted are of absolute value at most 1 and $(\rho N+\tau)/(\rho N+\rho\lambda) \le 1 + (1/2N)$.

$$\begin{aligned} \text{Set} \qquad W_{n,N}(x_1, \cdots, x_n) &= \det_{n\times n}\left(\frac{1}{N+\lambda} L_{N,-}\left(\frac{\sigma\pi x_i}{N+\lambda}, \frac{\sigma\pi x_i}{N+\lambda}\right)\right) \\ &= \det_{n\times n}\left(\frac{\sigma}{2(N+\lambda)} S_{\rho N+\tau}\left(\frac{\sigma\pi(x_i - x_j)}{N+\lambda}\right)\right) \end{aligned} \tag{3.10}$$

According to Lemma 5, we have

$$|W_{n,N}(x_1, \ldots, x_n)| \le \left(1 + \frac{1}{2N}\right)^n \le 2^n \tag{3.11}$$

LEMMA 10. *Let* $n \geq 2$, $N \geq 1$ *and* $\alpha \geq 0$ *and* $x \epsilon \mathbb{R}^n$ *with* $\max_{i,j} |x_i - x_j| \leq \alpha$. *Then*

$$|W_{n,N}(x_1, \cdots, x_n) - W_n(x_1, \ldots, x_n)| \leq n! \times n \times 2^{n-1} \times \left(\frac{1}{2N} + \frac{2\pi^2\alpha^2}{N^2} \right).$$

PROOF. The difference in question is the difference of determinants of $n \times n$ matrices, say $(a_{ij}\ )$ at (b_{ij}) which satisfy

$$|a_{ij} - b_{ij}| \leq \frac{1}{2N} + \frac{2\pi^2\alpha^2}{N^2} = \Delta \tag{3.12}$$

by Lemma 9, and

$$|a_{ij}| \quad \text{and} \quad |b_{ij}| \leq 2 = t \tag{3.13}$$

by Lemma 3.

So, if we expand the determinants as a sum over permutations, we see that the contribution of the permutation σ is at most

$$\left| \prod_i b_{i,\sigma(i)} - \prod_i a_{i,\sigma(i)} \right| = \left| \sum_{j=1}^{n} \left(\prod_{i<j} b_{i,\sigma(i)} \left(b_{j,\sigma(j)} - a_{j,\sigma(j)} \right) \right) \prod_{j<i} a_{i,\sigma(i)} \right| \leq n t^{n-1} \Delta \tag{3.14}$$

from which Lemma 10 follows.

We are ready to investigte the terms in part 1 of Theorem 2. We have

$$E(Z[n, F, G(N)]) = \frac{1}{N+\lambda} \quad \int_{[0,\sigma\pi]^n} F\left(\frac{(N+\lambda)x}{\sigma\pi}\right) d\mu_{n,N}(x)$$

(by Proposition 1)

$$= \frac{1}{n!(N+\lambda)} \quad \int_{[0,\sigma\pi]^n} F\left(\frac{(N+\lambda)x}{\sigma\pi} \right) \det_{n \times n}(L_N(x_i, x_j)) \times \prod_m \frac{dx_m}{\sigma\pi} \tag{3.15}$$

(by (53) of section 2).

To analyze (3.15), we introduce

$$\begin{aligned} E_-(Z[n, F, G(N)]) &= \frac{1}{(N+\lambda)} \quad \int_{[0,\sigma\pi]^n} F\left(\frac{(N+\lambda)x}{\sigma\pi} \right) d\mu_{-n,N}(x) \\ &= \frac{1}{n!(N+\lambda)} \quad \int_{[0,\sigma\pi]^n} \quad F\left(\frac{(N+\lambda)x}{\sigma\pi} \right) \det(L_{N,-}(x_i, x_j)) \prod_m \frac{dx_m}{\sigma\pi}. \end{aligned} \tag{3.16}$$

We will establish the following three Lemmas:

LEMMA 11.

$$|E(Z[n,F,G(N)]) - E_-(Z[n,F,G(N)])| \le \|F\|_\infty (8\alpha)^{n-1} 10 \log N/N \; .$$

LEMMA 12.

$$\left| E_-(Z[n,F,G(N)]) \; - \; \int_{[0,\alpha]^{n-1}(\text{order})} F(0,z) W_{n,N}(0,z) \prod_{j=2}^{n} dz_j \right|$$

$$\le 2(2\alpha)^n \|F\|_\infty / ((n-1)!N)$$

LEMMA 13.

$$\left| \int_{[0,\alpha]^{n-1}(\text{order})} F(0,z) W_{n,N}(0,z) \prod_{j=2}^{n} dz_j - E(n,F,univ) \right|$$

$$\le n^2 (2\alpha)^{n-1} \left(\frac{1}{2N} + \frac{2\pi^2\alpha^2}{N^2} \right) \|F\|_\infty \; .$$

One checks that these 3 Lemmas and the triangle inequality imply part 1 of Theorem 2. Now we have essentially proven Lemma 13, since according to the definition (4) and Lemma 10, the difference in question is at most

$$\int_{[0,\alpha]^{n-1}(\text{order})} n! n 2^{n-1} \left(\frac{1}{2N} + \frac{2\pi^2\alpha^2}{N^2} \right) \|F\|_\infty \prod_j dz_j$$

$$= n^2 (2\alpha)^{n-1} \left(\frac{1}{2N} + \frac{2\pi^2\alpha^2}{N^2} \right) \|F\|_\infty$$

as needed.

PROOF OF LEMMA 11. If $G(N)$ is $U(N)$ then $\mu_{n,N} = \mu_{-,n,N}$ and there is nothing to prove. In the other cases we have $\sigma = 1, \rho = 2$. We expand the $n \times n$ determinant of the $L_N(x,y)$ into $n!$ terms of the type $\operatorname{sgn}(\phi) \prod_i L_N(x_i, x_{\phi(i)})$ for ϕ in $\sum_n$. Writing

$$L_N(x,y) = L_{N,-}(x,y) + L_{N+}(x,y)$$

we expand the n-fold product $\prod_i L_N(x_i, x_{\phi(i)})$ into 2^n terms, corresponding to which factors we replace by $L_{N,-}(x,y)$ and which by $L_{N,+}(x,y)$. The choice of all "$-$" gives the $n \times n$ determinant of the $L_{N,-}(x,y)$: the remaining $(2^n - 1)n!$ terms are individually to be regarded as error terms. Thus the difference of the two integrals is a sum, with signs of $(2^n - 1)n!$ integrals of the form

$$\frac{1}{(N+\lambda)n!} \int_{[0,\pi]^n} F\left(\frac{(N+\lambda)x}{\pi}\right) \left(\prod_i L_{N,\pm}(x_i, x_{\phi(i)})\right) \prod_j \frac{dx_j}{\pi}$$

where in the product $\prod_j L_{N,\pm}(x_i, x_{\phi(i)})$ at least one $\pm$ is $+$. We choose one particular term $L_{N,+}(x_{i_o}, x_{\phi(i_o)})$ which has the $+$, and use the trivial estimate $\left|L_{N,\pm}(x,y)\right| \le N+1$, (Lemma 3), to deal with the $n-1$ other terms. Thus, each of the $(2^n-1)n!$ integrals is bounded in absolute value by one of the form

$$\frac{1}{n!N}(N+1)^{n-1} \int_{[0,\pi]^n} \left|F\left(\frac{(N+\lambda)x}{\pi}\right)\right| \left|L_{N,+}(x_{i_o}, x_{j_o})\right| \prod_j \frac{dx_j}{\pi}$$

Now $\mathrm{supp}(F) \le \alpha$, so $\left|F(\frac{(N+\lambda)x}{\pi})\right|$ is supported in the region

$$\Delta\left(n, \frac{\pi\alpha}{N}\right):\ \max_{i,j} |x_i - x_j| \le \frac{\pi\alpha}{N+\lambda} \le \frac{\pi\alpha}{N} .$$

Hence the above integral is, at most,

$$\frac{1}{n!N}(N+1)^{n-1}\|F\|_\infty \int\limits_{[0,\pi]^n \cap \Delta(n, \frac{\pi\alpha}{N})} |L_{N,+}(x_{i_o}, x_{j_o})| \prod_j \frac{dx_j}{\pi} .$$

Renumbering we may assume $i_o = 1$ and $j_o = 1$ or 2. We pass to coordinates

$$\begin{aligned} &x_1 \\ &\xi_j = x_j - x_1 \quad \text{for} \quad j = 2, \ldots, n . \end{aligned}$$

In these coordinates each $|\xi_j| \le \frac{\pi\alpha}{N}$, and so the domain of integration lies in the product region

$$[0,\pi] \times \left[\frac{-\pi\alpha}{N}, \frac{\pi\alpha}{N}\right]^{n-1} .$$

Thus

$$\int\limits_{[0,\pi]^n \cap \Delta(n, \frac{\pi\alpha}{N})} |L_{N,+}(x_1, x_{j_o})| \prod \frac{dx_j}{\pi} \le \int\limits_{[0,\pi]\times[\frac{-\pi\alpha}{N}, \frac{\pi\alpha}{N}]^{n-1}} |L_{N,+}(x_1, x_{j_o})| \frac{dx_1}{\pi} \prod_{j\ge 2} \frac{d\xi_j}{\pi}$$

$$= \int\limits_{[0,\pi]\times[\frac{-\pi\alpha}{N}, \frac{\pi\alpha}{N}]^{n-1}} |S_{2N+\tau}(2x_j + \xi_{j_o})| \frac{dx_1}{\pi} \prod_{j\ge 2} \frac{d\xi_j}{\pi} .$$

If $j_o = 1$, the integrand is a function of x_1 alone, and this is

$$= \left(\frac{2\alpha}{N}\right)^{n-1} \int_{[0,\pi]} |S_{2N+\tau}(2x_1)| \frac{dx_1}{2\pi}$$

$$= \left(\frac{2\alpha}{N}\right)^{n-1} \int_{[0,\pi]} \left|\frac{\sin(2N+\tau)x}{\sin x}\right| \frac{dx}{2\pi} .$$

If $j_o = 2$, the integrand is a function of x_1 and ξ_2 alone, and the integral is

$$\left(\frac{2\alpha}{N}\right)^{n-2} \int_{[\frac{-\pi\alpha}{N}, \frac{\pi\alpha}{N}]} \int_{[0,\pi]} |S_{2N+\tau}(2x+\xi)| \frac{dx}{2\pi} \frac{d\xi}{\pi}$$

$$\leq \left(\frac{2\alpha}{N}\right)^{n-1} \sup_{|\xi| \leq \frac{\alpha\pi}{N}} \int_{[0,\pi]} |S_{2N+\tau}(2x+\xi)| \frac{dx}{2\pi}$$

$$\leq \left(\frac{2\alpha}{N}\right)^{n-1} \sup_{|\xi| \leq \frac{\alpha\pi}{N}} \int_{[0,\pi]} \left|\frac{\sin(2N+\tau)(x+\xi/2)}{\sin(x+\xi/2)}\right| \frac{dx}{2\pi} .$$

In either case ($j_o = 1$ or 2) we apply the well known L^1–norm bound for the Dirichlet kernel [ ]

$$\int_0^{\pi} \left|\frac{\sin N(x+y)}{\sin(x+y)}\right| \frac{dx}{2\pi} \leq 2 + \log(N+1) \tag{3.17}$$

to conclude that

$$\int_{[0,\pi]^n \cap \Delta(n, \frac{\pi\alpha}{N})} |L_{N,+}(x_{i_o}, x_{j_o})| \prod_j \frac{dx_j}{\pi}$$

$$\leq \left(\frac{2\alpha}{N}\right)^{n-1} (2 + \log(2N+\tau-1))$$

$$\leq \left(\frac{2\alpha}{N}\right)^{n-1} 5 \log N .$$

This, and the above discussion, concludes the proof of Lemma 11.

We turn to Lemma 12.

PROOF OF LEMMA 12. Recall the definition of $E_-(Z[n,F,G(N)])$ in (3.16). If we make the change of variable

$$y = \frac{(N+\lambda)x}{\sigma\pi} \tag{3.18}$$

we get

$$E_-(Z(n,F,G(N)]) = \frac{1}{n!(N+\lambda)} \int_{[0,N+\lambda]^n} F(y) W_{n,N}(y) \prod_j dy_j \ . \tag{3.19}$$

A key point is that $W_{n,N}(y)$ is itself in $\mathcal{T}(n)$, i.e. it is both $\sum_n$–invariant (being of the form $\det_{n\times n}(f(y_i,y_j))$ and invariant by additive translations by all $\Delta_n(t) = (t,\dots,t)$, because $f(x,y)$ is of the form $g(x-y)$). The $\sum_n$–invariance allows us to write $E_-(Z(n,F,G(N)])$ as

$$\frac{1}{N+\lambda} \int\limits_{[0,N+\lambda]^n\,(\mathrm{order})} f(y) W_{n,N}(y) \prod_j dy_j \ .$$

Since $F(y)W_{n,N}(y)\epsilon\mathcal{T}_n$ we have

$$F(y)W_{n,N}(y) = F(0,y_2-y_1,\dots,y_n-y_1)W_{n,N}(0,y_2-y_1,\dots,y_n-y_1) \ .$$

Make the further change of variables

$$\begin{aligned} t &= y_1 \\ z_i &= y_i - y_1 \qquad , \quad i = 2,\dots,n \ . \end{aligned}$$

The integral for E_- becomes

$$\frac{1}{N+\lambda}\int_0^{N+\lambda} \int\limits_{[0,N+\lambda-t]^{n-1}\,(\mathrm{order})} F(0,z)W_{n,N}(0,z)\prod_{j=1}^n dz_j dt$$

$$= \frac{1}{N+\lambda}\int_0^{N+\lambda} g(t)dt$$

with

$$g(t) = \int\limits_{[0,N+\lambda-t]^{n-1}\,(\mathrm{order})} F(0,z)W_{n,N}(0,z)\prod_{j=2}^n dz_j \ .$$

Now F has supp $F \leq \alpha$, so for $z \epsilon [0, N+\lambda-t]^{n-1}$(order) the function $F(0,z)$ vanishes unless $z_n \leq \alpha$. So we may write the integral for $g(t)$ as

$$g(t) = \int\limits_{[0,\min(\alpha,N+\lambda-t)]^{n-1}(\text{order})} F(0,z) W_{n,N}(0,z) dz_2 \cdots dz_n \ .$$

Thus for $t \epsilon [0, N+\lambda]$

$$\begin{aligned} |g(t)| &\leq \int\limits_{[0,\alpha]^{n-1}(\text{order})} |F(0,z)W_{n,N}(0,z)| \, dz_2 \cdots dz_n \\ &\leq \frac{\alpha^{n-1}}{(n-1)!} \|F\|_\infty \|W_{n,N}\|_\infty \qquad (3.20) \\ &\leq \frac{2^n \alpha^{n-1}}{(n-1)!} \|F\|_\infty \ , \quad \text{by (11)} \ . \end{aligned}$$

In particular note that

$$\left| \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0,z) W_{n,N}(0,z) dz_2 \cdots dz_n \right| \leq \frac{2^n \alpha^{n-1}}{(n-1)!} \|F\|_\infty \ . \qquad (3.21)$$

To make the estimate claimed in Lemma 12, assume first that $N+\lambda > \alpha$: then we have

$$\frac{1}{(N+\lambda)} \int_0^{N+\lambda} g(t) dt = \frac{1}{N+\lambda} \int_0^{N+\lambda-\alpha} g(t) dt + \frac{1}{N+\lambda} \int_{N+\lambda-\alpha}^{N+\lambda} g(t) dt \qquad (3.22)$$

For $t \epsilon [0, N+\lambda-\alpha]$,

$$g(t) = \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0,z) W_{n,N}(0,z) dz_2 \cdots dz_n$$

so that the first term of (3.21) is

$$\frac{N+\lambda-\alpha}{N+\lambda} \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0,z)W_{n,N}(0,z)dz_2\cdots dz_n$$
$$= \left(1-\frac{\alpha}{N+\lambda}\right) \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0,z)W_{n,N}(0,z)dz_2\cdots dz_n \; . \tag{3.23}$$

We may estimate the second term in (3.23) by (using 3.20)

$$\frac{1}{N+\lambda}\int_{N+\lambda-\alpha}^{N+\lambda}|g(t)|\,dt \le \frac{\alpha}{N+\lambda}2^n\frac{\alpha^{n-1}}{(n-1)!}\,\|F\|_\infty \le \frac{(2\alpha)^n\,\|F\|_\infty}{N(n-1)!} \; .$$

Combining this with (3.23), we have

$$\left| E_-(Z[n,F,G(N)]) - \int\limits_{[0,\alpha]^{n-1}(\text{order})} F(0,z)W_{n,N}(0,z)dz_2\cdots dz_n \right| \le \frac{2(2\alpha)^n\|F\|_\infty}{(n-1)!N} \tag{3.24}$$

if $N+\lambda>\alpha$.

If $N+\lambda\le\alpha$, then the above holds trivially, since

$$|E_-(Z[n,F,G(N)])| \le \left|\frac{1}{N+\lambda}\int_0^{N+\lambda} g(t)dt\right| \le \frac{2^n\alpha^{n-1}}{(n-1)!}\|F\|_\infty$$

and we have already noted in (3.21) that the other term in Lemma 12 is also this small. This concludes the proof of Lemma 12, and hence, the proof of part 1 of Theorem 2, as well.

Proof of Part 3 of Theorem 2 (the variance): The variance of $Z[n,F,G(N)]$ may be expressed as

$$\mathrm{var}(Z) = E(Z^2[n,F,G(N)]) - E(Z[n,F,G(N)])^2 \; . \tag{3.25}$$

So we begin by analyzing the first term on the right hand side. According to its definition (see equation (3.2)) we have

$$(N+\lambda)^2Z^2 = \sum_{\substack{S\subset X(A)\\ |S|=|T|=n}}\sum_{T\subset X(A)} F\left(\frac{(N+\lambda)S}{\sigma\pi}\right)F\left(\frac{(N+\lambda)T}{\sigma\pi}\right) \; . \tag{3.26}$$

We may write this as

$$\sum_{\substack{S\subset\{1,\dots,N\}\\ T\subset\{1,\dots,N\}\\ |S|=|T|=n}} F\left(\frac{(N+\lambda)pr(S)(X)}{\sigma\pi}\right) F\left(\frac{(N+\lambda)pr(T)(X)}{\sigma\pi}\right) \tag{3.27}$$

where $pr(S) : \mathbb{R}^N \to \mathbb{R}^n$ is projection onto the n–coordinates defined by S. We break the sum (3.26) according to the value of $C := T \cup S$. For each subset C of $\{1, \dots, N\}$ of cardinality ℓ, define

$$h(C) = \sum_{C=T\cup S} F\left(\frac{(N+\lambda)pr(S)(X)}{\sigma\pi}\right) F\left(\frac{(N+\lambda)p(T)(X)}{\sigma\pi}\right) . \tag{3.28}$$

So

$$Z^2 = \frac{1}{N+\lambda} \sum_{n\le\ell\le 2n} \frac{1}{N+\lambda} \sum_{|C|=\ell} h(C) . \tag{3.29}$$

For each ℓ with $n \le \ell \le 2n$, it is tautological that the inner summand

$$\frac{1}{N+\lambda} \sum_{|C|=\ell} h(C)$$

is itself of the form $Z[\ell, H_\ell, G(N)](A)$ for H_ℓ the function on $\mathbb{R}^\ell$ defined by

$$H_\ell(x_1, \dots, x_\ell) = \sum_{\substack{C=T\cup S\\ |T|=|S|=n}} F(pr(S)(X))F(pr(T)(X)) . \tag{3.30}$$

Thus, we have

$$Z[n, F, G(N)]^2 = \sum_{n\le\ell\le 2n} \frac{1}{N+\lambda} Z[\ell, H_\ell, G(N)] \tag{3.31}$$

and hence

$$E(Z[n, F, G(N)]^2) = \sum_{n\le\ell\le 2n} \frac{1}{N+\lambda} E(Z[\ell, H_\ell, G(N)]) . \tag{3.32}$$

To make use of our analysis of expectations we need

LEMMA 14. *H_ℓ lies in $\mathcal{T}(\ell)$ and*

$$||H_\ell||_\infty \le \binom{\ell}{n}\binom{n}{\ell-n} ||F||_\infty^2 .$$

Moreover, if $\ell < 2n$, H_ℓ lies in $\mathcal{T}_o(\ell)$ and supp $(H_\ell) \le 2\alpha$.

PROOF. That $H_\ell \epsilon \mathcal{T}(\ell)$ follows from F being in $\mathcal{T}(n)$ and the definition of H_ℓ (eqn. (3.30)). That is the $\sum_\ell$ and $\Delta(t)$ invariance are immediate consequences of (3.30). Also the bound for $\|H_\ell\|_\infty$ follows from (3.30) by simply counting the number of possible S and T's contributing to the sum defining H. We first choose $T \subset \{1,\dots,\ell\}$, which can be done in $\binom{\ell}{n}$ ways. Having picked T, S may be any set of cardinality n which contains $C(T) = \{1,\dots,\ell\} - T$, a set of cardinality $\ell - n$. So picking S amounts to picking $S - C(T)$ which is any subset of cardinality $n - (\ell - n) = 2n - \ell$, of T. That can be done in $\binom{n}{2n-\ell} = \binom{n}{\ell-n}$ ways. So $\binom{\ell}{n}\binom{n}{\ell-n}$ gives the number of terms in the sum. Suppose now that $\ell < 2n$ we must show that supp $(H_\ell) \le 2\alpha$. In each of the terms $F(pr(T)(X))F(pr(S)(X))$ the two sets S and T cannot be disjoint (since $|S \cup T| < 2n$). So there is an index i_o which lies in both S and T. Any index j with $1 \le j \le \ell$ lies in either S or T. If j lies in S, then $F(pr(S)(X))$ vanishes if $|x_{i_o} - x_j| > \alpha$ because supp $F \le \alpha$. If j lies in T, then $F(pr(T)(X))$ vanishes if $|x_{i_o} - x_j| > \alpha$. So the product vanishes if $|x_{i_o} - x_j| > \alpha$ for any j. By the triangle inequality, $F(pr(T(X))F(pr(S)(X))$, and hence $H_\ell(X)$ itself vanishes if there are any two indices j and k such that $|x_k - x_j| > 2\alpha$. That is supp $H_\ell \le 2\alpha$.

Combining Lemma 14 with part (2) of Theorem 2, we get

PROPOSITION 15. *For $\ell < 2n$*

$$\left|\frac{1}{N+\lambda}E(Z[\ell, H_\ell, G(N)])\right| \le \|F\|_\infty^2 \binom{\ell}{n}\binom{n}{\ell-n} 2(2\alpha)^{\ell-1} \Big/ N(\ell-1)!\,.$$

¿From this, it is clear that the terms in (3.32) with $\ell < 2n$ will go to zero like $1/N$, as $N \to \infty$. We turn to a detailed look at the $\ell = 2n$ term,

$$\frac{1}{N+\lambda}E(Z[2n, H_{2n}, G(N)])\,. \tag{3.33}$$

LEMMA 16.

$$\left|\frac{1}{N+\lambda}E(Z[2n, H_{2n}, G(N)]) - E(Z[n, F, G(N)]^2)\right| \le 16\,\|F\|_\infty \binom{2n}{n}(4\alpha)^{2n-2} \Big/ N\,.$$

PROOF. According to Proposition 1, we have

$$\frac{1}{N+\lambda}E(Z[2n, H_{2n}, G(N)]) = \frac{1}{(N+\lambda)^2}\int_{[0,\sigma\pi]^{2n}} H_{2n}\left(\frac{(N+\lambda)x}{\sigma\pi}\right) d\mu_{2n,N}(x) \tag{3.34}$$

$$= \frac{1}{(N+\lambda)^2} \sum_{\substack{S\cup T=\{1,2,\dots,2n\}\\ |S|=|T|=n}} I(S,T) \tag{3.35}$$

where

$$I(S,T)=\int_{[0,\sigma\pi]^{2n}} F\left(\frac{(N+\lambda)pr(T)(x)}{\sigma\pi}\right)F\left(\frac{(N+\lambda)pr(T)(x)}{\sigma\pi}\right)d\mu_{2n,N}(x)\ . \tag{3.36}$$

For brevity of notation let

$$\begin{aligned} x(T)&=pr(T)(x)\\ f(x)&=F((N+\lambda)x/\sigma\pi)\ . \end{aligned} \tag{3.37}$$

So that in this notation

$$I(S,T)=\int_{[0,\sigma\pi]^{2n}} f(x(T))f(x(S))d\mu_{2n,N}(x)\ . \tag{3.38}$$

Because the measure $\mu_{2n,N}$ is $\sum_{2n}$–invariant and $\sum_{2n}$ acts transitively on the set of partitions (S,T) of $\{1,\dots,2n\}$ into disjoint sets of size n, we see that $I(S,T)$ is independent of (S,T). There are $\binom{2n}{n}$ such partitions. Fixing one such partition say $S=\{1,2,\dots,n\}$, $T=\{n+1,\dots,2n\}$ we have

$$\frac{1}{N+\lambda}E(Z[2n,H_{2n},G(N)])=\binom{2n}{n}\frac{1}{(N+\lambda)}I(S,T)\ . \tag{3.39}$$

Recall that the measure $\mu_{2n,N}$ is of the form (§2, eqn (53))

$$\frac{1}{(2n)!}\det{}_{2n\times 2n}(L_N(x_i,x_j))\prod_m \frac{dx_m}{\sigma\pi}\ .$$

Denote by

$$D_{2n,N}(x)=\det{}_{2n\times 2n}(L_N(x_i,x_j))\ . \tag{3.40}$$

Then from (3.39)

$$\begin{aligned} &\frac{1}{N+\lambda}E(Z[2n,H_{2n},G(N)])\\ &\qquad=\frac{1}{(n!)^2(N+\lambda)^2}\int_{[0,\sigma\pi]^{2n}} f(x(T))f(x(S))D_{2n,N}(x)\frac{dx_1}{\sigma\pi}\cdots\frac{dx_{2n}}{\sigma\pi} \end{aligned} \tag{3.41}$$

Now expand $D_{2n,N}(x)$ as a sum of $(2n)!$ terms of the form

$$\text{sgn }(\phi)\prod_i L_N(x_i,x_{\phi(i)})$$

indexed by $\phi \epsilon \sum_{2n}$. We distinguish two sorts of ϕ, those which respect the chosen partition (S,T), and those which do not. We group the corresponding terms

$$D_{\mathrm{resp},2n,N}(x) = \sum_{\text{respects } S \cup T} \mathrm{sgn}(\phi) \prod_i L_N(x_i x_{\phi(i)})$$

and

$$D_{\mathrm{noresp},2n,N}(x) = \sum_{\substack{\phi \\ \text{noresp } S \cup T}} \mathrm{sgn}(\phi) \prod_i L_N(x_i, x_{\phi(i)}).$$

Correspondingly the r.h.s. of (3.41) splits into a sum of two terms.

It is tautological that we have the product decomposition

$$D_{\mathrm{resp},2n,N}(x) = D_{n,N}(x(S)) D_{n,N}(x(T))$$

so that the first term in 3.41

$$\frac{1}{(n!(N+\lambda))^2} \int_{[0,\sigma\pi]^{2n}} f(x(T)) f(x(S)) D_{\mathrm{resp},2n,N}(x) \prod_{m=1}^{2n} \frac{dx_m}{\sigma\pi}$$

is the square of

$$\frac{1}{n!(N+\lambda)} \int_{[0,\sigma\pi]^n} f(x) D_{n,N}(x) \prod_{m=1}^{n} \frac{dx_m}{\sigma\pi} = E(Z[n,F,G(N)]) \, .$$

It follows that

$$\begin{aligned} &\frac{1}{N+\lambda} E(Z[2n, H_{2n}, G(N)]) - E(Z(n,F,G(N)]^2) \\ &\qquad = \frac{1}{(n!(N+\lambda))^2} \int_{[0,\sigma\pi]^{2n}} f(x(T)) f(x(S)) D_{\mathrm{noresp},2nN}(x) \prod_{i=1}^{2n} \frac{dx_i}{\sigma\pi} \end{aligned} \tag{3.42}$$

which will serve as our basis for proving the Lemma 16. Clearly, we must show that the absolute value of the r.h.s. of (3.42) is at most the r.h.s. of the inequality in the Lemma. For this it suffices to show that for each $\phi \epsilon \sum_{2n}$ which does not respect (S,T), of which there are most $(2n)!$, we have

$$\begin{aligned} &\left| \frac{1}{(n!(N+\lambda))^2} \int_{[0,\sigma\pi]^{2n}} f(x(T)) f(x(S)) \prod_i L_N(x_i x_{\phi(i)}) \prod_m \frac{dx_m}{\sigma\pi} \right| \\ &\qquad \leq \|F\|_\infty^2 \cdot 16 \cdot (4\alpha)^{2n-2} / N(n!)^2 \end{aligned} \tag{3.43}$$

To establish (3.43), look at the cycle decomposition of ϕ. At least one of its cycles contains elements of both S and of T. Renumbering, we may suppose that (S,T) is $(\{1,2,\dots,n\},\{n+1,\dots,n\})$, that both 1 and $n+1$ are both in the same ϕ–orbit, and that $\phi(1)=n+1$. Because the forward ϕ–orbit of $n+1$ does not stay in T, there is some $n+a>n$ and some $b\le n$ for which $\phi(n+a)=b$. We pay special attention to the two factors $L_N(x_1,x_{n+1})$, $L_N(x_{n+a},x_b)$ and use the trivial bound $|L_N(x_i,x_{\phi_i})|\le 2N$ (Lemma 3) for the other $2n-2$ factors. This yields that the L.H.S. of (3.43)

$$|LHS|\le \frac{(2N)^{2n-2}}{(n!(N+\lambda))^2}\int_{[0,\sigma\pi]^{2n}}|f(x(T))f(x(S))| \cdot |L_N(x_1,x_{n+1})L_N(x_{n+a},x_b)|\prod_i \frac{dx_i}{\sigma\pi}\ . \qquad (3.44)$$

Change variables to

$$\begin{aligned} s &= x_1 & \\ \xi_j &= x_j - x_1 & j=2,\dots,n \\ t &= x_{n+1} & \\ \eta_j &= x_{n+j}-x_{n+1} & j=2,\dots,n \end{aligned}$$

and put $\xi_1=\eta_1=0$.

In as much as supp $F\le\alpha$ we have $f(x(S))$ is supported in $\max_i|\xi_i|\le\frac{\sigma\pi\alpha}{N}$ and similarly $f(z(T))$ is supported in $\max_i|\eta_i|\le\frac{\sigma\pi\alpha}{N}$. Hence

$$\int_{[0,\sigma\pi]^{2n}}|f(x(T))f(x(S))|\,|L_N(x_1,x_{n+1})L_N(x_{n+a},x_b)|\prod\frac{dx_i}{\sigma\pi} \le \|F\|_\infty^2\int_{[\frac{-\sigma\pi\alpha}{N},\frac{\sigma\pi\alpha}{N}]^{2n-2}} Int_2(\xi,\eta)\prod\frac{d\xi_i}{\sigma\pi}\prod\frac{d\eta_j}{\sigma\pi} \qquad (3.45)$$

where

$$Int_2(\xi,\eta)=\int_{[0,\pi\sigma]^2}|L_N(s,t)L_N(t+\eta_a,s+\xi_b)|\,\frac{dsdt}{(\sigma\pi)^2}\ . \qquad (3.46)$$

It follows that

$$\int_{[0,\sigma\pi]^{2n}}|f(x(T))f(x(S))|\,|L_N(x_1,x_{n+1})L_N(x_{n+a},x_b)|\prod\frac{dx_i}{\sigma\pi} \le \|F\|_\infty^2\left(\frac{2\alpha}{N}\right)^{2n-2}\max_{\xi,\eta}|Int_2(\xi,\eta)|\ . \qquad (3.47)$$

But

$$|Int_2(\xi,\eta)| \leq 16 \int_{[0,4\pi]^2} |L_N(s,t)L_N(t+\eta_a, s+\xi_b)| \, \frac{ds}{4\pi}\frac{dt}{4\pi}$$

which by Cauchy–Schwartz is

$$\leq 16\|L_N(s,t)\|_2 \|L_N(t+\eta_a, s+\xi_b)\|_2$$

the L_2–norms being over $[0,4\pi]^2$ w.r.t. $\frac{ds\,dt}{16\pi^2}$. By Parseval applied to the definitions of $L_N(x,y)$ in §2, we have

$$\|L_N(S,t)\|_2 = \|L_N(S+\eta, t+\xi)\|_2 = \sqrt{N} \ .$$

Hence

$$|Int_2(\xi,\eta)| \leq 16N \ .$$

Applying this bound in (3.47) and that in (3.44) yields (3.43), and hence the Lemma.

We can now complete the proof of the variance bound in Theorem 2. Using (3.32), Proposition (15), and Lemma 16, we have

$$|\mathrm{var}(Z[n,F,G(N)])| \leq \frac{\|F\|_\infty^2}{N}\left(\sum_{\ell=n}^{2n-1}\binom{\ell}{n}\binom{n}{n-\ell} 2(2\alpha)^{\ell-1}\Big/ \ell! + 16\binom{2n}{n}(4\alpha)^{2n-2}\right) \ .$$

Now one checks that in the above ranges

$$\frac{\binom{\ell}{n}\binom{n}{\ell-n}}{\ell!} \leq 3 \ ,$$

also $\binom{2n}{n} \leq 2^{2n}$, so

$$\begin{aligned}
|\mathrm{var}(Z[n,F,G(N)])| &\leq \frac{\|F\|_\infty^2}{N}\left(6\sum_{\ell=n}^{2n-1}(2\alpha)^{\ell-1} + 2^{2n}16(4\alpha)^{2n-2}\right) \\
&\leq \frac{\|F\|_\infty^2}{N}\left(6n\left((2\alpha)^{n-1} + (2\alpha)^{2n-2}\right) + 2^{2n}\cdot 16(4\alpha)^{2n-2}\right) \\
&\leq \left(3(8\alpha)^{n-1} + 65(8\alpha)^{2n-2}\right)\frac{\|F\|_\infty^2}{N} \ .
\end{aligned}$$

This completes the proof of Theorem 2, and with it, this section.

4. Proofs of Theorems 0.1 and 0.2

Armed with the results of Section 3 we will prove the above two Theorems in this Section. Actually, in order to fully establish Theorem 0.2, we will need some sharp estimates on the sizes of the tails of the distributions μ_a (univ). The verification of these estimates is left for Section 5.

We begin with some combinatorics. Let $X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ be ordered, that is $x_1 \leq x_2 \leq x_3 \cdots \leq x_N$. We view these as N–points on a line (with possible multiplicities). For $s \geq 0$ the number of adjacent, or 0–separated, pairs of x_j's in X which are at most distance s apart will be denoted by $S_0(s, x_1, \ldots, x_N)$ or $S(s, X)$ or simply $S_0(s)$ if X is understood. That is

$$S_0(s, X) = |\{1 \leq j \leq N-1 : x_{j+1} - x_j \leq s\}| \,. \tag{4.1}$$

More generally, if $b \geq 0$ is an integer, the number of b–separated pairs which are at most s apart is denoted by

$$S_b(s, X) = |\{1 \leq j \leq N-b : x_{j+b+1} - x_j \leq s\}| \,. \tag{4.2}$$

The naive spacing measures $\mu_a(A,\ \text{naive})$ may be expressed in terms of the functions S_b. For $A \in G(N)$ and $X(A)$ its vector (ordered) of eigenvalue angles in $[0, \sigma\pi]^N$ (see Section 2), we have on interpreting the definitions

$$\mu_a(A,\ \text{naive},\ G(N))[0, s] = \frac{1}{N} S_{a-1}\left(s, \frac{N}{\sigma\pi} X(A)\right) \,. \tag{4.3}$$

In order to study S_b further, we express these functions in terms of the correlation functions Z. Let $C_b(s, X)$ denote the number of subsets of $\{x_1, \ldots, x_N\}$ of cardinality $b+2$ whose extreme points are a distance at most s apart. That is

$$C_b(s, X) = |\{1 \leq j_1 < j_2 \cdots < j_{b+2} \leq N | x_{j_{b+2}} - x_{j_1} \leq s\}| \tag{4.4}$$

$$= |\{B \subset \{1, \ldots, N\} | B = \{j_1, \ldots, j_{b+2}\}, \max_{j_r, j_s \in B} |x_{j_r} - x_{j_s}| \leq s\}| \,. \tag{4.5}$$

Both C_b and S_b vanish for $b >> 0$ (in fact $b > N-2$). There is a simple relation between the C's and S's.

LEMMA 1.

$$C_b(s) = \sum_{t \geq b} \binom{t}{b} S_t(s) \ .$$

PROOF. For each $t \geq b$ consider all tuples $1 \leq j_1 < j_2 \cdots < j_{b+2} \leq N$ with $j_{b+2} - j_1 - 1 = t$ and $x_{j_{b+2}} - x_{j_1} \leq s$. The two end points j_1, j_{b+2} contribute one unit to $S_t(s)$. On the other hand, for each subset B of $j_1 + 1,\ j_1 + 2\,, \cdots,\ j_1 + t (= j_{b+2} - 1)$, of size b we get a tuple (j_1, B, j_{b+2}) which conributes one unit to $C_b(s)$. There are $\binom{t}{b}$ such subsets and as we vary over all $t \geq b$, we get precisely the entire contribution to $C_b(s)$. This establishes the Lemma. Note that the sum is finite ($t \leq N - 2$).

COROLLARY 2. *In the indeterminate T, we have the identities*

$$\sum_{b \geq 0} C_b(s) T^b = \sum_{b \geq 0} S_b(s)(1+T)^b$$

and

$$\sum_{b \geq 0} S_b(s) T^b = \sum_{b \geq 0} C_b(s)(T-1)^b \ .$$

PROOF. The two are of course trivially equivalent. The first is equivalent to Lemma 1, as is clear, if we expand $(1+T)^b$ in the binomial expansion and collect terms.

If we equate like powers of T in the second identity in Corollary 2, we get

COROLLARY 3.

$$S_b(s) = \sum_{n \leq b} (-1)^{n-b} \binom{n}{b} C_n(s) \ .$$

On truncating the series above, we get inequalities for $S_b(s)$ as the following Lemma shows.

LEMMA 4. *For any integer $m \geq b$ we have*

$$S_b(s) \geq \sum_{n=b}^{m} (-1)^{n-b} \binom{n}{b} C_n(s) \qquad \text{if } m-b \ \text{ is odd}$$

and

$$S_b(s) \leq \sum_{n=b}^{m} (-1)^{n-b} \binom{n}{b} C_n(s) \qquad \text{if } m-b \text{ is even}\,.$$

PROOF. Fix $m \geq b$. According to Corollary 3, these inequalities will follow from

$$(-1)^{m+1-b} \sum_{n \geq m+1} (-1)^{n-b} \binom{n}{b} C_n(s) \geq 0$$

that is

$$\sum_{n\geq m+1}(-1)^{n-m-1}\binom{n}{b}C_n(s)\geq 0\ .$$

Now according to Lemma 2, this is the same as

$$\sum_{n\geq m+1}(-1)^{n-m-1}\binom{n}{b}\sum_{t\geq n}\binom{t}{n}S_t(s)\geq 0$$

that is

$$\sum_{t\geq m+1}S_t(s)\sum_{n=m+1}^{t}(-1)^{n-m-1}\binom{t}{n}\binom{n}{b}\geq 0\ .$$

Since $S_t(s)\geq 0$, it suffices to show that

$$\sum_{n=m+1}^{t}(-1)^{n-m-1}\binom{t}{n}\binom{n}{b}\geq 0$$

For $0\leq b\leq n\leq t$ we use the elementary identity

$$\binom{t}{n}\binom{n}{b}=\binom{t}{b}\binom{t-b}{n-b}$$

reducing us to show that

$$\binom{b}{t}\sum_{n=m+1}^{t}(-1)^{n-m-1}\binom{t-b}{n-b}\geq 0\ .$$

Setting $k=n-b$, $j=m+1-b$, $\ell=t-b$ we are left with showing that

$$\sum_{k=j}^{\ell}(-1)^{k-j}\binom{\ell}{k}\geq 0\ . \tag{4.6}$$

But

$$\sum_{k=j}^{\ell}(-1)^{k-j}\binom{\ell}{k}=\binom{\ell-1}{j-1} \tag{4.7}$$

which establishes (4.6) and hence the Lemma.

To see (4.7) we use Pascal's triangle identity

$$\binom{\ell}{k}=\binom{\ell-1}{k}+\binom{\ell-1}{k-1}$$

to write the $\ell.h.s.$ of (4.7) as

$$\sum_{k=j}^{\ell}(-1)^{k-j}\binom{\ell-1}{k}+\sum_{k=j}^{\ell}(-1)^{k-j}\binom{\ell-1}{k-1}$$

which telescopes to $\binom{\ell-1}{j-1}$.

With this we are set to establish Theorem 0.1 (bis). From (4.3) and Corollary 3, we have

$$\mu_a(A,\ \text{naive},\ G(N))[0,s]=\sum_{n\geq a-1}(-1)^{n-a+1}\binom{n}{a-1}\frac{C_n\left(s,\frac{NX(A)}{\sigma\pi}\right)}{N}\ . \tag{4.8}$$

According to the definitions of $C_n(s)$ in (4.5) and of $Z[n,F,G(N)]$ in equation (3.2) of Section 3, we have

$$\frac{1}{N}C_n(s,\frac{N}{\pi\sigma}X(A))=Z[n+2,F_{n+2,s},G(N)](A) \tag{4.9}$$

where

$$F_{m,s}(x_1,\dots,x_m)=\begin{cases}1 & \text{if } \max\limits_{j,k}|x_j-x_k|\leq s\\ 0 & \text{otherwise}\end{cases}\ . \tag{4.10}$$

Note that $F_{m,s}\in\mathcal{T}_0(m)$ and has

$$\left.\begin{aligned}&\text{supp}\ F_{m,s}\leq s\\ &\|F_{m,s}\|_\infty=1\end{aligned}\right\} \tag{4.11}$$

So we may apply the results of Section 3 and, in particular, Theorem 2 of that Section to $Z[n+2,F_{n+2,s},G(N)](A)$. Substituting (4.9) into (4.8) yields

$$\mu_a(A,\ \text{naive},G(N))[0,s]=\sum_{n\geq a-1}\binom{n}{a-1}Z[n+2,F_{n+2,s},G(N)](A)\ . \tag{4.12}$$

Taking expectations w.r.t. A over $G(N)$ yields

$$E(\mu_a(\text{naive},G(N)[0,s])=\sum_{n\geq a-1}(-1)^{n-a+1}\binom{n}{a-1}E(Z[n+2,F_{n+2,s},G(N)])\ . \tag{4.13}$$

According to Theorem 2 of Section 3, we have

$$|E(Z(m,F_{m,s},G(N)])|\leq\frac{2(2s)^{m-1}}{m-1} \tag{4.14}$$

and

$$\lim_{N\to\infty} E(Z[m, F_{m,s}, G(N)]) = E(Z[m, F_{m,s}, \text{ univ}]) \ . \tag{4.15}$$

Hence for s fixed we may apply the dominated convergence theorem in (4.13) to conclude that

$$\lim_{N\to\infty} E(\mu_a(\text{naive}, \ G(N)[0,s]) = \sum_{n\geq a-1} (-1)^{n-a+1} \binom{n}{a-1} E(Z[n+2, F_{n+2,s}, \text{univ}]) \tag{4.16}$$

which according to (3.4) of Section 3

$$= \sum_{n\geq a-1} (-1)^{n-a+1} \binom{n}{a-1} \frac{1}{(n+1)!} \int\limits_{[0,s]^{n+1}} F_{n+2}(0, x_2, \dots, x_{n+2}) \cdot W_{n+2}(0, x_2, \dots, x_{n+2}) dx_2 \dots dx_{n+2} \tag{4.17}$$

$$= \sum_{m\geq a} (-1)^{m-a} \binom{m-1}{a-1} \frac{1}{m!} \int\limits_{[0,s]^m} W_{m+1}(0, x_2, \dots, x_{m+1}) dx_2 \cdots dx_{m+1} \tag{4.18}$$

$$:= H_a(s) \ . \tag{4.19}$$

Now Lemma 8 of Section 3 asserts that $|W_m(x_1, \cdots x_m)| \leq 1$, so we see (again) that the series (4.18) is absolutely and rapidly convergent (for fixed s). And in fact one easily concludes that $H_a(s)$ is smooth in s. Moreover, since $E(\mu_a(A, \text{naive}, G(N))[0,s]$ is a positive measure (of mass $\frac{N-a}{N}$) it follows that $H_a(s)$ is nondecreasing as a function of $s \geq 0$. Hence

$$d\mu_a(s) = d\mu_a(\text{universal})(s) = \frac{dH_a}{ds}(s) \cdot ds \tag{4.20}$$

defines an absolutely continuous measure on $\mathbb{R}_{\geq 0}$. It is in fact a probability measure (i.e. has total mass equal to 1). The reason being that from the definition of $\mu_a(A, \text{naive}, G(N))$ we see that its mean is at most equal to a. Hence $E(\mu_a(\text{naive}, G(N))$ has mean at most equal to a. The total mass of $E(\mu_a(\text{naive}, G(N)))$ is $\frac{N-a}{N}$. Thus, for any $T > 0$

$$
\begin{aligned}
E(\mu_a(\text{naive}, G(N)[0 < T]) &= \frac{N-a}{N} - \int_T^\infty d(E(\mu_a(\text{naive}, G(N)))(t) \\
&\geq \frac{N-a}{N} - \frac{1}{T}\int_T^\infty td(E(\mu_a(\text{naive}, G(N)))(t) \\
&\geq \frac{N-a}{N} - \frac{1}{T}\int_0^\infty td(E(\mu_a(\text{naive}, G(N)))(t) \\
&\geq \frac{N-a}{N} - \frac{a}{T} \,.
\end{aligned}
$$

Letting $N \to \infty$ and applying (4.16) gives

$$
H_a(T) = \int_0^T d\mu_a(t) \geq 1 - \frac{a}{T} \,. \tag{4.21}
$$

It follows that $H_a(\infty) = 1$ and so μ_a is a probability measure and it satisfies the tail estimate (4.21). We will derive in Section 5 a much sharper estimate for the tail of μ_a, and will also show that μ_a indeed has mean a (the above ensure only that the mean of μ_a is at most a).

(4.16) asserts that for $s \geq 0$

$$
\lim_{N\to\infty} E(\mu_a(\text{naive}, G(N))[0, s] = \mu_a[0, s] \,. \tag{4.22}
$$

¿From this, (4.21) and an easy approximation argument it follows that for any bounded continuous function f on $\mathbb{R}_{\geq 0}$

$$
\lim_{N\to\infty} \int_0^\infty f(x)d(E(\mu_a(\text{naive}, G(N))(x) = \int_0^\infty f(x)d\mu_a(x) \,. \tag{4.23}
$$

Either (4.22) or (4.23) are the content of Theorem 0.2 bis, and in view of the comments at the end of Section 1, we have established Theorem 0.1 (with μ_a given by (4.20)).

We turn our attention now to Theorem 0.2, or rather its proof. Fix $a \geq 1$ an integer. For $s \geq 0$ let

$$
\Delta(s, A) = |\mu_a(A, \text{naive}, G(N))[0, s] - \mu_a[0, s]| \tag{4.24}
$$

be the deviation of the measure given to $[0,s]$ by $\mu_a(A)$ to that given by μ_a. By definition

$$D(\mu_a(A, \text{naive}, G(N)), \mu_a) = \sup_{0 \le s < \infty} \Delta(s, A)\ . \tag{4.25}$$

According to (4.3) and Lemma 4, we have for any $m \ge a-1$

$$\mu_a(A, \text{naive})[0,s] \le \frac{1}{N} \sum_{n=a-1}^{m} (-1)^{n-a+1} \binom{n}{a-1} C_n\left(s, \frac{NX(A)}{\sigma\pi}\right) \tag{4.26}$$

if $m-a+1$ is even and

$$\mu_a(A, \text{naive})[0,s] \ge \frac{1}{N} \sum_{n=a-1}^{m} (-1)^{n-a+1} \binom{n}{a-1} C_n\left(s, \frac{NX(A)}{\sigma\pi}\right) \tag{4.27}$$

if $m-a+1$ is odd.
In terms of Z we have

$$\mu_a(A, \text{naive})[0,s] \le \sum_{n=a-1}^{m} (-1)^{n-a+1} \binom{n}{a-1} Z[n+2, F_{n+2,s}, G(N)](A) \tag{4.28}$$

if $m-a+1$ is even, and

$$\mu_a(A, \text{naive})[0,s] \ge \sum_{n=a-1} (-1)^{n-a+1} \binom{n}{a-1} Z[n+2, F_{n+2,s,} G(N)](A) \tag{4.29}$$

if $m-a+1$ is odd.

Taking expectations in (4.28) and (4.29) and letting $N \to \infty$ we get (using Theorem 2 of Section 3 and (4.22))

$$\mu_a[0,s] \le \sum_{n=a-1}^{m} (-1)^{n-a+1} \binom{n}{a-1} Z[n+2, F_{n+2,s}, \text{univ}] \tag{4.30}$$

if $m-a+1$ is even, while

$$\mu_a[0,s] \ge \sum_{n=a-1}^{m} (-1)^{n-a+1} \binom{n}{a-1} Z[n+2, F_{n+2,s}, \text{univ}] \tag{4.31}$$

if $m-a+1$ is odd.

LEMMA 5. *For $s \geq 0$ and any integer $L \geq a-1$,*

$$\Delta(s,A) \leq \sum_{k=a-1}^{L} \binom{k}{a-1} \left| Z[k+2, F_{k+2,s}, G(N)](A) - Z[k+2, F_{k+2,s}, univ] \right|$$

$$+ \binom{L}{a-1} Z[L+2, F_{L+2,s}, univ] + \binom{L+1}{a-1} Z[L+3, F_{L+3,s}, univ] \, .$$

PROOF. Let $m \geq a$ and be of the same parity as $a-1$. Assume first that $\mu_a[0,s] \geq \mu_a(A, \text{naive})[0,s]$, then

$$0 \leq \mu_a[0,s] - \mu_a(A,\text{naive})[0,s] \leq \sum_{k=a-1}^{m} (-1)^{k-a+1} \binom{k}{a-1} Z[k+2, F_{s,k+2}, \text{univ}]$$

$$- \sum_{k=a-1}^{m-1} (-1)^{k-a+1} \binom{k}{a-1} Z[k+2, F_{k+2,s}, G(N)](A)$$

according to (4.28),..., (4.31)

$$\leq \binom{m}{a-1} Z[m+2, F_{m+2,s}, \text{univ}] + \sum_{k=a-1}^{m-1} \binom{k}{a-1} \Big| Z[k+2, F_{k+2,s}, \text{univ}]$$

$$- Z[k+2, F_{k+2,s}, G(N)](A) \Big| \, .$$

If $m = L$, this implies the asserted inequality. If $L = m+1$ is of the right parity, then the above inequality with m replaced by $m+2$ implies the claimed inequality.

Now suppose that $\mu_a(A)[0,s] \geq \mu_a[0,s]$, then

$$0 \leq \mu_a(A,\text{naive})[0,s] - \mu_a[0,s] \leq \sum_{k=a-1}^{m} (-1)^{k-a+1} \binom{k}{a-1} Z[k+2, F_{k+2,s} G(N)](A)$$

$$- \sum_{k=a-1}^{m+1} (-1)^{k-a+1} \binom{k}{a-1} Z[k+2, F_{k+2,s}, \text{univ}]$$

according to (4.28), (4.29) $\cdots$ (4.31)

$$\leq \binom{m+1}{a-1} Z[m+3, F_{s,m+3}, \text{univ}] + \sum_{k=a-1}^{m} \binom{k}{a-1} \Big| Z[k+2, F_{k+2,s}, G(N)](A)$$

$$- Z[k+2, F_{k+2,s}, \text{univ}] \Big| \, .$$

Now if L is one of m or $m+1$, the desired inequality follows from the above.

Invoking the bounds (3.1) and (3.2) of Theorem 2 of Section 3, the triangle inequality

$$\begin{aligned}\big|Z[k+2, F_{s,k+2}, G(N)](A) - Z[k+2, F_{s,k+2}, \mathrm{univ}]\big| \\ \leq \big|Z[k+2, F_{s,k+2}, G(N)](A) - E(Z[k+2, F_{s,k+2}, G(N)])\big| \\ + \big|E(Z[k+2, F_{s,k+2}, G(N)]) - Z[k+2, F_{s,k+2}, \mathrm{univ}]\big|\end{aligned}$$

and the inequality, $\binom{k}{a-1} \leq 2^k$, we get

Corollary 6. *For $L \geq a$,*

$$\begin{aligned}\Delta(s, A) \leq \sum_{k=a-1}^{L} 2^L \big|Z[k+2, F_{s,k+2}, G(N)](A) - E(Z[k+2, F_{s,k+2}, G(N)])\big| \\ + \frac{(4s)^{L+1}(s+1)}{(L+1)!} + \frac{10(s^{L+1}+1) \cdot 16^{L+1} \cdot L(s^2 + \log N)}{N} .\end{aligned}$$

In order to estimate the supremum over $s \geq 0$ of $\Delta(s, A)$ as required in (4.25), we proceed in the most direct (but crude) way. Let M be a second parameter, which like L, will be chosen to go to infinity as a function of N. M will be the number of points s_j, $j = 1, \ldots, M$ at which we invoke Corollary 6. Choose $0 < s_1 < s_2 \cdots < s_M < \infty$ such that

$$H_a(s_j) = \mu_a[0, s_j] = \frac{j}{M+1} \tag{4.32}$$

Since $H_a(s)$ is continuous and increasing with $H_a(0) = 0$ and $H_a(\infty) = 1$, such points s_j can be found (actually it is not difficult to see from the expression (4.18) for $H_a(s)$ that H_a is analytic in s*, and hence H_a is strictly increasing so that the s_j's are in fact unique!). Also set $s_0 = 0$, $s_{M+1} = \infty$. If $s_j \leq s \leq s_{j+1}$, then clearly

$$\Delta(s, A) \leq \frac{1}{M+1} + \max\{\Delta(s_j, A), \Delta(s_{j+1}, A)\} . \tag{4.33}$$

Hence

$$\sup_s \Delta(s, A) \leq \frac{1}{M+1} + \max_{0 \leq j \leq M}\{\Delta(s_j, A)\} . \tag{4.34}$$

---

*See Section 5

Applying Corollary 6 yields

$$\sup_s \Delta(s,A) \leq \frac{1}{M+1} + \frac{(4s_M)^{L+1}(s_M+1)}{(L+1)!} + \frac{10(s_M^{L+1}+1)\cdot 16^{L+1}\cdot L(s_M^2+\log N)}{N}$$
$$+ \max_{0\leq j\leq M} 2^L \sum_{k=a-1}^{L} \left|Z[k+2, F_{s_j,k+2}, G(N)](A) - E(Z[k+2, F_{s_j,k+2}, G(N)])\right. \tag{4.35}$$

$$\leq \frac{1}{M+1} + \frac{(4s_M)^{L+1}(s_M+1)}{L!} + \frac{10(s_M^{L+1}+1)\cdot 16^{L+1}\cdot L(s_M^2+\log N)}{N}$$
$$+ 2^L \sum_{j=0}^{M} \sum_{k=a-1}^{L} \left|Z[k+2, F_{s_j,k+2}, G(N)](A) - E(Z[k+2, F_{s_j,k+2}, G(N)])\right| \tag{4.36}$$

We integrate this inequality over $G(N)$ and apply Theorem 2 of Section 3 which asserts that

$$\int_{G(N)} \left|Z[k+2, F_{s_j,k+2} G(N)](A) - E(Z[k+2, F_{s_j,k+2}, G(N)])\right| dA$$
$$\leq \left(\int_{G(N)} |Z - E(Z)|^2 dA \right)^{1/2}$$
$$\leq \left(3(8s_j)^{k+1} + 65(8s_j)^{2k+2}\right)^{1/2} \Big/ \sqrt{N}\,. \tag{4.37}$$

This leads to (assuming $s_M > 1$) the main inequality valid for $L > 1,\ M > 1$;

$$\int_{G(N)} \left(\sup_s \Delta(s,A)\right) dA \leq \frac{1}{M+1} + \frac{(4s_M)^{L+1}(s_M+1)}{L!}$$
$$+ \frac{10(s_M^{L+1}+1)\cdot 16^{L+1}\cdot L(s_M^2+\log N)}{N} + \frac{10ML(16s_M)^{L+1}}{\sqrt{N}} \tag{4.38}$$

At this point we can directly establish the main Theorem 0.2 in non–quantitative form viz

$$\lim_{N\to\infty} \int_{G(N)} (\sup_s \Delta(s,A)) dA = 0\,. \tag{4.39}$$

Indeed for $\epsilon > 0$, choose $M+1 > \frac{1}{3\epsilon}$, then s_M is determined (indeed according to (4.21) $s_M \leq a(M+1)$). Next choose L sufficiently large so that the second term in (4.38) is less than $\epsilon/3$. Finally in terms of this M, s_M and L, we can clearly make the 3^{rd} plus the 4^{th} terms in (4.38) at most $\epsilon/3$ for N large enough, and hence the $\ell.h.s.$ is less than ϵ for N large enough. This establishes (4.39).

In order to estimate the right hand side of (4.38) quantitatively, and in particular to get an estimate of the type $N^{-\alpha}$ for some $\alpha > 0$, we need an upper bound on s_M. The upper bound $s_M = 0(M)$ in (4.21) and even an upper bound of $s_M = 0(\log M)$ does not suffice for this purpose. In Section 5 we will establish the estimate

$$s_M \leq 3a\sqrt{\log \frac{4Ma}{3}} \tag{4.40}$$

(see equation (5.56) Section 5).

Substituting this for s_M in (4.38), and choosing L and M so as to minimize the resulting quantity will lead to the desired estimate. We choose $M = N^{1/6}$ (or the integer part thereof to be more exact). For this choice

$$s_M \leq 3a\sqrt{\log \frac{4Ma}{3}} \leq 16a\sqrt{\log N} \tag{4.41}$$

if $M \geq a$. With such a choice the third plus the fourth terms in (4.38) is at most

$$\frac{20ML\left(16^2 a\sqrt{\log N}\right)^{L+3}}{\sqrt{N}} .$$

The second term is at most

$$\frac{\left(16^2 a\sqrt{\log N}\right)^{L+3}}{L!} .$$

Hence the integral in (4.38) satisfies

$$\int\limits_{G(N)} \left(\sup_s \Delta(s, A)\right) dA \leq N^{-1/6} + \left((16)^2 a\sqrt{\log N}\right)^{L+4} \left(\frac{1}{L!} + \frac{M}{\sqrt{N}}\right) \tag{4.42}$$

for $20L \leq \left(16^2 a\sqrt{\log M}\right)^{L+3}$, which is the case if $L \geq 20a$, for example.

Next choose

$$L! = \frac{N^{1/2}}{M} = N^{1/3} . \tag{4.43}$$

According to Stirlings series [ ], we have

$$\log L! = (L+1/2)\log L - L + \beta(L)$$

with

$$|\beta(L)| \leq 5 \qquad \text{if} \quad L \geq 2\ . \tag{4.44}$$

Hence for $L \geq 0$

$$\left.\begin{array}{rcc} \log N^{1/3} & \leq & L\log L \\ \log N^{1/3} & \geq & L \\ \text{and} \qquad\qquad & & \\ L & \geq & \dfrac{\log N^{1/3}}{\log\log N^{1/3}} \end{array}\right\} \ . \tag{4.45}$$

Let

$$R = \left(16^2 a (\log N)^{1/2}\right)^{L+4} \tag{4.46}$$

then

$$\begin{aligned} \log R &= (L+4)\log\left(16^2 a(\log N)^{1/2}\right) \\ &\leq L\log\left(\sqrt{\log N}\right) + L\log(16^2 a) + 4\log\left(16^2 a\sqrt{\log N}\right) \\ &\leq L\log(3L\log L)^{1/2} + L\log(16^2 a) + 4\log\left(16^2 a\sqrt{L\log L}\right) \\ &\leq \frac{1}{2}L\log L + 10\cdot L\log\log L \end{aligned} \tag{4.47}$$

if $\qquad L \geq 16^2 a^2 \quad .$

Given $\epsilon > 0$ we therefore have

$$\begin{aligned} \log R &\leq \left(\frac{1}{2}+\epsilon\right)[(L+1/2)\log L - L + \beta(L)] + 10L\log\log L + \frac{L}{2} - \frac{1}{4}\log L + \frac{\beta(L)}{2} \\ &\quad - \epsilon[(L+1/2)\log L - L + \beta(L)] \\ &\leq \left(\frac{1}{2}+\epsilon\right)\log N^{1/3} \quad , \end{aligned} \tag{4.48}$$

if $\epsilon L\log L \geq 11L\log\log L$, which will be the case if

$$L \geq e^{22/\epsilon} \tag{4.49}$$

That is to say that if the side inequalities are valid, then

$$R \leq N^{1/6+\epsilon} \quad .$$

This implies that

$$\int\limits_{G(N)} \left(\sup_s \Delta(s, A)\right) dA \leq N^{-1/6} + N^{1/6+\epsilon} N^{-1/3} \leq 2N^{-1/6+\epsilon} \ . \tag{4.50}$$

Checking the side conditions, one finds that (4.50) holds if

$$N \geq \exp(\max(16^4 a^4,\ \exp 22/\epsilon)) \quad . \tag{4.51}$$

This completes the proof of Theorem 2, bis, and hence of Theorem 2.

5. Tail estimates and Fredholm determinants

The primary goal of this section is to establish the estimates for the tails of the measure μ_a. This is done by expressing these measures in terms of certain Fredholm determinants. In this way the relation between the universal measures μ_a and Gaudin's "GUE" spacing measures is established. A crucial factorization see (5.41) gives a relation between the finite (i.e. N finite) level versions of these determinants for the unitary, orthogonal and symplectic cases. For the even orthogonal case, the determinant can be estimated from above directly using its representation as a multiple integral (see Proposition 7). This eventually leads us to the tail estimates (see Proposition 10). In the supplement to this section, we give a more conceptual proof of the relation between μ_a and the Gaudin measures for the special case of $U(N)$.

We begin with the series representation for μ_a, equation (4.17) of Section 4.

$$\mu_a[0, s] = \sum_{n=a-1}^{\infty} (-1)^{n-a+1} \binom{n}{a-1} \frac{1}{(n+1)!} \int_0^s \cdots \int_0^s W_{n+2}(0, x_2, \ldots, x_{n+2}) dx_2 \cdots dx_{n+2} \quad . \tag{5.1}$$

Set

$$G(T, s) = \sum_{n=0}^{\infty} T^n \int\limits_{0 \leq x_2 \leq x_3 \cdots \leq x_{n+2}} W_{n+2}(0, x_2, \ldots, x_{n+2}) dx_2 \cdots dx_{n+2} \tag{5.2}$$

$$= \sum_{n=0}^{\infty} \frac{T^n}{(n+1)!} \int_0^s \cdots \int_0^s W_{n+2}(0, x_2, \ldots, x_{n+2}) dx_2 \cdots dx_{n+2} \tag{5.2$'$}$$

Since $|W| \leq 1$ (Lemma 8, Section 3), the series defining G converges absolutely for $s \geq 0$ and all T. It is an entire function of T for each such s. Differentiating w.r.t. T, $(a-1)$

times and setting $T=-1$ yields

$$\left(\frac{\partial G}{\partial T}\right)^{a-1}\Bigg|_{T=-1} = (a-1)! \sum_{n=a-1}^{\infty} (-1)^{n-a+1} \binom{n}{a-1} \frac{1}{(n+1)!} \cdot$$

$$\int_0^s \cdots \int_0^s W_{n+2}(0, x_2, \ldots, x_{n+2}) dx_2 \cdots dx_{n+2} \ .$$

Hence

$$\mu_a[0,s] = \frac{1}{(a-1!)} \left(\frac{\partial}{\partial T}\right)^{a-1} G(T,s)\Bigg|_{T=-1} \ .$$

Equivalently

$$G(T,s) = \sum_{n=0}^{\infty} \mu_{n+1}[0,s](1+T)^n \ .$$

That is $\mu_{n+1}[0,s]$ is the n^{th} Taylor coefficient of $G(T,s)$ at $T=-1$.

LEMMA 1. *Let $n \geq 0$ be an integer and $s \geq 0$ and set*

$$e_n(s) = \int_0^s \cdots \int_0^s W_n(x_1, \ldots, x_n) dx_1 \cdots dx_n \ .$$

Then

(i) *e_n is the restriction to $\mathbb{R}_{\geq 0}$ of an entire function of s which is divisible by s^n and satisfies*

$$|e_n(s)| \leq |s|^n \ e^{n|s|\pi} \ n^{n/2}$$

(ii) *$e_1(s) = s$ and*

$$\frac{d}{ds} \ \frac{e_{n+2}(s)}{(n+2)!} = \frac{1}{(n+1)!} \int_0^s \cdots \int_0^s W_{n+2}(0, z_2, \ldots, z_{n+2}) dz_2 \cdots dz_{n+2}$$

PROOF.

(i) For $s > 0$ make the substitution $sy_j = x_j$. This yields

$$e_n(s) = s^n \int_0^1 \cdots \int_0^1 W_n(sy) dy_1 \cdots dy_n \ .$$

¿From this it is clear that $e_n(s)$ is divisible by s^n and entire, since $W_n(sy) = \det_{n\times n}\left(\frac{\sin \pi s(y_i - y_i)}{s\pi(y_i - y_j)}\right)$ is entire. Moreover, for $0 \leq y_i \leq 1$ and $0 \leq y_j \leq 1$ and $s \in \mathbb{C}$

$$\left|\frac{\sin \pi s(y_i - y_j)}{\pi s(y_i - y_j)}\right| \leq e^{\pi|s|} \ .$$

The inequality in (i) then follows from this and Hadamards inequality [Ha]

$$\left|\det_{n\times n}(a_{ij})\right| \le M^n n^{n/2} \tag{5.5}$$

where $M = \max_{i,j}|a_{ij}|$.

(ii) It suffices to verify the identity for $s > 0$. For $\epsilon \to 0, \epsilon > 0$, we have

$$\begin{aligned} e_n(s+\epsilon) - e_n(s) &= \int\limits_{[0,s+\epsilon]^n} W_n(x_1,\ldots,x_n)dx_1\cdots dx_n - \int\limits_{[0,s]^n} W_n(x_1,\ldots,x_n)dx_1\cdots dx_n \\ &= \epsilon\sum_{i=1}^n \int_0^s\cdots\int_0^s W_n(x_1,\ldots,x_{i-1},s,x_{i+1},\ldots,x_n)\prod_{j\neq i} dx_j + 0(\epsilon^2) \end{aligned}$$

Now use the $\sum_n$–invariance of W_n to rewrite the last as

$$\epsilon n\int_0^s\cdots\int_0^s W_n(s,x_2,\ldots,x_n)dx_2\cdots dx_n + 0(\epsilon^2)\ .$$

The invariance of W_n under $x \to -x$ and $x \to x + (t,t,\ldots,t)$ will take the integrand to $W_n(0, s-x_2,\ldots,s-x_n)$ under $x \to x - s$. Letting $\epsilon \to 0$ we get

$$\begin{aligned} \frac{de_n}{ds} &= n\int_0^s\cdots\int_0^s W_n(0,s-x_2,\ldots,s-x_n)dx_2\cdots dx_n \\ &= n\int_0^s\cdots\int_0^s W_n(0,x_2',\ldots,x_n')dx_2'\cdots dx_n' \end{aligned}$$

after setting $x_j' = s - x_j$. This proves the claim.

Corollary 2. *Let $E(T,s)$ be defined by*

$$E(T,s) = 1 + \sum_{n=1}^\infty \frac{e_n(s)}{n!}T^n \tag{5.6}$$

then E is entire in T and s and

$$\frac{\partial E}{\partial s} = T + T^2 G \tag{5.7}$$

PROOF. Part (i) of Lemma 1 shows ($n! \sim n^{n+1/2}e^{-n}\sqrt{2\pi}$) that the series defining $E(T,s)$ converges absolutely and uniformly on compacta in (s,T). Since each term is analytic in s and T, the entirety of E follows. To get the identity, we have

$$\begin{aligned}\frac{\partial E}{\partial s} &= \sum_{n=1}^{\infty} \frac{T^n}{n!}\,\frac{d}{ds} e_n(s) \\ &= \sum_{n=1}^{\infty} \frac{nT^n}{n!} \int_0^s \cdots \int_0^s W_n(0,z_2,\ldots,z_n)dz_2\cdots dz_n \\ &= T + T^2 G\,, \qquad\qquad \text{according to } (2')\ .\end{aligned}$$

In particular it follows from (5.3) and (5.7) that $H_a(s) = \mu_a[0,s]$ extends to an entire function of s, a fact which we mentioned in Section 4. Next we recognize $E(T,s)$ as Fredholm determinant. Writing out the definition of E

$$E(T,s) = 1 + \sum_{n=1}^{\infty} \frac{T^n}{n!} \int_0^s \cdots \int_0^s \det_{n\times n}(K(x_i,x_j))dx_1\cdots dx_n \tag{5.8}$$

with

$$K(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)} \tag{5.9}$$

or

$$E(T,s) = 1 + \sum_{n=1}^{\infty} \frac{T^n}{n!} \int_0^{\alpha+s} \cdots \int_0^{\alpha+s} \det_{n\times n}(K(x_i,x_j))dx_1\cdots dx_n \tag{5.10}$$

for any $\alpha \in \mathbb{R}$ (since $K(x,y) = K(x+\alpha, y+\alpha)$). The series (5.8) or (5.9) are precisely the Fredholm expansions of $\det(I + TK_{s,\alpha})$ where $K_{s,\alpha}$ is the integral operator with kernel (5.9), acting on $L^2([\alpha,\alpha+s],dx)$, see [W–W]. That is for any α,

$$E(T,s) = \det(I + TK_{s,\alpha})\ . \tag{5.11}$$

Write the Taylor expansion of E at $T=-1$ as

$$E(T,s) = \sum_{n=0}^{\infty} E_n(s)(1+T)^n \tag{5.12}$$

so that

$$E_n(s) = \frac{1}{n!}\left(\frac{\partial}{\partial T}\right)^n E(T,s)\Big|_{T=-1}\ . \tag{5.13}$$

We express μ_a in terms of the Fredholm determinant.

Proposition 3.

$$\mu_a[0,s] = 1 + \frac{d}{ds}\sum_{n=0}^{a-1}\frac{a-n}{n!}\left(\left(\frac{\partial}{\partial T}\right)^n E(T,s)\right)_{T=-1}$$

Proof. ¿From (5.7) we have

$$\begin{aligned}
G &= \frac{1}{T^2}\frac{\partial E}{\partial s} - \frac{1}{T} \\
&= \frac{1}{((T+1)-1)^2}\sum_{n=0}^{\infty}\frac{dE_n(s)}{ds}(1+T)^n + \frac{1}{1-(T+1)} \\
&= \sum_{\tau=0}^{\infty}\binom{-2}{\tau}(-1)^\tau (T+1)^\tau \sum_{n=0}^{\infty}\frac{dE_n}{ds}(1+T)^n + \sum_{n=0}^{\infty}(T+1)^n \\
&= \sum_{m=0}^{\infty}\beta_m(1+T)^m
\end{aligned} \tag{5.14}$$

where

$$\beta_m = \sum_{n=0}^{m}(m-n+1)\frac{dE_n}{ds} + 1\ . \tag{5.15}$$

However, according to (5.4) and (5.14)

$$\beta_n = \mu_{n+1}[0,s]\ .$$

Thus

$$\mu_a[0,s] = 1 + \sum_{n=0}^{a-1}(a-n)\frac{dE_n}{ds} \tag{5.16}$$

$$= 1 + \frac{d}{ds}\sum_{n=0}^{a-1}\frac{(a-n)}{n!}\left(\frac{\partial}{\partial T}\right)^n E(T,s)\bigg|_{T=1} \tag{5.17}$$

as claimed

If we differentiate (5.17) w.r.t. s, we have

$$d\mu_a = \left(\frac{d^2}{ds^2}\sum_{n=0}^{a-1}\frac{a-n}{n!}\left[\left(\frac{\partial}{\partial T}\right)^n \det(I+TK_s)\right]_{T=-1}\right) ds\ . \tag{5.18}$$

If we use the fact already established, that μ_a is a probability measure, then another way of writing (5.16) is

$$\mathrm{tail}_{\mu_a}(s) = -\sum_{n=0}^{a-1}(a-n)\frac{dE_n}{ds} \tag{5.19}$$

where, in general,

$$\mathrm{tail}_{\nu}(s) = \int_s^\infty d\nu(t)\ . \tag{5.20}$$

Finally, we record the special case of which we will make crucial use

$$\mathrm{tail}_{\mu_1}(s) = -\frac{d}{ds}E(-1,s) = -\frac{d}{ds}\det(I-K_s)\ . \tag{5.21}$$

Fredholm Determinants. There are a number of applications of the Fredholm determinant expression for μ_a. One such application is to write say $E_0(s) = E(-1,s)$ in terms of the eigenvalues of K_s. That is

$$E_0(s) = \det(I-K_s) = \prod_{j=0}^{\infty}(1-\lambda_j(s)) \tag{5.22}$$

where $\lambda_j(s)$ are the eigenvalues of the integral equation

$$\int_{-s/2}^{s/2} K(x,y)f(y)dy = \lambda f(x)\ . \tag{5.23}$$

The point is that the eigenfunctions of (5.23) are known special functions ("Prolate-spheroidal functions") and so for a given s the $\lambda_j(s)$'s can be efficiently computed (numerically). In this way $E_0(s)$ may be computed numerically, indeed the graphs in Figure 3 were obtained in this way. Also the formula (5.19), for example, shows that the measures μ_a coincide with the spacing measures "$p_2(a-1,s)$" obtained by Gaudin and Mehta [GAU], [MEH] for the Gaussian Unitary Ensemble. In the Appendix to the section we will give a more conceptual derivation of this equality. Thirdly, it has been shown by Jimbo et. al. [J–M—M–S] that $E_n(s)$ may be expressed in terms of Painlavé V transcendents and this allows one to bring in powerful tools from completely integrable systems to analyze the asymptotics of $E_n(s)$ as $s \to \infty$ (see [DEI]). We proceed more elementarily by exploiting another probabilistic interpretation of $E_n(s)$. Before turning to that we decompose $L^2([-s,s],dx)$ into even and odd functions.

Let K_+ and K_- be the even and odd parts of the kernel $K(x,y)$. That is

$$K_\pm(x,y) = K(x,y) \pm K(-x,y)\ . \tag{5.24}$$

These kernels may be used to define integral operators $K_{2s},\ K_{\pm,s}$ as follows:

$$K_{2s} = \text{ the integral operator with kernel } K(x,y) \text{ on } L^2([-s,s],dx) \tag{5.25}$$

$$K_{\pm,s} = \text{ the integral operator with kernel } K_\pm(x,y) \text{ on } L^2([0,s],dx)\ . \tag{5.26}$$

Now we have an orthogonal decomposition

$$\begin{gathered} L^2([-s,s],dx) = L^2([-s,s],dx)_{\text{even}} \oplus L^2([-s,s],dx))_{\text{odd}} \\ f = f_+ + f_-\ ,\ f_\pm = \frac{1}{2}(f(x) \pm f(-x)) \end{gathered} \tag{5.27}$$

Both of these subspaces are stable by the operator K_{2s} because $K(x,-y) = K(-x,y)$. Via the isometric isomorphisms

$$\frac{1}{\sqrt{2}}\text{Rest}\tau\colon L^2([-s,s],dx)_{\text{even}} \cong L^2([0,s],dx)$$
$$\frac{1}{\sqrt{2}}\text{Rest}\tau\colon L^2([-s,s],dx)_{\text{odd}} \cong L^2([0,s],dx)$$

we get isomorphisms

$$K_{2s}\Big|_{L^2([-s,s],dx)_{\text{even}}} \cong K_{+,s}\Big|_{L^2([0,s],dx)} \tag{5.28}$$

$$K_{2s}\Big|_{L^2([-s,s],dx)_{\text{odd}}} \cong K_{-,s}\Big|_{L^2([0,s],dx)}\ . \tag{5.29}$$

Define $E_\pm(T,s)$ by

$$\begin{aligned} E_\pm(T,s) &= 1 + \sum_{n=1}^\infty \frac{T^n}{n!}\int_0^s \cdots \int_0^s \det{}_{n\times n}(K_\pm(x_i,x_j))dx_1\cdots dx_n \\ &= \det(I + TK_{\pm,s}) = \det\left(I - TK_{2s}\Big|_{L^2([-s,s],dx)_\pm}\right) \end{aligned} \tag{5.30}$$

As before $E_\pm(T,s)$ is entire in (T,s) and in view of (5.27) and (5.29), we have the identity

$$E(T,2s) = E_+(T,s)E_-(T,s)\ . \tag{5.31}$$

As in (5.12) denote the expansions of $E_\pm$ at $T=-1$ by

$$E_\pm(T,s) = \sum_{n=0}^\infty (1+T)^n E_{n,\pm}(s)\ . \tag{5.32}$$

It turns out that $E(T,s)$, $E_\pm(T,s)$ are scaling limits of similar determinants defined at the level of $G(N)$.

Recall that for each integer $N \geq 1$, $S_N(x)$ is the function

$$S_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)} = \sum_{j=0}^{N-1} e^{i(N-1-2j)x/2} \, . \tag{5.33}$$

We define kernels $S_N(x,y)$ and $S_{\pm,N}(x,y)$ by

$$\left.\begin{aligned} S_N(x,y) &= S_N(x-y) \\ S_{\pm,N}(x,y) &= S_N(x,y) \pm S_N(-x,y) \end{aligned}\right\} \tag{5.34}$$

For $\alpha \in \mathbb{R}$ and $s \geq 0$ define the integral operators

$$\begin{aligned} K_{N,s,\alpha} = &\text{ the integral operator with kernel} \\ &S_N(x,y) \text{ on } L^2\left([\alpha, \alpha+s], \, \frac{dx}{2\pi}\right) \end{aligned} \tag{5.35}$$

$$\begin{aligned} K_{\pm,N,s} = &\text{ the integral operator with kernel} \\ &S_{\pm,N}(x,y) \text{ on } L^2\left([0,s], \, \frac{dx}{2\pi}\right) \\ = &\text{ the integral operator with kernel} \\ &\frac{1}{2} S_{\pm,N}(x,y) \text{ on } L^2\left([0,s], \frac{dx}{\pi}\right) \, . \end{aligned} \tag{5.36}$$

(For $s=0$, the spaces $L^2([\alpha,\alpha+s], dx/2\pi)$ and $L^2([0,s], dx/\pi)$ are the zero spaces). For fixed s and variable α all the operators $K_{N,s,\alpha}$ are isometrically equivalent. From the explicit formulas for their kernels, we see that the operators $K_{N,s,\alpha}$, $K_{\pm,N,s}$ are of finite rank ($\leq N$). More precisely, their images lie in the span of $\{e^{i(N-1-2j)x/2} | 0 \leq j \leq N-1\}$. Define the characteristic polynomials (finite rank Fredholm Determinants)

$$E(N,T,s) = \det(I + TK_{N,s,\alpha}) \tag{5.37}$$

$$E_\pm(N,T,s) = \det(I + TK_{\pm,N,s}) \, . \tag{5.38}$$

Explicitly, we have the formulas

$$E(N,T,s) = \sum_{k=0}^{\infty} \frac{T^k}{k!} \int_0^s \cdots \int_0^s \det_{k\times k}(S_N(x_i,x_j)) \frac{dx_1}{2\pi} \cdots \frac{dx_k}{2\pi} \tag{5.39}$$

$$E_\pm(N,T,s) = \sum_{k=0}^{\infty} \frac{T^k}{k!} \int_0^s \cdots \int_0^s \det_{k\times k}(S_{\pm,N}(x_i,x_j)) \frac{dx_1}{2\pi} \cdots \frac{dx_k}{2\pi} \tag{5.40}$$

with the convention that at 0×0 determinant is equal to 1. These are polynomials in T of degree at most N. Their coefficients are controlled by the following Lemma.

LEMMA 4. *For $N \geq 1$, $k \geq 1$ the functions $A_{k,N}(s)$ and $A_{\pm,k,N}(s)$ defined for $s \geq 0$ by*

$$A_{k,N}(s) = \int_0^s \cdots \int_0^s det_{k\times k}(S_N(x_i,x_j))\frac{dx_1}{2\pi}\cdots\frac{dx_k}{2\pi}$$

$$A_{\pm,k,N}(s) = \int_0^s \cdots \int_0^s det_{k\times k}(S_{\pm,N}(x_i,x_j))\frac{dx_1}{2\pi}\cdots\frac{dx_k}{2\pi}$$

are restrictions to $\mathbb{R}_{>0}$ of entire functions of s which satisfy

$$|A_{k,N}(s)| \leq \left|\frac{s}{2\pi}\right|^k k^{k/2}(2N)^k e^{kN|s|/2}$$

and

$$|A_{\pm,k,N}(s)| \leq \left|\frac{s}{2\pi}\right|^k k^{k/2}(2N)^k e^{kN|s|/2}$$

for $s \in \mathbb{C}$.

PROOF. At $s = 0$, all these vanish. For $s > 0$ we change variable; $x_j = sy_j$ and the integrals become

$$\left(\frac{s}{2\pi}\right)^k \int_0^1 \cdots \int_0^1 \det_{k\times k}(S_N(sy_i,sy_j))dy_1\cdots dy_k$$

and

$$\left(\frac{s}{2\pi}\right)^k \int_0^1 \cdots \int_0^1 \det_{k\times k}(S_{\pm,N}(sy_i,sy_j))dy_1\cdots dy_k\ .$$

¿From this it is clear that $A_{k,N}$ and $A_{\pm,k,N}$ are entire in s. Also from (5.33) it is clear that

$$|S_N(x)| \leq N\exp(N|x|/2) \text{ for } x \in \mathbb{C}\ .$$

So the claimed inequalities follow from Hadamard's inequality (5.5).

Exactly as in the case of $E(T,s)$ we have an orthogonal decomposition of $L^2([-s,s],\ dx/2\pi)$ into even and odd parts which gives

$$E(N,T,2s) = E_+(N,T,s)E_-(N,T,s)\ . \tag{5.41}$$

Following a well known calculation due to Gaudin and Mehta [GAU], [MEH] we give a probabilistic interpretation of the Taylor coefficients of $E(N,T,s)$, $E_+(N,T,s)$, and $E_-(N,T,s)$ at $T=-1$.

Recall (Section 2) that to each $A \in G(N)$ we have N eigenvalue angles, that is $X(A) \in [0, \sigma\pi]^N$. We call these $\phi_1, \ \phi_2, \ldots, \phi_N$. For $n \geq 0$ an integer, and $0 \leq s < \sigma\pi$ we let $P_n(s, G(N))$ be the probability that an $A \in G(N)$ has exactly n of its eigenvalue–angles lie in $[0, s]$. That is

$$P_n(s, G(N)) = \mathrm{Haar}_{G(N)} \left\{ A \in G(N) \middle| \phi_j(A) \in [0, s], \text{ for exactly } n \text{ indices } j \in \{1, \cdots N\} \right\} \tag{5.42}$$

PROPOSITION 5. *$P_n(s, G(N))$ is given in terms of $E_\pm(N, s, T)$ by the following table:*

$\underline{G(N)}$	$\underline{P_n(s, G(N))}$
$U(N)$	$\left(\left(\frac{d}{dT} \right)^n \frac{1}{n!} E\big(N, T, s\big) \right)_{T=-1}$
$SO(2N+1)$	$\left(\left(\frac{d}{dT} \right)^n \frac{1}{n!} E_-\big(2N, T, s\big) \right)_{T=-1}$
$USP(2N)$ *or* $0_-(2N+2)$	$\left(\left(\frac{d}{dT} \right)^n \frac{1}{n!} E_-\big(2N+1, T, s\big) \right)_{T=-1}$
$SO(2N)$	$\left(\left(\frac{d}{dT} \right)^n \frac{1}{n!} E_+\big(2N-1, T, S\big) \right)_{T=-1}$.

PROOF. Recall that the kernels $L_N(x, y)$ attached to $G(N)$ are given by the following table (see Section 2):

$\underline{G(N)}$	$\underline{L_N(x, y)}$
$U(N)$	$S_N(x, y) e^{i(N-1)(x-y)/2}$
$SO(2N+1)$	$1/2\big(S_{2N}(x-y) - S_{2N}(x+y)\big)$
$USP(2N)$ or $0_-(2N+2)$	$1/2\big(S_{2N+1}(x-y) - S_{2N+1}(x+y)\big)$
$SO(2N)$	$1/2\big(S_{2N-1}(x-y) + S_{2N-1}(x+y)\big)$

In the $U(N)$ case, the integral operator with kernel $L_N(x, y)$ is unitarily conjugate (by multiplication by $e^{i(N-1)x/2}$ to the one with kernel $S_N(x, y)$ on $L^2([\alpha, \alpha+s], dx/2\pi)$. So Proposition 5 will follow from:

PROPOSITION 5 BIS. *Let $N \geq 2$ and let $G(N)$ be one of $U(N)$, $SO(2N+1)$, $USP(2N)$, $SO(2N)$, $0_-(2N+2)$, and $s \in [0, \sigma\pi]$. Then*

$$P_n(s, G(N)) = \left(\left(\frac{d}{dT} \right)^n \frac{1}{n!} \det \left(I + TL_N(x,y) \Big|_{L^2([0,s], dx/\sigma\pi)} \right) \right)_{T=-1}$$

PROOF. We begin with the case $n = 0$, to give the idea. In this case, $P_0(s, G(N))$ is the probability that $A \in G(N)$ has all its eigenvalue angles $\phi_j \in (s, \sigma\pi]$. By of Section 2, this probability is equal to

$$\int\limits_{(s,\sigma\pi]^N} d\mu_{N,N}(x) = \frac{1}{N!} \int\limits_{(s,\sigma\pi]^N} \det{}_{N\times N}\big(L_N(x_i,x_j)\big) \prod_m \frac{dx_m}{\sigma\pi} \tag{5.43}$$

Let I denote the characteristic function of $[0, s]$ (s is fixed). The $\ell.h.s.$ of (5.43) is

$$\int\limits_{[0,\sigma\pi]^N} \prod_i \big(1 - I(x_i)\big) d\mu_{N,N}(x) \ .$$

Expanding out the product, this is equal to

$$\sum_{k=0}^{N} (-1)^k \int\limits_{[0,\sigma\pi]^N} \left(\sum_{\substack{\text{subsets } J \\ |J|=k}} \prod_{j \epsilon J} I(x_j) \right) d\mu_{N,N}(x) \ .$$

Applying of Section 3, this becomes

$$\begin{aligned} &\sum_{k=0}^{N} (-1)^k \int\limits_{[0,\sigma\pi]^k} \prod_{i=1}^{k} I(x_i) d\mu_{k,N}(x_1, \ldots, x_k) \\ &= \sum_{k=0}^{N} (-1)^k \int_0^s \cdots \int_0^s d\mu_{k,N}(x_1, \cdots x_k) \\ &= \sum_{k=0}^{N} (-1)^k \frac{1}{k!} \int_0^s \cdots \int_0^s \det{}_{k\times k}\big(L_N(x_i,x_j)\big) \prod_{m=1}^{k} \frac{dx_m}{\sigma\pi} \ . \end{aligned}$$

This is the value at $T=-1$ of

$$\begin{aligned}
&\sum_{k=0}^{N}\frac{T^k}{k!}\int_0^s\cdots\int_0^s \det{}_{k\times k}\big(L_N(x_i,x_j)\big)\prod_{m=1}^{k}\frac{dx_m}{\sigma\pi}\\
&\quad=\sum_{k=0}^{\infty}\frac{T^k}{k!}\int_0^s\cdots\int_0^s \det{}_{k\times k}\big(L_N(x_i,x_j)\big)\prod_{m=1}^{k}\frac{dx_m}{\sigma\pi}\\
&\quad=\det\left(I+TL_N(x,y)\Big|_{L^2\left([0,s],\ \frac{dx}{\sigma\pi}\right)}\right)\quad .
\end{aligned}$$

This is what was claimed in the case $n=0$.

Turning to the general case $n>0$, we denote by A_k the quantity

$$A_k=\int_0^s\cdots\int_0^s \det{}_{k\times k}\big(L_N(x_i,x_j)\big)\prod_{m=1}^{k}\frac{dx_m}{\sigma\pi}\ .$$

If $n>N$, then there is nothing to prove, since $P_n(s,G(N))=0$, there being no $A\in G(N)$ with n eigenangles while the r.h.s. of the identity to be proven, is 0 since the determinant is a polynomial in T of degree at most N. For $n=N$, $P_N(s,G(N))$ equals

$$\begin{aligned}
\int\limits_{[0,\sigma\pi]^N} d\mu_{N,N}(x)&=\frac{1}{N!}\int\limits_{[0,\sigma\pi]^N}\det{}_{N\times N}\big(L_N(x_i,x_j)\big)\prod_{m=1}^{N}\frac{dx_m}{\sigma\pi}\\
&=\frac{A_N}{N!}\\
&=\left(\frac{d}{dT}\right)^N\frac{1}{N!}\left(\sum_{0\le\ell\le N}T^\ell\frac{A\ell}{\ell!}\right)\\
&=\left(\frac{d}{dT}\right)^N\frac{1}{N!}\det\left(I+TL_N(x,y)\Big|_{L^2\left([0,s],\frac{dx}{\sigma\pi}\right)}\right)\\
&=\left(\left(\frac{d}{dT}\right)^N\frac{1}{N!}\det\left(I+TL_N(x,y)\Big|_{L^2\left([0,s],\frac{dx}{\sigma\pi}\right)}\right)\right)_{T=-1}
\end{aligned}$$

as claimed.

We are left with $1 \leq n \leq N-1$. In this case, the probability in question is

$$\frac{1}{n!(N-n)!} \int\limits_{[0,s]^n \times (s,\sigma\pi]^{N-n}} D_{N,N}(x) \prod_{m=1}^{N} \frac{dx_m}{\sigma\pi}$$

where

$D_{n,N}(x_1,\ldots,x_n) = \det_{n\times n}\big(L_N(x_i,x_j)\big)$. So the above

$$= \frac{1}{n!(N-n)!} \int\limits_{[0,s]^n \times [0,\sigma\pi]^{N-n}} \prod_{i=n+1}^{N} \big(1 - I(x_i)\big) D_{N,N}(x) \prod_{m} \frac{dx_m}{\sigma\pi}$$

$$= \frac{1}{n!(N-n)!} \sum_{k=0}^{N-n} (-1)^k \sum_{\substack{J\subset\{n+1,\ldots,N\}\\ |J|=k}} \times$$

$$\int\limits_{[0,s]^n \times [0,\sigma\pi]^{N-n}} \prod_{j\in J} I(x_j) D_{N,N}(x) \prod_{m} \frac{dx_m}{\sigma\pi} \ .$$

Using the $\sum_N$–invariance of $D_{N,N}(x_1,\ldots,x_N)$ the integral above is independent of the particular choice of the subset J of the given cardinality k, so its equal to

$$\int\limits_{[0,s]^{n+k} \times [0,\sigma\pi]^{N-n-k}} D_{N,N}(x) \prod_{m} \frac{dx_m}{\sigma\pi} \ .$$

Now apply Lemma   to integrate out the unrestricted variables x_i, $i > n+k$ yielding

$$(N-n-k)! \int\limits_{[0,s]^{n+k}} D_{n+k,N}(x_1,\ldots,x_{n+k}) \prod_{m=1}^{n+k} \frac{dx_m}{\sigma\pi} = (N-n-k)! A_{n+k} \ .$$

Thus $P_n(s, G(N))$ is equal to

$$\begin{aligned} \frac{1}{n!(N-n)!} \sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} (N-n-k)! A_{n+k} &= \sum_{k=0}^{N-n} (-1)^k \ A_{n+k}/\, n!k! \\ &= \sum_{k=0}^{\infty} (-1)^k \ A_{n+k}/\, n!k! \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{n} \frac{A_{n+k}}{(n+k)!} \end{aligned}$$

which is the value at $T=-1$ of

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} T^k \binom{n+k}{n} \frac{A_{n+k}}{(n+k)!} \\
&= \left(\frac{d}{dT}\right)^n \frac{1}{n!} \sum_{k=0}^{\infty} \frac{T^{n+k} A_{n+k}}{(n+k)!} \\
&= \left(\frac{d}{dT}\right)^n \frac{1}{n!} \sum_{\ell=n}^{\infty} \frac{T^{\ell} A_{\ell}}{\ell!} \\
&= \left(\frac{d}{dT}\right)^n \frac{1}{n!} \sum_{\ell=0}^{\infty} \frac{T^{\ell} A_{\ell}}{\ell!} \\
&= \left(\left(\frac{d}{dT}\right)^n \frac{1}{n!} \det\left(I + TL_N(x,y)\right)\Big|_{L^2\left([0,s],\frac{dx}{\sigma\pi}\right)}\right)_{T=-1}
\end{aligned}
$$

This completes the proof of Proposition 5, bis.

As a consequence of Proposition 5, we learn that the degree in T of the polynomials $E(N,T,s)$, $E_-(2N+1,T,s)$, $E_+(2N-1,T,s)$, $E_-(2N,T,s)$ and $E_+(2N,T,s)$ are all equal to N. Indeed, this follows from the interpretation of the T derivatives of the first four as $P_n(s,G(N))$. For $n>N$, $P_n(s,G(N)) \equiv 0$ while for $n=N$ and $s=\sigma\pi$, $P_n(\sigma\pi,G(N))=1$. The case of the degree of the fifth follows from the first and the fourth and the identity (5.41). In fact, the identity (5.41), together with Proposition 5, implies the following striking relation.

Corollary 6. *For $N \geq 2$ and $s \epsilon [0,\pi]$*

$$
\begin{aligned}
P_0(2s, U(2N-1)) &= P_0(s, SO(2N)) \times P_0(s, USP(2N-2)) \\
&= P_0(s, SO(2N)) \times P_0(s, O_-(2N)) \, .
\end{aligned}
$$

More generally for each $n \geq 0$.

$$
\begin{aligned}
P_n(2s, U(2N-1)) &= \sum_{a+b=n} P_a(s, SO(2N)) P_b(s, UPS(2N-2)) \\
&= \sum_{a+b=n} P_a(s, SO(2N)) P_b(s, O_-(2N)) \, .
\end{aligned}
$$

The probabilistic interpretation shows that $0 \leq E(N,-1,s) \leq 1$ and also $0 \leq E_{\pm}(N,-1,s) \leq 1$. For $E_+(2N-1,-1,s)$ one can give a much better upper bound by direct examination of the integral defining $P_0(s, SO(2N))$. This is the analogue of the estimation by Mehta [MEH2] for the related probability for the Gaussian orthogonal ensemble. The following is the crucial inequality which allows us to suitably bound the tails of μ_a.

PROPOSITION 7. *For $N \geq 2$ and $s \epsilon [0, \pi]$*

$$P_0(s, SO(2N)) \leq \left(1 - \left(\frac{s}{\pi}\right)^2\right)^{N^2-N} .$$

PROOF. To prove this estimate, we go back to the very first (and standard) expression for the Haar measure on $SO(2N)$, viz of Section 2. We have

$$P_0(s, SO(2N)) = \frac{2}{N!}\int_s^\pi \cdots \int_s^\pi \prod_{i<j}(2\cos x_i - 2\cos x_j)^2 \frac{dx_1}{2\pi} \cdots \frac{dx_N}{2\pi} .$$

Using the identity $2\cos x_i = 2 - 4\sin^2(x_i/2)$, we can rewrite this as

$$\frac{2}{N!}\int_s^\pi \cdots \int_s^\pi \prod_{i<j}(4\sin^2(x_i/2) - 4\sin^2(x_j/2))^2 \frac{dx_1}{2\pi} \cdots \frac{dx_N}{2\pi} .$$

Setting $\theta_j = x_j/2$, this becomes

$$\frac{2}{N!}\int_{s/2}^{\pi/2} \cdots \int_{s/2}^{\pi/2} \prod_{i<j}(4\sin^2\theta_i - 4\sin^2\theta_j)^2 \frac{d\theta_1}{\pi} \cdots \frac{d\theta_N}{\pi} .$$

Changing variables again, set $\omega_i = \sin\theta_i$. Over the range of integration we have

$$\frac{2\theta}{\pi} \leq \sin\theta \leq \theta .$$

Now

$$d\omega_i = \cos\theta_i d\theta_i = \sqrt{1 - \sin^2\theta_i d\theta_i}$$

so

$$d\theta_i = \frac{d\omega_i}{\sqrt{1-\omega_i^2}} .$$

The integral becomes

$$\frac{2}{\pi^N N!}\int_{\sin s/2}^{1} \cdots \int_{\sin s/2}^{1} \prod_{i<j}\left(4\omega_i^2 - 4\omega_j^2\right)^2 \prod_m \frac{d\omega_m}{\sqrt{1-\omega_m^2}} .$$

Applying the above inequality, i.e. $s/\pi \leq \sin s/2 \leq s/2$, to control the domain of integration, we get

$$\frac{2}{\pi^N N!}\int_{s/2}^{1}\cdots\int_{s/2}^{1}\prod_{i<j}\left(4\omega_i^2-4\omega_j^2\right)^2\prod_m\frac{d\omega_m}{\sqrt{1-\omega_m^2}} \leq P_0(s,SO(2N))$$
$$\leq \frac{2}{\pi N!}\int_{s/\pi}\cdots\int_{s/\pi}\prod_{i<j}\left(4\omega_i^2-4\omega_j^2\right)^2\prod_m\frac{d\omega_m}{\sqrt{1-\omega_m^2}} . \tag{5.44}$$

All three are of the form $I(y)$ where

$$I(y)=\int_y^1\cdots\int_y^1\prod_{i<j}\left(4\omega_i^2-4\omega_j^2\right)^2\prod_m\frac{d\omega_m}{\sqrt{1-\omega_m^2}} . \tag{5.45}$$

In particular

$$P_0(s,SO(2N))=\frac{2}{\pi^N N!}I(\sin s/2) . \tag{5.45$'$}$$

Now $P_0(0,SO(2N)=1$, hence

$$\frac{2}{\pi^N N!}I(0)=1 . \tag{5.46}$$

To further estimate $I(y)$ for $0<y<1$, we make the substitution

$$\eta_i^2=\omega_i^2-y^2 . \tag{5.47}$$

Then as ω_i runs from y to 1, η_i runs from 0 to $\sqrt{1-y^2}$. Also

$$\frac{d\omega_i}{\sqrt{1-\omega_i^2}}=\frac{d\omega_i^2}{2\omega_i\sqrt{1-\omega_i^2}}=\frac{d\eta_i^2}{2\omega_i\sqrt{1-y^2-\eta_i^2}}=\frac{\eta_j}{\omega_i}\frac{d\eta_i}{\sqrt{1-y^2-\eta_i^2}} .$$

So

$$I(y)=\int_0^{\sqrt{1-y^2}}\cdots\int_0^{\sqrt{1-y^2}}\prod_{i<j}\left(4\eta_i^2-4\eta_j^2\right)^2\prod_m\frac{\eta_m d\eta_m}{\omega_m\sqrt{1-y^2-\eta_m^2}} .$$

Yet one more change of variables, set $\eta_i^2=z_i^2(1-y^2)$. Then

$$\frac{d\eta_i}{\sqrt{1-\delta^2-\eta_i^2}}=\frac{dz_i}{\sqrt{1-z_i^2}} .$$

So

$$I(y)=\int_0^1\cdots\int_0^1\left(\prod_{i<j}\left(1-y^2\right)\right)^2$$
$$\cdot\prod_{i<j}\left(4z_i^2-4z_j^2\right)^2\prod_m\frac{\eta_m}{\omega_m}\frac{dz_m}{\sqrt{1-z_m^2}} .$$

But $\eta_i/\omega_i < 1$ so that from the last

$$I(y) \leq \left(1-y^2\right)^{N^2-N} I(0) .$$

Hence, from (5.46)

$$\frac{2}{\pi^N N!} I(y) \leq \left(1-y^2\right)^{N^2-N} .$$

Taking $y = s/\pi$ we get from (5.45′)

$$P_0(s, SO(2N)) \leq \left(1-\left(s/\pi\right)^2\right)^{N^2-N}, \qquad \text{as desired.}$$

Armed with the inequality of Proposition 7, we want to let $N \to \infty$ with an appropriate scaling of s, to get a corresponding bound for $E_+(-1, s)$. The scaling limits in question are described by the following Proposition.

PROPOSITION 8. *For any $\alpha \in \mathbb{R}$, the three sequences of entire functions of (T, s), $E(N, T, 2\pi s/N + \alpha)$, $E_+(N, T, 2\pi s/N + \alpha)$ and $E_-(N, T, 2\pi s/N + \alpha)$, converge to the entire functions $E(T, s), E_+(T, s)$ and $E_-(T, s)$ respectively, uniformly on compact subsets of $\mathbb{C}^2$.*

PROOF. For N large, the ratio $\frac{N}{N+\alpha} \leq 2$. For any such N the coefficient of $T^k/k!$ in any of the Taylor series is bounded by

$$\left(4|s|e^{2\pi|s|}k^{1/2}\right)^k$$

as is immediate from Lemma 4. The same bound holds for the coefficient $T^k/k!$ in $E(T, s)$ or $E_\pm(T, s)$ (Lemma 1). Using these estimates, we see that given $m > 0$ and $\epsilon > 0$, there exists L such that in the region $(|T| \leq m, |s| \leq m)$, each of the functions in question, namely $E(T, s)$ or $E_\pm(T, s)$ or one of its finite N approximants with N large enough, is approximated within ϵ by the sum of its first L terms as a series in T with coefficients functions of s. So we need only prove that the coefficients of individual powers of T converge uniformly on compact subsets of the s–plane. Let us do this explicitly for say $E_\pm(T, s)$, the $E(T, s)$ case being entirely similar. Fix $k \geq 1$. The coefficient $e_{\pm,k}(s)$ of $T^k/k!$ in $E_\pm(T, s)$ is

$$\int_0^1 \cdots \int_0^1 \det_{k\times k}\left(sK_\pm\left(sy_i - sy_j\right)\right) dy_1 \cdots dy_k .$$

The coefficient of $T^k/k!$ in $E_\pm(N,T,2s\pi/N+\alpha)$ is

$$\int_0^1 \cdots \int_0^1 \det_{k\times k}\left(\frac{s}{N+\alpha} S_{\pm,N}\left(\frac{2\pi s y_i}{N+\alpha}, \frac{2\pi s y_i}{N+\alpha}\right)\right) dy_1 - dy_k \ .$$

The determinant is integrated over a compact region, so it suffices to check uniform convergence for $y \epsilon [0,1]^k$ and s in a compact, of the above determinant to

$$\det_{k\times k}\big(sK_\pm(sy_i, sy_j)\big) \ .$$

For this it suffices for the individual entries

$$\frac{s}{N+\alpha} S_{\pm,N}\left(\frac{2\pi s y_i}{N+\alpha}, \frac{2\pi s y_j}{N+\alpha}\right)$$

to converge to $sK_\pm(sy_i, sy_j)$. This reduces to the elementary well known limit

$$\lim_{N\to\infty} \frac{1}{N+\alpha} S_N\left(\frac{2\pi s}{N+\alpha}\right) = \lim_{N\to\infty} \frac{\sin\left(\frac{\pi s N}{N+\alpha}\right)}{(N+\alpha)\sin(\pi s/N+\alpha)} = \frac{\sin \pi s}{\pi s} \ ,$$

which completes the proof of Proposition 8.

¿From Propositions 6 and 8 we have for $s \geq 0$ and $n \geq 0$,

$$\lim_{N\to\infty} P_n\left(\frac{2\pi s}{N}, U(N)\right) = E_n(s) \tag{5.48}$$

$$\lim_{N\to\infty} P_n\left(\frac{\pi s}{N+1/2}, SO(2N+1)\right) = E_{-,n}(s) \tag{5.49}$$

$$\lim_{N\to\infty} P_n\left(\frac{\pi s}{N}, USP(2N)\right) = \lim_{N\to\infty} P_n\left(\frac{\pi s}{N+1}, 0_-(2N+2)\right) = E_{-,n}(s) \tag{5.50}$$

$$\lim_{N\to\infty} P_n\left(\frac{\pi s}{N}, SO(2N)\right) = E_{+,n}(s) \ . \tag{5.51}$$

In particular, since P_n's are probabilities, we learn from (5.48), (5.49), or (5.50) that

$$0 \leq E_0(s) \leq 1 \quad \text{and} \quad 0 \leq E_{-,0}(s) \leq 1 \ . \tag{5.52}$$

While from (5.51) and Proposition 7

$$E_{+,0}(s) \leq \lim_{N\to\infty} \big(1-(s/N)^2\big)^{N^2-N} = e^{-s^2} \tag{5.53}$$

(5.52), (5.53) and (5.31) imply that for $s \geq 0$

$$E_0(2s) \leq e^{-s^2} \ . \tag{5.54}$$

We can now establish the main tail estimate for μ_1:

PROPOSITION 9. *For $s \geq 0$*

$$tail_{\mu_1}(s) \leq \frac{4}{3} e^{-s^2/8} .$$

PROOF. ¿From (5.21) we have

$$\text{tail}_{\mu_1}(s) = -\frac{dE_0}{ds} .$$

Thus

$$\begin{aligned} E_0(s) - E_0(s+1) &= \int_s^{s+1} -\frac{dE_0}{ds} ds \\ &= \int_s^{s+1} \text{tail}_{\mu_1}(s) ds \\ &\geq \text{tail}_{\mu_1}(s+1) \end{aligned}$$

(tail(s) is nonincreasing in s). So

$$E_0(s) \geq E_0(s) - E_0(s+1) \geq \text{tail}_{\mu_1}(s+1) ,$$

hence applying (5.54) we get

$$\text{tail}_{\mu_1}(s+1) \leq e^{-s^2/4} . \tag{5.55}$$

This bound for $s \geq 1$ combined with the trivial bound $\text{tail}_{\mu_1}(s) \leq 1$ easily imply the bound asserted in Proposition 9.

The above estimate for tail_{μ_1} implies a similar estimate for $\text{tail}_{\mu_a}(s)$ as follows:

PROPOSITION 10. *For $a \geq 1$ and $s \geq 0$*

$$tail_{\mu_a}(s) \leq \frac{4a}{3} \exp\left(-\frac{s^2}{8a^2}\right) .$$

PROOF. For any $A \in U(N)$ with eigenvalues $\phi \leq \phi_1 \leq \phi_2 \cdots \leq \phi_N < 2\pi$ we have

$$\left| \left\{ 1 \leq j \leq N \middle| \frac{N}{2\pi}\left(\phi_{j+a}(A) - \phi_j(A)\right) \geq a\, s \right\} \right| \leq a \left| \left\{ 1 \leq j \leq N \middle| \frac{N}{2\pi}\left(\phi_{j+1}(A) - \phi_j(A)\right) \geq s \right\} \right| .$$

Dividing by N integrating w.r.t. A and letting $N \to \infty$ yields

$$\begin{aligned} \mathrm{tail}_{\mu_a}(a\,s) &\leq a\ \mathrm{tail}_{\mu_1}(s) \\ &\leq \frac{a4}{3} e^{-s^2/8a^2} \ . \end{aligned}$$

With this we can finally establish the desired bound for s_M that was used in the proof in Section 4. That is, if s_M is such that $\mu_a[0, s_M] = 1 - 1/M$, or $\mathrm{tail}_{\mu_a}(s_M) = \frac{1}{M}$, then according to Proposition 10

$$\frac{1}{M} \leq \frac{4a}{3} e^{-s_M^2/8a^2} \ .$$

That is

$$s_M \leq 3a\sqrt{\log \frac{4Ma}{3}} \ . \tag{5.56}$$

Supplement to Section 5

In the supplement to this Section we give a more conceptual proof of the relation between the universal measures μ_a and Fredholm determinants (viz Proposition 3). In fact, in the special case of $U(N)$, the measure $E(\mu_a(A, U(N))$ can be expressed as a (finite) Fredholm determinant for each N. In the limit as $N \to \infty$ this relation reproduces Proposition 3. We also pursue the probabilities $P_n(s, G(N))$ further and establish that the mean of μ_a is indeed a.

We begin by introducing some probability measures which are closely related to $P_n(s, G(N))$ and their scaling limits. As before, if $A \in G(N)$ we let

$$0 \leq \phi_1 \leq \phi_2 \ldots \leq \phi_N \leq \sigma\pi \tag{S.1}$$

denote its sequence of eigen–angles. We renormalize these angles by setting

$$\theta_n = \frac{(N+\lambda)\phi_n}{\sigma\pi} \tag{S.2}$$

for $n = 1, \ldots, N$. So

$$0 \leq \theta_1 \leq \theta_2 \cdots \leq \theta_N \leq N + \lambda \tag{S.3}$$

Note that the normalization of multiplying by $(N+\lambda)/\sigma\pi$ is not the only natural one. One could normalize so that the mean of ϕ_n over $G(N)$ is equal to 1. Asymptotically as

$N \to \infty$, we will see that this amounts to the normalization $\widetilde{\theta}_n = \alpha_n N$ for suitable constants α_n. The limiting distributions of the θ_n's are not universal which is an interesting and useful feature.

It is worth pointing out that unlike the normalized spacings $\Delta_{j,n}$, the above normalized angles of an $A \in G(N)$ depends on A as an element of $G(N)$, not just as an element of the ambient unitary group. For example, in $SO(2N+1)$ every element A has 1 as an eigenvalue, and so there is a shift in numbering. $\theta_n(A \text{ in } SO(2N+1)) = \theta_{n+1}(A \text{ in } U(2N+1))$ for $1 \le n \le N$. Similarly in $O_-(2N+2)$ every element A has both ± 1 as eigenvalues, and again there is a shift of numbering, $\theta_n(A \text{ in } O_-(2N+2)) = \theta_{n+1}(A \text{ in } U(2N+2))$ for $1 \le n \le N$. In the case of A in $USP(2N)$ or $SO(2N)$ the eigenvalues 1 occurs with an even multiplicity $2k$ and there is a shift depending on this multiplicity. $\theta_n(A \text{ in } SO(2N)$ or $USP(2N)) = \theta_{n+k}(A \text{ in } U(2N), 1 \le n \le N$. A redeeming feature is that for $G(N)$ one of $U(N)$, $USP(2N)$, or $SO(2N)$ the set of A's for which 1 is not an eigenvalue is of full measure, while for $SO(2N+1)$, or $O_-(2N+2)$, the set of A's for which 1 is an eigenvalue is of full measure.

Define the probability measures $\nu_n(G(N))$ "the n^{th} normalized eigenvalue distribution" by

$$\nu_n(G(N)) = (\theta_n)_*\big(\mathrm{Haar}_{G(N)}\big)\ . \tag{S.4}$$

That is, the cumulative distribution function, $CDF_{\nu_n}(G(N))$ is given by

$$CDF_{\nu_n(G(N))}(s) = \mathrm{Haar}_{G(N)}\big\{A \in G(N))\big|\theta_n \le s\big\}\ . \tag{S.5}$$

The measures $\nu_n(G(N))$ are closely related to the probabilities $P_n(s, G(N))$.

Lemma 1. *For $n \ge 2$ and $G(N)$ any one of $U(N)$, $SO(2N+1)$, $USP(2N)$, $SO(2N)$, $O_-(2N+2)$, and $s \in (0, N+\lambda)$ we have*

(1) $P_0\big(\frac{s\sigma\pi}{N+\lambda},\ G(N)\big) = tail_{\nu_1(G(N))}(s) := 1 - CDF_{\nu_1(G(N))}(s)$

(2) *For* $1 \le n \le N-1$

$$P_n\big(\frac{s\sigma\pi}{N+\lambda},\ G(N)\big) = tail_{\nu_{n+1}(G(N))}(s) - tail_{\nu_n(G(N))}(s)$$

(3) $P_N\big(\frac{s\sigma\pi}{N+\lambda}, G(N)\big) = CDF_{\nu_N(G(N))}(s)$

(4) $tail_{\nu_n(G(N))}(s) = \sum_{j=0}^{n-1} P_j\ \big(\frac{s\sigma\pi}{N+\lambda},\ G(N)\big)$

PROOF. The set $\{A \in G(N) | \theta_n > s\}$ is equal to $\{A \in G(N)$ with at most $n-1$ normalized angles in $[0,s]\}$

$$= \coprod_{j=0}^{n-1} \{A \in G(N) \text{ with exactly } j \text{ normalized angles in } [0,s]\} .$$

Its complement $\{A \in G(N) | \theta_n \leq s\}$

$$= \coprod_{j \geq n} \{A \in G(N) \text{ with exactly } j \text{ normalized angles in } [0,s]\} .$$

Taking Haar measure of these sets, and recalling the definition (5.42) and that for $j > N$, $P_j\left(\frac{s\sigma\pi}{N+\lambda}, G(N)\right) = 0$ (the set being empty) gives our assertions.

In view of Proposition 5, we have

COROLLARY 2. *For $N \geq 2$, $G(N)$ and s as above, we have*

$$\left(\left(\frac{d}{dT}\right)^n \frac{1}{n!}\right) E_{\mathrm{sign}\epsilon}\left(\rho N + \tau,\ T,\ \frac{s\sigma\pi}{N+\lambda}\right)\Big|_{T=-1}$$

$$= \begin{cases} tail_{\nu_1(G(N))}(s) & \textit{if } n = 0 \\ tail_{n+1}(G(N))(s) - tail_{\nu_n(G(N))}(s) & \textit{if } 1 \leq n \leq N-1 \\ CDF_{\nu_N(G(N))}(s) & \textit{if } n = N \\ 0 & \textit{if } n > N \end{cases} .$$

We turn now to the scaling limits as $N \to \infty$, that is the limit of $\nu_n(G(N))$ as $N \to \infty$.

COROLLARY 3. *For $s \geq 0$, $n \geq 1$ and $G(N)$ any one of $U(N)$, $SO(2N+1)$, $USP(2N)$, $SO(2N)$, $O_-(2N+2)$ we have*

$$(1) \quad \lim_{N \to \infty} tail_{\nu_n(G(N))}(s) = \sum_{j=0}^{n-1} E_{\mathrm{sgn}(\epsilon),j}(s)$$

$$(2) \quad \lim_{N \to \infty} CDF_{\nu_n(G(N))}(s) = \sum_{j \geq n} E_{\mathrm{sgn}(\epsilon),j}(s) .$$

PROOF. (1) follows from Lemma 1, Corollary 2, and Proposition 6 and 8 (or rather their consequences (5.48) ... (5.51)). (2) follows from (1) and the relation

$$1 = \sum_{n=0}^{\infty} E_{\mathrm{sgn}(\epsilon),n}(s) \tag{S.6}$$

The last being an immediate from (5.12) and (5.32) evaluated when $T = 0$ (and $\det I$ being equal to 1). Note in passing that the pre–limit version of (S.6), viz

$$1 = \sum_{n=0}^{\infty} P_n(s, G(N)),\ 0 \leq s \leq \sigma\pi \qquad \text{S.7}$$

comes from simply interpreting the statement that the sets $\{A \in G(N) | A$ has exactly n eigenangles in $[0, s]\}$ are a partition of $G(N)$.

PROPOSITION 4. *Let ν_n and $\nu_{\pm,n}(n \geq 1)$ be given by*

$$CDF_{\nu_n}(s) = 1 - \sum_{j=0}^{n-1} E_j(s)$$

$$CDF_{\nu_{\pm,n}}(s) = 1 - \sum_{j=0}^{n-1} E_{\pm,j}(s)\,,$$

$s \geq 0$. Then ν_n, $\nu_{\pm,n}$ are probability measures on $[0, \infty)$ and

(1) $\lim_{N\to\infty} \nu_n(U(N)) = \nu_n$

(2) $\lim_{N\to\infty} \nu_n(SO(2N+1)) = \nu_{-,n}$

(3) $\lim_{N\to\infty} \nu_n(USP(2N)) = \nu_{-,n}$

(4) $\lim_{N\to\infty} \nu_n(SO(2N)) = \nu_{+,n}$

(5) $\lim_{N\to\infty} \nu_n(O_-(2N+2)) = \nu_{-,n}$.

The convergence in these being convergence in measure, i.e. Convergence of the CDF's for each s.

REMARK. The probability measures ν_n, $\nu_{\pm,n}$ are all different. This is the nonuniversal feature mentioned earlier and which is useful in distinguishing the $G(N)$–limits. Their means (which are not equal to 1) are an artifact of our scaling in (2). As mentioned above, one can scale in (2) by $\alpha_{n,N}^{-1}$ where $\alpha_{n,N}$ is the mean of ϕ_n over $G(N)$. Doing so will lead to the measure $\tilde{\nu}_n$ and $\tilde{\nu}_{\pm,n}$ where $\tilde{\nu}$ is the measure ν rescaled to have mean equal to n. Graphs of $\tilde{\nu}_n$ and $\tilde{\nu}_{\pm,n}$ are displayed in Figures 1, 2 and 3.

PROOF OF PROPOSITION 4. In fact we have established everything in the Proposition except that the ν's are probability measures. A priori, being the limits of probability measures on $\mathbb{R}_{\geq 0}$, the $\nu_{\epsilon,n}$'s are nonnegative measures of total mass at most equal to 1.

What needs proof is that they do indeed have total mass 1 or in view of the formulae in their definition in Proposition 4, that for any fixed $n \geq 0$:

$$\left.\begin{aligned} &\lim_{s\to\infty} E_n(s) = 0 \\ &\lim_{s\to\infty} E_{\pm,n}(s) = 0 \end{aligned}\right\} \qquad \text{S.8}$$

The next few pages will be concerned with a proof of (S.8). We give a proof for $E_n(s)$ and this proof works equally well for $E_{\pm,n}(s)$.

Recall that

$$E(T,s) = \det(I + TK_s) \qquad \text{S.9}$$

where

$$K_s = K\Big|_{L^2[-s/2,s/2]} \ . \qquad \text{S.10}$$

¿From a general analysis of such self–adjoint integral operators K with entire kernels (see [H–T]) we know that the eigenvalues of K_s are real and if they are denoted by $\lambda_j(s)$, $j = 1,\ 2, \ldots$ with

$$|\lambda_1| \geq |\lambda_2| \ldots \qquad \text{S.11}$$

then

$$|\lambda_j(s)| \leq C_s \exp(-\frac{j \log j}{8}) \ . \qquad \text{S.12}$$

Here C_s is a constant depending on s. The product

$$E(T,s) = \prod_{j=1}^{\infty} (1 + T\lambda_j(s)) \qquad \text{S.13}$$

is therefore rapidly convergent. We claim that

(A) for any $s > 0$ and any j, $0 < \lambda_j(s) < 1$, and
(B) for any fixed j, $\lim_{s\to\infty} \lambda_j(s) = 1$.

To see these, consider first the operator $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Taking Fourier transform we see that K is isometric with $\widetilde{K} : L^2(\widehat{\mathbb{R}}) \to L^2(\widehat{\mathbb{R}})$ where

$$\widetilde{K}\hat{f}(\xi) = I_{[-1,1]}(\xi)\hat{f}(\xi) \qquad \text{S.14}$$

I_A being the characteristic function of the set A. Hence K is isometric to the projection operator $\widetilde{K}$ which projects $L^2(\widehat{\mathbb{R}}) \to L^2[-1,1]$ by truncating the functions outside $[-1,1]$.

Thus, K is a projector ($K^2 = K,\ K = K^*,\ \|K\| = 1$) and its spectrum is $\{0,1\}$ and in fact each of 0 and 1 is of infinite multiplicity. Let $L_s : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$L_s = P_s K P_s \tag{S.15}$$

where P_s is the projector of $L^2(\mathbb{R})$ to $L^2[-s/2, s/2]$. So

$$\left.\begin{aligned} L_s\Big|_{L^2[-s/2,s/2]} &= K_s \\ L_s\Big|_{(L^2[-s/2,s/2])^\perp} &= 0 \end{aligned}\right\} \tag{S.16}$$

Now it is clear that as $s \to \infty$

$$L_s \to K \qquad \text{strongly} \tag{S.17}$$

(i.e. $L_s f \to K f$ for any $f \epsilon L^2(\mathbb{R})$). Let λ be an eigenvalue of K_s i.e.

$$(K_s f)(x) = \lambda f(x) \qquad \text{for} \quad x \epsilon [-s/2, s/2] \quad (f \neq 0)\ .$$

Extend f to $\mathbb{R}$ by setting it equal to zero outside $[-s/2, s/2]$, then $L_s f = \lambda f$, and hence

$$\lambda \langle f, f \rangle = \langle P_s K P_s f, f \rangle = \langle K(P_s f), (P_s f) \rangle\ . \tag{S.18}$$

Since K is projector, we have that for any $g \not\equiv 0$

$$0 \leq \frac{\langle Kg, g \rangle}{\langle g, g \rangle} \leq 1 \tag{S.19}$$

Moreover, equality occurs on the left iff $g \epsilon$ (Range $K)^\perp$ and equality on the right iff $g \epsilon$ Range K. Hence from (S.18) we see that $0 \leq \lambda \leq 1$. If $\lambda = 1$, then in fact $P_s f \epsilon$ Range (K) which implies that $\widehat{P_s f}(\xi)$ is supported in $[-1, 1]$. This means that both $(P_s f)(x)$ and $\widehat{P_s f}(\xi)$ are of compact support (and not $\equiv 0$) which is a contradiction. Hence $\lambda = 1$ is not possible. If $\lambda = 0$, then $P_s f \epsilon$ (Range $K)^\perp$ which implies that $\widehat{P_s f}(\xi) \equiv 0$ for $-1 \leq \xi \leq 1$. But $\widehat{P_s f}(\xi)$ is the Fourier Transform of a function of compact support, and so is entire (and is not $\equiv 0$). This contradicts the above so that $\lambda = 0$ is also not allowed. Hence $0 < \lambda < 1$, and we have established (A) above.

To prove (B), we use (S.17). From general functional analysis it follows that since $L_s \to L$ strongly, every point of spectrum (K) is a limit of spectrum (L_s) as $s \to \infty$ (see [R–S]). Hence $\lambda_1(s) \to 1$ as $s \to \infty$. In fact, the above is true with multiplicities as well (precisely the spectral projections $E_{[1-\epsilon,1+\epsilon]}(L_s) \to E_{[1-\epsilon,1+\epsilon]}(L)$ for any $\epsilon > 0$ (see [R–S]), since the latter is infinite dimensional it follows that the former must grow unboundedly in dimension as $s \to \infty$). Hence $\lambda_j(s) \to 1$ as $s \to \infty$ for any fixed j. This proves (B).

We can now complete the proof of (8).

For $|T+1| \leq 1$ we have, (using A),

$$|1 + T\lambda_j(s)| \leq |1 - \lambda_j(s)| + |\lambda_j(s) + T\lambda_j(s)| \leq 1 - \lambda_j(s) + \lambda_j(s)|1+T| \leq 1 \, .$$

Hence for $n \geq 0$ fixed, and $|1+T| \leq 1$

$$|E(T,s)| = \prod_{j=1}^{\infty} |1 + T\lambda_j(s)| \leq \prod_{j=1}^{n+1} |1 + T\lambda_j(s)| \, .$$

Let $\epsilon > 0$, then for $|1+T| = \epsilon$ we have, (using B),

$$\lim_{s\to\infty} \max_{|T+1|=\epsilon} |E(T,s)| \leq \lim_{s\to\infty} \max_{|T+1|=\epsilon} \prod_{j=1}^{n+1} |1 + T\lambda_j(s)| = \epsilon^{n+1} \, . \qquad \text{S.20}$$

Applying Cauchy's inequalities to the analytic function (of T) $E(T,s)$ with expansion at $T = -1$ and $|1+T| = \epsilon$, yields

$$|E_n(s)| \leq \max_{|T+1|=\epsilon} \frac{|E(T,s)|}{\epsilon^n} \, .$$

Hence from (S.20)

$$\lim_{s\to\infty} |E_n(s)| \leq \epsilon$$

ϵ is arbitrary, and so we have proven (8), and with it also Proposition (4).

The measures ν_n, that is the scaling limits of the n^{th} eigenvalue of the $U(N)$'s, are related in a simple way to the universal spacing measures $\mu_a = \mu_a(\text{univ})$ as follows:

Proposition 5.

(i)
$$\frac{d}{ds} CDF_{\nu_1} = 1 - CDF_{\mu_1} = tail_{\mu_1}$$

(ii) For $n \geq 2$

$$\frac{d}{ds} CDF_{\nu_n} = CDF_{\mu_{n-1}} - CDF_{\mu_n}$$

(iii) For $n \geq 1$

$$tail\mu_n = \sum_{j=1}^{n} \frac{d}{ds} CDF_{\nu_j}$$

(iv)

$$tail_{\mu_n}(s) = -\sum_{j=0}^{n-1} (n-j) \frac{dE_j}{ds} \ .$$

PROOF. The relation (iv) was already established in (5.19). Substituting the expression for $CDF_{\nu_n}(s)$ in Proposition 4 in (iii) one sees that (iii) and (iv) are equivalent. (i) and (iii) follow by inverting the relation (iii).

If we integrate (iv) of Proposition 5 from s to ∞ and use (8), we get

$$\int_s^\infty \mathrm{tail}_{\mu_n}(t)dt = -\sum_{j=0}^{n-1}(n-j)(E_j(\infty) - E_j(s)) = \sum_{j=0}^{n-1}(n-j)E_j(s) \ .$$

Setting $s = 0$ and using the relations $E_0(0) = 1$ and $E_j(0) = 0$ for $j \geq 1$ which are clear from definitions (5.8) and (5.13), yields

$$\int_0^\infty \mathrm{tail}_{\mu_n}(t)dt = n \qquad \text{for} \quad n \geq 1 \tag{S.21}$$

With this, we can establish that the mean of μ_n is indeed n (this was promised after (4.21). In view of (S.21), there is a sequence $x_j \to \infty$ s.t.

$$\lim_{j\to\infty} x_j \ \mathrm{tail}_{\mu_n}(x_j) = 0 \ . \tag{S.22}$$

Hence, if

$$M_n = \lim_{j\to\infty} \int_0^{x_j} x \ d\mu_n(x) \left(= \int_0^\infty x \, d\mu_n(x) \right)$$

then

$$\begin{aligned} M_n &= \lim_{j\to\infty} - \int_0^{x_j} x \frac{d}{dx} \mathrm{tail}_{\mu_n}(n) dx \\ &= \lim_{j\to\infty} \left(-x \ \mathrm{tail}_{\mu_n}(x) \Big|_0^{x_j} + \int_0^{x_j} \mathrm{tail}_{\mu_n}(x) dx \right) \\ &= \int_0^\infty \mathrm{tail}_{\mu_n}(x) dx \ . \end{aligned}$$

¿From (S.21) it follows that $M_n = n$, i.e. that μ_n has mean n.

To end the supplement, we give a direct and more conceptual proof of the relation between the μ_n's and the ν_n's (and hence between the μ_n's and $E_n(s)$). The relation can be derived at the finite level for the family $U(N)$ (all of this is special to this particular family). Consider the product group $U(1) \times U(N)$. For $1 \leq n \leq N$, let $F_n : U(1) \times U(N) \to \mathbb{R}_{\geq 0}$ be defined by $F_n(e^{i\phi}, A)$ is equal to the normalized distance from $e^{i\phi}$ to the n^{th} eigenvalue of A which one encounters starting from $e^{i\phi}$ and walking counter clockwise about the unit circle, measuring distances so that the circumference is equal to N.

Lemma 6. *The direct image of Haar measure on $U(1) \times U(N)$ under F_n is $\nu_n(U(N))$.*

Proof. In terms of the n^{th} normalized angle $\theta_n(A)$ attached to $A \in U(N)$, we have

$$F_n(e^{i\phi}A) = \theta_n(e^{-i\phi}A)\,.$$

So if we denote by $\pi : U(1) \times U(N) \to U(N)$ the surjective homomorphism $\pi(e^{i\phi}, A) = e^{-i\phi}A$, then we have the diagram

$$\begin{array}{ccc} U(1) \times U(N) & \xrightarrow{F_n} & \mathbb{R} \\ \pi \Big\uparrow & & \\ U(N) & & \theta_n \end{array}$$

Since $\pi_\star\ \mathrm{Haar}_{U(1)\times U(N)} = \mathrm{Haar}_{U(N)}$ the assertion follows from the transitivity of direct image

$$F_n \star \mathrm{Haar}_{U(1)\times U(N)} = \theta_n \star \pi_\star \mathrm{Haar}_{U(1)\times U(N)} = \theta_n \star \mathrm{Haar}_{U(N)} = \nu_n(U(N))\,.$$

Recall that we defined $\mu_n(U(N))$ to be

$$\mu_n(U(N)) = E(\mu_n(A, U(N)), U(N))\,.$$

Proposition 7. *For any $N \geq 2$, and $s \geq 0$*

$$tail_{\nu_1(U(N))}(s) = \int_s^\infty tail_{\mu_1(U(N))}(t)dt$$

Proof. First of all both measures $\mu_1(U(N))$ and $\nu_1(U(N))$ are supported in $[0, N]$ so both sides vanish if $s > N$. So suppose that $0 \leq s \leq N$. We first express both sides as integrals over $U(N)$ and then we show that these have the same integrand.

We begin with $\mathrm{tail}_{\nu_1(U(N))}(s)$. We view $\nu_1(U(N))$ as $F_1 \star (\mathrm{Haar}_{U(1)\times U(N)})$, thus by definition

$$\begin{aligned}\mathrm{tail}_{\nu_1(U(N))}(s) = \nu_1(U(N))(s,\infty) &= \mathrm{Haar}_{U(1)\times U(N)} \quad \left\{(e^{i\phi},A)|\ \text{first eigenvalue past } e^{i\phi} \text{ is at normalized distance} > s \text{ past } e^{i\phi}\right\} \\ &= \mathrm{Haar}_{U(1)\times U(N)} \quad \left\{(e^{i\phi},A)|\ A \text{ has no eigenvalue in } [\phi,\phi+\tfrac{2\pi s}{N}]\right\} \\ &= \int_{U(N)} \mathrm{Haar}_{U(1)} \quad \left\{[\phi|A \text{ has no eigenvalue in } [\phi,\phi+\tfrac{2\pi s}{N}]\right\} dA\end{aligned}$$

On the other hand by definition

$$\mathrm{tail}_{\mu_1(U(N))}(t) = \int_{U(N)} \mathrm{tail}_{\mu_1(A,U(N))}(t)dA\ .$$

Hence

$$\int_s^\infty \mathrm{tail}_{\mu_1(U(N))}(t)dt = \int_{U(N)} \left(\int_s^\infty \mathrm{tail}_{\mu_1(A,U(N))}(t)dt\right) dA\ . \qquad \text{S.24}$$

So the putative identity will follow, if we show the integrands in (S.23) and (S.24) coincide. The set $U(N)^{\mathrm{reg}}$ of regular elements in $U(N)$, i.e. those with N distinct eigenvalues, is of full measure. So it suffices to show, that for each such regular A, we have

$$\mathrm{Haar}_{U(1)} \quad \{\phi|A \text{ has no eigenvalue in } [\phi,\phi+\tfrac{2\pi s}{N}]\} = \int_s^\infty \mathrm{tail}_{\mu_1(A,U(N))}(t)dt\ . \qquad \text{S.25}$$

Now for a regular element A with

$$0 \le \phi_1 < \phi_2 \ldots < \phi_N < 2\pi,\ \phi_{N+1} = 2\pi + \phi_N\ ,$$

let

$$S_i = \frac{N}{2\pi}(\phi_{i+1}-\phi_i), \quad i = 1,\ \ldots N$$

be the normalized spacings of A. Consider the intervals

$$S_i = (\phi_i,\phi_{i+1}] \subset U(1),\ i=1,\ \ldots N\ .$$

Thus s_i = normalized length of S_i, and $U(1)$ is the disjoint union of the S_i's. Hence

$$\begin{aligned}&\left\{\phi|A \text{ has no eigenvalue in } [\phi,\phi+\frac{2\pi s}{N}]\right\} \\ &= \coprod_i \left\{\phi\epsilon S_i|A \text{ has no eigenvalue in } [\phi,\phi+\frac{2\pi s}{N}]\right\}\ .\end{aligned}$$

But for ϕ in S_i the first eigenvalue after ϕ is ϕ_{i+1} so the condition that A have no eigenvalue in $[\phi, \phi + \frac{2\pi s}{N}]$ is the condition that the entire interval $[\phi, \phi + \frac{2\pi s}{N}]$ lie in the interior of S_i. Now S_i has Haar length s_i/N, while $[\phi, \phi + \frac{2\pi s}{N}]$ has length s/N. So unless $s_i > s$ there are no intervals $[\phi, \phi + \frac{2\pi s}{N}]$ of Haar length s/N in the interior of S_i. If $s_i > s$, then there are such intervals and their starting points can be any ϕ in $(\phi_i, \phi_{i+1} - \frac{2\pi s}{N})$, an interval of Haar length $\frac{s_i - s}{N}$. Thus we see that

$$\left\{\phi | A \text{ has no eigenvalue in } [\phi, \phi + \frac{2\pi s}{N}]\right\} = \coprod_{i, s_i > s} \left(\phi_i, \phi_{i+1} - \frac{2\pi s}{N}\right) .$$

Taking Haar measure, we get

$$\mathrm{Haar}_{U(1)} \left\{\phi | A \text{ has no eigenvalue in } [\phi, \phi + \frac{2\pi s}{N}]\right\} = \frac{1}{N} \sum_{i, s_i > s} (s_i - s) . \qquad \text{S.26}$$

Finally we compute the r.h.s. of (S.25) in terms of the s_i's

$$\begin{aligned} \mathrm{tail}_{\mu_1(A, U(N))}(t) &= \mu_1(A, U(N))(t, \infty) \\ &= \frac{1}{N} \left|\{i | s_i > t\}\right| \\ &= \frac{1}{N} \sum_i I_{[0, s_i)}(t) . \end{aligned}$$

Hence

$$\begin{aligned} \int_s^\infty \mathrm{tail}_{\mu_1(A, U(N))}(t) dt &= \int_s^\infty \frac{1}{N} \sum_i I_{[0, s_i]}(t) dt \\ &= \frac{1}{N} \sum_{i, s_i > s} (s_i - s) \end{aligned} \qquad \text{S.27}$$

Since (S.26) coincides with (S.27) we have established (S.25), and with it Proposition 7.

We now state and prove the extenison of Proposition 7 for $n \geq 2$.

PROPOSITION 8. *For* $N > k \geq 2, s \geq 0$

$$tail_{\nu_k(U(N))}(s) = \int_s^\infty \left(tail_{\mu_k(U(N))}(t) - tail_{\mu_{k-1}(U(N))}(t)\right) dt .$$

PROOF. We proceed as in the proof of Proposition 7.

The finite N identities in Propositions 7 and 8 lead to similar relations in the $N \to \infty$ limit. Since the mean of $\mu_1(U(N))$ is equal to 1, we can rewrite the relation in Proposition 7 as

$$CDF_{\nu_1(U(N))}(s) = \int_0^s \mathrm{tail}_{\mu_1(U(N))}(t) dt \qquad \text{S.28}$$

As $N \to \infty$ the $\ell.h.s.$ converges to $CDF_{\nu_1}(s)$ according to Proposition 7. On the other hand as $N \to \infty$, we know from Theorem 1 that $CDF_{\mu_1(U(N))}(t)$ converges to a limit called $CDF_{\mu_1}(t)$. By the dominated convergence theorem, we conclude that the $r.h.s.$ of (S.28) converges to $\int_0^s \mathrm{tail}_{\mu_1}(t)dt$. Hence, we have

$$CDF_{\nu_1}(s) = \int_0^s \mathrm{tail}_{\mu_1}(t)dt\ .$$

Similarly the rest of Propostion 5, and hence also (5.19), can be established this way.

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