FURTHER REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS

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Abstract. We construct a new class of representations of the canonical commutation relations, which generalizes previously known classes. We perturb the infinitesimal generator of the initial Fock representation (*i.e.* the free quantum field) by a function of the field which is square-integrable with respect to the associated Gaussian measure. We characterize which such perturbations lead to representations of the canonical commutation relations. We provide conditions entailing the irreducibility of such representations, show explicitly that our class of representations subsumes previously studied classes, and give necessary and sufficient conditions for our representations to be unitarily equivalent, resp. quasi-equivalent, with Fock, coherent or quasifree representations.

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I. Introduction

The canonical commutation relations (henceforth the CCR) were initially introduced in 1927 by Dirac as generalizations of Heisenberg's commutation relations in order to discuss radiation theory [10]. Since then, the CCR have proven to be of central importance in the quantum description of bosonic systems. When rigorous mathematical analysis of the CCR in the case of infinitely many degrees of freedom - the case of relevance to quantum statistical mechanics and quantum field theory - began in the 1950's, it was quickly realized that there are uncountably infinitely many unitarily inequivalent irreducible representations of the CCR [12], in striking contrast to the unique unitary equivalence class in the case of finitely many degrees of freedom [20], and that the choice of proper representation is crucial in any physical application. This last point deserves emphasis. It has become clear from rigorous study of concrete models in constructive quantum field theory that bosonic systems with identical kinematics but physically distinct dynamics require inequivalent representations of the CCR. Roughly speaking, the kinematical aspects determine the choice of CCR-algebra, whereas the dynamics fix the choice of the representation of the given CCR-algebra in which to make the relevant, perturbation-free computations. (It is also believed - and proven in a number of indicative special cases - that perturbation series in one representation provide divergent and at best asymptotic approximations to the physically relevant quantities in another representation, when the two representations are unitarily inequivalent.)

Many classes of representations of the CCR have been rigorously studied in the literature - we mention, in particular, the Fock, coherent, quasifree (or symplectic), infinite product, and, more recently, quadratic [26][22][29] and polynomial representations [30]. In this paper, we wish to construct and prove certain physically relevant properties of a broad class of representations of the CCR containing all of the above (except possibly the infinite product representations) as special cases.

We suggest heuristically the nature of these representations with the following easily describable examples. Let $\{q_k, p_k\}_{k=1}^{\infty}$ be a system of densely defined operators acting on the complex Hilbert space $\mathcal{K}$ and satisfying

$$[q_j, p_k] = i\delta_{jk} \cdot \mathbb{1} \quad ,$$

(along with $[q_j, q_k] = 0$ and $[p_j, p_k] = 0$) and for the standard annihilation and creation operators $a_k \equiv \frac{1}{\sqrt{2}}(q_k + ip_k)$ and $a_k^\dagger \equiv \frac{1}{\sqrt{2}}(q_k - ip_k)$ let there be a vector $\Omega \in \mathcal{K}$ such that $a_k\Omega = 0$ for all $k \in \mathbb{N}$. In other words, we consider the Fock representation of a bosonic system with infinitely many degrees of freedom (in the so-called basis-dependent formulation). In this setting, the coherent, resp. quasifree, quadratic or polynomial canonical transformations can be written as

$$
\begin{aligned}
&(coherent) && q_k \mapsto q_k, && p_k \mapsto p_k + \lambda_k \cdot \mathbb{I} \quad ,\\
&(quasifree) && q_k \mapsto q_k, && p_k \mapsto p_k + \sum_{k_1} \lambda_{kk_1}\, q_{k_1} \quad ,\\
&(quadratic) && q_k \mapsto q_k, && p_k \mapsto p_k + \sum_{k_1,k_2} \lambda_{kk_1k_2} \; : q_{k_1} q_{k_2} : \quad ,\\
&(polynomial) && q_k \mapsto q_k, && p_k \mapsto p_k + \sum_{k_1,\dots,k_n} \lambda_{kk_1\cdots k_n} \; : q_{k_1} \cdots q_{k_n} : \quad ,
\end{aligned}
$$

with all coefficients totally symmetric in the indices.[1] The canonical transformations of general degree we construct in this paper are of the form

$$
q_k \mapsto q_k, \qquad p_k \mapsto p_k + \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\cdots k_m} \; : q_{k_1} \cdots q_{k_m} : \quad . \tag{1.1}
$$

We emphasize that this generalization opens up a much larger class of representations of the CCR - representations, which differ from the original Fock representation by an extremely singular perturbation. Aside from the mathematical interest in this extension of the representation theory of an important class of C^*-algebras, there are advantages for the quantum theory of bosonic systems with infinitely many degrees of freedom. As alluded to earlier, such representations allow rigorous treatment of interacting systems with extremely singular "bare" interaction. One may therefore handle systems with new classes of dynamics (for a suggestive treatment of the connection between representation and dynamics, see the discussion in [1]). This will entail that the divergences which typically arise in perturbation theory can be avoided, since the very need for a perturbation expansion from the "wrong" representation is obviated. Furthermore, as briefly shown in the final section of [22], but which is well-known to theoretical physicists, the Hamilton operator, which represents the total energy of the system, can often be simplified and in certain situations even be diagonalized by such canonical transformations. This leads to significant computational advantages. We postpone addressing these issues in detail to a later publication.

In Chapter II, the general setting of this work will be more precisely specified. The transformations of general degree will be constructed in Chapter III, and those which lead to regular representations of the CCR will be characterized in a number of different manners. Convenient sufficient conditions assuring the irreducibility of such representations will also be given. It will be shown in Chapter IV that the class of representations constructed in Chapter III subsumes the previously studied classes discussed above. In Chapter V we shall restrict our attention to the computationally more manageable canonical transformations of finite degree and shall prove in that context necessary and

[1] In point of fact, the polynomial representations of the CCR constructed in [30] were constructed globally and not via their infinitesimal generators as is done in this paper.

sufficient conditions for such representations to be quasi-equivalent to Fock, coherent or quasifree representations. For our purposes, the primary interest of these results is in the violation of the stated necessary and sufficient conditions - for then one has representations of bosonic systems of infinitely many degrees of freedom which describe *new* physics. However, in this paper we do not try to study the new physical content of these representations. Moreover, as an aside, the previously known conditions for unitary equivalence of quasifree states follow as a very special case of the results in Chapter V. The significantly less transparent conditions for unitary equivalence with the quadratic representations constructed in [22] will not be given here, though they are known. (See the end of Chapter V for a brief discussion.)

As this paper is an extension of [22] and employs a number of the results and arguments from that paper, we shall maintain the same notational conventions, as detailed in the next chapter. Early versions of some of the results presented in this paper appeared in the *Diplomarbeit* [11].

II. Notation and General Setting

We begin with an arbitrary real nondegenerate symplectic space (H, σ) with an associated regular Weyl system $(\mathcal{K}, W(f))$ consisting of a complex Hilbert space $\mathcal{K}$ and a mapping $W : H \mapsto \mathcal{U}(\mathcal{K})$ from H into the group $\mathcal{U}(\mathcal{K})$ of unitary operators on $\mathcal{K}$ which satisfies the following axioms [16]:

$$W(f)W(g) = e^{-i\sigma(f,g)/2}W(f+g), \quad \forall f, g \in H, \tag{2.1}$$

$$W(f)^* = W(-f), \quad \forall f \in H, \tag{2.2}$$

and

$$\mathrm{I\!R} \ni t \mapsto W(tf) \in \mathcal{B}(\mathcal{K}) \quad \text{is weakly continuous for all} \quad f \in H. \tag{2.3}$$

($\mathcal{B}(\mathcal{K})$ denotes the set of all bounded linear operators $A : \mathcal{K} \mapsto \mathcal{K}$.) Condition (2.3) entails that the map $\mathrm{I\!R} \ni t \mapsto W(tf) \in \mathcal{B}(\mathcal{K})$ is actually strongly continuous, hence by Stone's Theorem one knows that for each $f \in H$ there exists a self-adjoint operator $\Phi(f)$ on $\mathcal{K}$ such that $W(tf) = e^{it\Phi(f)}$ for all $t \in \mathrm{I\!R}$ and by (2.1) the map $f \mapsto \Phi(f)$ is (real) linear. In fact, there exists a dense domain of vectors $D_W \subset \mathcal{K}$ which is a core of and left invariant by every $\Phi(f)$ [23][13]; it is on this domain that the linearity just mentioned can be verified. On this domain one also verifies that the generators satisfy the CCR:

$$\Phi(f)\Phi(g) - \Phi(g)\Phi(f) = i\sigma(f,g)\mathbb{1}, \quad \forall f, g \in H. \tag{2.4}$$

We therefore also call the Weyl system $(\mathcal{K}, W(f))$ and its associated generators as above a regular representation of the CCR over (H, σ). In the physical literature such infinitesimal generators $\Phi(f)$ are called field operators, and we shall use this language in the following.

We shall denote by $\mathcal{A}(H,\sigma)$ the C^*-algebra on $\mathcal{K}$ generated by the operators $\{W(f) \mid f \in H\}$. As the notation indicates, the algebra $\mathcal{A}(H,\sigma)$ does not depend on the choice of representation of the Weyl operators $\{W(f) \mid f \in H\}$ [36] or Theorem 5.2.8 in [6]. $\mathcal{A}(H,\sigma)$ is a simple, nonseparable C^*-algebra [16][36]. There are, in fact, many different C^*-algebras one can associate with the CCR (see [14] for a discussion of some of the alternatives), and the one we have chosen is minimal in the sense of set containment [19]; but, for practical purposes the choice is immaterial, since one is generally interested in a von Neumann algebra which is 'generated' by the C^*-algebra, and all the C^*-algebras discussed in [14] (realized concretely on a given representation space) have the same weak closure. In the following it shall be understood that one choice of σ has been fixed, and we shall write $\mathcal{A}(H)$ instead of $\mathcal{A}(H,\sigma)$.

For any real linear map $J : H \mapsto H$ satisfying $\sigma(Jf,Jg) = \sigma(f,g)$, $-\sigma(Jf,f) > 0$ $(f \neq 0)$, and $J^2 = -\mathbb{1}$, one can introduce a complex structure on H as follows[2] [14]: $(\alpha + i\beta)f \equiv \alpha f + \beta J f$, for all $f \in H$, and $\alpha, \beta \in \mathbb{R}$. Moreover, $\langle f,g\rangle_{\mathcal{H}} \equiv -\sigma(Jf,g) + i\sigma(f,g)$ defines a scalar product on H such that $(H, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a complex preHilbert space [14], the completion of which we shall denote by $\mathcal{H}$. If, on the other hand, one begins with a complex Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, then with $\sigma(f,g) = \Im m\langle f,g\rangle_{\mathcal{H}}$, $(\mathcal{H},\sigma)$ is a real symplectic space of the sort with which we began, and with $\langle f,g\rangle \equiv \Re e\langle f,g\rangle_{\mathcal{H}}$, then $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a real Hilbert space.

With a choice of a σ-admissible complex structure J on (H,σ), there is an important representation of $\mathcal{A}(H)$ called the Fock representation. This is given as the GNS-representation $\{\mathcal{K}, \pi_J, \Omega\}$ associated with the state [18] ω_J determined on $\mathcal{A}(H)$ by

$$\omega_J(W(f)) \equiv e^{\sigma(Jf,f)/4}. \tag{2.5}$$

Given such a representation, one can define the following 'annihilation' and 'creation' operators:

$$a(f) \equiv \frac{1}{\sqrt{2}}(\Phi(f) + i\Phi(Jf)), \qquad a^\dagger(f) \equiv \frac{1}{\sqrt{2}}(\Phi(f) - i\Phi(Jf)), \tag{2.6}$$

where $\pi_J(W(tf)) = e^{it\Phi(f)}$. One has then $a(f)\Omega = 0$ for all $f \in H$.

For the purposes of this paper, we shall assume that a choice of σ-admissible complex structure J has been made on (H,σ) and held fixed, so we have the complex one-particle space $\mathcal{H}$ and a corresponding Fock representation. Since as sets $H = \mathcal{H}$, we shall distinguish notationally the vector f viewed as an element of the real Hilbert space H from the same vector, denoted as $\tilde{f}$, viewed as an element of the complex Hilbert space $\mathcal{H}$. If $\{\tilde{e}_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis in $\mathcal{H}$, then the set $\{e_k, Je_k\}_{k\in\mathbb{N}}$ forms a symplectic orthonormal system

[2] It should be mentioned that such a J does not necessarily exist for arbitrary choice of symplectic space (H,σ) [28].

in H, in particular

$$\sigma(e_k, e_l) = \sigma(Je_k, Je_l) = 0 \quad , \quad \sigma(e_k, Je_l) = \delta_{kl} \quad ,$$
$$\langle e_k, e_l \rangle = \langle Je_k, Je_l \rangle = \delta_{kl} \quad , \quad \langle e_k, Je_l \rangle = 0 \quad .$$

In this paper, whenever a choice of symplectic orthonormal basis $\{e_k, Je_k\}_{k\in\mathrm{I\!N}}$ has been made, the Hilbert subspace of H generated by $\{e_k\}_{k\in\mathrm{I\!N}}$ will be denoted by V.

With the choice of complex structure made as above, the corresponding Fock state $\omega_J : \mathcal{A}(H) \mapsto \mathbb{C}$ now satisfies

$$\omega_J(W(f)) = e^{-\|f\|^2/4},$$

and the associated GNS-space may be represented by the symmetric Fock space $\mathcal{F}_+(\mathcal{H})$. We recall that the Fock space $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$ ($\mathcal{H}^0 = \mathbb{C}$, $\mathcal{H}^n$ the n-fold tensor product of $\mathcal{H}$ with itself), and that $\mathcal{F}_+(\mathcal{H})$ is the totally symmetric subspace $\bigoplus_{n=0}^{\infty} P_+\mathcal{H}^n$ of $\mathcal{F}(\mathcal{H})$, where P_+ is the projection

$$P_+(\tilde{f}_1 \otimes \tilde{f}_2 \otimes \cdots \otimes \tilde{f}_n) = \frac{1}{n!} \sum_{\pi} \tilde{f}_{\pi(1)} \otimes \tilde{f}_{\pi(2)} \otimes \cdots \otimes \tilde{f}_{\pi(n)}$$

($\tilde{f}_i \in \mathcal{H}$, π: permutation on the set $\{1, 2, \ldots, n\}$). The projection operator $P_n : \mathcal{F}(\mathcal{H}) \mapsto \mathcal{H}^n$ projects onto the n-particle subspace. The vector $\Omega \equiv (1, 0, 0, \ldots) \in \mathcal{F}_+(\mathcal{H})$ is the Fock vacuum. $\mathcal{F}_0$ is the finite-particle subspace, i.e. the linear span of the ranges of $\{P_n\}_{n\in\mathrm{I\!N}}$. The elements of $\mathcal{F}_0$ will be called finite-particle vectors. As is well-known, the GNS-representation for the Fock state may be identified with this Fock space representation.

With $a(\tilde{f})$ the usual annihilation operator in $\mathcal{F}_+(\mathcal{H})$ for $\tilde{f} \in \mathcal{H}$ and the adjoint operator $a^*(\tilde{f})$ (an extension of) the corresponding creation operator, then the linear self-adjoint operator $\Phi_S(\tilde{f}) \equiv \frac{1}{\sqrt{2}} \overline{(a(\tilde{f}) + a^*(\tilde{f}))}$ (the bar denotes the closure of the operator) is called the Segal field operator in $\mathcal{F}_+(\mathcal{H})$. If we also view $\tilde{f} \in \mathcal{H}$ as an element of H, then we have $\Phi_S(\tilde{f}) = \Phi(f)$, after identifying $\mathcal{A}(H)$ with $\pi_J(\mathcal{A}(H))$ (since they are isomorphic). $\mathcal{F}_0$ is a core for $\Phi(f)$, and for $\varphi \in \mathcal{F}_0$, $\Phi(f)$ satisfies the bound $\|\Phi(f)\varphi\| \leq \sqrt{2}\sqrt{n_\varphi + 1}\|f\|\|\varphi\|$, where n_φ equals the smallest $n \in \mathrm{I\!N}$ such that $P_N\varphi = 0$ for all $N > n$. If $\{f_n\}_{n=1}^{\infty}$ converges in H to f, then for every $\varphi \in \mathcal{F}_0$ the sequence $\{\Phi(f_n)\varphi\}_{n=1}^{\infty}$ converges in $\mathcal{F}_+(\mathcal{H})$ to $\Phi(f)\varphi$.

To make a notational connection to the discussion in Chapter I and in keeping with common harmonic oscillator conventions, if $\{e_k, Je_k\}$ is a symplectic orthonormal basis in H, then the 'position operator' and the 'momentum operator' corresponding to the k-th degree of freedom are given by

$$q_k \equiv \Phi_S(\tilde{e}_k) = \Phi(e_k) \quad , \quad p_k \equiv \Phi_S(i\tilde{e}_k) = \Phi(Je_k) \quad .$$

A crucial technical tool in this paper is the use of Q-space techniques. Let $\{e_k, Je_k\}$ be a symplectic orthonormal basis. Moreover, let $x = (x_1, x_2, \ldots)$ be a point in $Q \equiv \times_{k=1}^{\infty} \mathrm{I\!R}$, and Σ be the σ-algebra generated by the cylinder sets of Q with Lebesgue measurable base. Then $\mu = \otimes_{k=1}^{\infty} \mu_k$, where each μ_k is the Gaussian measure $d\mu_k = \pi^{-\frac{1}{2}} e^{-x_k^2}\, dx_k$, is a probability measure on (Q, Σ).

It is well-known that there exists a unitary map S of $\mathcal{F}_+(\mathcal{H})$ onto $L^2(Q, d\mu)$ such that [32][35][25]

$$S\Omega_0 = 1 \quad \text{and} \quad S\, P_+\, (\tilde{e}_{k_1} \otimes \tilde{e}_{k_2} \otimes \cdots \otimes \tilde{e}_{k_r}) = (r!)^{-\frac{1}{2}} (\sqrt{2})^r : x_{k_1} x_{k_2} \cdots x_{k_r} : \,,$$

$$S\Phi(e_k)S^{-1} \equiv q_k = x_k \mathbb{1} = \text{ (multiplication by) } \quad x_k, \quad \text{and}$$

$$S\Phi(Je_k)S^{-1} \equiv p_k = \frac{1}{i}\frac{\partial}{\partial x_k} + i x_k \mathbb{1} \,,$$

where the operator equations are understood to hold on the dense set $S\mathcal{F}_0$. We shall drop the symbol S, when the identification is clear.

Let $\underline{k} = \{k_1, k_2, \ldots, k_r\}$ be a multiple index in $\mathbb{N}^{(r)} = \times_{j=1}^{r} \mathbb{N}$, $\{\lambda_{\underline{k}}\}_{\underline{k} \in \mathbb{N}^{(r)}}$ a sequence of real numbers, totally symmetric in $\underline{k}$, $\sum_{\underline{k}} \lambda_{\underline{k}}^2 < \infty$, and $I_n = \{\underline{k} \mid \max\{k_1, k_2, \ldots, k_r\} \le n\}$. Then

$$f_n^{(r)} = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}\ e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_r} \ \in H^{(r)} \,,$$

with $\|f_n^{(r)}\|^2 = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2$. Since with $m < n$, we have $\|f_n^{(r)} - f_m^{(r)}\|^2 = \sum_{\underline{k} \in I_n \setminus I_m} \lambda_{\underline{k}}^2$ and $\sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2$ converges with $n \to \infty$, it follows that

$$f_n^{(r)} \ \longrightarrow\ f^{(r)} = \sum_{\underline{k}} \lambda_{\underline{k}}\ e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_r} \quad \text{in } H^{(r)} \ .$$

Consider the sequence of operators

$$A(f_n^{(r)}) = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}\ : \Phi(e_{k_1}) \Phi(e_{k_2}) \cdots \Phi(e_{k_r}) : \,,$$

the Q–space realization of which is given by

$$A_n = A(f_n^{(r)}) = \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}\ : x_{k_1} x_{k_2} \cdots x_{k_r} : \mathbb{1} \quad .$$

Note that since it is a polynomial, $A_n = A_n(x)$ is in $L^2(Q, d\mu)$ with

$$\|A_n\|^2 = 2^{-r}\ r! \sum_{\underline{k} \in I_n} \lambda_{\underline{k}}^2 = 2^{-r}\ r!\ \|f_n^{(r)}\|^2 \quad ,$$

$$\text{and} \qquad \|A_n - A_m\|^2 = 2^{-r}\ r! \quad \|f_n^{(r)} - f_m^{(r)}\|^2$$

(cf. the proof of Lemma I.18 of [35]). Therefore, $A_n(x)$ converges in $L^2(Q, d\mu)$, and we shall call the a.e.-defined limit $A(x) = \sum_{\underline{k}} \lambda_{\underline{k}}\ : x_{k_1} x_{k_2} \cdots x_{k_r} :$, which up to a factor of $\sqrt{r!}\,(\sqrt{2})^{-r}$ corresponds to $f^{(r)}$.

The advantage of the Q–space formulation is that all functions of the elements of $\{\Phi(e_k) \mid k \in \mathbb{N}\}$ become multiplication operators on $L^2(Q, d\mu)$. $A_n(x)$ and $A(x)$ are measurable, real-valued functions on Q which are finite almost everywhere with respect to μ. So with $D(A) \equiv \{\varphi \mid A\varphi \in L^2(Q, d\mu)\}$ (similarly for $D(A_n)$), $(A_n\varphi)(x) = A_n(x)\varphi(x)$ and $(A\varphi)(x) = A(x)\varphi(x)$ are self-adjoint operators (cf. [24], VIII.3 Proposition 1). Thus, for every $f^{(r)} \in P_+ H^{(r)}$, $A(f^{(r)})$ represents a well-defined self-adjoint multiplication operator.

III. A General Class of Representations of the CCR

In Section 3.1 we shall use Q-space techniques to construct our general class of representations of the CCR, as motivated above. These techniques were already employed to establish some of the results proven in [22] about quadratic representations of the CCR. Our representations of general degree of the CCR will be defined in terms of linear maps Λ from the one-particle space H into Q-space itself. We shall give a number of characterizations of those Λ which yield representations of the CCR, *i.e.* which determine canonical transformations of the field operators. It will be shown in Section 3.2 that when such Λ are bounded, then the resulting representation is irreducible. We shall also explain how unbounded Λ can lead to reducible representations.

3.1 Canonical Transformations of Arbitrary Degree.

We shall use the basic facts that the set $\{ : x_{k_1} \ldots x_{k_m} : \mid k_1, \ldots , k_m \in \mathbb{N} \}$ is an orthogonal basis in $L^2(Q, d\mu)$ and that

$$\| \sum_{k_1,\ldots,k_m} \lambda_{k_1 \ldots k_m} : x_{k_1} \ldots x_{k_m} : \|_2^2 = \sum_{k_1,\ldots,k_m} \lambda_{k_1 \ldots k_m}^2 \frac{m!}{2^m} \quad , \tag{3.1.1}$$

for arbitrary $\lambda_{k_1 \ldots k_m} \in \mathbb{R}$ symmetric in the indices $k_1 \ldots k_m$ (see Section 4.3 in [22]). Thus, standard arguments entail the following lemma.

Lemma 3.1.1. *Let $F = \sum_{k_1,\ldots,k_m} c_{k_1 \ldots k_m} : x_{k_1} \ldots x_{k_m} : \in L^2(Q, d\mu)$ with complex numbers $c_{k_1 \ldots k_m}$ symmetric in the indices $k_1 \ldots k_m$. Then*

$$\langle F, : x_{k_1} \ldots x_{k_m} : \rangle = c_{k_1 \ldots k_m} \frac{m!}{2^m} \quad ,$$

and F is real-valued (a.e. μ) if and only if $c_{k_1 \ldots k_m} \in \mathbb{R}$ for all $k_1, \ldots , k_m \in \mathbb{N}$.

We also state a well-known fact about the Q-space representation of Fock space. Recall that for $f \in V$, $x(f)\mathbb{1} = \Phi(f)$, given our stated convention of dropping the unitary S.

Lemma 3.1.2. *The set $\mathcal{G}$ equal to the linear span of $\{e^{ix(f)} \mid f \in V\}$ is a core for the number operator N and for $\Phi(g)$, given any $g \in H$.*

In the next proposition we relate the operators $\Phi(f)$ and $\Phi(f) + F\mathbb{1}$, for any $F \in L^2(Q, d\mu)$, via a unitary transformation.

Proposition 3.1.3. *If $F \in L^2(Q, d\mu)$ is real-valued (μ a.e.), then the operator $\Phi(f) + F\mathbb{1}$, is essentially self-adjoint on $L^\infty(Q, d\mu)$, when $f \in V$, and on $e^{-iG}\mathcal{G}$, when $f \notin V$ (G will be defined shortly). Furthermore, if $f \notin V$, then the closure of the corresponding operator is given in terms of the self-adjoint $\Phi(f)$ by*

$$\overline{\Phi(f) + F\mathbb{1}} = e^{-iG} \Phi(f) e^{iG} \quad , \tag{3.1.2}$$

for any choice of (μ a.e.) real-valued $G \in L^2(Q, d\mu)$ such that $\partial G/\partial x(f_2) = F$, where f_2 is determined uniquely by f by the decomposition $f = f_1 + Jf_2$, $f_1, f_2 \in V$.

Proof. By Lemma 4.3.1 in [22], $L^\infty(Q, d\mu)$ is a core for the corresponding multiplication operator for every $F \in L^2(Q, d\mu)$. If $f \in V$, then the operator $\Phi(f) + F\mathbb{1}$ is symmetric, $D(\Phi(f)+F\mathbb{1}) = D(\Phi(f)) \cap D(F\mathbb{1})$, and, on $L^\infty(Q, d\mu) \subset L^2(Q, d\mu)$, it is equal to the operator corresponding to multiplication by the L^2-function $x(f) + F$, *i.e.* $\Phi(f) + F\mathbb{1}$ is essentially self-adjoint on $L^\infty(Q, d\mu)$.

One may assume that $f \notin V$ and therefore $f = Je_1 + v$ for a suitable $v \in V$ (after choosing the basis $\{e_k \mid k \in \mathbb{N}\}$ appropriately). There exist suitable G_n, $n \in \mathbb{N} \cup \{0\}$, in the subspace of $L^2(Q, d\mu)$, which includes the constant functions and the set $\{: x_{k_1} \dots x_{k_m} : \mid k_1, \dots, k_m \geq 2\}$, such that

$$F = \sum_{n=0}^{\infty} : x_1^n : G_n \quad .$$

Set $G = \sum_{n=0}^{\infty} \frac{1}{n+1} : x_1^{n+1} : G_n$. The series converges with respect to the L^2-norm, *i.e.* $G \in L^2(Q, d\mu)$, since

$$\begin{aligned}
\|G\|_2^2 &= \sum_n \frac{1}{(n+1)^2} \| : x_1^{n+1} : \|_2^2 \, \|G_n\|_2^2 \\
&\overset{(3.1.1)}{=} \sum_n \frac{1}{(n+1)^2} 2^{-n-1} (n+1)! \|G_n\|_2^2 \\
&\leq \sum_n 2^{-n} n! \|G_n\|_2^2 \\
&= \sum_n \| : x_1^n : \|_2^2 \, \|G_n\|_2^2 \\
&= \|F\|_2^2 < \infty \quad ,
\end{aligned}$$

and G, G_n, $n \in \mathbb{N}$, are (μ-a.e.) real-valued, by Lemma 3.1.1.

Note that $\partial : x_1^k : /\partial x_1 = k : x_1^{k-1} :$ is a polynomial of degree $k-1$ with leading term kx_1^{k-1}. It is evident that $\mathcal{G} \subset L^\infty(Q, d\mu)$, but one has, furthermore, $e^{-iG}\mathcal{G} \subset D(\Phi(f))$ and $\Phi(f)e^{-iG}\varphi = e^{-iG}(\Phi(f) - F\mathbb{1})\varphi$, for all $\varphi \in \mathcal{G}$, since

$$e^{-i\sum_{n=0}^{m} \frac{1}{n+1} : x_1^{n+1} : G_n} \to e^{-iG}$$

strongly as $m \to \infty$ (see the proof of Lemma 4.3.1 in [22]) and

$$\begin{aligned}
\Phi(f)e^{-i\sum_{n=0}^{m} \frac{1}{n+1} : x_1^{n+1} : G_n}\varphi &= \left(\frac{\partial}{i\partial x_1} + ix_1 + x(v)\right) e^{-i\sum_{n=0}^{m} \frac{1}{n+1} : x_1^{n+1} : G_n}\varphi \\
&= e^{-i\sum_{n=0}^{m} \frac{1}{n+1} : x_1^{n+1} : G_n}\left(\Phi(f) - \sum_{n=0}^{m} : x_1^n : G_n\right)\varphi \\
&\to e^{-iG}(\Phi(f) - F\mathbb{1})\varphi
\end{aligned}$$

with respect to the L^2-norm. Therefore,

$$\begin{aligned} e^{iG}(\Phi(f)+F\mathbb{1})e^{-iG}\varphi &= e^{iG}\Phi(f)e^{-iG}\varphi + F\varphi \\ &= (\Phi(f)-F\mathbb{1})\varphi + F\varphi \\ &= \Phi(f)\varphi \quad , \end{aligned}$$

for any $\varphi \in \mathcal{G}$. Since $\mathcal{G}$ is a core for the self-adjoint operator $\Phi(f)$, the symmetric operator $\Phi(f)+F\mathbb{1}$ is essentially self-adjoint on $e^{-iG}\mathcal{G}$ and (3.1.2) holds.

To see the truth of the final assertion of the proposition, note that if $G_0 \in L^2(Q,d\mu)$ satisfies $\frac{\partial}{\partial x_1}G_0 = F$, then $G-G_0$ is an element of the subspace generated by $\{: x_{k_1}\cdots x_{k_m} :\mid k_1,\ldots,k_m \geq 2\} \subset L^2(Q,d\mu)$, *i.e.* $\Phi(f)$ commutes with $e^{i(G-G_0)}$. □

We therefore see that for any real-linear, densely defined $\Lambda : H \mapsto L^2(Q,d\mu)$ we obtain self-adjoint transforms of the field operators $\Phi(f)$:

$$\Phi_\Lambda(f) \equiv \overline{\Phi(f)+\Lambda f\mathbb{1}} \quad , \quad f \in D(\Lambda) \quad ,$$

where the closure is understood to be taken on $D(\Phi(f)) \cap D(\Lambda f\mathbb{1})$. It is, of course, not true in general that the operators $\{\Phi_\Lambda(f) \mid f \in D(\Lambda)\}$ form a representation of the CCR. We shall concentrate upon those that do.

Definition 3.1.4. *Let $\mathcal{L}$ be the set of all real-linear, densely defined maps from H to $L^2(Q,d\mu)$, and let $\mathcal{L}_{CCR} \subset \mathcal{L}$ be the subset of $\mathcal{L}$ consisting of elements Λ such that*

$$\pi_\Lambda(W(f)) \equiv e^{i\Phi_\Lambda(f)} \quad , \quad f \in D(\Lambda) \quad ,$$

defines a regular representation $(\pi_\Lambda, L^2(Q,d\mu))$ of the CCR-algebra $\mathcal{A}(D(\Lambda))$.

Note that, by Corollary 4.1.2 in [22], this definition coincides in the special case considered in [22] with the definition made there of $\mathcal{L}_{CCR}$.

Hence, each $\Lambda \in \mathcal{L}_{CCR}$ induces a canonical transformation on the quantum fields $\Phi(f) \mapsto \Phi_\Lambda(f)$, which itself is exponentiable to yield a regular representation of the algebra $\mathcal{A}(D(\Lambda))$. We now wish to characterize the members of this set $\mathcal{L}_{CCR}$ and shall do so in more than one way. Let $P_n : L^2(Q,d\mu) \mapsto L^2(Q,d\mu)$, $n \in \mathbb{N}$, be the orthogonal projection onto the subspace of n-particle vectors and $P : H \mapsto H$ be the orthogonal projection onto the subspace JV.

Theorem 3.1.5. *Let $\Lambda \in \mathcal{L}$. Then $\Lambda \in \mathcal{L}_{CCR}$ if and only if Λh is a (μ a.e.) real-valued function, for each $h \in D(\Lambda)$, and*

$$\langle \Lambda f, a^*(Pg)\psi\rangle = \langle \Lambda g, a^*(Pf)\psi\rangle \quad , \tag{3.1.3}$$

for arbitrary $f,g \in D(\Lambda)$ and $\psi \in D(a^*(Pf)) \cap D(a^*(Pg))$. The assertion still holds if (3.1.3) is replaced by

$$a(Pf)P_n\Lambda g = a(Pg)P_n\Lambda f \quad , \tag{3.1.4}$$

for all $f, g \in D(\Lambda)$ *and* $n \in \mathbb{N}$.

Proof. Assume that $\Lambda \in \mathcal{L}_{CCR}$. Then because the field operator $\Phi_\Lambda(h)$ must be self-adjoint, the function Λh must be real-valued (μ a.e.). Equations (3.1.3) and (3.1.4) are trivial if $f, g \in V$, so that $Pf = 0 = Pg$. Hence, one may assume that $f = Je_1 + v_1$ and $g = c_1 Je_1 + c_2 Je_2 + v_2$ for suitable $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2 \in V$, after choosing the basis $\{e_k, Je_k \mid k \in \mathbb{N}\}$ appropriately. Differentiating the equation

$$\langle e^{it\Phi_\Lambda(f)}\varphi, e^{is\Phi_\Lambda(g)}\psi\rangle = e^{its\sigma(f,g)}\langle e^{is\Phi_\Lambda(g)}\varphi, e^{it\Phi_\Lambda(f)}\psi\rangle$$

with respect to s and t and evaluating at $t = 0 = s$, one can conclude, using Theorem VIII.7 in [24] that

$$\langle \Phi_\Lambda(f)\varphi, \Phi_\Lambda(g)\psi\rangle = \langle \Phi_\Lambda(g)\varphi, \Phi_\Lambda(f)\psi\rangle + i\sigma(f,g)\langle\varphi,\psi\rangle \quad ,$$

for any $\varphi, \psi \in \mathcal{G} \subset D(\Phi_\Lambda(f)) \cap D(\Phi_\Lambda(g))$. Since Λf and Λg are μ a.e. real-valued, one has

$$\begin{aligned}
0 &= \langle \Phi_\Lambda(f)\Omega, \Phi_\Lambda(g)\psi\rangle - \langle \Phi_\Lambda(g)\Omega, \Phi_\Lambda(f)\psi\rangle - i\sigma(f,g)\langle\Omega,\psi\rangle \\
&= \langle \Phi(f)\Omega, \Phi(g)\psi\rangle + \langle \Lambda f\Omega, \Phi(g)\psi\rangle + \langle \Phi(f)\Omega, \Lambda g\psi\rangle + \langle \Lambda f\Omega, \Lambda g\psi\rangle \\
&\quad - \langle \Phi(g)\Omega, \Phi(f)\psi\rangle - \langle \Lambda g\Omega, \Phi(f)\psi\rangle - \langle \Phi(g)\Omega, \Lambda f\psi\rangle - \langle \Lambda g\Omega, \Lambda f\psi\rangle \\
&\quad - i\sigma(f,g)\langle\Omega,\psi\rangle \\
&= \langle \Phi(f)\Omega, \Phi(g)\psi\rangle - \langle \Phi(g)\Omega, \Phi(f)\psi\rangle - i\sigma(f,g)\langle\Omega,\psi\rangle \\
&\quad + \langle \Lambda f\Omega, \Phi(g)\psi\rangle + \langle \Phi(f)\Omega, \Lambda g\psi\rangle - \langle \Phi(g)\Omega, \Lambda f\psi\rangle - \langle \Lambda g\Omega, \Phi(f)\psi\rangle \\
&\quad + \langle \Lambda f\Omega, \Lambda g\psi\rangle - \langle \Lambda g\Omega, \Lambda f\psi\rangle \\
&= \langle \Lambda f\Omega, \Phi(g)\psi\rangle + \langle \Phi(f)\Omega, \Lambda g\psi\rangle - \langle \Phi(g)\Omega, \Lambda f\psi\rangle - \langle \Lambda g\Omega, \Phi(f)\psi\rangle \\
&= \langle \Lambda f, (\Phi(g) - \overline{\Phi(g)\Omega}\mathbb{1})\psi\rangle - \langle \Lambda g, (\Phi(f) - \overline{\Phi(f)\Omega}\mathbb{1})\psi\rangle \quad .
\end{aligned}$$

By using

$$\begin{aligned}
(\Phi(f) - \overline{\Phi(f)\Omega}\mathbb{1})\psi &= (\Phi(Je_1) - \overline{\Phi(Je_1)\Omega}\mathbb{1})\psi = (\Phi(Je_1) + ix_1)\psi \\
&= (\Phi(Je_1) + i\Phi(e_1))\psi = \sqrt{2}a^*(Je_1)\psi \\
&= \sqrt{2}a^*(Pf)\psi
\end{aligned}$$

and the similar equality $(\Phi(g) - \overline{\Phi(g)\Omega}\mathbb{1})\psi = \sqrt{2}a^*(Pg)\psi$, one can conclude that (3.1.3) is fulfilled for $\psi \in \mathcal{G}$.

The next step is to prove (3.1.3) for polynomials $\psi \in L^2(Q, d\mu)$. Since $\mathcal{G}$ is a core for N (Lemma 3.1.2), for arbitrary $k_1, \ldots, k_m \in \mathbb{N}$ there exists a sequence $\{\psi_n\} \subset \mathcal{G}$ which converges in $L^2(Q, d\mu)$ to $: x_{k_1} \cdots x_{k_m} :$ such that also the sequence $\{N\psi_n\}$ converges in $L^2(Q, d\mu)$ to $N : x_{k_1} \cdots x_{k_m} :$. For arbitrary $h \in H$ one has

$$\begin{aligned}
&\|a^*(h)(\psi_n - : x_{k_1} \cdots x_{k_m} :)\|^2 \le \\
&\qquad \|h\|^2 \langle (\psi_n - : x_{k_1} \cdots x_{k_m} :), (N+1)(\psi_n - : x_{k_1} \cdots x_{k_m} :)\rangle \to 0 \quad ,
\end{aligned}$$

as $n \to \infty$, so that (3.1.3) holds for $\psi =: x_{k_1} \cdots x_{k_m} :$. It follows that

$$\begin{aligned} \langle \Lambda f, a^*(Pg)\psi \rangle &= \sum_{n=0}^{\infty} \langle P_n \Lambda f, P_n a^*(Pg)\psi \rangle = \sum_{n=1}^{\infty} \langle P_n \Lambda f, a^*(Pg) P_{n-1}\psi \rangle \\ &= \sum_{n=1}^{\infty} \langle \Lambda f, a^*(Pg) P_{n-1}\psi \rangle = \sum_{n=1}^{\infty} \langle \Lambda g, a^*(Pf) P_{n-1}\psi \rangle \\ &= \langle \Lambda g, a^*(Pf)\psi \rangle \quad , \end{aligned} \tag{3.1.5}$$

for $\psi \in D(a^*(Pf)) \cap D(a^*(Pg))$. The chain of equalities (3.1.5) establishes the asserted equivalence of (3.1.3) and (3.1.4).

Assume now that (3.1.3) holds for arbitrary $f, g \in D(\Lambda)$ and that Λh is μ a.e. real-valued, for each $h \in D(\Lambda)$. The assertion in this direction will be established in part by appealing to another characterization of $\Lambda \in \mathcal{L}_{CCR}$ appearing in Theorem 3.1.6, which will be presented subsequently. In particular, here it will be shown that there exists an extension Λ' of Λ containing V in its domain of definition and satisfying the hypothesis of Theorem 3.1.6. It will then follow from Theorem 3.1.6 that both Λ' and Λ are contained in $\mathcal{L}_{CCR}$.

Let $g \in V \cap D(\Lambda)$ be arbitrary but fixed. The set $\{Pf \mid f \in D(\Lambda)\}$ is dense in JV, so the set $\{a^*(Pf)\psi \mid \psi \in D(a^*(Pf)) \cap D(a^*(Pg)), f \in D(\Lambda)\}$ is dense in the orthogonal complement of the set $\{\Omega\}$. Therefore, Λg must be a multiple of Ω, *i.e.* a constant function, whenever $Pg = 0$, *i.e.* whenever $g \in V$ (use the hypothesis (3.1.3)). Thus, the restriction of Λ to $V \cap D(\Lambda)$ determines a linear form $\ell : V \cap D(\Lambda) \mapsto \mathbb{R}$. But ℓ has a linear extension ℓ' to V, so Λ has a linear extension Λ' to $D(\Lambda) + V$ such that $\Lambda' \mid V = \ell'$ and such that (3.1.3) holds with Λ replaced by Λ' (both sides of (3.1.3) under this replacement are equal to zero for the additional vectors $g \in V$ and constant Λg). It shall be established that this extension Λ' fulfills the hypothesis of Theorem 3.1.6.

For arbitrary $f_1, \ldots, f_m \in JD(\Lambda') \cap V$, one has

$$a^*(f_2) : x(f_3) \cdots x(f_m) := \sqrt{2} : x(f_2) x(f_3) \cdots x(f_m) : \quad ,$$

so that

$$\begin{aligned} \langle \Lambda' J f_1, : x(f_2) \cdots x(f_m) : \rangle &= \frac{1}{\sqrt{2}} \langle \Lambda' J f_1, a^*(f_2) : x(f_3) \cdots x(f_m) : \rangle \\ &= -\frac{i}{\sqrt{2}} \langle \Lambda' J f_1, a^*(J f_2) : x(f_3) \cdots x(f_m) : \rangle \\ &\overset{(3.1.3)}{=} -\frac{i}{\sqrt{2}} \langle \Lambda' J f_2, a^*(J f_1) : x(f_3) \cdots x(f_m) : \rangle \\ &= \langle \Lambda' J f_2, : x(f_1) x(f_3) \cdots x(f_m) : \rangle \quad , \end{aligned}$$

and the conditions (i)-(iii) of Theorem 3.1.6 are satisfied for Λ'. □

We now give the announced second characterization of $\Lambda \in \mathcal{L}_{CCR}$.

Theorem 3.1.6. *If $\Lambda \in \mathcal{L}_{CCR}$, then there exists an extension $\Lambda' \in \mathcal{L}_{CCR}$ of Λ with $V \subset D(\Lambda')$. Moreover, if $V \subset D(\Lambda)$, then $\Lambda \in \mathcal{L}_{CCR}$ is equivalent to the following three conditions:*

(i) The functions Λh are μ a.e. real-valued, for all $h \in D(\Lambda)$.

(ii) For arbitrary $f_1, \dots, f_m \in JD(\Lambda) \cap V$,

$$\langle \Lambda J f_1, : x(f_2)x(f_3)\cdots x(f_m) :\rangle = \langle \Lambda J f_2, : x(f_1)x(f_3)\cdots x(f_m) :\rangle \quad . \tag{3.1.6}$$

(iii) Λf is a (real) constant function for all $f \in V$.

Proof. The proof of Theorem 3.1.5 shows that $\Lambda \in \mathcal{L}_{CCR}$ implies that there exists an extension Λ' of Λ which fulfills conditions (i)-(iii) and $V \subset D(\Lambda')$. It therefore remains only to prove that $V \subset D(\Lambda)$ and (i)-(iii) imply $\Lambda \in \mathcal{L}_{CCR}$. By applying a suitable coherent transformation, it may be assumed that $\langle \Lambda h, \Omega \rangle = 0$, for all $h \in D(\Lambda)$. By choosing a suitable basis $\{e_k \mid k \in \mathbb{N}\}$ of V, it may also be assumed that $\{Je_k \mid k \in \mathbb{N}\} \subset D(\Lambda)$.

Now let $f, g \in D(\Lambda)$ be arbitrary but fixed. One can choose the basis of V such that $f = c_1 J e_1 + v_1$, $g = c_2 J e_1 + c_3 J e_2 + v_2$, with $c_1, c_2, c_3 \in \mathbb{R}$ and $v_1, v_2 \in V$. There are suitable $\lambda_{k_1 \dots k_m} \in \mathbb{R}$ $(m, k_1, \dots, k_m \in \mathbb{N})$ such that $\lambda_{k_1 \dots k_m}$ is symmetric in $k_2, \dots, k_m$ and such that

$$\Lambda J e_{k_1} = \sum_m \sum_{k_2, \dots, k_m} \lambda_{k_1 \dots k_m} : x_{k_2} \cdots x_{k_m} : \quad ,$$

for arbitrary $k_1 \in \mathbb{N}$. The $\lambda_{k_1 \dots k_m}$ are symmetric in the indices according to Lemma 3.1.1 and (3.1.6):

$$\begin{aligned}
\lambda_{k_1 k_2 k_3 \dots k_m} &= \frac{2^m}{m!} \langle \Lambda J e_{k_1}, : x_{k_2} x_{k_3} \cdots x_{k_m} :\rangle \\
&= \frac{2^m}{m!} \langle \Lambda J e_{k_2}, : x_{k_1} x_{k_3} \cdots x_{k_m} :\rangle \\
&= \lambda_{k_2 k_1 k_3 \dots k_m} \quad .
\end{aligned}$$

The function defined by

$$G = \sum_m \sum_{\{1,2\} \cap \{k_1, \dots, k_m\} \neq \emptyset} \frac{\lambda_{k_1 \dots k_m}}{m} : x_{k_1} \cdots x_{k_m} : \quad ,$$

is an L^2-function, since by (3.1.1) one has

$$
\begin{aligned}
\|G\|_2^2 &= \sum_m \sum_{\{1,2\}\cap\{k_1,\dots,k_m\}\neq\emptyset} \frac{\lambda^2_{k_1\dots k_m}}{m^2}\frac{m!}{2^m} \\
&\le \sum_m \sum_{1\in\{k_1,\dots,k_m\}} \frac{\lambda^2_{k_1\dots k_m}}{m^2}\frac{m!}{2^m} + \sum_m \sum_{2\in\{k_1,\dots,k_m\}} \frac{\lambda^2_{k_1\dots k_m}}{m^2}\frac{m!}{2^m} \\
&\le \sum_m \sum_{l_2,\dots,l_m} m\frac{\lambda^2_{1l_2\dots l_m}}{m^2}\frac{m!}{2^m} + \sum_m \sum_{l_2,\dots,l_m} m\frac{\lambda^2_{2l_2\dots l_m}}{m^2}\frac{m!}{2^m} \\
&= \sum_m \sum_{l_2,\dots,l_m} \lambda^2_{1l_2\dots l_m}\frac{(m-1)!}{2^m} + \sum_m \sum_{l_2,\dots,l_m} \lambda^2_{2l_2\dots l_m}\frac{(m-1)!}{2^m} \\
&= \frac{1}{2}\|\sum_m \sum_{l_2,\dots,l_m} \lambda_{1l_2\dots l_m} :x_{l_2}\cdots x_{l_m}:\|_2^2 \\
&\quad + \frac{1}{2}\|\sum_m \sum_{l_2,\dots,l_m} \lambda_{2l_2\dots l_m} :x_{l_2}\cdots x_{l_m}:\|_2^2 \\
&= \frac{1}{2}\|\Lambda J e_1\|_2^2 + \frac{1}{2}\|\Lambda J e_2\|_2^2 < \infty \quad .
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial x_1} G &= \lim_{m_0\to\infty} \sum_{m=1}^{m_0}\sum_{j=1}^{m}\sum_{k_i, i\neq j} \frac{\lambda_{k_1\dots k_{j-1}1k_{j+1}\dots k_m}}{m} :x_{k_1}\cdots x_{k_{j-1}}x_{k_{j+1}}\cdots x_{k_m}: \\
&= \sum_m \sum_{l_2,\dots,l_m} \lambda_{1l_2\dots l_m} :x_{l_2}\cdots x_{l_m}: \\
&= \Lambda J e_1 \quad .
\end{aligned}
$$

But Λv_1 is a constant function and $\langle \Lambda v_1, \Omega\rangle = 0$, so $\Lambda v_1 = 0$ and $c_1 \frac{\partial}{\partial x_1} G = \Lambda f$. Prop. 3.1.3 then entails

$$
\Phi_\Lambda(f) = e^{-iG}\Phi(f)e^{iG} \quad .
$$

Similarly, one has

$$
\frac{\partial}{\partial x_2} G = \Lambda J e_2 \quad ,
$$

and, since

$$
\begin{aligned}
\frac{\partial}{\partial x(c_2e_1+c_3e_2)} G &= c_2\frac{\partial}{\partial x_1}G + c_3\frac{\partial}{\partial x_2}G \\
&= c_2\Lambda J e_1 + c_3 \Lambda J e_2 \\
&= \Lambda g
\end{aligned}
$$

(recall that $\frac{\partial}{\partial x(c_2e_1+c_3e_2)} = \sqrt{2}(a(c_2e_1+c_3e_2)) = \sqrt{2}(c_2a(e_1)+c_3a(e_2)) = c_2\frac{\partial}{\partial x_1} + c_3\frac{\partial}{\partial x_2}$ and use $\Lambda v_2 = 0$, as well), Prop. 3.1.3 also implies

$$
\Phi_\Lambda(g) = e^{-iG}\Phi(g)e^{iG} \quad .
$$

Hence, one has $\Phi_\Lambda(f+g) = e^{-iG}\Phi(f+g)e^{iG}$, which implies

$$\begin{aligned} e^{i\Phi_\Lambda(f)}e^{i\Phi_\Lambda(g)} &= e^{-iG}e^{i\Phi(f)}e^{i\Phi(g)}e^{iG} \\ &= e^{-iG}e^{-\frac{i}{2}\sigma(f,g)}e^{i\Phi(f+g)}e^{iG} \\ &= e^{-\frac{i}{2}\sigma(f,g)}e^{i\Phi_\Lambda(f+g)} \quad , \end{aligned}$$

and the proof is complete. □

It follows from Theorem 3.1.5 that each $\Lambda \in \mathcal{L}_{CCR}$ is of the form $\Lambda = \Lambda_l + \Lambda_q$, where $\Lambda_l \in \mathcal{L}_{CCR}$ is linear, that is to say, the associated field operators $\Phi_{\Lambda_l}(f)$, $f \in D(\Lambda_l)$, are of the form $\Phi(g) + c\mathbb{I}$, for suitable $g \in H$ and $c \in \mathbb{R}$. In other words, the transformation $\Phi(f) \mapsto \Phi_{\Lambda_l}(f)$ is one of the inhomogeneous linear canonical transformations studied by Shale [33] and Berezin [4], among many others. See Chapter IV for a more detailed discussion. It then follows that, with a suitable choice of linear Λ_l, the operator $\Lambda_q \in \mathcal{L}_{CCR}$ satisfies $P_0\Lambda_q f = 0 = P_1\Lambda_q f$, for any $f \in D(\Lambda_q)$. The set of such operators Λ_q will be denoted by $\mathcal{L}^q_{CCR}$. The superscript q is chosen because the degree of such transformations is quadratic or higher. The structure of the linear elements of $\mathcal{L}_{CCR}$ will be discussed in Chapter IV. Here we shall consider the elements of $\mathcal{L}^q_{CCR}$. The following result generalizes Prop. 3.3.4 in [22] from the quadratic case to this general setting.

Proposition 3.1.7. *Let $\Lambda \in \mathcal{L}^q_{CCR}$. Then there exists a unique maximal extension $\Lambda_{max} \in \mathcal{L}^q_{CCR}$ of Λ.*

Proof. By Theorem 3.1.6, it may be assumed that $V \subset D(\Lambda)$, and by suitably choosing the basis $\{e_k \mid k \in \mathbb{N}\}$ of V, it may also be assumed that $\{e_k, Je_k \mid k \in \mathbb{N}\} \subset D(\Lambda)$. If $f = f_1 + Jf_2 \in D(\Lambda)$ with $f_1, f_2 \in V$, then Lemma 3.1.1 and Theorem 3.1.6 entail

$$\begin{aligned} \Lambda f &= \sum_{m,k_1,\dots,k_m} \frac{2^m}{m!}\langle \Lambda J f_2, :x_{k_1}x_{k_2}\cdots x_{k_m}:\rangle :x_{k_1}\cdots x_{k_m}: \\ &= \sum_{m,k_1,\dots,k_m} \frac{2^m}{m!}\langle \Lambda J e_{k_1}, :x(f_2)x_{k_2}\cdots x_{k_m}:\rangle :x_{k_1}\cdots x_{k_m}: \quad . \end{aligned}$$

If $\Lambda' \in \mathcal{L}^q_{CCR}$ is an extension of Λ and $f = f_1 + Jf_2 \in D(\Lambda')$, with $f_1, f_2 \in V$, then

$$\begin{aligned} \infty &> \|\Lambda' f\|^2 \\ &= \|\sum_{m,k_1,\dots,k_m} \frac{2^m}{m!}\langle \Lambda J e_{k_1}, :x(f_2)x_{k_2}\cdots x_{k_m}:\rangle :x_{k_1}\cdots x_{k_m}: \|^2 \\ &\overset{(3.1.1)}{=} \sum_{m,k_1,\dots,k_m} \frac{2^m}{m!}|\langle \Lambda J e_{k_1}, :x(f_2)x_{k_2}\cdots x_{k_m}:\rangle|^2 \quad . \end{aligned}$$

Hence, define Λ_{max} by

$$D(\Lambda_{max}) = \{f + Jg \mid f, g \in V, \sum_{m,k_1,\dots,k_m} \frac{2^m}{m!}|\langle \Lambda J e_{k_1}, :x(g)x_{k_2}\cdots x_{k_m}:\rangle|^2 < \infty\}$$

and

$$\Lambda_{max} f = \sum_{m,k_1,\dots,k_m} \frac{2^m}{m!} \langle \Lambda J e_{k_1}, : x(f_2) x_{k_2} \cdots x_{k_m} : \rangle : x_{k_1} \cdots x_{k_m} : \quad ,$$

for $f = f_1 + J f_2 \in D(\Lambda_{max})$, $f_1, f_2 \in V$. Since all other assertions are now clear, it remains only to show that $\Lambda_{max} \in \mathcal{L}_{CCR}$. But for $f_1, \dots, f_m \in JD(\Lambda_{max}) \cap V$ (with $f_l = \sum_k c_{lk} e_k$), one sees

$$\begin{aligned}
&\langle \Lambda_{max} J f_1, : x(f_2) \cdots x(f_m) : \rangle \\
&= \sum_{k_2,\dots,k_m} c_{2k_2} \cdots c_{mk_m} \langle \Lambda_{max} J f_1, : x(e_{k_2}) \cdots x(e_{k_m}) : \rangle \\
&= \sum_{k_2,\dots,k_m} c_{2k_2} \cdots c_{mk_m} \langle \Lambda J e_{k_m}, : x(f_1) x(e_{k_2}) \cdots x(e_{k_{m-1}}) : \rangle \\
&= \sum_{k_m} c_{mk_m} \langle \Lambda J e_{k_m}, : x(f_1) x(f_2) \cdots x(f_{m-1}) : \rangle \\
&= \sum_{k_m} c_{mk_m} \langle \Lambda J e_{k_m}, : x(f_2) x(f_1) \cdots x(f_{m-1}) : \rangle \\
&= \langle \Lambda_{max} J f_2, : x(f_1) x(f_3) \cdots x(f_m) : \rangle \quad .
\end{aligned}$$

Theorem 3.1.6 then implies $\Lambda_{max} \in \mathcal{L}_{CCR}$. □

We then can use this result to show, as in Section 3.3 of [22] for the quadratic case, that any $\Lambda \in \mathcal{L}^q_{CCR}$ has an explicit form, which leads to a convenient "standard" form for the corresponding field operators $\Phi_\Lambda(f)$.

Proposition 3.1.8. *For each $\Lambda \in \mathcal{L}^q_{CCR}$ there exist an orthonormal basis $\{e_k \mid k \in \mathbb{N}\}$ of V and real numbers $\lambda_{kk_1\dots k_m}$ totally symmetric in the indices such that all of the following conditions are satisfied:*

$$\sum_{m,k_1,\dots,k_m} \lambda^2_{kk_1\dots k_m} \frac{m!}{2^m} < \infty \quad ,$$

for any $k \in \mathbb{N}$; moreover, if $\Lambda' \in \mathcal{L}^q_{CCR}$ is defined on the linear span of $\{e_k, Je_k \mid k \in \mathbb{N}\}$ by $\Lambda' e_k = 0$ and $\Lambda' J e_k = \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\dots k_m} : x_{k_1} \cdots x_{k_m} :$, then $\Lambda \subset \Lambda'_{max}$.

Proof. Once again, it may be assumed that $V \subset D(\Lambda)$ and $\{Je_k \mid k \in \mathbb{N}\} \subset D(\Lambda)$. Set

$$\lambda_{kk_1\dots k_m} = \frac{2^m}{m!} \langle \Lambda J e_k, : x_{k_1} \cdots x_{k_m} : \rangle \quad .$$

Then, by Lemma 3.1.1, one finds

$$\begin{aligned}
\infty &> \|\Lambda J e_k\|^2 \\
&= \| \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\dots k_m} : x_{k_1} \cdots x_{k_m} : \|^2 \\
&= \sum_{m,k_1,\dots,k_m} \lambda^2_{kk_1\dots k_m} \frac{m!}{2^m} \quad .
\end{aligned}$$

Prop. 3.1.7 then completes the proof. □

Since Λ_{max} uniquely exists, one may consider $\Lambda \in \mathcal{L}^q_{CCR}$ as being defined on a symplectic orthonormal basis $\{e_k, Je_k\}_{k\in\mathbb{N}}$ such that $\Lambda e_k = 0$ and $\Lambda Je_k = \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\dots k_m} : x_{k_1} \cdots x_{k_m} :$. Then one has

$$q_k \equiv \Phi(e_k) \mapsto \Phi_\Lambda(e_k) = \Phi(e_k) = q_k \quad ,$$

and

$$\begin{aligned} p_k \equiv \Phi(Je_k) \mapsto \Phi_\Lambda(Je_k) &= \overline{p_k + \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\dots k_m} : x_{k_1} \cdots x_{k_m} : \mathbb{1}} \\ &= \overline{p_k + \sum_{m,k_1,\dots,k_m} \lambda_{kk_1\dots k_m} : q_{k_1} \cdots q_{k_m} :} \quad . \end{aligned}$$

In short, the standard form of a canonical transformation of arbitrary degree is that given in equation (1.1). This standard form, along with being physically more transparent, was useful in the special case of quadratic transformations in [22] to establish results concerning the unitary equivalence of such representations with the Fock representation. Though we do not prove such results here for transformations of arbitrary degree, we *shall* give necessary and sufficient conditions for unitary, resp. quasi-, equivalence between Fock, coherent, and quasifree representations and representations of finite degree in Chapters IV and V.

3.2 Irreducibility of the Representation.

In this section we shall show that for bounded $\Lambda \in \mathcal{L}_{CCR}$, the corresponding representation π_Λ of the CCR is irreducible. If Λ is unbounded, then it can occur that π_Λ is reducible. We begin with a technical lemma concerning the closability and continuity properties of $\Lambda \in \mathcal{L}_{CCR}$ such that $P_0\Lambda$ is the zero operator, *i.e.* such that the range of Λ is orthogonal to the vacuum vector Ω. Note that this is true of each $\Lambda \in \mathcal{L}_{CCR}$, up to a linear transformation.

Lemma 3.2.1. *Let $\Lambda \in \mathcal{L}_{CCR}$ with $P_0\Lambda \subset 0$. Then Λ is closable and $\overline{\Lambda} \in \mathcal{L}_{CCR}$. Furthermore, for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset D(\Lambda)$ such that $f_n \to f \in D(\overline{\Lambda})$ and $\Lambda f_n \to \overline{\Lambda} f$ as $n \to \infty$, then the operators $\{e^{i\Phi_\Lambda(f_n)}\}_{n\in\mathbb{N}}$ converge strongly to $e^{i\Phi_{\overline{\Lambda}}(f)}$ as $n \to \infty$.*

Proof. It will first be shown that such Λ are closable. Once again, it may be assumed that $V \subset D(\Lambda)$ and $\{Je_k \mid k \in \mathbb{N}\} \subset D(\Lambda)$. Let $\{g_n\}$ in $D(\Lambda)$ be a sequence such that $g_n = h_n + Jh'_n$, with $h_n, h'_n \in V$, $g_n \to 0$ and $\Lambda g_n \to F$, for some $F \in L^2(Q, d\mu)$. Then for arbitrary $m, k_1, \dots, k_m \in \mathbb{N}$, Theorem 3.1.6 implies

$$\begin{aligned} \langle \Lambda g_n, : x_{k_1} \cdots x_{k_m} :\rangle &= \langle \Lambda Jh'_n, : x_{k_1} x_{k_2} \cdots x_{k_m} :\rangle \\ &= \langle \Lambda Je_{k_1}, : x(h'_n) x_{k_2} \cdots x_{k_m} :\rangle \\ &\to 0 \quad , \end{aligned}$$

as $n \to \infty$. Hence, $F = 0$ and Λ is closable. Since $\overline{\Lambda}$ fulfills (3.1.4), Theorem 3.1.5 entails $\overline{\Lambda} \in \mathcal{L}_{CCR}$.

In addressing the final assertion in the lemma, one may assume that $V \subset D(\Lambda)$, $\Lambda = \overline{\Lambda}$, $f = Je_1 + v$, with $v \in V$, and

$$\Lambda f = \sum_m \sum_{l=0}^{\infty} \sum_{k_1,\dots,k_m \geq 2} c_{lk_1\dots k_m} : x_1^l x_{k_1} \cdots x_{k_m} : \quad ,$$

for suitable $c_{lk_1\dots k_m}$, symmetric in the indices $k_1 \dots k_m$. Set

$$G_n = \sum_{\substack{2\leq k_1,\dots,k_m\leq n \\ m+l\leq n}} c_{lk_1\dots k_m} \frac{1}{l+1} : x_1^{l+1} x_{k_1} \cdots x_{k_m} :$$

and

$$G = \sum_{m,l,k_1,\dots,k_m} c_{lk_1\dots k_m} \frac{1}{l+1} : x_1^{l+1} x_{k_1} \cdots x_{k_m} : \quad .$$

Note that the proof of Prop. 3.1.3 implies that $G \in L^2(Q, d\mu)$. Then one has

$$\begin{aligned} \Phi_\Lambda(f_n) e^{-iG_m}\varphi &= \Phi(f_n) e^{-iG_m}\varphi + \Lambda f_n e^{-iG_m}\varphi \\ &\to \Phi(f) e^{-iG_m}\varphi + \Lambda f e^{-iG_m}\varphi \\ &= \Phi_\Lambda(f) e^{-iG_m}\varphi \quad , \end{aligned}$$

for $\varphi \in \mathcal{G}$. But the set $\{e^{-iG_m}\varphi \mid m \in \mathbb{N}, \varphi \in \mathcal{G}\}$ is contained in the domain of the strong graph limit of the sequence $\{\Phi_\Lambda(f_n)\}$, and this strong graph limit is a symmetric and closed operator (see, *e.g.*, Theorem VIII.27 in [24]). Furthermore, this strong graph limit acts upon the elements of this set as $\Phi_\Lambda(f)$. As in the proof of Prop. 3.1.3, one may conclude that $e^{-iG_m}\varphi$ converges to $e^{-iG}\varphi$ as $m \to \infty$ and, since $\frac{\partial}{\partial x_1} G_m$ converges to Λf in the L^2-norm,

$$\Phi_\Lambda(f) e^{-iG_m}\varphi = e^{-iG_m}(\Phi(f) + \Lambda f \mathbb{1} - \frac{\partial}{\partial x_1} G_m \mathbb{1})\varphi \to e^{-iG}\Phi(f)\varphi \quad .$$

In other words, the set $e^{-iG}\mathcal{G}$ is contained in the domain of the strong graph limit of the sequence $\{\Phi_\Lambda(f_n)\}$, and it acts upon this set as $\Phi_\Lambda(f) = e^{-iG}\Phi(f)e^{iG}$. But $\mathcal{G}$ is a core for $\Phi(f)$, so it follows that $\Phi_\Lambda(f)$ is, in fact, the strong graph limit of $\{\Phi_\Lambda(f_n)\}$. The final assertion of the lemma then follows from Theorems VIII.21 and VIII.26 in [24]. □

We now provide a sufficient condition on Λ which entails that π_Λ is irreducible.

Theorem 3.2.2. *Let $\Lambda \in \mathcal{L}_{CCR}$. If a dense subset of V is contained in $D(\overline{\Lambda - P_0\Lambda})$, then π_Λ is irreducible. In particular, π_Λ is irreducible for bounded $\Lambda - P_0\Lambda$, resp. for bounded Λ.*

Proof. Let $T : L^2(Q, d\mu) \mapsto L^2(Q, d\mu)$ be bounded and assume

$$[T, e^{i\Phi_\Lambda(g)}] = 0 \quad ,$$

for any $g \in D(\Lambda)$. Then

$$[T, e^{i\Phi(g)}] = 0 \quad , \tag{3.2.1}$$

for g in a dense subset of V. But Lemma 3.2.1 implies that equation (3.2.1) holds for all $g \in V$. Since $\{e^{ix(f)} \mid f \in V\}$ generates the maximally abelian von Neumann algebra $L^\infty(Q, d\mu)$, T may be identified with an element of $L^\infty(Q, d\mu)$ (or, more accurately, with the corresponding multiplication operator).

Now, Prop. 3.1.3 implies that for any $g \in D(\Lambda) \setminus V$ there exists a $G \in L^2(Q, d\mu)$ such that

$$e^{i\Phi_\Lambda(g)} = e^{-iG} e^{i\Phi(g)} e^{iG} \quad .$$

But $[T, e^{iG}] = 0$ entails equation (3.2.1) for $g \in D(\Lambda) \setminus V$ and, therefore, by Lemma 3.2.1, also for any $g \in H$. Since any Fock representation is irreducible, it follows that T is a multiple of the identity. □

We wish to show that there do exist $\Lambda \in \mathcal{L}_{CCR}$ such that π_Λ is reducible. To set this up properly, we first prove the following lemma.

Lemma 3.2.3. *If $\mathcal{K}$ is a Hilbert space and $T : \mathcal{K} \supset D(T) \mapsto \mathcal{K}$ is an unbounded densely defined operator, then there exists a densely defined $S \subset T$ such that $\overline{R(S)} \neq \mathcal{K}$.*

Proof. The equality $D(T^*) = \mathcal{K}$ would imply the boundedness of T^* and therefore of T. Thus, there exists an $f \in \mathcal{K} \setminus D(T^*)$, with which one may define the operator S as the restriction of T to

$$D(S) = \{g \in D(T) \mid \langle f, Tg \rangle = 0\} \quad .$$

$f \notin D(T^*)$ entails the existence of a unit vector $f_n \in D(T)$ such that

$$\lim_{n \to \infty} |\langle f, Tf_n \rangle| \to \infty \quad .$$

It may be assumed that $f_n \notin D(S)$. Let $g \in D(T)$ be arbitrary. Then

$$g - \frac{\langle f, Tg \rangle}{\langle f, Tf_n \rangle} f_n \in D(S) \quad \text{and} \quad g - \frac{\langle f, Tg \rangle}{\langle f, Tf_n \rangle} f_n \to g \quad ,$$

as $n \to \infty$. Hence, $g \in \overline{D(S)}$. Since $D(T)$ is dense in $\mathcal{K}$, this establishes that $\overline{D(S)} = \mathcal{K}$. □

We can now show that there exists a linear $\Lambda \in \mathcal{L}_{CCR}$ such that π_Λ is reducible. In fact, to each linear element Λ of $\mathcal{L}_{CCR}$ there corresponds a symplectic transformation, which is unbounded if Λ is unbounded (see [26]). According to Lemma 3.2.3, we may restrict the domain of any unbounded symplectic transformation to a set which is still dense in H in such a manner that the range of the restriction is not dense in H. However, the representation induced by a symplectic transformation is irreducible if and only if the range of the symplectic operator is dense in H, as we shall see in the next chapter (Lemma 4.1).

We also wish to point out that Corollary 4.1.2 in [22] is false as stated; in particular, the claim of irreducibility does not follow. As discussed in [11], the argument sketch given in [22] tacitly assumed that $\overline{\Lambda}$ is defined on a proper standard basis, which certainly follows if $\overline{\Lambda} = \Lambda_{max}$, and hence also if Λ is bounded, but which is not true in general. Florig also provides an example of a *reducible* quadratic representation (see Section 2.3 in [11]).

IV. Quasifree States and Linear Canonical Transformations

In this chapter we shall restrict our attention to quasifree states on $\mathcal{A}(H)$ and the associated representations. The notion of quasifree state was introduced by D.W. Robinson [27] in his study of the ground state of the Bose gas. It was shown in [18] that such (pure) states can be obtained by Bogoliubov transformations of a Fock state, hence making it clear that the class of representations (commonly called symplectic representations) studied by Segal [32] and Shale [33], among others, essentially coincided with the quasifree representations. We wish to show that the GNS representation of any pure quasifree state is unitarily equivalent to one of the representations π_Λ constructed in the previous chapter, for a suitable choice of $V \subset H$ and a linear $\Lambda \in \mathcal{L}_{CCR}$. The polynomial representations constructed globally under certain boundedness restrictions in [30] are clearly included in the representations of finite degree (special cases in the class constructed in Chapter III) discussed in more detail in Chapter V. Since it is evident that the coherent representations and, certainly, the quadratic representations of [22] are also subsumed in the class of representations presented in Chapter III, we see that our methods serve to unify the approaches to these various classes of representations, as well as to extend them to arbitrary degree.

To elucidate more clearly what we mean by linear canonical transformations, we recall two well-studied classes of representations of the CCR - the coherent and the quasifree representations. If $\{\mathcal{K}, W(f)\}$ is a representation of $\mathcal{A}(H)$ and $l : H \to \mathbb{R}$ is a linear map, then

$$\pi_l(W(f)) \equiv \hat{W}(f) \equiv e^{i\,l(f)}W(f) \quad , \quad f \in H \quad ,$$

determines a representation of $\mathcal{A}(H)$ generally called a coherent representation if $\{\mathcal{K}, W(f)\}$ is a Fock representation (see, *e.g.* [31]). This leads to the relationship $\hat{\Phi}(f) = \Phi(f) + l(f)\mathbb{1}$, $f \in H$ between the generators of the representations. One may equivalently start with a Fock state ω_J on $\mathcal{A}(H)$ with associated representation $\{\mathcal{K}, \pi_J\}$ and define a coherent state ω_l by

$$\omega_l(W(f)) \equiv \omega_J(W(f))e^{i\,l(f)} \quad , \quad f \in H \quad .$$

Of course, the GNS representation of $\mathcal{A}(H)$ corresponding to ω_l is given on $\mathcal{K}$ by

$$\pi_l(W(f)) \equiv e^{i\,l(f)}\pi_J(W(f)) \quad , \quad f \in H \quad .$$

From Theorem 3.1 of [26] (but see also [34][4]) it follows that the representations π_l and π_J are unitarily equivalent if and only if the map $l : H \mapsto \mathbb{R}$ is bounded.

Quasifree (or symplectic) representations can be obtained from a given (Fock) representation of the CCR-algebra $\mathcal{A}(H)$ by a Bogoliubov transformation

$$\Phi_T(f) \equiv \Phi(Tf) = \Phi(f) + \Phi((T - \mathbb{1})f) \quad ,$$

$$\text{(or equivalently} \quad \pi_T(W(f)) \equiv \pi_J(W(Tf)) \quad ,$$

using a symplectic operator T, i.e. one leaving the symplectic form invariant: $\sigma(Tf, Tg) = \sigma(f, g)$, for all $f, g \in H$. It is known [33] that the representations π_J and π_T are unitarily equivalent if and only if the operator $\mathbb{1} - |T|$ is Hilbert-Schmidt. (See Theorem 3.2 in [26] for a basis-dependent formulation of this result.)

The canonical transformations associated with the coherent and quasifree representations constitute the inhomogeneous linear group of canonical transformations, which is discussed in detail in [4].

Given a dense subspace $H_0 \subset H$, we shall show that all the quasifree states on $\mathcal{A}(H_0)$ can be obtained from the Fock state ω_J by symplectic transformations $\Lambda \in \mathcal{L}_{CCR}$. To begin, we consider pure quasifree states. It is known that, in our terminology, that a pure quasifree state is, up to a coherent transformation, a Fock state [18]. In order to be more precise, we need to introduce some notation.

Let ω'_F be a Fock state on $\mathcal{A}(H_0)$, so there exists an associated scalar product s' on H_0 [17], with respect to which the completion of H_0 will be denoted by H' (and the induced scalar product on H' will again be called s'). The scalar product s' is such that the symplectic form σ determining $\mathcal{A}(H)$ is continuous with respect to s', when restricted to H_0 [17]. Hence, the restriction of σ to H_0 extends uniquely to a nondegenerate symplectic bilinear form σ' on H'. Moreover, there exists an operator $J' : H' \mapsto H'$ which induces a complex structure on H' [14], so that, in particular,

$$s'(f, g) = -\sigma'(J'f, g) \quad ,$$

for all $f, g \in H'$. Since H_0 is dense in H, resp. H', with respect to s, resp. s', and σ and σ' are nondegenerate, we may assume $f = f' \in H \cap H'$, whenever

$$\sigma(f, h) = \sigma'(f', h) \quad f \in H \,,\, f' \in H' \,,\, \forall h \in H_0 \quad . \tag{4.1}$$

There is no loss of generality, since we do not assume that $\sigma(f_1, f_2) = \sigma'(f_1, f_2)$ for $f_1, f_2 \in H \cap H'$.

The existence of a symplectic $T : H \mapsto H$ with $s'(f, f) = s(Tf, Tf)$, for all $f \in H$, was already proven for $H = H_0 = H'$ in [18]. We shall need to generalize this result. First, we characterize symplectic maps and irreducible symplectic representations.

Lemma 4.1. *An operator $T : H \supset D(T) \mapsto H$ is symplectic (with respect to σ), i.e. $\sigma(Tf, Tg) = \sigma(f, g)$ for all $f, g \in D(T)$, if and only if*

$$-JT^{-1}J \subset T^*$$

(it is not assumed here that T^{-1}, resp. T^*, is necessarily densely defined). A self-adjoint operator T is symplectic if and only if $-JT^{-1}J = T$. For symplectic T, the representation π_T defined by

$$\pi_T(W(f)) = e^{i\Phi(Tf)} \quad , \quad f \in D(T) \quad ,$$

is irreducible if and only if $\overline{R(T)} = H$.

Proof. Let T be symplectic and $g \in D(T)$. Then one has for $f \in D(JT^{-1}J)$

$$\begin{aligned} s(-JT^{-1}Jf, g) &= \sigma(-T^{-1}Jf, g) = -\sigma(Jf, Tg) \\ &= s(f, Tg) = s(T^*f, g) \quad . \end{aligned}$$

Hence, $-JT^{-1}J \subset T^*$. The converse follows from the equalities

$$\begin{aligned} \sigma(f, g) &= s(JT^{-1}Tf, g) = s(T^*JTf, g) \\ &= s(JTf, Tg) = \sigma(Tf, Tg) \quad , \end{aligned}$$

for all $f, g \in D(T)$. If T is symplectic and self-adjoint, then T^{-1} is densely defined (T's null space is trivial), so, by the above result, T is a self-adjoint extension of $-JT^{-1}J = -J(T^*)^{-1}J = (-JT^{-1}J)^*$, and thus $T = -JT^{-1}J$.

Turning to the characterization of irreducible symplectic representations, if $g \in R(T)^\perp$, then $e^{i\Phi(Jg)} \in \pi_T(\mathcal{A}(R(T)))'$, so that π_T is reducible if $\overline{R(T)} \neq H$. Assume now that $\overline{R(T)} = H$. If the sequence $\{f_n\}_{n\in\mathbb{N}} \subset H$ converges to $f \in H$, then $\{e^{i\Phi(f_n)}\}_{n\in\mathbb{N}}$ converges to $e^{i\Phi(f)}$ strongly (use Lemma 3.1.1 with $\Lambda = 0$). Thus, an element of the commutant of $\pi_T(\mathcal{A}(R(T)))$ must commute with the elements of $\pi_J(\mathcal{A}(H))$, which is itself a Fock representation and, hence, irreducible. □

With this in hand, we can now generalize the mentioned result of Manuceau and Verbeure.

Proposition 4.2. *Given the above-established notation, there exists a subspace $H_1 \supset H_0$ of $H \cap H'$ such that the following conditions are fulfilled. If one defines an operator K by $K \subset J'$ and $D(K) = \{f \in H_1 \mid J'f \in H_1\}$, then $-JK : H \supset D(-JK) \mapsto H$ is a symplectic (with respect to σ) positive self-adjoint (with respect to s) operator, and $T = (-JK)^{1/2}$ is a symplectic transformation with $D(T) = H_1$ and*

$$s'(f, g) = s(Tf, Tg) \quad , \quad \forall f, g \in H_1 \quad .$$

Proof. By using Zorn's Lemma, it is easy to show that there exists a subspace $H_1 \supset H_0$ of $H \cap H'$ such that H_1 is maximal with the property

$$\sigma(f,g) = \sigma'(f,g) \quad , \quad \forall f,g \in H_1 \quad . \tag{4.2}$$

Consider the restriction s'_{H_1} of the positive quadratic form s' determined by the form core $Q(s'_{H_1}) = H_1$. Because of the aforesaid maximality, H_1 is closed with respect to the norm

$$\|f\| = \sqrt{s'(f,f) + s(f,f)} \quad , \quad f \in H \cap H' \quad .$$

The quadratic form s'_{H_1} determines a self-adjoint operator $A : H \supset D(A) \mapsto H$ (use, *e.g.* Theorem VIII.15 in [24]), and the closure of $D(A)$ with respect to the above norm $\|\cdot\|$ is H_1. Hence,

$$\sigma(f, JAg) = s(f, Ag) = s'_{H_1}(f,g) = \sigma'(f, J'g) \quad , \quad f,g \in D(A) \subset H_1$$

is also true for $f \in H_1$. But the equality $\sigma(f, JAg) = \sigma'(f, J'g)$, for any $f \in H_1 \supset H_0$ entails the equality $JAg = J'g \in H \cap H'$ (see (4.1)) and, thus, by the maximality of H_1, the equality $JAg = J'g \in H_1$, for $g \in D(A)$. According to the definition of K, one has $Kg = J'g$, for $g \in D(K)$, hence $Ag = -JJ'g = -JKg$, for $g \in D(A)$. From (4.2) one sees

$$\begin{aligned} s(f, -JKg) &= \sigma(f, Kg) = \sigma'(f, Kg) \\ &= -\sigma'(Kf, g) = -\sigma(Kf, g) = s(-JKf, g) \quad , \end{aligned}$$

for $f,g \in D(-JK) \subset H_1$ (so $Kf, Kg \in H_1$), thus the operator $-JK \supset A$ is symmetric. Hence, $-JK = A$ is positive and self-adjoint, and one can define the positive self-adjoint operator $T = (-JK)^{1/2}$. The equality $s'(f,g) = s(Tf, Tg)$, which holds for all $f,g \in D(A)$, is therefore still true for $f,g \in D(T) = H_1$, which is the closure of $D(A)$ with respect to the norm $\|\cdot\|$.

The operator $-JK$ is symplectic with respect to σ, since

$$\sigma(-JKf, -JKg) = \sigma(Kf, Kg) = \sigma'(Kf, Kg) = \sigma'(f,g) = \sigma(f,g) \quad ,$$

for $f,g \in D(-JK) \subset H_1$ (so $Kf, Kg \in H_1$).

It remains to prove that T is symplectic. From [33] and Lemma 4.1, one can decompose H and $-JK$ as follows:

$$H = U \oplus JU \quad , \quad -JK = L \oplus -JL^{-1}J \quad ,$$

with $L : U \mapsto U$ self-adjoint and $0 \le L \le \mathbb{1}$. Therefore, $T = L^{1/2} \oplus -JL^{-1/2}J = -JT^{-1}J$ is symplectic, by Lemma 4.1. □

This permits us to characterize pure quasifree states. (If the proof is not yet clear, then read the first few lines of the proof of Prop. 4.4.)

Corollary 4.3. *Let H_0 be a dense subspace of H. Each pure quasifree state ω on $\mathcal{A}(H_0)$ has a characteristic function of the form*

$$\omega(W(f)) = e^{il(f) - \frac{s(Tf,Tf)}{4}} \quad , \quad f \in H_0 \quad ,$$

for some linear form $l : H_0 \mapsto \mathrm{I\!R}$ and a symplectic positive self-adjoint operator $T : H_0 \subset D(T) \mapsto H$.

With this result, we can characterize general quasifree states.

Proposition 4.4. *Let H_0 be a dense subspace of H. Each quasifree state ω on $\mathcal{A}(H_0)$ has a characteristic function of the form*

$$\omega(W(f)) = e^{il(f) - \frac{s(Tf,Tf)}{4}} \quad , \quad f \in H_0 \quad ,$$

for some linear form $l : H_0 \mapsto \mathrm{I\!R}$ and a symplectic operator $T : H_0 \subset D(T) \mapsto H$.

Proof. By [18], ω has a characteristic function of the form

$$\omega(W(f)) = e^{il(f) - \frac{s'(f,f)}{4}} \quad , \quad f \in H_0 \quad ,$$

for some scalar product s' on a Hilbert space $M \supset H_0$. The symplectic form σ can be continuously (with respect to s') extended to a bilinear form σ' on M (see inequality (2) in [18]). Let $P : M \mapsto M$ be the orthogonal projection onto the closure of $\{f \in M \mid \sigma(f,g) = 0, \forall g \in M\}$ and set $Q = \mathbb{1} - P$. The inequality (see, once again, (2) in [18])

$$|\sigma'(f,g)|^2 = |\sigma'(Qf,Qg)|^2 \leq s'(Qf,Qf)s'(Qg,Qg) \quad ,$$

for all $f, g \in M$, implies that one can define a state ω' on $\mathcal{A}(H_0)$ by

$$\omega'(W(f)) = e^{-s'(Qf,Qf)/4} \quad ,$$

for all $f \in H_0$, by Prop. 10 in [18]. According to Prop. 11 in the same paper, the state ω' is also primary, since the restriction of σ' to $QM \times QM$ is nondegenerate. The discussion in Section IV of [18] also implies the existence of a scalar product s_0 on H_0 associated with a pure state on $\mathcal{A}(H_0)$ such that

$$s_0(f,f) \leq s'(Qf,Qf) \quad ,$$

for all $f \in H_0$. Thus, there exists a symplectic operator $S : H_0 \mapsto H$ such that

$$s(Sf,Sf) = s_0(f,f) \leq s'(Qf,Qf) \leq s'(f,f) \quad ,$$

for all $f \in H_0$, using Corollary 4.3. Furthermore, from Theorem VIII.15 in [24] one has

$$s'(f,g) = s(A^{\frac{1}{2}}f, A^{\frac{1}{2}}g) \quad , \forall f,g \in D(A^{\frac{1}{2}}) \supset H_0 \quad ,$$

for a suitable self-adjoint A. But then $s(A^{\frac{1}{2}}\cdot, A^{\frac{1}{2}}\cdot) - s(S\cdot, S\cdot)$ is a positive quadratic form on H_0, so, by appealing once again to Theorem VIII.15 in [24], there exists a positive self-adjoint operator $B : H \supset D(B) \mapsto H$ such that

$$s(Bf, Bf) = s(A^{\frac{1}{2}}f, A^{\frac{1}{2}}f) - s(Sf, Sf) \quad , \tag{4.3}$$

for all $f \in H_0$.

Now define isometries (with respect to s) $U, V : H \mapsto H$ by

$$Ue_k = e_{3k}\,,\; UJe_k = Je_{3k} \quad \text{and} \quad Ve_k = e_{3k+1}\,,\; VJe_k = Je_{3k+2} \quad , k \in \mathbb{N} \quad ,$$

and set $T = US + VB$. Then the equalities (by definition, U commutes with J)

$$\begin{aligned}\sigma(Tf, Tg) &= \sigma((US + VB)f, (US + VB)g) = \sigma(USf, USg) \\ &= \sigma(Sf, Sg) = \sigma(f, g) \quad ,\end{aligned}$$

for all $f, g \in D(T)$, entail that T is symplectic. The claim then follows after noting that (4.3) implies

$$s(Tf, Tf) = s(Sf, Sf) + s(Bf, Bf) = s(A^{\frac{1}{2}}f, A^{\frac{1}{2}}f) = s'(f, f) \quad ,$$

for all $f \in H_0$. □

This permits us to prove the result announced at the beginning of the chapter.

Theorem 4.5. *Let π be the GNS-representation associated to a pure quasifree state on $\mathcal{A}(H_0)$, where H_0 is a dense subspace of H. For a suitable choice of $V \subset H$, there exists a linear $\Lambda \in \mathcal{L}_{CCR}$ such that π is unitarily equivalent to π_Λ.*

Proof. From the above discussion, it may be assumed that there exist a positive self-adjoint operator $T : H \supset D(T) \mapsto H$ with $H_0 \subset D(T)$ and a linear form $l : H_0 \mapsto \mathbb{R}$ such that

$$\pi(W(f)) = e^{i\Phi(Tf) + il(f)} \quad ,$$

for all $f \in H_0$. From the proof of Prop. 4.2 it is clear that H and T decompose as

$$H = U \oplus JU \quad , \quad T = A \oplus -JA^{-1}J \quad ,$$

with $A : U \mapsto U$ self-adjoint and satisfying $0 \leq A \leq \mathbb{1}$. Set

$$U_1 = \{(\mathbb{1} + JA)\varphi \mid \varphi \in U\} \subset H \quad .$$

It is easy to see that $JU_1 \subset U_1^\perp$. With $\varphi \in U$, the inclusions

$$\varphi = (\mathbb{1} + JA)\frac{1}{\mathbb{1} + A^2}\varphi - J(\mathbb{1} + JA)\frac{A}{\mathbb{1} + A^2}\varphi \in U_1 \oplus JU_1$$

and

$$J\varphi \in J(U_1 \oplus JU_1) = U_1 \oplus JU_1$$

imply $H = U_1 \oplus JU_1$. The restriction of T to U_1 is then an isometry, since

$$T(\mathbb{1} + JA)\varphi = (A + (-JA^{-1}J)JA)\varphi = J(-JA + \mathbb{1})\varphi \quad ,$$

for all $\varphi \in U$, and

$$\begin{aligned}\|T(\mathbb{1} + JA)\varphi\|^2 &= \|J(-JA + \mathbb{1})\varphi\|^2 = \|JA\varphi\|^2 + \|\varphi\|^2 \\ &= \|(\mathbb{1} + JA)\varphi\|^2 \quad .\end{aligned}$$

One can similarly prove that $H = \overline{TU_1} \oplus J\overline{TU_1}$, so there exist a unitary (considering $\mathcal{H}$ instead of H) $W : H \mapsto H$ such that WT is the identity on U_1. The equalities

$$0 = \sigma(WTf, WTg) - \sigma(f, g) = \sigma(f, (WT - \mathbb{1})g) \quad ,$$

for $f \in U_1$, $g \in D(T)$, entail $R(WT - \mathbb{1}) \subset U_1$. Choose now $V = U_1$ and define $\Lambda : H_0 \mapsto L^2(Q, d\mu)$ by $\Lambda f = l(f) + x((WT - \mathbb{1})f)$ for any $f \in H_0$. Then $\Lambda \in \mathcal{L}_{CCR}$ and $\Phi_\Lambda(f) = \Phi(WTf) + l(f)\mathbb{1}$. According to [33], since $|W| - \mathbb{1} = 0$ is a Hilbert-Schmidt operator, there exists a unitary L such that $L\Phi(f)L^* = \Phi(Wf)$, for all $f \in H$. In particular, one has $L\Phi(Tf)L^* = \Phi(WTf)$, for all $f \in D(T)$. It is therefore clear that the representation π_Λ corresponding to $\Phi_\Lambda(f)$ is unitarily equivalent to the representation π given by

$$\pi(W(f)) = e^{i\Phi(Tf) + il(f)} \quad , \quad f \in H_0 \quad .$$

□

To close this chapter, we give a characterization of our linear canonical transformations. Recall that $P : H \mapsto H$ is the orthogonal projection onto the subspace JV.

Proposition 4.6. *To every linear $\Lambda \in \mathcal{L}_{CCR}$ there corresponds a linear form $l : D(\Lambda) \mapsto \mathbb{R}$ and a symmetric operator $S : V \supset D(S) = JPD(\Lambda) \mapsto V$ such that*

$$\Lambda f = x(SJPf) + l(f) \quad , \tag{4.4}$$

for every $f \in D(\Lambda)$. Moreover, each such pair (l, S) defines a linear $\Lambda \in \mathcal{L}_{CCR}$ in this manner, with $D(\Lambda) = D(SJP)$.

Proof. Let $\Lambda \in \mathcal{L}_{CCR}$. From Theorem 3.1.6 it may be assumed that $V \subset D(\Lambda)$ and that Λ has the form given in (4.4). Thus, one sees that

$$\Phi_\Lambda(f) = \Phi(f) + \Lambda f \mathbb{1} = \Phi(f) + (x(SJPf) + l(f))\mathbb{1} = \Phi(f + SJPf) + l(f)\mathbb{1} \quad ,$$

since $R(S) \subset V$. But since $\Lambda \in \mathcal{L}_{CCR}$, these operators must satisfy the CCR; hence, the operator $\mathbb{1} + SJP$ must be symplectic. The resultant equalities

$$\begin{aligned}\sigma(f,g) &= \sigma(f + SJPf, g + SJPg) \\ &= \sigma(f,g) + \sigma(f, SJPg) + \sigma(SJPf, g) \\ &= \sigma(f,g) + \sigma(Pf, SJPg) + \sigma(SJPf, g) \\ &= \sigma(f,g) + s(JPf, SJPg) - s(SJPf, JPg) \quad , \end{aligned} \tag{4.5}$$

for all $f, g \in D(\Lambda)$, imply $s(JPf, SJPg) = s(SJPf, JPg)$, in other words, S is a symmetric operator. The same computation (4.5) shows that if S is symmetric, then (4.4) defines an element $\Lambda \in \mathcal{L}_{CCR}$. □

V. Canonical Transformations of Finite Degree

In this chapter we restrict our attention to the computationally simpler canonical transformations of arbitrary but finite degree.[3] We provide necessary and sufficient conditions on the mapping Λ of finite degree so that the associated representation π_Λ of the CCR is unitarily equivalent to a Fock, a coherent or a quasifree representation. As we show, these results contain the previously known conditions [33][9] for the unitary equivalence of irreducible quasifree representations. The case of unitary equivalence with a quadratic representation is briefly indicated at the end of the chapter. We emphasize that when the conditions isolated in this chapter are violated, one has a representation of the CCR which can describe bosonic systems with infinitely many degrees of freedom manifesting physics *different* from that describable by the Fock, coherent, quasifree or quadratic representations.

Definition 5.1. *Let $\mathcal{L}^{(n)}_{CCR}$ denote the set of elements Λ of $\mathcal{L}_{CCR}$ such that*

$$\Lambda = \sum_{i=0}^{n} P_i \Lambda \neq \sum_{i=0}^{n-1} P_i \Lambda \quad .$$

Such elements and the corresponding canonical transformations and representations will be said to be of degree n.

Using the equality (3.1.1) given above and the estimate (4.3.5) given in [22], the following lemma can be easily proven.

Lemma 5.2. *There exist real constants $C_{lm} > 0$ such that*

$$\|\varphi_l \varphi_m\|_2 \leq C_{lm} \|\varphi_l\|_2 \, \|\varphi_m\|_2 \quad ,$$

for all $\varphi_l \in R(\sum_{j \leq l} P_j)$ and $\varphi_m \in R(\sum_{j \leq m} P_j)$. Therefore, the finite-particle vectors are contained in the domain of the field operators $\Phi_\Lambda(f)$, for all $f \in D(\Lambda)$, whenever $\Lambda \in \mathcal{L}^{(n)}_{CCR}$.

Another straightforward fact we shall need is given in the next lemma.

[3]These are the counterparts in our approach to the polynomial representations of [30]. Note, however, that due to the boundedness assumptions made in [30], which do not need to be made here, we shall be discussing a larger class of representations. In any case, the questions treated below are not addressed in [30].

Lemma 5.3. *Let $\Lambda \in \mathcal{L}^{(n)}_{CCR}$. Then one has*

$$P_m \Phi_\Lambda(f) \subset P_m \Phi_\Lambda(f) \sum_{k \leq n+m+1} P_k \quad ,$$

for any $m \in \mathbb{N} \cup \{0\}$ and $f \in D(\Lambda)$.

Proof. For arbitrary $\varphi \in D(\Phi_\Lambda(f))$ and $\psi \in L^2(Q, d\mu)$, one sees from Lemma 5.2 that

$$\begin{aligned} \langle \psi, P_m \Phi_\Lambda(f) \varphi \rangle &= \langle \Phi_\Lambda(f) P_m \psi, \varphi \rangle \\ &= \langle \sum_{k \leq \max\{n+m, m+1\}} P_k \Phi_\Lambda(f) P_m \psi, \varphi \rangle \\ &= \langle \psi, P_m \Phi_\Lambda(f) \sum_{k \leq n+m+1} P_k \varphi \rangle \quad . \end{aligned}$$

□

We give a characterization of the existence of strong graph limits of sequences of field operators in representations of degree n.

Lemma 5.4. *Let $\Lambda \in \mathcal{L}^{(n)}_{CCR}$ and $\{f_m\}_{m \in \mathbb{N}} \subset D(\Lambda)$. There exist vectors $g_m, h_m \in V$ such that $f_m = g_m + J h_m$, for every $m \in \mathbb{N}$. The strong graph limit of the sequence of operators $\{\Phi_\Lambda(f_m)\}$ exists if and only if the sequences $\{h_m\}$ and $\{x(g_m) + \Lambda f_m\}$ converge in their respective Hilbert spaces. If this strong graph limit exists and the sequence $\{f_m\}$ converges, then also the sequence $\{\Lambda f_m\}$ converges.*

Proof. Assume that the strong graph limit indicated exists (the other direction can be proven using the argument of Lemma 3.2.1). Since this limit is densely defined, for arbitrary $\epsilon > 0$ there exists a sequence $\{\varphi_m\}_{m \in \mathbb{N}}$ such that both it and the sequence $\{\Phi_\Lambda(f_m)\varphi_m\}_{m \in \mathbb{N}}$ converge and such that $\|\varphi_m - \Omega\|_2 < \epsilon$, for all $m \in \mathbb{N}$. Hence, the sequence $\{P_0 \varphi_m\}_{m \in \mathbb{N}}$ converges and one has $\|P_0 \varphi_m\| > 1 - \epsilon$. One may therefore choose the sequence $\{\varphi_m\}_{m \in \mathbb{N}}$ such that, in addition, one has $P_0 \varphi_m = \Omega$, for all $m \in \mathbb{N}$. There also exists a real constant C such that $C \geq C_{kl}$, for $k, l \leq 2n+2$, where the constants C_{kl} are those evoked in Lemma 5.2, and thus

$$\|\Phi(f)\varphi\|_2 \leq C \|f\| \, \|\varphi\|_2 \quad ,$$

for arbitrary φ in the range of the projection $\sum_{l \leq 2n+2} P_l$ and $f \in H$. One also has the following estimate, using previously established notation.

$$\begin{aligned}
\|\Phi_\Lambda(f_m)\Omega\|_2 &= \|\Phi(f_m)\Omega + \Lambda f_m\|_2 \\
&= \|\Phi(Jh_m)\Omega + x(g_m) + \Lambda f_m\|_2 \\
&= \|ix(h_m) + x(g_m) + \Lambda f_m\|_2 \\
&= (\|x(h_m)\|_2^2 + \|x(g_m) + \Lambda f_m\|_2^2)^{1/2} \\
&\overset{(3.1.1)}{=} (\frac{1}{2}\|h_m\|_2^2 + \|x(g_m) + \Lambda f_m\|_2^2)^{1/2} \\
&\geq \frac{1}{2}\max\{\|h_m\|, \|x(g_m) + \Lambda f_m\|_2\} \\
&\geq \frac{1}{4}(\|h_m\| + \|x(g_m) + \Lambda f_m\|_2) \quad . \qquad (5.1)
\end{aligned}$$

Of course, also the sequence $\{\sum_{j=0}^n P_j \Phi_\Lambda(f_m)\varphi_m\}_{m\in\mathbb{N}}$ converges. Now consider the estimate (obtained using Lemma 5.3)

$$\begin{aligned}
&\|\sum_{j=0}^{n+1} P_j\Phi_\Lambda(f_m)\varphi_m - \Phi_\Lambda(f_m)\Omega\|_2 = \|\sum_{j=0}^{n+1} P_j\Phi_\Lambda(f_m)\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 \\
&\leq \|\sum_{j=0}^{n+1} P_j\Phi(Jh_m)\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 + \|\sum_{j=0}^{n+1} P_j(x(g_m) + \Lambda f_m)\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 \\
&\leq \|\Phi(Jh_m)\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 + \|(x(g_m) + \Lambda f_m)\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 \\
&\leq C(\|h_m\| + \|x(g_m) + \Lambda f_m\|_2)\,\|\sum_{l=1}^{2n+2} P_l\varphi_m\|_2 \\
&\leq C(\|h_m\| + \|x(g_m) + \Lambda f_m\|_2)\,\epsilon \\
&\overset{(5.1)}{\leq} \|\Phi_\Lambda(f_m)\Omega\|_2\, 4C\epsilon \quad .
\end{aligned}$$

From this estimate, one sees that the boundedness of the sequence $\{\sum_{j=0}^{n+1} P_j\Phi_\Lambda(f_m)\varphi_m\}_{m\in\mathbb{N}}$ entails the boundedness of the sequence $\{\|\Phi_\Lambda(f_m)\Omega\|_2\}_{m\in\mathbb{N}}$. Thus, for any $\delta > 0$, there exist convergent sequences $\{\varphi_m\}_{m\in\mathbb{N}}$ and $\{\sum_{j=0}^{n+1} P_j\Phi_\Lambda(f_m)\varphi_m\}_{m\in\mathbb{N}}$ such that

$$\|\sum_{j=0}^{n+1} P_j\Phi_\Lambda(f_m)\varphi_m - \Phi_\Lambda(f_m)\Omega\|_2 < \delta \quad ,$$

for all $m \in \mathbb{N}$. This implies the convergence of the sequence $\{\Phi_\Lambda(f_m)\Omega\}$. It may therefore be concluded that the sequences $\{h_m\}_{m\in\mathbb{N}}$ and $\{x(g_m) + \Lambda f_m\}_{m\in\mathbb{N}}$ converge (use (5.1) with f_m replaced by $f_{m_1} - f_{m_2}$). $\square$

It is easy to see that the following lemma is true. We simply record it here for later reference.

Lemma 5.5. *Let π be a representation of a C^*-algebra $\mathcal{A}$, $\{A_n\}$ a sequence of elements of $\mathcal{A}$, and k a cardinal number. The strong graph limit of the operator sequence $\{\pi(A_n)\}$ exists and is densely defined if and only if the strong graph limit of the operator sequence $\{k\pi(A_n)\}$ exists and is densely defined, where $k\pi$ is the direct sum of k copies of π.*

We can finally prove our characterization of the quasi-equivalence of a representation π_Λ of degree n with the original Fock representation π_J. This generalizes Theorem 4.1 in [22], which was restricted to the case $n = 2$. Of course, if π_Λ is irreducible (see Theorem 3.2.2), then quasi-equivalence implies unitary equivalence.

Theorem 5.6. *Let $\Lambda \in \mathcal{L}^{(n)}_{CCR}$. The representation π_Λ is quasi-equivalent to the restriction of the Fock representation π_J to $\mathcal{A}(D(\Lambda))$ if and only if the operator $\overline{\Lambda}$ is Hilbert-Schmidt.*

Proof. Assume that the representations π_Λ and $\pi_J \mid \mathcal{A}(D(\Lambda))$ are quasi-equivalent. Let $f \in H$ and $\{f_n\}_{n\in\mathbb{N}} \subset D(\Lambda)$ be arbitrary with $\{f_n\}_{n\in\mathbb{N}}$ converging to f. The convergence of the sequence $\{\Phi(f_n)\varphi\}_{n\in\mathbb{N}}$ to $\Phi(f)\varphi$ for finite-particle vectors φ entails that $\Phi(f)$ is the strong graph limit of $\{\Phi(f_n)\}_{n\in\mathbb{N}}$ (see Theorem VIII.27 in [24]). As quasi-equivalence is the same as unitary equivalence up to multiplicity (see *e.g.* Theorem 2.4.26 in [5]), Lemma 5.5 implies that the strong graph limit of $\{\Phi_\Lambda(f_n)\}_{n\in\mathbb{N}}$ exists, so that, by Lemma 5.4, the sequence $\{\Lambda f_n\}_{n\in\mathbb{N}}$ converges. This entails that Λ is bounded, and thus, by Lemma 3.2.1, one may assume that $D(\Lambda) = H$. According to Theorem 5.2.14 in [6] there must exist a dense subset $\mathcal{K}$ of $L^2(Q, d\mu)$ such that

$$\sum_{k=1}^{\infty} \|(\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k))\varphi\|_2^2 < \infty \quad , \tag{5.2}$$

for all $\varphi \in \mathcal{K}$, since a densely defined number operator exists. Thus, one must have

$$\begin{aligned} &\sum_{k=1}^{\infty} \|\sum_{l=0}^{n} P_l(\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k))\varphi\|_2^2 \\ &= \sum_{k=1}^{\infty} \|\sum_{l=0}^{n} P_l(\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k)) \sum_{m=0}^{2n+1} P_m\varphi\|_2^2 \\ &< \infty \quad , \end{aligned}$$

and, since the finite-particle vectors are in the domain of the number operator of the Fock representation π_J,

$$\sum_{k=1}^{\infty} \|\sum_{l=0}^{n} P_l(\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k) - \sqrt{2}a(e_k)) \sum_{m=0}^{2n+1} P_m\varphi\|_2^2 < \infty \quad . \tag{5.3}$$

For each $\epsilon > 0$ there exists a vector $\varphi_\epsilon \in \mathcal{K}$ such that $\|\varphi_\epsilon - \Omega\|_2 < \epsilon$; and set

$$C \equiv \max_{r,s \leq 2n+1} C_{rs}$$

(see Lemma 5.2). Then, since $\|(\Lambda e_k + i\Lambda Je_k)P_0\varphi_\epsilon\|_2 = \|(\Lambda e_k + i\Lambda Je_k)\|_2\|P_0\varphi_\epsilon\|_2$ ($P_0\varphi_\epsilon$ is just a constant function), one has the estimate

$$\begin{aligned}
&\|\sum_{l=0}^{n} P_l(\Phi_\Lambda(e_k) + i\Phi_\Lambda(Je_k) - \sqrt{2}a(e_k))\sum_{m=0}^{2n+1} P_m\varphi_\epsilon\|_2 \\
&= \|\sum_{l=0}^{n} P_l(\Lambda e_k + i\Lambda Je_k)\sum_{m=0}^{2n+1} P_m\varphi_\epsilon\|_2 \\
&\geq \|\sum_{l=0}^{n} P_l(\Lambda e_k + i\Lambda Je_k)P_0\varphi_\epsilon\|_2 - \sum_{m=1}^{2n+1}\|\sum_{l=0}^{n} P_l(\Lambda e_k + i\Lambda Je_k)P_m\varphi_\epsilon\|_2 \\
&\geq \|(\Lambda e_k + i\Lambda Je_k)P_0\varphi_\epsilon\|_2 - \sum_{m=1}^{2n+1}\|(\Lambda e_k + i\Lambda Je_k)P_m\varphi_\epsilon\|_2 \\
&\geq \|\Lambda e_k + i\Lambda Je_k\|_2\|P_0\varphi_\epsilon\|_2 - \sum_{m=1}^{2n+1} C\|\Lambda e_k + i\Lambda Je_k\|_2\|P_m\varphi_\epsilon\|_2 \\
&\geq (1-\epsilon)\|\Lambda e_k + i\Lambda Je_k\|_2 - (2n+1)C\epsilon\|\Lambda e_k + i\Lambda Je_k\|_2 \\
&= (1-\epsilon-(2n+1)C\epsilon)\|\Lambda e_k + i\Lambda Je_k\|_2 \quad .
\end{aligned} \tag{5.4}$$

Choosing $\epsilon > 0$ such that $1-\epsilon-(2n+1)C\epsilon \geq \frac{1}{2}$, one sees that

$$\sum_k \|\Lambda e_k + i\Lambda Je_k\|_2^2 < \infty \quad ,$$

so that Λ is a Hilbert-Schmidt operator (Λe_k and ΛJe_k are a.e. real-valued):

$$\sum_k (\|\Lambda e_k\|_2^2 + \|\Lambda Je_k\|_2^2) = \sum_k \|\Lambda e_k + i\Lambda Je_k\|_2^2 \quad .$$

The asserted sufficiency of the condition will now be proven. Without loss of generality, one may assume that $\Lambda = \overline{\Lambda}$ and $P_0\Lambda = 0$. There exist real constants $\lambda_{kk_1\cdots k_m}$ symmetric in the indices, such that (using (3.1.1))

$$\Lambda Je_k = \sum_{m,k_1,\ldots,k_m} \lambda_{kk_1\cdots k_m} : x_{k_1}\cdots x_{k_m} :$$

and

$$\sum_k \sum_{m,k_1,\ldots,k_m} \lambda_{kk_1\cdots k_m}^2 \frac{m!}{2^m} = \sum_k \|\Lambda Je_k\|_2^2 < \infty \quad .$$

Setting

$$G = \sum_{m,k_1,\ldots,k_m} \frac{\lambda_{k_1\cdots k_m}}{m} : x_{k_1}\cdots x_{k_m} : \quad \in L^2(Q,d\mu) \quad ,$$

one sees that

$$e^{-iG}\Phi(f)e^{iG} = \Phi(f) \quad ,$$

for all $f \in V$, as well as

$$e^{-iG}\Phi(Je_k)e^{iG} = \Phi_\Lambda(Je_k) \quad ,$$

for all $k \in \mathrm{I\!N}$, by Prop. 3.1.3. This then demonstrates that

$$e^{-iG}\Phi(g)e^{iG} = \Phi_\Lambda(g) \quad ,$$

for all g in the linear span of $\{Je_k \mid k \in \mathrm{I\!N}\}$. Lemma 3.2.1 completes the proof. □

And next we give a characterization of the quasi-equivalence of a representation of degree n with a quasifree representation.

Theorem 5.7. *Let $\Lambda \in \mathcal{L}^{(n)}_{CCR}$ and π be a GNS-representation of a pure quasifree state on $\mathcal{A}(D(\Lambda))$. There exists a symplectic operator $K : D(\Lambda) \mapsto H$ such that $\Phi_{P_1\Lambda}(f) = \Phi(Kf)$, for all $f \in D(\Lambda)$. The representations π_Λ and π are quasi-equivalent if and only if the following conditions are fulfilled:*

(i) $\pi_{(P_0+P_1)\Lambda}$ and π are quasi-equivalent.

(ii) The closure of the restriction of $(\Lambda - P_0\Lambda - P_1\Lambda)_{max}$ to the range of K is Hilbert-Schmidt.

Proof. By Lemma 4.1 and Corollary 4.4, it may be assumed that π is of the form

$$\pi(W(f)) = e^{i(\Phi(Tf)+l(f))} \quad , \quad f \in D(\Lambda) \quad ,$$

for a symplectic operator $T : D(\Lambda) \mapsto H$ with dense range and a linear form $l : D(\Lambda) \mapsto \mathrm{I\!R}$. After a coherent transformation, one may assume that l is the zero mapping. Let the representations π and π_Λ be quasi-equivalent. Then the restriction of π_J to $\mathcal{A}(R(T))$ and the representation π', defined by

$$\pi'(W(f)) = e^{i\Phi_\Lambda(T^{-1}f)} \quad , \quad f \in R(T) \quad ,$$

are quasi-equivalent. The linear part of the field operators associated with π' is, up to constants, equal to $\Phi_{P_1\Lambda}(T^{-1}f)$. There exists a symplectic transformation $S : R(T) \mapsto H$ such that $\Phi(Sf) = \Phi_{P_1\Lambda}(T^{-1}f)$. As in the proof of Theorem 5.6, one may extend π' to a representation of $\mathcal{A}(H)$ which is quasi-equivalent to π_J. This extension will also be denoted by π'. The operator $\Lambda \circ T^{-1}$ is bounded; consequently S is bounded, as well. It may be assumed that $S = \overline{S}$.

Proceeding as in the proof of Theorem 5.6, one needs to consider the linear part of the annihilation operators associated with π' separately. Furthermore, the operators $a(e_k)$ in equation (5.3) must be replaced by somewhat different terms, discussed below. Let $\Phi'(f)$, $f \in H$, denote the field operators associated with π'.

The linear part of the associated annihilation operators $\frac{1}{\sqrt{2}}(\Phi'(e_k) + i\Phi'(Je_k))$ is given by

$$\begin{aligned}\frac{1}{\sqrt{2}}(\Phi(Se_k) + i\Phi(SJe_k)) &= \frac{1}{2}(a(Se_k) + a^*(Se_k) + ia(SJe_k) + ia^*(SJe_k)) \\ &= \frac{1}{2}(a((S - JSJ)e_k) + a^*((S + JSJ)e_k)) \quad .\end{aligned} \tag{5.5}$$

The operators $S - JSJ$ and J commute, so that, if $S - JSJ = UA$ is the polar decomposition, then U may be considered as a unitary operator on $\mathcal{H}$. Therefore, (5.3) is still fulfilled if $a(e_k)$ is replaced by $a(Ue_k)$.

The bound

$$\|(S - JSJ - 2U)f\| \leq \|(S + JSJ)f\| \quad , \tag{5.6}$$

will be proven for every $f \in H$. Note that for an arbitrary unit vector $f \in H$, one has

$$\begin{aligned}\|Af\|^2 &= \|(S - JSJ)f\|^2 \\ &= \|Sf\|^2 + \|JSJf\|^2 - 2\langle Sf, JSJf\rangle \\ &= \|Sf\|^2 + \|SJf\|^2 + 2\sigma(Sf, SJf) \\ &= \|Sf\|^2 + \|SJf\|^2 + 2\sigma(f, Jf) \\ &= \|Sf\|^2 + \|SJf\|^2 + 2 \quad .\end{aligned}$$

Similarly, it follows that

$$\|(S + JSJ)f\|^2 = \|Sf\|^2 + \|SJf\|^2 - 2 \quad .$$

From the bound

$$1 = |\sigma(SJf, Sf)| \leq \|SJf\| \, \|Sf\|$$

it follows that

$$\|Af\|^2 \geq \|Sf\|^2 + \frac{1}{\|Sf\|^2} + 2 \geq 4 \quad ,$$

and $A \geq 2\mathbb{1}$. This proves (5.6), as $A \geq 2\mathbb{1}$ and $U^*U = \mathbb{1}$ imply

$$\begin{aligned}\|(S - JSJ - 2U)f\|^2 &= \|(A - 2\mathbb{1})f\|^2 \\ &= \langle f, (A^2 - 4A + 4\mathbb{1})f\rangle \\ &\leq \langle f, (A^2 - 8\mathbb{1} + 4\mathbb{1})f\rangle \\ &= \|(S - JSJ)f\|^2 - 4 \\ &= \|(S + JSJ)f\|^2 \quad .\end{aligned}$$

One can then proceed as in the proof of Theorem 5.6 to find suitable Hilbert-Schmidt conditions - one must simply modify (5.5) by the term $-a(Ue_k)$, resp., replace $a(e_k)$ in (5.3) by $a(Ue_k)$. In the counterpart to (5.4) one finds the additional terms

$$\|a^*((S+JSJ)e_k)P_0\varphi_\epsilon\|_2 \overset{(3.1.1)}{=} \|(S+JSJ)e_k\| \, \|P_0\varphi_\epsilon\|_2 \quad ,$$

$$\|a^*((S+JSJ)e_k)\sum_{l=1}^{2n+1} P_l\varphi_\epsilon\|_2 \le C\epsilon\|(S+JSJ)e_k\| \quad ,$$

and

$$\begin{aligned}\|a((S-JSJ-2U)e_k)\sum_{l=1}^{2n+1} P_l\varphi_\epsilon\|_2 & \\ &\le C\|(S-JSJ-2U)e_k\| \, \|\sum_{l=1}^{2n+1} P_l\varphi_\epsilon\|_2 \\ &\overset{(5.6)}{\le} \quad C\|(S+JSJ)e_k\|\epsilon \quad .\end{aligned}$$

It may therefore be concluded that $S+JSJ$ is a Hilbert-Schmidt operator. Furthermore, the constant part of the field operator $\Phi'(f)$ is a bounded linear form, so by Lemma 5.8 it may be assumed without loss of generality that it is trivial. Thus, the representations π and $\pi_{P_0\Lambda+P_1\Lambda}$ are quasi-equivalent. The closure of the operator $(\Lambda - P_0\Lambda - P_1\Lambda)\circ T^{-1}$ is also Hilbert-Schmidt. That the mapping $S^{-1} : R(S) \mapsto D(S) = H$ is bounded is implied by the bound

$$\|Sf\| \, \|S\| \, \|f\| \ge |\sigma(Sf, SJf)| = \|f\|^2 \quad ,$$

for every $f \in H$, as well as

$$\|Sf\| \ge \frac{1}{\|S\|}\|f\| \quad .$$

Hence, recalling $S = \overline{KT^{-1}}$, the operator

$$\begin{aligned}\overline{(\Lambda - P_0\Lambda - P_1\Lambda)\circ T^{-1}}\circ S^{-1} &= \overline{(\Lambda - P_0\Lambda - P_1\Lambda)\circ T^{-1}\circ T\circ K^{-1}} \\ &= \overline{(\Lambda - P_0\Lambda - P_1\Lambda)_{max}\circ P\circ K^{-1}} \\ &= \overline{(\Lambda - P_0\Lambda - P_1\Lambda)_{max} \mid R(K)}\end{aligned}$$

is Hilbert-Schmidt.

To prove that the stated Hilbert-Schmidt conditions imply the desired quasi-equivalence, it is sufficient to show that the representation admits a densely defined number operator, in other words, by Theorem 5.2.14 in [6] it suffices to show that for any finite-particle vector $\varphi = \sum_{l=0}^{m} P_l\varphi$, and any orthonormal basis $\{f_k, Jf_k \mid k \in \mathbb{N}\}$ of H, one has

$$\sum_k \|(\Phi'(f_k) + i\Phi'(Jf_k))\varphi - \sqrt{2}a(Uf_k)\varphi\|_2^2$$
$$\leq D^{(m)}(\|S + JSJ\|_{HS}^2 + \|\overline{(\Lambda - P_0\Lambda - P_1\Lambda)_{max} \mid R(K)}\|_{HS}^2)\, \|\varphi\|_2^2$$
$$< \infty \quad ,$$

for a constant $D^{(m)} \in \mathbb{R}$ which does not depend on the choice of the basis (the Hilbert-Schmidt norm of an operator is denoted by $\|\cdot\|_{HS}$). This can be done by using arguments already employed above (see Lemma 5.2 and equations (5.5) and (5.6)). It follows then that

$$\sum_k \|(\Phi'(f_k) + i\Phi'(Jf_k))\varphi\|^2$$

is bounded by a finite real number, which only depends on m, $\|\varphi\|_2$, $\|S+JSJ\|_{HS}^2$, and $\|\overline{(\Lambda - P_0\Lambda - P_1\Lambda)_{max} \mid R(K)}\|_{HS}^2$. □

Taken together, Theorems 5.6 and 5.7 give conditions so that our higher order representations are *inequivalent* to the well-studied class of representations associated with inhomogeneous linear canonical transformations, and hence so that they can describe bosonic systems with *different* physics.

Lemma 5.8. *Let π be a GNS-representation of a quasifree state ω on $\mathcal{A}(H_0)$, where H_0 is a dense subspace of $\mathcal{H}$. There exists a linear form $k : H_0 \mapsto \mathbb{R}$ and a scalar product s' on H_0 such that*

$$\omega(W(f)) = e^{ik(f) - \frac{s'(f,f)}{4}} \quad ,$$

for all $f \in H_0$ [18]. If $l : H_0 \mapsto \mathbb{R}$ is a linear form, then the representation π_l of $\mathcal{A}(H_0)$ defined by $\pi_l(W(f)) = e^{il(f)}\pi(W(f))$ is quasi-equivalent to π if and only if l is bounded with respect to s'.

Proof. By Corollary 4.4, it may be assumed that the linear form k is trivial and that $\pi(W(f)) = e^{i\Phi(Tf)}$, $f \in H_0$, for a symplectic mapping $T : H_0 \mapsto H$. Suppose the indicated representations are quasi-equivalent. The convergence $f_n \xrightarrow{s'} 0$ implies the convergence $Tf_n \xrightarrow{s} 0$, so that the strong graph limit of $\{\Phi(Tf_n)\}$ is 0. The assumed quasi-equivalence implies also that 0 is the strong graph limit of the sequence $\{\Phi(Tf_n) + l(f_n)\}$. Appealing to Lemma 5.4, one concludes that also $\{l(f_n)\}$ converges to 0.

Assume now that the linear form l is bounded with respect to s', so that $l(f) = s'(g, f)$ for some element g in the completion of H_0 with respect to s'. There exists a sequence $\{g_n\} \subset H_0$ converging to g with respect to s'; hence the sequence $\{Tg_n\}$ converges to some element $h \in H$ with respect to s. Then, for $f \in H_0$, one sees

$$\begin{aligned} e^{i\Phi(Jh)}\Phi(Tf)e^{-i\Phi(Jh)} &= \Phi(Tf) - \sigma(Jh,Tf)\mathbb{1} \\ &= \Phi(Tf) + s(h,Tf)\mathbb{1} \\ &= \Phi(Tf) + \lim_n s(Tg_n,Tf)\mathbb{1} \\ &= \Phi(Tf) + s'(g,f)\mathbb{1} \\ &= \Phi(Tf) + l(f)\mathbb{1} \quad . \end{aligned}$$

□

We provide an extension to Prop. 3.1.7 for irreducible representations of finite degree.

Proposition 5.9. *Let $\Lambda \in \mathcal{L}^q_{CCR} \cap \mathcal{L}^{(n)}_{CCR}$ determine an irreducible representation π_Λ of $\mathcal{A}(D(\Lambda))$. If $l : H_0 \mapsto \mathrm{I\!R}$ is a linear form such that the representation π_l of $\mathcal{A}(H_0)$ defined by $\pi_l(W(f)) = e^{il(f)}\pi_\Lambda(W(f))$ is unitarily equivalent to π_Λ, then l has the form $l(f) = \sigma(g,f)$, for all $f \in D(\Lambda)$, for some $g \in D(\Lambda_{max})$. Therefore, the set $D(\Lambda_{max})$ is the maximal extension of the test function space $D(\Lambda)$.*

Proof. Let $U : L^2(Q,d\mu) \mapsto L^2(Q,d\mu)$ be unitary such that

$$U^*\Phi_\Lambda(f)U = \Phi_\Lambda(f) + l(f)\mathbb{1} \quad ,$$

for all $f \in D(\Lambda)$. Then the operator $\pi_{\Lambda_{max}}(W(g))U^*\pi_{\Lambda_{max}}(W(-g))U$ commutes with the elements of $\pi_\Lambda(\mathcal{A}(D(\Lambda)))$, for all $g \in D(\Lambda_{max})$. By the assumed irreducibility, it must be a multiple of the identity. Thus, for each $g \in D(\Lambda_{max})$ one has

$$U^*\pi_{\Lambda_{max}}(W(g))U = e^{ic_g}\pi_{\Lambda_{max}}(W(g)) \quad ,$$

for suitable $c_g \in \mathrm{I\!R}$.

It shall next be shown that $e^{ic_g} = e^{il'(g)}$, for a suitable linear form $l' : D(\Lambda_{max}) \mapsto \mathrm{I\!R}$. But the equality $e^{ic_{t_1 g}}e^{ic_{t_2 g}} = e^{ic_{(t_1+t_2)g}}$ implies $e^{ic_{tg}} = e^{itk}$, for all $t \in \mathrm{I\!R}$ and a suitable $k \in \mathrm{I\!R}$. Thus, it has been shown that

$$U^*\Phi_{\Lambda_{max}}(tg)U = \Phi_{\Lambda_{max}}(tg) + tk\mathbb{1} \quad ,$$

in other words,

$$U^*\pi_{\Lambda_{max}}(W(f))U = e^{il'(f)}\pi_{\Lambda_{max}}(W(f)) \quad ,$$

for all $f \in D(\Lambda_{max})$, with $l' : D(\Lambda_{max}) \mapsto \mathrm{I\!R}$ linear. Hence, there exists an extension $l' : D(\Lambda_{max}) \mapsto \mathrm{I\!R}$ of l such that the representation $\pi_{l'}$ of $\mathcal{A}(D(\Lambda_{max}))$ defined by $\pi_{l'}(W(f)) = e^{il'(f)}\pi_{\Lambda_{max}}(W(f))$, for $f \in D(\Lambda_{max})$, is unitarily equivalent to $\pi_{\Lambda_{max}}$. Therefore, it may be assumed that $l = l'$ and $\Lambda = \Lambda_{max}$.

Let $\{e_k, Je_k \mid k \in \mathbb{N}\} \subset D(\Lambda)$ be an orthonormal symplectic basis in H. For each $f \in D((P_m\Lambda)_{max})$, there exist real constants $\lambda_{k_1\cdots k_m}(f)$, which are totally symmetric in the indices, such that

$$(P_m\Lambda)_{max} f = \sum_{k_1,\dots,k_m} \lambda_{k_1\cdots k_m}(f) : x_{k_1}\cdots x_{k_m} : \quad .$$

If $\{f_n\}$ is a convergent sequence in V, then the strong graph limit of $\{\Phi(f_n)\}$ exists, so that also the strong graph limit of $\{\Phi(f_n) + l(f_n)\}$ exists. Therefore, the restriction of l to V is bounded. Set $l_k \equiv l(e_k)$, $h \equiv \sum_k l_k J e_k \in H$ and

$$A_l(f) \equiv \Phi_\Lambda(f) - \sum_m \sum_{k_1,\dots,k_m \le l} \lambda_{k_1\cdots k_m}(f) : x_{k_1}\cdots x_{k_m} : \mathbb{1} \quad ,$$

with $D(A_l(f)) = D(\Phi_\Lambda(f)) \cap (\cap_{k_1,\dots,k_m \le l} D(x_{k_1}\cdots x_{k_m}\mathbb{1}))$. Then the operator $A_l(f)$ is symmetric and the sequence $\{A_l(f)\varphi\}_{l\in\mathbb{N}}$ converges to $\Phi(f)\varphi$ for every $\varphi \in \mathcal{G}$. Since $\mathcal{G}$ is a core for $\Phi(f)$ (Lemma 3.1.2), it follows from Theorem VIII.27 of [24] that $\Phi(f)$ is the strong graph limit of $\{A_l(f)\}_{l\in\mathbb{N}}$.

With $U^* x_k U = x_k + l_k$ and $p_l(f)$ suitable polynomials of degree $n-2$, one sees therefore that the strong graph limit of

$$\begin{aligned}
&U^* A_l(f) U \\
&= U^*\Phi_\Lambda(f)U - \sum_{m=2}^{n} \sum_{k_1,\dots,k_m \le l} \lambda_{k_1\cdots k_m}(f) U^* : x_{k_1}\cdots x_{k_m} : U \\
&= \Phi_\Lambda(f) + (l(f) + p_l(f) - \sum_{k_1,\dots,k_{n-1} \le l} \lambda_{k_1\cdots k_{n-1}}(f) : x_{k_1}\cdots x_{k_{n-1}} :)\mathbb{1} \\
&\quad - \sum_{k_1,\dots,k_n \le l} (\lambda_{k_1\cdots k_n}(f) : x_{k_1}\cdots x_{k_n} : + \lambda_{k_1\cdots k_n}(f)\, n\, l_{k_n} : x_{k_1}\cdots x_{k_{n-1}} :)\mathbb{1}
\end{aligned} \tag{5.7}$$

is $U^*\Phi(f)U$. According to the proof of Lemma 5.4, the existence of the indicated strong graph limit entails the convergence of

$$\begin{aligned}
p_l(f) \; &- \sum_{k_1,\dots,k_{n-1} \le l} \lambda_{k_1\cdots k_{n-1}}(f) : x_{k_1}\cdots x_{k_{n-1}} : \\
&- \sum_{k_1,\dots,k_n \le l} (\lambda_{k_1\cdots k_n}(f) : x_{k_1}\cdots x_{k_n} : + \lambda_{k_1\cdots k_n}(f)\, n\, l_{k_n} : x_{k_1}\cdots x_{k_{n-1}} :) \quad .
\end{aligned}$$

Following the argument of Lemma 3.1, it can also be shown that the strong graph limit of $\{U^* A_l(f) U\}_{l\in\mathbb{N}}$ is of the form $\overline{\Phi(f) + F\mathbb{1}}$, for a suitable $F \in R(\sum_{k=0}^{n-1} P_k) \subset L^2(Q, d\mu)$. Hence, there exists a $\Lambda' \in \mathcal{L}_{CCR}^{(n-1)}$ such that

$$U^*\Phi(f)U = \Phi_{\Lambda'}(f) \quad ,$$

for all $f \in D(\Lambda) = D(\Lambda')$.

According to Theorem 5.6, Λ' is a Hilbert-Schmidt operator. Hence one sees that

$$\begin{aligned}
\infty &> \sum_k \|P_{n-1}\Lambda'(Je_k)\|_2^2 \\
&= \sum_k \| \sum_{k_1,\dots,k_n} \lambda_{k_1\cdots k_n}(Je_k) l_{k_n} n : x_{k_1}\cdots x_{k_{n-1}} : \|_2^2 \\
&\overset{(3.1.1)}{=} \sum_k \sum_{k_1,\dots,k_{n-1}} (\sum_{k_n} \lambda_{k_1\cdots k_n}(Je_k) l_{k_n} n)^2 \, \frac{(n-1)!}{2^{n-1}} \\
&= n^2 \frac{(n-1)!}{2^{n-1}} \sum_k \sum_{k_1,\dots,k_{n-1}} (\sum_{k_n} \lambda_{k_1\cdots k_n}(Je_k) l_{k_n})^2 \\
&\overset{(Lemma\, 3.1.1)}{=} n \frac{n!}{2^{n-1}} \sum_k \sum_{k_1,\dots,k_{n-1}} (\sum_{k_n} \frac{2^n}{n!} \langle \Lambda Je_k, : x_{k_1}\cdots x_{k_{n-1}} x_{k_n} : \rangle l_{k_n})^2 \\
&= 2n \sum_{k,k_1,\dots,k_{n-1}} \frac{2^n}{n!} \langle \Lambda Je_k, : x_{k_1}\cdots x_{k_{n-1}} x(-Jh) : \rangle^2 \quad ,
\end{aligned}$$

which entails $h \in D((P_n\Lambda)_{max})$ (see the proof of Prop. 3.1.7).

There exists a $G \in L^2(Q, d\mu) \cap R(P_{n+1})$ such that $\Phi_{(P_n\Lambda)_{max}}(h) = e^{-iG}\Phi(h)e^{iG}$. By replacing U by $e^{iG}Ue^{-iG}$ and $\Phi_\Lambda(f)$ by $e^{iG}\Phi_\Lambda(f)e^{-iG}$, which is the same as replacing Λ by another element of $\mathcal{L}_{CCR}$ differing from Λ only in the component of degree n, it may be assumed that $(P_n\Lambda)_{max}h = 0$ and hence that the range of the mapping $(P_n\Lambda)_{max}$ lies in the subspace of $L^2(Q, d\mu)$ generated by

$$\{: x(f_1)\cdots x(f_n) : \mid \sigma(h, f_1) = \ldots = \sigma(h, f_n) = 0, \, f_1, \dots, f_n \in V\}$$

(use (3.1.3) and the fact that $a(h)(P_n\Lambda)_{max}f = 0$, for $f \in D((P_n\Lambda)_{max})$). The operator $e^{i\Phi(h)}$ commutes with these elements. From the definition of h, the adjoint actions of $e^{i\Phi(h)}$ and U on the field operators $\Phi_\Lambda(f) = \Phi(f)$, $f \in V$, are identical, inducing the same coherent canonical transformation. Hence, $e^{-i\Phi(h)}U$ commutes with $e^{ix(f)}$, for all $f \in V$, and thereby may be identified with the multiplication operator corresponding to some suitable element of $L^\infty(Q, d\mu)$, as in the proof of Theorem 3.2.2. Thus, one has

$$U^*((P_n\Lambda)_{max}f)U = (P_n\Lambda)_{max}f \quad ,$$

for all $f \in D((P_n\Lambda)_{max})$, and it therefore follows that

$$U^*\Phi_{(\Lambda - P_n\Lambda)_{max}}(f)U = \Phi_{(\Lambda - P_n\Lambda)_{max}}(f) + l(f)\mathbb{1} \quad ,$$

for all $f \in D(\Lambda_{max})$.

Next, one can consider $\Lambda - P_n\Lambda$ instead of Λ and prove $h \in D((P_{n-1}\Lambda)_{max})$. Repeating this process finitely many times, one concludes that $h \in D(\Lambda_{max})$ and

$$U^*\Phi(f)U = \Phi(f) + l(f)\mathbb{1} \quad ,$$

for all $f \in D(\Lambda_{max})$, with U a suitable unitary. As before, the boundedness of l follows, and thus, again, the existence of a $g \in H$ such that $l(f) = \sigma(g, f)$, for all $f \in H$, is assured. But then the equalities

$$\sigma(g, e_k) = l(e_k) = l_k = -\sigma(h, e_k) \quad ,$$

for all $k \in \mathbb{N}$, imply that $g + h \in V$ and, thus, $g \in D(\Lambda)$, since $V \subset D(\Lambda)$ and $h \in D(\Lambda)$.

□

Theorem 5.10. *Let $\Lambda \in \mathcal{L}^{(n)}_{CCR}$ and π be a GNS-representation of a quasifree state on $\mathcal{A}(D(\Lambda))$. Assume that π_Λ and $\pi_{P_1\Lambda}$ are irreducible (see Theorem 3.2.2). The representations π_Λ and π are quasi-equivalent if and only if the following conditions are fulfilled:*
(i) $\pi_{(P_0+P_1)\Lambda}$ and π are quasi-equivalent.
(ii) The closure of the operator $(\mathbb{1} - P_0 - P_1)\Lambda$ is Hilbert-Schmidt.

Proof. ($\Leftarrow$) Since $\pi_{P_1\Lambda}$ is irreducible and since quasi-equivalence is an equivalence relation, after applying a suitable linear transformation, it may be assumed that $P_0\Lambda \subset 0$ and $P_1\Lambda \subset 0$, and then Theorem 5.6 yields the quasi-equivalence of π and π_Λ.

($\Rightarrow$) Now let π and π_Λ be quasi-equivalent. Let s' be the scalar product associated to π and H' be the completion of $D(\Lambda)$ with respect to s', as above. By Corollary 4.4, it may further be assumed that

$$\pi(W(f)) = e^{il(f)}e^{i\Phi(Tf)} \quad , \quad f \in D(\Lambda) \quad ,$$

for a symplectic $T : D(\Lambda) \mapsto H$ and a linear form $l : D(\Lambda) \mapsto \mathbb{R}$. By applying a suitable coherent transformation, one has $l \subset 0$. By Lemma 5.5, if a sequence $\{f_n\}$ in $D(\Lambda)$ converges with respect to s', then the strong graph limit of $\{\Phi(Tf_n)\}$ exists, as does the strong graph limit of $\{\Phi_\Lambda(f_n)\}$. From Lemma 5.4 and $P_1\Lambda \subset 0$, one notes that the sequences $\{f_n\}$ and $\{\Lambda f_n\}$ also converge, so it may be assumed that $D(\Lambda)$ is closed with respect to s', *i.e.* T is closed. Theorem 5.9 entails that each coherent transformation of π_Λ inducing a quasi-equivalent representation is of the form

$$\Phi_\Lambda(f) \mapsto \Phi_\Lambda(f) + \sigma(f, g)\mathbb{1} \quad ,$$

for all $f \in D(\Lambda)$ and some suitable $g \in H$. From Lemma 5.8 one concludes that such coherent transformations of π are of the form

$$\Phi(Tf) \mapsto \Phi(Tf) + s'(f, g)\mathbb{1} \quad ,$$

for all $f \in D(\Lambda)$ and some suitable $g \in H$. Thus, the assumed quasi-equivalence implies that, for arbitrary $f \in D(\Lambda)$ and arbitrary but fixed $g \in D(\Lambda)$, one has

$$s(Tf, Tg) = s'(f, g) = \sigma(f, g') = s(f, -Jg') \quad ,$$

for a suitable $g' \in H$. Hence, $Tg \in D(T^*)$, for all $g \in D(T) = D(\Lambda)$, which implies consecutively the boundedness of T (since T is closed) and then of Λ.

Let $T\mid_V = U|T\mid_V|$ be the polar decomposition of the restriction $T\mid_V$. By considering the quasifree quasi-equivalent representations $\pi \mid \mathcal{A}(V)$ and $\pi_\Lambda \mid \mathcal{A}(V)$, one may conclude that $|T\mid_V| - \mathbb{1}$ is a Hilbert-Schmidt operator. Since T is symplectic, one has $\sigma(f,g) = 0$, for all $f,g \in R(U)$, so that $JR(U) \subset R(U)^\perp$. Let $P : H \mapsto H$, resp. $Q : H \mapsto H$, be the orthogonal projection onto $R(U)$, resp. $JR(U)$. Note that $|T\mid_V|$ is invertible, since for any $0 \neq f \in H$ there exists a $g \in H$ such that $\sigma(Tf,Tg) = \sigma(f,g) \neq 0$. Hence, one can define the operator S on H by

$$S = U|T\mid_V|^{-1}U^* - JU|T\mid_V|U^*J + (\mathbb{1} - P - Q) \quad .$$

By Lemma 4.1, $S = -JS^{-1}J$ is symplectic. Furthermore, the operator $|S| - \mathbb{1} = S - \mathbb{1}$ is Hilbert-Schmidt, so we may replace T by ST, since quasi-equivalence is an equivalence relation. Note that $T\mid_V = U$ is an isometry. By replacing T with WT for a suitable W, which is unitary when viewed as an operator on $\mathcal{H}$, it may further be assumed that $R(U) \subset V$.

There exist real constants $\lambda_{k_1 \ldots k_m}(f)$, symmetric in the indices, such that

$$\Lambda f = \sum_{m,k_1,\ldots,k_m} \lambda_{k_1\ldots k_m}(f) : x_{k_1} \cdots x_{k_m} : \quad , \quad f \in H \quad .$$

As in the proof of Theorem 5.8, one applies the inverse transformation of Λ to π_Λ and π. One then uses the strong graph limit of the sequence

$$\{\Phi_\Lambda(f) - \sum_m \sum_{k_1,\ldots,k_m \leq l} \lambda_{k_1\ldots k_m}(f) : x_{k_1} \cdots x_{k_m} : \mathbb{1}\}_{l \in \mathbb{N}} \quad ,$$

which is $\Phi(f)$, to show that π_J is quasi-equivalent to a representation π' defined by

$$\pi'(W(f)) = e^{i\Phi'(f)} \quad , f \in H \quad ,$$

where (note that $x(Ue_k) = \Phi(Te_k)$)

$$\Phi'(f) = \overline{\Phi(Tf) - \sum_{m,k_1,\ldots,k_m} \lambda_{k_1\ldots k_m}(f) : x(Ue_{k_1}) \cdots x(Ue_{k_m}) : \mathbb{1}} \quad . \tag{5.8}$$

As in the proof of Theorem 5.6, one can derive Hilbert-Schmidt conditions. In equation (5.3), one only has to replace $a(e_k)$ by $a(Ue_k)$, since the linear term of the annihilation operators $\frac{1}{\sqrt{2}}(\Phi'(e_k) + i\Phi'(Je_k))$ is, by (5.8),

$$\begin{aligned}\frac{1}{\sqrt{2}}(\Phi(Te_k) + i\Phi(TJe_k)) &= \frac{1}{\sqrt{2}}(\Phi(Ue_k) + i\Phi(TJe_k)) \\ &= a(Ue_k) + \frac{i}{\sqrt{2}}\Phi((TJ - JU)e_k) \quad .\end{aligned}$$

In the counterpart to (5.4), one obtains the additional terms

$$\|\Phi((TJ-JU)e_k)P_0\varphi_\epsilon\|_2 \overset{(3.1.1)}{=} \frac{1}{\sqrt{2}}\|(TJ-JU)e_k\|\,\|P_0\varphi_\epsilon\|_2 \quad ,$$

and

$$\begin{aligned}\|\Phi((TJ-JU)e_k)\sum_{l=1}^{2n+1} P_l\varphi_\epsilon\|_2 &\leq C\|(TJ-JU)e_k\|\|\sum_{l=1}^{2n+1} P_l\varphi_\epsilon\|_2 \\ &\leq C\epsilon\|(TJ-JU)e_k\| \quad .\end{aligned}$$

One may therefore conclude that the closure of the mapping $\Lambda - (P_0 + P_1)\Lambda$ is Hilbert-Schmidt, thus implying the quasi-equivalence of π_Λ and $\pi_{(P_0+P_1)\Lambda}$ and hence the quasi-equivalence of π and $\pi_{(P_0+P_1)\Lambda}$. □

A further immediate consequence is the following result.

Corollary 5.11. *Let $\Lambda \in \mathcal{L}^q_{CCR} \cap \mathcal{L}^{(n)}_{CCR}$ determine an irreducible representation π_Λ of $\mathcal{A}(D(\Lambda))$. π_Λ is quasi-equivalent to a GNS-representation of a quasifree state if and only if $\overline{\Lambda}$ is a Hilbert-Schmidt operator. If $\Lambda \in \mathcal{L}^{(n)}_{CCR}$ is bounded, then π_Λ is quasi-equivalent to a GNS-representation of a quasifree state if and only if $\overline{\Lambda - P_1\Lambda}$ is a Hilbert-Schmidt operator.*

Proof. Under the given hypothesis, one has $P_1\Lambda \subset 0$. Thus, with Theorem 3.2.2, the stated assertions follow at once from the previous Theorem. □

The results in this paper can be used to provide an alternative proof to a well-known criterion for the unitary equivalence of two pure quasifree states [33][9]. Recall that pure quasifree states are Fock.

Theorem 5.12. *Let $\pi = \pi_J$ and $\pi' = \pi_{J'}$ be two pure quasifree states on $\mathcal{A}(H_0)$, where H_0 is a dense subspace of H. Then they are unitarily equivalent if and only if $J - J'$ is Hilbert-Schmidt with respect to s, the scalar product on H associated with π_J (equivalently, Hilbert-Schmidt with respect to s').*

Proof. Using notation already established in Chapter IV, it follows easily from Lemma 5.4 that if two Fock states $\omega_F = \omega_J$ and $\omega_F' = \omega_{J'}$ on $\mathcal{A}(H_0)$ are unitarily equivalent then the associated scalar products s and s' are equivalent - this entails $H = H'$. By Prop. 4.4, Theorems 4.5 and 5.7, the operator (employing the notation of Prop. 4.2)

$$(|T| - \mathbb{1})(|T| + \mathbb{1}) = |T|^2 - \mathbb{1} = -JK - \mathbb{1} = -JJ' - \mathbb{1} \quad ,$$

resp. $J' - J = J(-JJ' - \mathbb{1})$, is Hilbert-Schmidt with respect to s, resp. s'. □

As in Theorem 5.12, one can use the results of Chapters IV and V, particularly Theorems 4.5 and 5.10 and Prop. 4.4, to give an alternative proof of the criteria characterizing the quasi-equivalence of quasifree states (see [33][9][8][2][3] for increasingly general results), but we shall not give the details here.

In [11] are given necessary and sufficient conditions so that two irreducible quadratic representations (see [22]) are unitarily equivalent. One could generalize that result in order to obtain necessary and sufficient conditions so that an irreducible quadratic representation and an irreducible representation of finite degree are unitarily equivalent. But, as these conditions are not particularly transparent, we shall not present them here.

Finally, we mention that in this paper and in the previous ones, [26][22], the choice of the complex structure, and thus the choice of the Fock representation, has been held fixed. It is therefore of interest to point out that in [11] the unitary equivalence of two quadratic representations constructed from different Fock representations is characterized. Those arguments can also be generalized to the case of the representations of finite degree discussed in this chapter, but once again the conditions which emerge are not particularly edifying.

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