LAGRANGIAN GRAPHS, MINIMIZING MEASURES AND MAÑÉ'S CRITICAL VALUES

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Abstract. Let L be a convex superlinear Lagrangian on a closed connected manifold M. We consider critical values of Lagrangians as defined by R. Mañé in [13]. We show that the critical value of the lift of L to a covering of M equals the infimum of the values of k such that the energy level k bounds an exact Lagrangian graph in the cotangent bundle of the covering. As a consequence we show that up to reparametrization, the dynamics of the Euler-Lagrange flow of L on an energy level that contains minimizing measures with nonzero homology can be reduced to Finsler metrics. We also show that if the Euler-Lagrange flow of L on the energy level k is Anosov, then k must be strictly bigger than the critical value $c_u(L)$ of the lift of L to the universal covering of M. It follows that given $k < c_u(L)$, there exists a potential ψ with arbitrarily small C^2-norm such that the energy level k of $L + \psi$ possesses conjugate points.

1. Introduction

Let M be a closed connected smooth manifold and let $L : TM \to \mathbb{R}$ be a smooth convex superlinear Lagrangian. This means that L restricted to each T_xM has positive definite Hessian and that for some Riemannian metric we have that

$$\lim_{|v|\to\infty} L(x,v)/|v| = \infty,$$

uniformly on $x \in M$. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian associated to L and let $\mathcal{L} : TM \to T^*M$ be the Legendre transform $(x,v) \mapsto \partial L/\partial v(x,v)$. Since M is compact, the extremals of L give rise to a complete flow $\phi_t : TM \to TM$ called the Euler-Lagrange flow of the Lagrangian. Using the Legendre transform we can push forward ϕ_t to obtain another flow ϕ_t^* which is the Hamiltonian flow of H with respect to the canonical symplectic structure of T^*M. Recall that the *energy* $E : TM \to \mathbb{R}$ is defined by

$$E(x,v) = \frac{\partial L}{\partial v}(x,v).v - L(x,v).$$

Since L is autonomous, E is a first integral of the flow ϕ_t.

A very interesting aspect of the dynamics of the Euler-Lagrange flows is given by those orbits or invariant measures that satisfy some global variational properties. Research on these special orbits goes back to M. Morse [17] and G.A. Hedlund [10] and has reappeared in recent years in the work of V. Bangert [1], M.J. Dias Carneiro [6], A. Fathi [8, 9], R. Mañé [13, 14] and J. Mather [15, 16]. For autonomous systems, like the ones we are considering, these distinguished orbits and measures have the remarkable property of living on certain

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energy levels related to minimal values of the action. This link was discovered by M.J. Dias Carneiro [6] and later exploited and enhanced by Mañé in his unfinished manuscript [13] (for proofs of Mañé's results in [13] we refer to [3, 4]).

In order to sate our results we need to recall the main concepts introduced by Mather in [15].

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra of TM that have compact support and are invariant under the flow ϕ_t. Let $H_1(M,\mathbb{R})$ be the first real homology group of M. Given a closed one-form ω on M and $\rho \in H_1(M,\mathbb{R})$, let $<\omega,\rho>$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its *homology* is defined as the unique $\rho(\mu) \in H_1(M,\mathbb{R})$ such that

$$<\omega,\rho(\mu)>=\int \omega\, d\mu,$$

for all closed one-forms on M. The integral on the right-hand side is with respect to μ with ω considered as a function $\omega : TM \to \mathbb{R}$. The function $\rho : \mathcal{M}(L) \to H_1(M,\mathbb{R})$ is surjective [15]. The homology of an invariant measure is essentially Schwartzman's asymptotic cycle [20].

The *action* of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L\, d\mu.$$

Finally we define the function $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$ by

$$\beta(\gamma) = \inf\{A_L(\mu) :\ \rho(\mu) = \gamma\}.$$

The function β is *convex* and *superlinear* and the infimum can be shown to be a *minimum* [15] and the measures at which the minimum is attained are called *minimizing measures*. In other words, $\mu \in \mathcal{M}(L)$ is a minimizing measure iff

$$\beta(\rho(\mu)) = A_L(\mu).$$

Let us recall how the convex dual $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ of β is defined. Since β is convex and superlinear we can set

$$\alpha([\omega]) = \max\{<\omega,\gamma> -\beta(\gamma) :\ \gamma \in H_1(M,\mathbb{R})\},$$

where ω is any closed one-form whose cohomology class is $[\omega]$. The function α is also convex and superlinear. It is not difficult to see that [15]

$$\alpha([\omega]) = -\min\left\{\int (L-\omega)\, d\mu :\ \mu \in \mathcal{M}(L)\right\}. \tag{1}$$

Let $\mathcal{M}^{\omega}(L)$ denote the set of all minimizing measures μ such that $[\omega]$ is the slope of a supporting hyperplane through $(\rho(\mu),\beta(\rho(\mu)))$. M.J. Dias Carneiro proved [6] that if μ is a minimizing measure in $\mathcal{M}^{\omega}(L)$, then its support is contained in the energy level k with $k = \alpha([\omega])$.

The purpose of the present paper is to present a new geometric way of describing Mather's α function and Mañé's notion of critical value of a Lagrangian [13]. This approach has many interesting applications that we describe below. Recall that a smooth one form ω is a section of the bundle $T^*M \mapsto M$. Let $G_\omega \subset T^*M$ be the graph of ω. It is well known that G_ω is a Lagrangian submanifold of T^*M if and only if ω is closed. When ω is exact we shall say that G_ω is an *exact Lagrangian graph.* Let us define a function $h : H^1(M,\mathbb{R}) \to \mathbb{R}$ by

$$h(q) = \inf\{k \in \mathbb{R} :\ H^{-1}(-\infty,k) \text{ contains a Lagrangian graph } G_\omega \text{ with } [\omega] = q\}.$$

Theorem 1.1. *$\alpha \equiv h$.*

Theorem 1.1 is a consequence of a more general theorem to be stated below and whose proof is based on the notion of critical values introduced by Mañé in [13] and that we now recall. The action of the Lagrangian L on an absolutely continuous curve $u : [a,b] \to M$ is defined by

$$A_L(u) = \int_a^b L(u(t), \dot{u}(t))\, dt.$$

Given two points, x_1 and x_2 in M, denote by $\mathcal{C}(x_1, x_2)$ the set of absolutely continuous curves $u : [0,T] \to M$, with $u(0) = x_1$ and $u(T) = x_2$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : M \times M \to \mathbb{R}$ by

$$\Phi_k(x_1, x_2) = \inf\{A_{L+k}(u) :\ u \in \mathcal{C}(x_1, x_2)\}.$$

Mañé showed [13, 3] that there exists $c(L) \in \mathbb{R}$ such that

- if $k < c(L)$, then $\Phi_k(x_1, x_2) = -\infty$, for all x_1 and x_2;
- if $k \geq c(L)$, then $\Phi_k(x_1, x_2) > -\infty$ for all x_1 and x_2 and Φ_k is a Lipschitz function;
- if $k \geq c(L)$, then
$$\Phi_k(x_1, x_3) \leq \Phi_k(x_1, x_2) + \Phi_k(x_2, x_3),$$
for all x_1, x_2 and x_3 and
$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) \geq 0,$$
for all x_1 and x_2;
- if $k > c(L)$, then for $x_1 \neq x_2$ we have
$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) > 0.$$

Observe that in general the action potential Φ_k is **not** symmetric, however defining $d_k : M \times M \to \mathbb{R}$ by

$$d_k(x_1, x_2) = \Phi_k(x_1, x_2) + \Phi_k(x_2, x_1),$$

the properties above say that d_k is a metric for $k > c(L)$ and a pseudometric for $k = c(L)$. The number $c(L)$ is called the *critical value* of L.

It is important for our purposes to indicate that the results above also hold for coverings of M, i.e. suppose $\widehat{M}$ is a covering of M with covering projection p. Take the lift of the Lagrangian L to $\widehat{M}$ which is given by

$$\widehat{L}(x, v) = L(p(x), dp(v)).$$

Then we define for each $k \in \mathbb{R}$ the action potential $\widehat{\Phi}_k$ just as above and the results hold for $\widehat{L}$. Thus we have a critical value for $\widehat{L}$.

Moreover, if M_1 and M_2 are coverings of M such that M_1 covers M_2, then

$$c(L_1) \leq c(L_2), \tag{2}$$

where L_1 and L_2 denote the lifts of the Lagrangian L to M_1 and M_2 respectively. Note that if M_1 is a *finite* covering of M_2 then $c(L_1) = c(L_2)$.

Among all possible coverings of M there are two distinguished ones; the universal covering which we shall denote by $\widetilde{M}$, and the abelian covering which we shall denote by $\overline{M}$. The latter is defined as the covering of M whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \mapsto H_1(M, \mathbb{R})$. When $\pi_1(M)$ is abelian, $\widetilde{M}$ is a finite covering of $\overline{M}$.

The universal covering of M gives rise to the critical value

$$c_u(L) \stackrel{\text{def}}{=} c(\widetilde{L}),$$

and the abelian covering of M gives rise to the critical value

$$c_a(L) \stackrel{\text{def}}{=} c(\overline{L}).$$

¿From inequality (2) it follows that

$$c_u(L) \leq c_a(L),$$

but in general the inequality may be strict as it was shown in [19].

Mañé [13, 3] established a connection between the critical values of a Lagrangian and α, the convex dual of Mather's β function. He showed that

$$c(L) = -\min\left\{\int L\,d\mu : \mu \in \mathcal{M}(L)\right\}, \tag{3}$$

and therefore combining (1) and (3) we obtain the remarkable equality

$$c(L-\omega) = \alpha([\omega]), \tag{4}$$

for any closed one-form ω whose cohomology class is $[\omega]$.

Finally, Mañé defined the *strict critical value* of L as

$$c_0(L) \stackrel{\text{def}}{=} \min\{c(L-\omega) : \ [\omega] \in H^1(M,\mathbb{R})\} = -\beta(0).$$

It was shown in [19] that the strict critical value of L equals the critical value of the lift of L to the abelian covering of M, that is, $c_a(L) = c_0(L)$.

For a given covering let us define $g(\widehat{L})$ as follows:

$$g(\widehat{L}) = \inf\{k \in \mathbb{R} : \widehat{H}^{-1}(-\infty, k) \text{ contains an exact Lagrangian graph}\},$$

where $\widehat{H}$ is the Hamiltonian associated with $\widehat{L}$.

In Section 2 we shall show

Theorem 1.2. *$c(\widehat{L}) = g(\widehat{L})$.*

Let us explain why Theorem 1.1 follows from Theorem 1.2. Since $\alpha([\omega]) = c(L-\omega)$, it sufficies to show that $g(L-\omega) = h([\omega])$. All the closed one forms in the class $[\omega]$ are given by $\omega + df$ where ω is fixed and f ranges among all smooth functions. Observe that the Hamiltonian associated with $L-\omega$ is $H_\omega(x,p) \stackrel{\text{def}}{=} H(x, p+\omega_x)$ and therefore $H^{-1}(-\infty,k)$ contains the Lagrangian graph $G_{\omega+df}$ if and only if $H_\omega^{-1}(-\infty,k)$ contains the exact Lagrangian graph G_{df} and thus $g(L-\omega) = h([\omega])$.

Note that saying that $\widehat{H}^{-1}(-\infty,k)$ contains an exact Lagrangian graph is equivalent to saying that there exist strict smooth subsolutions $f : \widehat{M} \to \mathbb{R}$ of the Hamilton-Jacobi equation $\widehat{H}(x, d_x f) = k$. In fact when $k = c(L)$ a recent result of A. Fathi [8] says that there exist Lipschitz viscosity solutions of the Hamilton-Jacobi equation $H(x, d_x f) = c(L)$ and that $c(L)$ is the only real number for which this happens.

Theorem 1.2 has the following interesting corollary whose proof will also be given in Section 2.

Corollary 1.3. *If $k > c(\widehat{L})$, then it is possible to see the dynamics of $\phi_t|_{\widehat{E}^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriatedly chosen Finsler metric on $\widehat{M}$.*

In particular, if we take $k > c_0(L)$ then it is possible to see the dynamics of $\phi_t|_{E^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriatedly chosen Finsler metric on M. Simply apply the previous corollary to the Lagrangian $L - \omega$ where ω is a closed one form such that $c_0(L) = c(L - \omega)$.

Finally, let us describe another application of Theorem 1.2. Let $\pi : TM \to M$ denote the canonical projection and, if $(x, v) \in TM$, let $V(x, v)$ denote the vertical fibre at (x, v) defined as usual as the kernel of $d\pi_{(x,v)} : T_{(x,v)}TM \to T_xM$. Let us set

$$e = \max_{x \in M} E(x, 0) = -\min_{x \in M} L(x, 0).$$

Note that the energy level $E^{-1}(k)$ projects *onto* the manifold M if and only if $k \geq e$ and for any $k > e$, the energy level $E^{-1}(k)$ is a smooth closed hypersurface of TM that intersects each tangent space T_xM in a sphere containing the origin in its interior. It is quite easy to check that the inequality $e \leq c_u(L)$ always holds, but in general the inequality may be strict (cf. [19]). An *Anosov energy level* is a regular energy level on which the flow ϕ_t is an Anosov flow. G.P. Paternain and M. Paternain showed in [18] that Anosov energy levels are free of conjugate points and that they must project onto the whole manifold thus generalizing a well known result of Klingenberg [11] for geodesic flows (cf. also [12]). Conjugate points, means, as usual, pair of points $(x_1, v_1) \neq (x_2, v_2) = \phi_t(x_1, v_1)$ such that $d\phi_t(V(x_1, v_1))$ intersects $V(x_2, v_2)$ non-trivially. Moreover in [19] they showed that if there exists k such that for all $k' \geq k$, the energy level k' is Anosov, then $k > c_u(L)$. In Section 3 we shall complete these results by showing:

Theorem 1.4. *If the energy level k is Anosov, then*

$$k > c_u(L).$$

In [19], G.P. Paternain and M. Paternain gave examples of Anosov energy levels k with $k < c_0(L)$ on surfaces of genus greater or equal than two. Observe that an energy level k such that $c_u(L) < k < c_0(L)$ has the following remarkable property: it can reparametrized as a Finsler geodesic flow on the universal covering but **not** on M itself!

We also have the following corollary of Theorem 1.4 whose proof will also be given in Section 3.

Corollary 1.5. *Given a convex superlinear Lagrangian L, $k < c_u(L)$ and $\varepsilon > 0$ there exists a smooth function $\psi : M \to \mathbb{R}$ with $|\psi|_{C^2} < \varepsilon$ and such that the energy level k of $L + \psi$ possesses conjugate points.*

We remark that if k is a regular value of the energy such that $k < e$, then the energy level k always contains conjugate points (cf. Proposition 3.1 in Section 3), therefore in the light of the previous discussion it is natural to pose the following:

Problem: Is it true that if $k < c_u(L)$, then the energy level k possesses conjugate points?

After writing the first draft of this paper and send it to Prof. Albert Fathi we learned from him that he had also discovered Theorem 1.1. He stated the result in the following variational way

$$\alpha([0]) = \inf_f \sup_{x \in M} H(d_x f),$$

where $f : M \to \mathbb{R}$ is an arbitrary C^1 function. His proof was similar to ours and relies in his weak KAM theorem proved in [8]. Ours, as we explained before, is based on the related notion of action potential introduced by Mañé.

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2. Proof of Theorem 1.2 and Corollary 1.3

Theorem 1.2 will be an immediate consequence of Lemma 2.1 and Proposition 2.3 below.

Lemma 2.1. *If there exists a C^1 function $f : \widehat{M} \to \mathbb{R}$ such that $\widehat{H}(df) < k$, then $k > c(\widehat{L})$.*

Proof: recall that

$$\widehat{H}(x,p) = \max_{v \in T_x \widehat{M}} \{p(v) - \widehat{L}(x,v)\}.$$

Since $\widehat{H}(df) < k$ it follows that for all $(x,v) \in T\widehat{M}$,

$$d_x f(v) - \widehat{L}(x,v) < k.$$

Therefore if $u : [0,T] \to \widehat{M}$ is any absolutely continuous closed curve with $T > 0$ we have

$$\int_0^T (L(u,\dot{u}) + k)\, dt = \int_0^T (L(u,\dot{u}) + k + d_u f(\dot{u}))\, dt > 0,$$

and thus $k > c(\hat{L})$.

□

Lemma 2.2. *Let $k \geq c(\widehat{L})$. If $f : \widehat{M} \to \mathbb{R}$ is differentiable at $x \in \widehat{M}$ and satisfies*

$$f(y) - f(x) \leq \widehat{\Phi}_k(x,y)$$

for all y in a neighbourhood of x, then $H(d_x f) \leq k$.

Proof: let $u(t)$ be a differentiable curve on $\widehat{M}$ with $(u(0), \dot{u}(0)) = (x,v)$. Then

$$\limsup_{t \to 0^+} \frac{f(u(t)) - f(x)}{t} \leq \liminf_{t \to 0^+} \frac{1}{t} \int_0^t [\widehat{L}(u,\dot{u}) + k]\, ds.$$

Hence $d_x f(v) \leq \widehat{L}(x,v) + k$ for all $v \in T_x \widehat{M}$ and thus

$$\widehat{H}(x, d_x f) = \max_{v \in T_x \widehat{M}} \{d_x f(v) - \widehat{L}(x,v)\} \leq k.$$

□

Proposition 2.3. *For any $k > c(\widehat{L})$ there exists $f \in C^\infty(\widehat{M}, \mathbb{R})$ such that $\widehat{H}(df) < k$.*

Proof: we shall explain first how to prove the proposition in the case of $\widehat{M} = M$ and then we will lift the construction to an arbitrary covering $\widehat{M}$.

Set $c = c(L)$. Fix $q \in M$ and let $g(x) = \Phi_c(q,x)$. By the triangle inequality, we have that

$$g(y) - g(x) \leq \Phi_c(x,y) \text{ for all } x, y \in M.$$

By the previous lemma, $H(d_x g) \leq c$ at any point $x \in M$ where $g(x)$ is differentiable.

We proceed to regularize g. We can assume that $M \subseteq \mathbb{R}^N$. Let U be a tubular neighbourhood of M in $\mathbb{R}^N$, and $\rho : U \to M$ a C^∞ projection along the normal bundle. Extend $g(x)$ to U by $\overline{g}(z) = g(\rho(z))$. Then $\overline{g}(z)$ is also Lipschitz.

Extend the Lagrangian to U by $\mathbb{L}(z,v) = L(\rho(z), d_z\rho(v)) + \frac{1}{2}\,|v - d_z\rho(v)|^2$. Then the corresponding Hamiltonian satisfies $\mathbb{H}(z, p \circ d_z\rho) = H(\rho(z), p)$ for $p \in T^*_{\rho(z)}M$. At any point of differentiablity of $\overline{g}$, we have that $d_z\overline{g} = d_{\rho(z)}g \circ d_z\rho$, and $\mathbb{H}(d_z\overline{g}) = H(d_{\rho(z)}g) \le c$.

Let $\varepsilon > 0$ be such that

(a) The 3ε-neighbourhood of M in $\mathbb{R}^N$ is contained in U.
(b) If $x \in M$, $(y,p) \in T^*\mathbb{R}^N = \mathbb{R}^{2N}$, $\mathbb{H}(y,p) \le c$ and $d_{\mathbb{R}^N}(x,y) < \varepsilon$ then $\mathbb{H}(x,p) < k$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\psi(x) \ge 0$, support$(\psi) \subset (-\varepsilon, \varepsilon)$ and $\int_{\mathbb{R}^N} \psi(|x|)\,dx = 1$. Let $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be $K(x,y) = \psi(|x-y|)$. Let N_ε be the ε-neighbourhood of M in $\mathbb{R}^N$. Define $f : N_\varepsilon \to \mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}^N} \overline{g}(y)\,K(x,y)\;dy\,.$$

Then f is C^∞ on N_ε.

Observe that $\partial_x K(x,y) = -\partial_y K(x,y)$. Since $\overline{g}(y)$ is Lipschitz, it is differentiable at Lebesgue almost every point of U (Rademacher's Theorem, cf. [7]). Moreover it is weakly differentiable (cf. [7, Section 4.2.3]), that is, for any C^∞ function $\varphi : U \to \mathbb{R}$ with compact support

$$\int_{\mathbb{R}^N} (\varphi\; d\overline{g} + \overline{g}\; d\varphi)\;dx = 0\,.$$

Hence

$$-\int_{\mathbb{R}^N} \overline{g}(y)\;\partial_y K(x,y)\;dy = \int_{\mathbb{R}^N} K(x,y)\;d_y\overline{g}\;dy.$$

Now, since

$$d_x f = \int_{\mathbb{R}^N} \overline{g}(y)\;\partial_x K(x,y)\;dy,$$

we obtain

$$d_x f = \int_{\mathbb{R}^N} K(x,y)\;d_y\overline{g}\;dy.$$

¿From the choice of $\varepsilon > 0$ we have that $\mathbb{H}(x, d_y\overline{g}) < k$ for almost every $y \in \mathrm{supp}K(x,\cdot)$ and $x \in M$. Since $K(x,y)\,dy$ is a probability measure, by Jensen's inequality

$$H(d_x f) \le \mathbb{H}(d_x f) \le \int_{\mathbb{R}^N} \mathbb{H}(x, d_y\overline{g})\,K(x,y)\;dy < k\,.$$

for all $x \in M$.

Now, suppose that $\widehat{M}$ is a covering of M. Fix $q \in \widehat{M}$ and set $g(x) = \widehat{\Phi}_{c(\widehat{L})}(x,q)$ We can regularize our function g similarly as we now explain. For $\widehat{x} \in \widehat{M}$ let x be the projection of $\widehat{x}$ to M and let μ_x be the Borel probability measure on M defined by

$$\int_M \varphi\,d\mu_x = \int_{\mathbb{R}^n} (\varphi \circ \rho)(y) K(x,y)\,dy,$$

for any continuous function $\varphi : M \to \mathbb{R}$. Then the support of μ_x satisfies

$$\mathrm{supp}(\mu_x) \subset \{y \in M \,:\, d_M(x,y) < \varepsilon\}.$$

Let $\widehat{\mu}_{\widehat{x}}$ be the lift of μ_x with $\mathrm{supp}(\widehat{\mu}_{\widehat{x}}) \subset \{\widehat{y} \in \widehat{M} \,:\, d_{\widehat{M}}(\widehat{x},\widehat{y}) < \varepsilon\}$. Then we have

$$\frac{d}{d\widehat{x}} \int_{\widehat{M}} \varphi\,d\widehat{\mu}_{\widehat{x}} = \int_{\widehat{M}} d_{\widehat{y}}\varphi\,d\widehat{\mu}_{\widehat{x}}(\widehat{y}),$$

for any weakly differentiable fucntion $\varphi : \widehat{M} \to \mathbb{R}$. The same arguments as above show that

$$f(\widehat{x}) = \int_{\widehat{M}} g(\widehat{y})\, d\widehat{\mu}_{\widehat{x}}(\widehat{y}),$$

satisfies $H(d_{\widehat{x}} f) < k$.

$\square$

Let us prove now Corollary 1.3. If $k > c(\widehat{L})$, then $\widehat{H}^{-1}(-\infty, k)$ contains an exact Lagrangian graph. This means that there exists a smooth function $f : \widehat{M} \to \mathbb{R}$ such that $\widehat{H}(x, d_x f) < k$. Therefore the new Hamiltonian $\widehat{H}_{df}(x,p) \stackrel{\text{def}}{=} \widehat{H}(x, p + d_x f)$ is such that $\widehat{H}_{df}^{-1}(-\infty, k)$ contains the zero section of T^*M. Let $\varphi : T^*M \to T^*M$ be the map $\varphi(x,p) = (x, p + d_x f)$. Observe that the Hamiltonian flow ϕ_t^* of $\widehat{H}$ and the Hamiltonian flow ψ_t of $\widehat{H}_{df}$ are related by $\psi_t \circ \varphi = \varphi \circ \phi_t^*$. Define now a new Hamiltonian G on $T^*\widehat{M}$ minus the zero section such that G takes the value one on $\widehat{H}_{df}^{-1}(k)$ and such that $G(x, \lambda p) = \lambda^2 G(x,p)$ for all positive λ. Since G is positively homogeneous of degree two and convex in p, it follows that the Legendre transform $\mathcal{L}_G$ associated to G is a diffeomorphism from $T\widehat{M}$ minus the zero section to $T^*\widehat{M}$ minus the zero section. Therefore the Hamiltonian G induces a Finsler metric on $\widehat{M}$ simply by taking $G \circ \mathcal{L}_G$.

Since by definition $G^{-1}(1) = \widehat{H}_{df}^{-1}(k)$ it follows that Hamiltonian flows of G and $\widehat{H}_{df}^{-1}(k)$ conicide up to reparametrization on the energy level $G^{-1}(1) = \widehat{H}_{df}^{-1}(k)$ and therefore Euler-Lagrange solutions of $\widehat{L}$ with energy k are reparametrizations of unit speed geodesics of $G \circ \mathcal{L}_G$.

$\square$

3. Proof of Theorem 1.4 and Corollary 1.5

Suppose that the energy level k is Anosov and set $\Sigma \stackrel{\text{def}}{=} H^{-1}(k)$. Let $\pi : T^*M \to M$ denote the canonical projection. G.P Paternain and M. Paternain proved in [18] that Σ must project onto the whole manifold M and that the weak stable foliation $\mathcal{W}^s$ of ϕ_t^* is *transverse* to the fibres of the fibration by $(n-1)$-spheres given by

$$\pi|_\Sigma : \Sigma \to M.$$

Let $\widetilde{M}$ be the universal covering of M. Let $\widetilde{\Sigma}$ denote the energy level k of the lifted Hamiltonian $\widetilde{H}$. We also have a fibration by $(n-1)$-spheres

$$\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}.$$

Let $\widetilde{\mathcal{W}^s}$ be the lifted foliation which is in turn a weak stable foliation for the Hamiltonian flow of $\widetilde{H}$ restricted to $\widetilde{\Sigma}$. The foliation $\widetilde{\mathcal{W}^s}$ is also transverse to the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}$. Since the fibres are compact a result of Ehresman (cf. [2]) implies that for every $(x,p) \in \widetilde{\Sigma}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)} : \widetilde{\mathcal{W}^s}(x,p) \to \widetilde{M},$$

is a covering map. Since $\widetilde{M}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}^s}(x,p)$ is simply connected. Consequently, $\widetilde{\mathcal{W}^s}(x,p)$ intersects each fibre of the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{M}$ at just one point. In other words, each leaf $\widetilde{\mathcal{W}^s}(x,p)$ is the graph of a one form. On the other hand it is well know that the weak stable leaves of an Anosov energy level are Lagrangian submanifolds. Since any closed one form in the universal covering must be exact, it follows that each leaf $\widetilde{\mathcal{W}}^s(x,p)$ is an exact Lagrangian graph. The theorem now follows from Lemma 2.1 and the fact that there exists $\varepsilon > 0$ such that for all $k' \in (k-\varepsilon, k+\varepsilon)$ the energy level k' is Anosov.

□

Let us prove now Corollary 1.5.

Suppose now that there exists $\epsilon > 0$ such that for every ψ with $|\psi|_{C^2} < \epsilon$, the energy level k of $L + \psi$ has no conjugate points. The main result in [5] says that in this case the energy level k of L must be Anosov thus contradicting Theorem 1.4.

□

Proposition 3.1. *If k is a regular value of the energy such that $k < e$, then the energy level k has conjugate points.*

Proof: if an orbit does not have conjugate points then there exist along it two subbundles called the *Green subbundles*. They have the following properties: they are invariant, Lagrangian and they have dimension $n = \dim M$. Moreover, they are contained in the same energy level as the orbit and they do not intersect the vertical subbundle (cf. [4]). If k is a regular value of the energy with $k < e$, then $\pi(E^{-1}(k))$ is a manifold with boundary and at the boundary the vertical subspace is completely contained in the energy level. Therefore the orbits that begin at the boundary must have conjugate points, because at the boundary two n-dimensional subspaces contained in the energy level (which is $(2n-1)$-dimensional) must intersect.

□

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