[twoside,art12]article =7in=8.5in -1.5pc-1.5pc .5pc myheadings ARTHUR JAFFE QUANTUM HARMONIC ANALYSIS AND GEOMETRIC INVARIANTS

section .subsection .equation

Quantum Harmonic Analysis and Geometric InvariantsSupported in part by the Department of Energy under Grant DE-FG02-94ER-25228 and by the National Science Foundation under Grant DMS-94-24344. Arthur Jaffe Harvard University September 1997

document empty abstract We develop two topics in parallel and show their inter-relation. The first centers on the notion of a fractional-differentiable structure on a commutative or a non-commutative space. We call this study *quantum harmonic analysis*. The second concerns homotopy invariants for these spaces and is an aspect of non-commutative geometry.

We study an algebra $\mathfrak{A}$, which will be a Banach algebra with unit, represented as an algebra of operators on a Hilbert space $\mathcal{H}$. In order to obtain a geometric interpretation of $\mathfrak{A}$, we define a derivative on elements of $\mathfrak{A}$. We do this in a Hilbert space context, taking da as a commutator $da = [Q, a]$. Here Q is a basic self-adjoint operator with discrete spectrum, increasing sufficiently rapidly that $\exp(-\beta Q^2)$ has a trace whenever $\beta > 0$.

We can define fractional differentiability of order μ, with $0 < \mu \leq 1$, by the boundedness of $(Q^2 + I)^{\mu/2} a (Q^2 + I)^{-\mu/2}$. Alternatively we can require the boundedness of an appropriate smoothing (Bessel potential) of da. We find that it is convenient to assume the boundedness of $(Q^2 + I)^{-\beta/2} da (Q^2 + I)^{-\alpha/2}$, where we choose $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$. We show that this also ensures a fractional derivative of order $\mu = 1 - \beta$ in the first sense. We define a family of interpolation spaces $\mathfrak{J}_{\beta,\alpha}$. Each such space is a Banach algebra of operator, whose elements have a fractional derivative of order $\mu = 1 - \beta > 0$.

We concentrate on subalgebras $\mathfrak{A}$ of $\mathfrak{J}_{\beta,\alpha}$ which have certain additional covariance properties under a group $\mathbb{Z}_2 \times \mathfrak{G}$ acting on $\mathcal{H}$ by a unitary representation $\gamma \times U(g)$. In addition, the derivative Q is assumed to be $\mathfrak{G}$-invariant. The geometric interpretation flows from the assumption that elements of $\mathfrak{A}$ possess an arbitrarily small fractional derivative. We study homotopy invariants of $\mathfrak{A}$ in terms of equivariant, entire cyclic cohomology. In fact, the existence of a fractional derivative on $\mathfrak{A}$ allows the construction of the cochain $\tau^{\rm JLO}$, which plays the role of the integral of differential forms. We give a simple expression for a homotopy invariant $\mathfrak{Z}^Q(a;g)$, determined by pairing $\tau^{\rm JLO}$, with a $\mathfrak{G}$-invariant element $a \in \mathfrak{A}$, such that a is a square root of the identity. This invariant is $\mathfrak{Z}^Q(a;g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} {\rm Tr}\left(\gamma U(g) a e^{-Q^2 + itda}\right) dt$.

This representation of the pairing is reminiscent of the heat-kernel representation for an index. In fact this quantity is an invariant, in the following sense. We isolate a simple condition on a family $Q(\lambda)$ of differentiations that yields a continuously-differentiable family $\tau^{\rm JLO}(\lambda)$ of cochains. Since $\mathfrak{Z}^Q(a;g)$ need not be an integer, continuity of $\tau^{\rm JLO}(\lambda)$ in λ is insufficient to prove the constancy of the pairing. However the existence of the derivative leads to the existence of the homotopy. As $\langle \tau, a \rangle$ vanishes for τ a coboundary, and as $d\tau^{\rm JLO}(\lambda)/d\lambda$ is a coboundary, our condition on $Q(\lambda)$ ensures that $\mathfrak{Z}^{Q(\lambda)}(a;g)$ is independent of λ. Hence it is a homotopy invariant.

The theory of $\mathfrak{Z}^Q(a;g)$ reduces to the study of the Radon transform of sequences of certain functions. The fractional differentiability properties of elements of $\mathfrak{A}$ translate into properties of the asymptotics of the sequences of Radon transforms. The condition that $\tau^{\rm JLO}$ fit into the framework of entire cyclic cohomology translates to the existence of some fractional derivative for functions in the algebra under study, and in particular the assumption $\alpha + \beta < 1$. Thus the study of fractionally-differentiable structures dove-tails naturally with the theory of homotopy invariants.

In our study of quantum harmonic analysis, we introduce spaces $\mathcal{T}(-\beta, \alpha)$ of operator-valued distributions. These spaces are bounded, linear operators between Sobolev spaces. The elements of the interpolation spaces, the Banach algebras $\mathfrak{J}_{\beta,\alpha}$, have derivatives da which belong to the spaces $\mathcal{T}(-\beta, \alpha)$. For a certain range of β and α, we extend the theory of the Radon transform from products of regularized, bounded operators to products of regularized, operator-valued distributions.

We sometimes wish to evaluate such an invariant at the endpoint of an interval such as $\lambda \in (0, 1]$, where $\mathfrak{Z}^{Q(\lambda)}(a;g)$ becomes singular as $\lambda \to 0$. We discuss in brief a procedure to regularize the endpoint, and a method to recover $\mathfrak{Z}^{Q(\lambda)}(a;g)$ fully from certain partial information at the endpoint.

Finally, we generalize this approach to cover the case when Q can be split into the sum of "independent" parts $Q_1 + Q_2$, such that $(Q_1 + Q_2)^2 = (Q_1)^2 + (Q_2)^2$. Here we assume that Q_1 and $(Q_2)^2$ are $\mathfrak{G}$-invariant, but not necessarily Q_2. With further assumptions on a, the most important being that $(Q_1)^2 - (Q_2)^2$ commutes with a, we obtain a modified formula for an invariant, namely $\mathfrak{Z}^{\{Q_j\}}(a; g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \mathrm{Tr}\left(\gamma U(g) a e^{-(Q_1^2 + Q_2^2)/2 + it d_1 a}\right) dt$.