ON THE CREATION OF CONJUGATE POINTS FOR HAMILTONIAN SYSTEMS

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Abstract. For a fixed Hamiltonian H on the cotangent bundle of a compact manifold M and a fixed energy level e, we prove that the set $\mathcal{A}_e$ of potentials ϕ on M such that the Hamiltonian flow of $H+\phi$ is Anosov, is the interior in the C^2 topology of the set $\mathcal{B}_e$ of potentials such that the flow has no conjugate points.

1. Introduction

Let M be a closed connected riemannian manifold and T^*M its cotangent bundle. By a Hamiltonian on T^*M we shall mean a C^∞ function $H : T^*M \to \mathbb{R}$ satisfying the following conditions:

(a) *Convexity:* For all $q \in M$, $p \in T_q^*M$, the Hessian matrix $\frac{\partial^2 H}{\partial p_i\, \partial p_j}(q,p)$ (calculated with respect to linear coordinates on T_q^*M) is positive definite.

(b) *Superlinearity:*
$$\lim_{|p|\to\infty} \frac{H(q,p)}{|p|} = +\infty\,.$$

The *Hamiltonian equation* for H is defined as
$$\begin{aligned} q' &= H_p\,, \\ p' &= -H_q \end{aligned} \tag{1}$$
where H_q and H_p are the partial derivatives with respect to q and p. Observe that the Hamiltonian function H is constant along the solutions of (1). Its level sets $\Sigma = H^{-1}\{e\}$ are called *energy levels* of H. Then the compacitness of M and the superlinearity hypothesis imply that the energy levels are compact. Since the Hamiltonian vector field (1) is Lipschitz, the solutions of (1) are defined on all $\mathbb{R}$. Denote by ψ_t the corresponding *Hamiltonian flow* on T^*M.

Let $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $\mathbb{V}(\theta) = \ker(d\pi(\theta))$. Two points θ_1, $\theta_2 \in T^*M$ are said *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(\mathbb{V}(\theta_1)) \cap \mathbb{V}(\theta_2) \neq \{0\}$.

For $e \in \mathbb{R}$, let $\mathcal{A}_e$ be the set of $\phi \in C^\infty(M)$ such that the flow of $H+\phi$ is Anosov in $(H+\phi)^{-1}(e)$ and let $\mathcal{B}_e$ be the set of $\phi \in C^\infty(M)$ such that $(H+\phi)^{-1}(e)$ has no conjugate points. As is well known $\mathcal{A}_e$ is open in C^k topology and $\mathcal{B}_e$ is closed.

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On the other hand G. and M. Paternain [5] have shown that $\mathcal{A}_e$ is contained in $\mathcal{B}_e$. The purpose of this paper is to prove the following

Theorem 1. *The interior of $\mathcal{B}_e$ in the C^2 topology is $\mathcal{A}_e$.*

This theorem is an extension to the Hamiltonian setting of a result of R. O. Ruggiero for the geodesic flow [6]. Our main tools are the Green bundles, Theorem 2 below describing $\mathcal{A}_e$ inside $\mathcal{B}_e$, and the index form in the Lagrangian setting.

2. Green bundles and hyperbolicity

The Green bundles were constructed for the geodesic flow by Green in [4] using the Jacobi equation. For Hamiltonian systems they were constructed in [3], where proofs for the statements in this section can be found.

Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two $d\psi_t$-invariant Lagrangian subbundles $\mathbb{E}$, $\mathbb{F} \subset T(T^*M)$ along the orbit of θ given by

$$\mathbb{E}(\theta) = \lim_{t\to+\infty} d\psi_{-t}\Big(\mathbb{V}(\psi_t(\theta))\Big)\,,$$
$$\mathbb{F}(\theta) = \lim_{t\to+\infty} d\psi_{t}\Big(\mathbb{V}(\psi_{-t}(\theta))\Big)\,.$$

Moreover, $\mathbb{E}(\theta)\cup\mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta)\cap\mathbb{V}(\theta) = \mathbb{F}(\theta)\cap\mathbb{V}(\theta) = \{0\}$, $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ and $\dim\mathbb{E}(\theta) = \dim\mathbb{F}(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the Hamiltonian vectorfield and $\Sigma = H^{-1}\{e\}$.

Fix a riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two Lagrangian subspaces: the vertical subspace $\mathbb{V}(\theta) = \ker(d\pi(\theta))$ and the horizontal subspace $\mathbb{H}(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M \times T^*_{\pi(\theta)}M$, $\xi \mapsto (d\pi(\theta)\,\xi\,,\,\nabla_{\pi\xi}\,\theta)$, we can identify $\mathbb{H}(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $\mathbb{V}(\theta) \approx \{0\}\times T^*_{\pi(\theta)}M \approx T_{\pi(\theta)}M$. Let $E \subset T_\theta T^*M$ be an n-dimensional subspace such that $E\cap\mathbb{V}(\theta) = \{0\}$. Then E is the graph of some linear map $S : \mathbb{H}(\theta)\to\mathbb{V}(\theta)$. It can be verified that E is Lagrangian if and only if in symplectic coordinates S is symmetric.

Take $\theta \in T^*M$ and $\xi \in T_\theta T^*M = \mathbb{H}(\theta)\oplus\mathbb{V}(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Writing $d\psi_t(\xi) = \big(h(t), v(t)\big)$, we obtain the Hamiltonian Jacobi equations

$$\begin{aligned} \dot h &= \quad H_{pq}\,h + H_{pp}\,v\,, \\ \dot v &= -H_{qq}\,h - H_{qp}\,v\,, \end{aligned} \tag{2}$$

where the covariant derivatives are evaluated along $\pi(\psi_t(\theta))$, and H_{qq}, H_{qp}, H_{pp} and H_{qq} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $\left(\frac{\partial^2 H}{\partial q_i \partial q_j}\right)$, $\left(\frac{\partial^2 H}{\partial q_i\,\partial p_j}\right)$, $\left(\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right)$ and $\left(\frac{\partial^2 H}{\partial p_i\,\partial q_j}\right)$. Moreover, since the Hamiltonian H is convex, then H_{pp} is positive definite.

Let E be a Lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon,\varepsilon[$ we have that $d\psi_t(E)\cap\mathbb{V}(\psi_t(\theta)) = \{0\}$. Then we can write $d\psi_t(E) = \operatorname{graph} S(t)$, where $S(t) : \mathbb{H}(\psi_t\,\theta)\to\mathbb{V}(\psi_t\,\theta)$ is a symmetric map. That is, if $\xi \in E$

then

$$d\psi_t(\xi) = \Big(h(t), S(t)\,h(t)\Big)\,.$$

Using equation (2) we have that

$$\dot{S}h + S(H_{pq}h + H_{pp}Sh) = -H_{qq}h - H_{qp}Sh\,.$$

Since this holds for all $h \in \mathbb{H}(\psi_t(\theta))$ we obtain the Ricatti equation:

$$\dot{S} + SH_{pp}S + SH_{pq} + H_{qp}S + H_{qq} = 0\,. \tag{3}$$

Suppose there are no conjugate points in the interval $[-T, T]$, we can construct matrix solutions $(Z_T(t), V_T(t))$, $(Z_{-T}(-t), V_{-T}(-t))$, $0 \le t \le T$ to the Jacobi equation with $Z_{\pm T}(0) = I$, $Z_T(T) = Z_{-T}(-T) = 0$. Let $K_{\pm T}(t) = V_{\pm T}(t)Z_{\pm T}(t)^{-1}$ be the corresponding solutions to the Ricatti equation. Let $E_T(\theta) = d\psi_{-T}(\mathbb{V}(\psi_T(\theta)))$ and $F_T(\theta) = d\psi_T(\mathbb{V}(\psi_{-T}(\theta)))$, then $d\psi_t(E_T(\theta)) = \operatorname{graph} K_T(t)$ and $d\psi_t(F_T(\theta)) = \operatorname{graph} K_{-T}(t)$. If there are no conjugate points along all the orbit we have that $K_T(0) \to \mathbb{S}(\theta)$, and $K_{-T}(0) \to \mathbb{U}(\theta)$ as $T \to +\infty$; where $\mathbb{S}(\theta)$, $\mathbb{U}(\theta)$ are the the symmetric solutions of the Ricatti equation (3) corresponding to the Green bundles $\mathbb{E}(\theta) = \operatorname{graph}(\mathbb{S}(\theta))$ and $\mathbb{F}(\theta) = \operatorname{graph}(\mathbb{U}(\theta))$.

The following theorem relates the hyperbolicity of the flow ψ_t with the transversality of the Green bundles.

Theorem 2. *Let $\Sigma = H^{-1}\{e\}$ be a regular energy level without conjugate points. Then the following statements are equivalent:*

(a) *$\psi_t|_\Sigma$ is Anosov.*
(b) *For all $\theta \in \Sigma$, $\mathbb{E}(\theta)$ and $\mathbb{F}(\theta)$ are transversal in $T_\theta\Sigma$.*
(c) *For all $\theta \in \Sigma$, $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta)\rangle$.*
(d) *If $\theta \in \Sigma$, $v \in T_\theta\Sigma$, $v \notin \langle X(\theta)\rangle$ then* $\displaystyle\sup_{t\in\mathbb{R}} |d\psi_t(\theta)\cdot v| = +\infty$.

Where $\langle X(\theta)\rangle \subset T_\theta\Sigma$ is the 1-dimensional subspace generated by the vector field $X(\theta)$ of ψ.

3. The Index Form

The index form is more naturally defined in the Lagrangian setting. Since the Hamiltonian is convex in the fibers we can use the Legendre transform to obtain a Lagrangian $L : TM \to \mathbb{R}$ (see [1]). The Euler-Lagrange equations

$$\begin{aligned} \tfrac{d}{dt}\, x &= v \\ \tfrac{d}{dt}\, L_v &= L_x \end{aligned} \tag{4}$$

are equivalent to the Hamilton equations (1). We derive the Jacobi equation in this Lagrangian setting. Let $\gamma(t)$ be a solution of the Euler-Lagrange equation. Considering a variation $f(s,t)$ of $\gamma(t) = f(0,t)$ made of solutions $t \mapsto f(s,t)$ of the Euler-Lagrange equation and taking the covariant derivative $\frac{D}{ds}$, we obtain the *Jacobi equation*

$$\tfrac{D}{dt}\,(L_{vx}\,k + L_{vv}\,k') = L_{xx}\,k + L_{xv}\,k'\,, \tag{5}$$

where $k = \frac{\partial f}{\partial s}(0,t)$, $\dot{k} = \frac{D}{dt}\frac{\partial f}{\partial s}$ and the derivatives of L are evaluated on $\gamma(t) = f(0,t)$. Here we have used that $\frac{D}{ds}\frac{\partial F}{\partial t} = \frac{D}{dt}\frac{\partial F}{\partial s}$ for the variation map $F(s,t) = L_v\big(f(s,t), \frac{\partial f}{\partial s}(s,t)\big) \in T^*M$, where $\frac{D}{ds}$ and $\frac{D}{dt}$ are the covariant derivatives on the riemannian manifold T^*M. The linear operators L_{xx}, L_{xv}, L_{vv} coincide with the corresponding matrices of partial derivatives in local coordinates. The solutions of (5) satisfy

$$D\varphi_t\big(k(0), \dot{k}(0)\big) = \big(k(t), \dot{k}(t)\big) \in T_{\gamma(t)}(TM)\,,$$

where φ_t is the Lagrangian flow on TM.

Let Ω_T be the set of continuous piecewise C^2 vectorfields ξ along $\gamma_{[0,T]}$. Define the *index form* on Ω by

$$I(\xi,\eta) = \int_0^T \Big(\dot{\xi}\, L_{vv}\, \dot{\eta} + \dot{\xi}\, L_{vx}\, \eta + \xi\, L_{xv}\, \dot{\eta} + \xi\, L_{xx}\, \eta\Big)\, dt\,, \tag{6}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s} \in \Omega_T$.

The following expresion of the index form is taken from [3] and originally due to Clebsch [2]. Let $\theta \in TM$ and suppose that the orbit of θ, $\varphi_t(\theta)$, $0 \le t \le T$ does not have conjugate points. Let $Z(t)$, $V(t)$ be a matrix solution of the Hamiltonian Jacobi equation (2) such that $\det Z(t) \neq 0$. Let $\eta \in \Omega_T$ and define $\zeta \in \Omega_T$ by $\eta(t) = Z(t)\,\zeta(t)$. Then the integrand of $I(\eta,\eta)$ in (6) is $(Z\zeta')^*(H_{pp})^{-1}Z\zeta' + [(Z\zeta)^*V\zeta]'$. Since η and ζ are continuous, we have that

$$I(\eta,\eta) = \int_0^T (Z\zeta')^*\, (H_{pp})^{-1}\, Z\zeta'\, dt + (Z\zeta)^*\, V\zeta\Big|_0^T\,. \tag{7}$$

Corollary 1. *If $\theta \in T^*M$ and the segment $\{\,\psi_t(\theta)\,|\, t \in [0,T]\,\}$ has no conjugate points then the index form is positive definite on*

$$\Gamma_T = \Big\{\, \xi : [0,T] \to TM \,\Big|\, \xi(t) \in T_{\pi\psi_t\theta}M,\ \xi \text{ is piecewise } C^2,\ \xi(0) = 0,\ \xi(T) = 0 \,\Big\}\,.$$

Proof:

Let $\xi \in \Gamma_T$, $\xi \neq 0$. Write $\xi(t) = Z(t)\zeta(t)$. Since $\det Z(t) \neq 0$, $\zeta(0) = 0$, $\zeta(T) = 0$ and $\zeta(t) \not\equiv 0$, then $\zeta' \not\equiv 0$. Now use formula (7).

□

One can extend formula (7) using different solutions (Z_i, V_i) of the Jacobi equation on disjoint subintervals $\{]t_i, t_{i+1}[: i = 0, \dots, N\}$ of [0,T]. Then

$$I(\eta,\eta) = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} (Z_i\zeta_i')^*\, (H_{pp})^{-1}\, Z_i\zeta_i'\, dt + [(Z_i\zeta_i)^*\, V_i\zeta_i]_{t_i^+}^{t_{i+1}^-}. \tag{8}$$

where $\eta|_{]t_i,t_{i+1}[} = Z_i\zeta_i$.

4. Proof of Theorem 1

Let $\varphi \in \mathcal{B}_e \setminus \mathcal{A}_e$ and $\varepsilon > 0$. We can assume that $\varphi = 0$. By theorem 2 there are $\theta \in \Sigma$, $v \in T_\theta\Sigma$ such that $v \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta) \setminus \langle X(\theta)\rangle$. Let $\gamma(t) = \pi\psi_t(\theta)$. Since $X(\theta) \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ and $\dot\gamma(0) = d\pi(X(\theta)) \neq 0$, we can assume that $|d\pi(v)| = 1$ and $\langle d\pi(v), \dot\gamma(0)\rangle = 0$. Let $C > 0$ be such that the segment $\gamma|_{[-C,C]}$ is injective, and let $T > C$.

Let $Z_{\pm T}(t), V_{\pm T}(t), K_{\pm T}(t)$, $0 \le t \le T$ and $E_T(\theta)$, $F_T(\theta)$ be as in section 2.

Let $\zeta_T \in E_T(\theta)$, $\eta_T \in F_T(\theta)$ be such that $d\pi\zeta_T = d\pi\eta_T = d\pi v$.

Define $\xi_T : [-T, T] \to TM$ by

$$\xi_T(t) = \begin{cases} d\pi d\psi_t(\eta_T) = Z_{-T}(t)d\pi(v) & \text{if} \quad t \in [-T, 0] \\ d\pi d\psi_t(\zeta_T) = Z_T(t)d\pi(v) & \text{if} \quad t \in [0, T]. \end{cases} \tag{9}$$

By (8)

$$\begin{aligned} I(\xi_T, \xi_T) &= \xi_T(T)^* K_T(T)\xi_T(T) - \xi_T(-T)^* K_{-T}(-T)\xi_T(-T) \\ &\quad + \xi_T(0)^*(K_{-T}(0) - K_T(0))\xi_T(0) \\ &= d\pi(v)^*(K_{-T}(0) - K_T(0))d\pi(v) \end{aligned} \tag{10}$$

There exist $\lambda \in (0, C/2)$ and $T_0 > 0$ such that $\left|\overline{\xi}_T(t)\right|^2 > \frac{1}{2}$ for $|t| \le \lambda$ and $T > T_0$, where $\overline{\xi}_T(t)$ is the component of $\xi(t)$ orthogonal to $\dot\gamma(t)$.

Since $K_T(0) \to \mathbb{S}(\theta)$, $K_{-T}(0) \to \mathbb{U}(\theta)$ as $T \to \infty$ and $v = (d\pi(v), \mathbb{S}(\theta)d\pi(v)) = (d\pi(v), \mathbb{U}(\theta)d\pi(v))$, we have that $|(K_{-T}(0) - K_T(0))d\pi(v)| \to 0$ as $T \to \infty$. Fix T sufficiently large to have $I(\xi_T, \xi_T) < \varepsilon\lambda/4$. ¿From now on we omit the subscript T in ξ.

Choose coordinates $\overline{x} = (x_1, \dots, x_n) : U \to [-C, C] \times \mathbb{R}^{n-1}$ on a tubular neighbourhood of $\gamma([-C, C])$ such that $\overline{x}\circ\gamma([-C, C]) = [-C, C] \times \{\mathbf{0}\}$ and $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$ is an orthogonal frame over the points of $\gamma([-C, C])$.

Define $\widetilde{H} = H + \phi$ with $\phi(x_1, \dots, x_n) = \frac{1}{2} f(x_1/C) f(|\mathbf{x}|/\delta)\, \varepsilon\, |\mathbf{x}|^2$, where $\mathbf{x} = (x_2, x_3, \dots, x_n)$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ bump function with support in $[-1, 1]$ which is 1 on $[-\frac{1}{2}, \frac{1}{2}]$. Thus $\widetilde{H}_p = H_p$, $\widetilde{L} = L - \phi$, $\widetilde{L}_v = L_v$, $\widetilde{L}_{vv} = L_{vv}$, $\widetilde{L}_{vx} = L_{vx}$. Since

$$\phi_x = \frac{1}{2}\varepsilon \left(C^{-1} f'(x_1/C) f(|\mathbf{x}|/\delta)\, |\mathbf{x}|^2,\ f(x_1/C)[f'(|\mathbf{x}|/\delta)\, |\mathbf{x}|/\delta + 2\, f(|\mathbf{x}|/\delta)]\mathbf{x} \right),$$

and $\phi_{xx} = \frac{1}{2}\varepsilon \begin{bmatrix} a(\overline{x}) & \boldsymbol{b}(\overline{x}) \\ \boldsymbol{b}(\overline{x})^t & \mathbb{A}(\overline{x}) \end{bmatrix}$ where

$$a = C^{-2} f''(x_1/C) f(|\mathbf{x}|/\delta)\, |\mathbf{x}|^2, \quad \boldsymbol{b} = C^{-1} f'(x_1/C)[f'(|\mathbf{x}|/\delta)\, |\mathbf{x}|/\delta + 2\, f(|\mathbf{x}|/\delta)]\mathbf{x},$$

$$\mathbb{A} = f\left(\frac{x_1}{C}\right) \left([f''(|\mathbf{x}|/\delta)/\delta^2 + (3/\delta|\mathbf{x}|)\, f'(|\mathbf{x}|/\delta)]\mathbf{x} \otimes \mathbf{x} + [f'(|\mathbf{x}|/\delta)\, |\mathbf{x}|/\delta + 2\, f(|\mathbf{x}|/\delta)]I \right),$$

we have $|\phi_x| \le c\,\delta\,\varepsilon$, $|\phi_{xx}| \le c\,\varepsilon$ and then

$$\left|\widetilde{X}(\theta) - X(\theta)\right| \le c\,\delta\,\varepsilon \qquad \left|D\widetilde{X}(\theta) - DX(\theta)\right| \le c\,\varepsilon$$

where c is a constant. Using (6), we have in tubular coordinates

$$\widetilde{I}(\xi,\xi) = I(\xi,\xi) + \int_{-T}^{T} \left(\dot\xi(L_{vv}(\widetilde{\psi}_t(\theta)) - L_{vv}(\psi_t(\theta)))\dot\xi + 2\dot\xi(L_{vx}(\widetilde{\psi}_t(\theta)) - L_{vx}(\psi_t(\theta)))\xi\right)\, dt$$
$$+ \int_{-T}^{T} \xi\left(\widetilde{L}_{xx}(\widetilde{\psi}_t(\theta)) - \widetilde{L}_{xx}(\psi_t(\theta)) + \widetilde{L}_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta))\right)\xi\, dt. \tag{11}$$

Since $\lim_{\delta\to 0}\widetilde{\psi}_t(\theta) = \psi_t(\theta)$ uniformly on $t\in[-T,T]$, for $\delta > 0$ sufficiently small we have

$$\widetilde{I}(\xi,\xi) \le I(\xi,\xi) + \int_{-T}^{T} \xi\left(\widetilde{L}_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta))\right)\xi\, dt + \varepsilon\lambda/4.$$

Let $\Delta(t) = \xi(t)(\widetilde{L}_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta)))\xi(t)$. If $\gamma(t)\in \overline{x}^{-1}([-C,C]\times\{\mathbf{0}\})$,

$$\Delta(t) = -\xi(t)\phi_{xx}(\gamma(t))\xi(t) = -\varepsilon\,\xi(t)\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & f(x_1\circ\gamma(t)/C)\end{bmatrix}\xi(t). \tag{12}$$

Then

$$\Delta(t) = \begin{cases} -\varepsilon f(x_1\circ\gamma(t)/C)\left|\overline{\xi}\right|^2 & \text{if} \quad \gamma(t)\in\overline{x}^{-1}([-C,C]\times\{\mathbf{0}\}) \\ 0 & \text{if} \quad \gamma(t)\notin B(C,\delta)\end{cases} \tag{13}$$

where $B(C,\delta) = \{q\in M : \overline{x}(q)\in[-C,C]\times B_\delta(\mathbf{0})\}$. Therefore

$$\int_{-T}^{T}\Delta(t)\,dt < -\varepsilon\lambda + \int_{D(T,C,\delta)}\Delta(t)\,dt$$

where $D(T,C,\delta) = ([-T,T]\setminus[-C,C])\cap\gamma^{-1}(B(C,\delta))$.

Lemma 1. *If the orbit $\gamma(t)$ is not periodic*

$$\lim_{\delta\to 0}\int_{D(T,C,\delta)}\Delta(t)\,dt = 0$$

Proof:

If the segment $\gamma|_{[-T,T]}$ is injective, then we can choose $\delta > 0$ sufficiently small to have $D(T,C,\delta) = \emptyset$ and hence the integral vanishes.

Suppose that $([-T,T]\setminus[-C,C])\cap\gamma^{-1}(\gamma([-C,C])) = \{s_1,\dots,s_N\}$. Let $t_1,\dots,t_N\in[-C,C]$ with $\gamma(s_i) = \gamma(t_i)$. Let $\overline{\gamma}_i'$ be the component of $(\overline{x}\circ\gamma)'(s_i)$ orthogonal to $(\overline{x}\circ\gamma)'(t_i)$. Let $r > 0$ be such that $|\overline{\gamma}_i'| > 2\,r$ for $i = 1,\dots,N$ and define $h_i(t) = \langle\overline{x}\circ\gamma(t),\overline{\gamma}_i'\rangle/|\overline{\gamma}_i'|$ for $|t-s_i|$ small. Then $\exists\alpha > 0$ such that $h_i'(t) > r$ and $|h_i(t)| > |t-s_i|\,r$ for $|t-s_i| < \alpha$. Thus $\gamma(t)\notin B(C,\delta)$ if $\delta/r < |t-s_i| < \alpha$ and so

$$\int_{D(T,C,\delta)}\Delta(t)\,dt = \sum_{i=1}^{N}\int_{s_i-(\delta/r)}^{s_i+(\delta/r)}\Delta(t)\,dt. \tag{14}$$

The lemma follows.

□

Lemma 2. *If the orbit $\gamma(t)$ is periodic, then*

$$\lim_{\delta\to 0}\int_{D(T,C,\delta)}\Delta(t)\,dt \le 0.$$

Proof: Let τ be the period of γ. Defining

$$D'(T,C,\delta) = \Big([-T,T] \setminus \bigcup_{m=0}^{\infty} [m\tau - C, m\tau + C]\Big) \cap \gamma^{-1}(B(C,\delta)),$$

for $k = [T + C/\tau]$ we have

$$\int_{D(T,C,\delta)} \Delta(t)\, dt = \int_{D'(T,C,\delta)} \Delta(t)\, dt + \sum_{m=1}^{k} \int_{m\tau - C}^{m\tau + C} \Delta(t)\, dt.$$

As in Lemma 1 one proves that $\lim_{\delta\to 0} \int_{D'(T,C,\delta)} \Delta(t)\, dt = 0$.

By (13) we have $\sum_{m=1}^{k} \int_{m\tau-C}^{m\tau+C} \Delta(t)\, dt \le 0$. $\square$

Thus, for $\delta > 0$ sufficiently small we have $\widetilde{I}(\xi,\xi) \le I(\xi,\xi) - \varepsilon\lambda + \varepsilon\lambda/4 < 0$ and corollary 1 implies that $H + \phi$ has conjugate points in $(H+\phi)^{-1}(e)$.

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