On the Use of the Melnikov Integral in the Arnold Diffusion Problem

Michael Rudnev
Applied Mathematics 217-50
Caltech
Pasadena, CA 91125

Stephen Wiggins
Applied Mechanics and
Control and Dynamical Systems 116-81
Caltech
Pasadena, CA 91125 *

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Abstract

In this note we want to point out a number of difficulties of arithmetic nature with the the so-called Melnikov integral (i.e., first order perturbation theory) as a measure of the splitting distance between the stable and unstable manifolds of tori in perturbations of a-priori stable integrable Hamiltonian systems with three or more degrees-of-freedom. We do this by considering a specific example which illustrates a number of the issues. We show that it is possible to introduce additional assumptions on the frequencies of the tori so that the Melnikov integral is the dominant term in the perturbation series for the distance between the stable and unstable manifolds of the torus. However, even when the Melnikov integral can be used to estimate the splitting distance, we show that even more difficulties arise when one uses it to determine if the manifolds intersect transversely, which is a key ingredient for constructing transition chains in the Arnold diffusion problem.

1 Introduction

In 1964 Arnold published an example of a three degree of freedom Hamiltonian system containing a mechanism for a global instability of the action variables which has come to be known as *"Arnold diffusion"*. The purpose of this note is to highlight one of the main difficulties in proving that this mechanism occurs in some generic sense. The difficulties are best motivated by considering Arnold's original example.

Arnold [1964] considered a Hamiltonian of the form

$$H = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) + \mu\varepsilon\left(\cos x - 1\right)\left(\sin\varphi_1 + \cos\varphi_2\right), \tag{1}$$

where $x \in T^1$, $y \in \mathbb{R}$, $(I_1, I_2) \in \mathbb{R}^2$, $(\varphi_1, \varphi_2) \in T^2$, and ε and μ are parameters. The Hamiltonian obtained by setting $\mu = 0$ is referred to as the *unperturbed Hamiltonian*, which one recognizes as the Hamiltonian for three uncoupled systems: a pendulum and two *rotors* described by two action-angle pairs of variables. Hamilton's canonical equations for this system are given by

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$$
\begin{aligned}
\dot{x} &= \tfrac{\partial H}{\partial y} = y, \\
\dot{y} &= -\tfrac{\partial H}{\partial x} = \varepsilon \sin x + \mu\varepsilon \sin x \left(\sin\varphi_1 + \cos\varphi_2\right), \\
\dot{\varphi}_1 &= \tfrac{\partial H}{\partial I_1} = I_1, \\
\dot{I}_1 &= -\tfrac{\partial H}{\partial \varphi_1} = -\mu\varepsilon \cos\varphi_1 \left(\cos x - 1\right), \\
\dot{\varphi}_2 &= \tfrac{\partial H}{\partial I_2} = 1, \\
\dot{I}_2 &= -\tfrac{\partial H}{\partial \varphi_2} = \mu\varepsilon \sin\varphi_2 \left(\cos x - 1\right).
\end{aligned} \tag{2}
$$

We want to describe the geometry of the phase space for $\mu = 0$. In this case the system has a four dimensional normally hyperbolic invariant manifold given by

$$\mathcal{M}_0 = \{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, x = y = 0\}.$$

whose five dimensional stable and unstable manifolds coincide along

$$W^s\left(\mathcal{M}_0\right) = W^u\left(\mathcal{M}_0\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0\right\}.$$

$\mathcal{M}_0$ is foliated by a family of two-tori, and the trajectories on the two-tori are given by

$$
\begin{aligned}
\varphi_1(t) &= I_1 t + \varphi_{10}, \\
\varphi_2(t) &= t + \varphi_{20}.
\end{aligned}
$$

We denote these two-tori by $\mathcal{T}(I_1)$.

Recall that the dynamics is restricted to lie in the five dimensional energy surface given by

$$h = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right),$$

and the four dimensional $\mathcal{M}_0$ intersects the five dimensional energy surface in a three dimensional set. This three dimensional set, denoted by $\mathcal{M}_0^h$, is filled out by a family of two tori, and each two-torus has three-dimensional stable and unstable manifolds that coincide and are given by

$$W^s\left(\mathcal{T}(I_1)\right) = W^u\left(\mathcal{T}(I_1)\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0,\; I_1 \text{ fixed}\right\}.$$

The stable and unstable manifolds of a given torus are often referred to as the *whiskers* of the torus. Moreover, $W^s\left(\mathcal{T}(I_1)\right)$ intersects $W^u\left(\mathcal{T}(I_1)\right)$ non-transversely, which is a situation that we do not expect to persist for μ small, but nonzero, and it is to this situation that we now turn.

The particular form of the perturbation in (1) is such that it does not affect $\mathcal{M}_0$ or any of the tori on it. Nevertheless, for any given torus the perturbation may cause the stable and unstable manifolds to intersect transversely. Moreover, because the tori are arbitrarily close on $\mathcal{M}_0^h$, *and are not isolated by the energy integral for systems with three or more degrees-of-freedom*, the stable and unstable manifolds of nearby tori may intersect and, consequently, give rise to a mechanism for a global drift in the trajectories. Following Lochak [1995], we refer to this as the *Arnold mechanism* and we now describe it in a bit more detail. The tori can be denoted by the parameter I_1. Arnold showed that one could find a sequence of tori, $\{I_1^i\}$, $i = 1, \ldots, N$, with neighboring tori sufficiently close, i.e. $|I_1^i - I_1^{i+1}| = O(|\mu|^2)$, such that the unstable manifold of $\mathcal{T}_\mu(I_1^i)$ transversely intersects the stable manifold of $\mathcal{T}_\mu(I_1^{i+1})$, $i = 1, \ldots, N-1$. This sequence of whiskered tori is said to form a *transition chain*, and Arnold uses it to prove the following theorem.

Theorem 1.1 (Arnold) *For every ε, $r > 0$ there exists a $\mu_0 > 0$ such that for all $0 < \mu \le \mu_0$ there are invariant tori $\mathcal{T}_\mu(I_1)$ and $\mathcal{T}_\mu(I_1')$, with $|I_1 - I_1'| > r$, which are connected by a transition chain.*

He then shows that the transition chain is shadowed by a true orbit of the system and in this way one obtains an $O(1)$ drift in the I_1 variable over some finite time interval.

In carrying out this program in general there are two main difficulties that must be overcome. One is concerned with the problem of the splitting of separatrices, and the other is concerned with the construction of transition chains and shadowing orbits. It is the former problem with which this note is concerned.

2 Classes of Systems Studied

Before proceeding it is important to clarify the nature of the types of near-integrable Hamiltonian systems under consideration and what one means by the "Arnold diffusion" problem itself (see Lochak [1995]). We feel obliged to do so, for lately a number of papers have appeared, namely Xia [1993], [1994], and Moeckel [1996], which claim to study and establish "Arnold's diffusion" in certain celestial mechanics problems. However, these papers do not really deal with the phenoma under consideration here (nor do they prove the existence of any global instabilities). In particular, the very first passage of Arnold [1964] emphasizes that the unperturbed part of the system under consideration is expressed in action-angle variables, with the actions being first integrals, and no hyperbolicity present. The three papers, mentioned above consider systems that are not of this type.

In addition, we want to make the following point, implicit in Arnold [1964]:

- The Hamiltonian under consideration is real-analytic in all its variables, and the unperturbed part satisfies certain non-degeneracy properties.

Broadly speaking, there are two types of systems which, following terminology introduced by Gallavotti, we refer to as *a-priori stable* or *a-priori unstable*. This classification is with respect to the structure of the *unperturbed system*. The coordinates with respect to which the unperturbed system is characterized are also used to characterize the perturbation.

A-Priori Stable Systems

This is the perturbation theory for these systems, that was called by Poincaré "the general problem of dynamics" [1] We can describe *a-priori stable systems* in several ways.

1. All trajectories of the unperturbed system are quasiperiodic and Lyapunov stable.
2. The unperturbed system is expressed in terms of action-angle variables.
3. There is no "hyperbolicity" in the unperturbed system. All orbits are purely "elliptic" in stability type.

Hence, the Hamiltonian for an a-priori stable system is given by

$$H(\vec{I},\vec{\varphi},\mu) = H_0(\vec{I}) + \mu H_1(\vec{I},\vec{\varphi},\mu), \quad (\vec{I},\vec{\varphi}) \in \mathbb{R}^n \times T^n, \tag{3}$$

with associated Hamiltonian vector field

$$\begin{aligned}
\dot{\vec{I}} &= -\mu \frac{\partial H_1}{\partial \vec{\varphi}}(\vec{I},\vec{\varphi},\mu), \\
\dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial \vec{I}}(\vec{I}) + \mu \frac{\partial H_1}{\partial \vec{I}}(\vec{I},\vec{\varphi},\mu),
\end{aligned}$$

where μ is the perturbation parameter.

[1] Poincaré, H. New Methods of Celestial Mechanics, *History of Modern Physics and Astronomy*, **13**, I23, 22, American Institute of Physics.

A-Priori Unstable Systems

We can also describe *a-priori unstable systems* in several ways.

1. Some trajectories of the unperturbed system are saddle-type, with exponentially growing components.
2. The unperturbed system cannot be expressed entirely in terms of action-angle variables. There must be auxiliary variables that describe the saddle type trajectories.
3. "Hyperbolicity" is present in the unperturbed system.

Hence, an example of a Hamiltonian for an a-priori unstable system would be

$$H(y,x,\vec{I},\vec{\varphi},\mu) = H_0(\vec{I}) + \tilde{H}_0(y,x) + \mu H_1(y,x,\vec{I},\vec{\varphi},\mu), \quad (y,x) \in \mathbb{R}^2, \quad (\vec{I},\vec{\varphi}) \in \mathbb{R}^{n-1} \times T^{n-1}, \tag{4}$$

with associated Hamiltonian vector field

$$\begin{aligned} \dot{y} &= -\frac{\partial \tilde{H}_0}{\partial x}(y,x) - \mu \frac{\partial H_1}{\partial x}(y,x,\vec{I},\vec{\varphi},\mu), \\ \dot{x} &= \frac{\partial \tilde{H}_0}{\partial y}(y,x) - \mu \frac{\partial H_1}{\partial y}(y,x,\vec{I},\vec{\varphi},\mu), \\ \dot{\vec{I}} &= -\mu \frac{\partial H_1}{\partial \vec{\varphi}}(y,x,\vec{I},\vec{\varphi},\mu), \\ \dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial \vec{I}}(\vec{I}) + \mu \frac{\partial H_1}{\partial \vec{I}}(y,x,\vec{I},\vec{\varphi},\mu). \end{aligned} \tag{5}$$

Notice that for the unperturbed system ($\mu = 0$) the $x - y$ component of the vector field decouples from the rest. One introduces hyperbolicity into the unperturbed system by imposing an assumption, such as the following.

Assumption 2.1 (Hyperbolicity in the Unperturbed System) *For the system*

$$\begin{aligned} \dot{y} &= -\frac{\partial \tilde{H}_0}{\partial x}(y,x), \\ \dot{x} &= \frac{\partial \tilde{H}_0}{\partial y}(y,x), \end{aligned}$$

has a hyperbolic equilibrium point at $(y,x) = (y_0,x_0)$ that is connected to itself by a homoclinic orbit $(y^h(t),x^h(t))$, i.e.,

$$\lim_{|t|\to\infty} (y^h(t),x^h(t)) = (y_0,x_0).$$

The consequences of this assumption for the full phase space of the unperturbed system is that the set

$$\mathcal{M}_0 = \left\{ (y,x,\vec{I},\vec{\phi}) \,|\, y = y_0, x = x_0 \right\},$$

is a $2n-2$ dimensional normally hyperbolic invariant manifold that coincide along a branch of their $2n-1$ dimensional stable and unstable manifolds given by

$$W^s(\mathcal{M}_0) \cap W^u(\mathcal{M}_0) = \left\{ (y,x,\vec{I},\vec{\phi}) \,|\, y = y^h(t), x = x^h(t),\, t \in \mathbb{R} \right\}.$$

Moreover, $\mathcal{M}_0$ is foliated by a $n-1$ parameter family of $n-1$-tori, denoted $\tau(\vec{I})$, each having n dimensional stable and unstable manifolds (the "whiskers") that coincide along a branch. (A number of other examples of a-priori unstable systems can be found in Wiggins [1988].)

With respect to the separatrix splitting problem, the main difference between the two classes of systems are that the whiskers are part of the *unperturbed* problem for a-priori unstable systems, and they are *created* by the perturbation in a-priori stable systems (which are often referred to as *weakly hyperbolic*). In other words, for a-priori stable systems, one has to deal with the singular perturbation theory versus non-singular for a-priori unstable systems. Consequentially, in the homoclinic splitting problem for the former, one has to develop the perturbation theory in all orders, whereas for the latter the first order will generically suffice. We will address this point more fully in the following sections. Not surprisingly, a-priori unstable systems are much easier to deal with, so it becomes important to develop certain perturbative technique for them and further hope that at least part of it will digest the singular weak hyperbolicity of a-priori stable systems. The following idea goes back to Poincaré.

2.1 The Two-Parameter Stratagem

Many problems have more than one small parameter, but are such that if all the parameters are set equal to zero then the problem is a-priori stable. Poincaré used a two-parameter setting to study the periodically forced pendulum. Arnold's original example is also of this form. One can sometimes make such problems appear as a-priori unstable problems by selecting a particular parameter as the perturbation parameter (μ in Arnold's example), and let all the remaining parameters remain fixed, but nonzero. In this way the unperturbed problem obtained by setting only the selected small parameter be zero may have the form of an a-priori *unstable* system, although the hyperbolicity in this case is weak. An example of such a system would be:

$$H(y,x,\vec{I},\vec{\varphi}) = H_0(\vec{I}) + \varepsilon\tilde{H}_0(y,x) + \mu H_1(y,x,\vec{I},\vec{\varphi},\mu,\varepsilon), \quad (y,x)\in\mathbb{R}^2, \quad (\vec{I},\vec{\varphi})\in\mathbb{R}^{n-1}\times T^{n-1}, \tag{6}$$

with associated Hamiltonian vector field

$$\begin{aligned}
\dot{y} &= -\varepsilon\frac{\partial\tilde{H}_0}{\partial x}(y,x) - \mu\frac{\partial H_1}{\partial x}(y,x,\vec{I},\vec{\varphi},\mu,\varepsilon),\\
\dot{x} &= \varepsilon\frac{\partial\tilde{H}_0}{\partial y}(y,x) - \mu\frac{\partial H_1}{\partial y}(y,x,\vec{I},\vec{\varphi},\mu,\varepsilon),\\
\dot{\vec{I}} &= -\mu\frac{\partial H_1}{\partial\vec{\varphi}}(y,x,\vec{I},\vec{\varphi},\mu,\varepsilon),\\
\dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial\vec{I}}(\vec{I}) + \mu\frac{\partial H_1}{\partial\vec{I}}(y,x,\vec{I},\vec{\varphi},\mu,\varepsilon),
\end{aligned}$$

where we view μ as the perturbation parameter, and ε as small and fixed. The terms multiplied by ε are assumed to introduce hyperbolicity into the unperturbed system in the same way that the analogous terms account for hyperbolicity in the Hamiltonian (4). This type of two-parameter problem has been considered by a number of people (e.g. Arnold [1964], Xia [1993], [1994], Chierchia and Gallavotti [1994]), and is motivated by Arnold's original example.

Naturally, carrying out proofs and estimates in this setting will result in some smallness condition on μ in terms of the parameters of what is considered as the "unperturbed problem", which now includes ε. In particular, Arnold [1964] requires that for $\varepsilon > 0$ one has $|\mu| \sim \exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)$ or less, which makes it severely non-generic. Nevertheless, if it is possible to extend his results for the case $\mu \sim \varepsilon^p$ for a finite power p, the model becomes much more general, for the reasons that we will further expose.

Now we explain how hyperbolicity can be created by the perturbation in the a-priori stable system (3). Arnold's example is considered to be a model for the dynamics near a multiplicity one resonance surface, and it is believed by some that global motion along these multiplicity one resonance surfaces is influenced by transition chains.

For the unperturbed Hamiltonian (3) consider a *multiplicity one* or *simple* resonance, i.e. there exists *one* independent integer vector, $\vec{k}$, such that

$$\vec{k}\cdot\partial_{\vec{I}}H_0(\vec{I}) = 0. \tag{7}$$

For nondegenerate $H_0(\vec{I})$ (7) defines a codimension one surface in the action space or, equivalently, a hyperplane in the frequency space. Near this *resonance surface* the Hamiltonian can be transformed to a normal form of the following form

$$H(y,x,\vec{I},\vec{\varphi},\sqrt{\varepsilon}) = h_0(\vec{I},\varepsilon) + P(y,x,\vec{I},\varepsilon) + \mu F(y,x,\vec{I},\vec{\varphi},\sqrt{\varepsilon}), \tag{8}$$

where $(y,x) \in \mathbb{R}^2$, $(\vec{I},\vec{\varphi}) \in \mathbb{R}^{n-1} \times T^{n-1}$, and $|\mu| \le \mu_0 = \mathcal{O}(\varepsilon^d)$, the latter being such that $d \ge \frac{3}{2}$. Precise statements on the size of domains and analyticity conditions can be found in Rudnev and Wiggins [1997a]. We regard μ as the perturbation parameter in (8). Then the important point in relation to Arnold's example is that under certain non-degeneracy conditions, for fixed $\vec{I}$ and ε, $P(y,x,\vec{I},\varepsilon)$ is the Hamiltonian of a one degree-of-freedom Hamiltonian system whose phase portrait is qualitatively that of a simple pendulum. In particular, it has hyperbolic equilibria connected by homoclinic orbits. Thus, we see that (8) has the structure of the Hamiltonian of a pendulum coupled to $n-1$ action-angle pairs, or *rigid rotors*. This is the way in which one can view Arnold's example as a model for the global dynamics near a multiplicity one resonance surface. This naturally raises the question "is Arnold's mechanism of transition chains a generic mechanism causing the drift of trajectories along a multiplicity one resonance surface?", to which the answer is not known. This example also brings up two other points.

- The two-parameter stratagem is often useful for artificially decomposing the problem into various components in a way that allows one to examine separately the role that each component plays with respect to some phenomenon of interest, e.g., separatrix splitting. However, the parameters may be functionally related, or one may want to consider a functional relationship in order to estimate the size in parameter space where a certain phenomenon occurs. The nature of this functional relationship will play a crucial role in the justification of the Melnikov integral as the dominant term in the splitting distance.

- In this particular example (i.e., dynamics near a multiplicity one resonance surface) the perturbation contains an infinite number of harmonics (the perturbation in Arnold's example contained two). These multifrequency perturbations will lead to small divisors in the perturbation series for the splitting distance which can make it very difficult to estimate the relative sizes of various terms in the perturbation series.

2.2 Arnold's diffusion versus Arnold's mechanism

In this section we will continue further establishing the terminology, the task, which, however didactic, we feel obliged to pay heed to, although it has already been pointed out by Arnold [1964] and further by Chierchia and Gallavotti [1994], Lochak [1995], Rudnev and Wiggins [1997b].

In the Introduction we have described Arnold's mechanism, and clearly it requires the presence of some hyperbolicity, however weak (a-priori stable systems near simple resonances) or strong (a-priori unstable systems). Nevertheless, "Arnold's diffusion" will pertain to the former class of systems only. Recall, that with a certain geometric assumption, such as convexity about the unperturbed energy surface, the Nekhoroshev theorem will guarantee that the "speed of diffusion", or the quantity inverse to the time it takes for the actions, which used to be the first integrals for the unperturbed problem, to evolve by $O(1)$, be exponentially small in the perturbation parameter, with the exponent depending upon the number of degrees of freedom. In other words, at least one aspect of the "Arnold diffusion" problem shall go beyond all orders of canonical perturbation theory. Arnold's mechanism appears to be so far the easiest way, at least, to imagine such an evolution of the action variables. Besides, different aspects of it can be treated analytically reasonably well.

This is undoubtedly one of the most important and difficult problems in the theory of near-integrable Hamiltonian systems. Furthermore, using the term "Arnold's diffusion" we will imply "Arnold's mechanism for a-priori stable systems". The rest of this section will delineate what we consider the main problems.

We remark that establishing Arnold's mechanism for a-priori *unstable systems* is considerably much easier, for it is almost entirely the application of canonical perturbation theory to a finite (first) order. It was successfully studied in detail by Chierchia and Gallavotti [1994]. Bernard [1996] studied Arnold's example with $\varepsilon = 1$ and showed that the drift along the orbits shadowing the transition chains can be as fast as the square of the perturbation parameter μ. In general, due to the two above mentioned works, Arnold's mechanism for the a-priori unstable systems can be deemed rather well understood, although certain questions pertaining to its establishment for general systems still remain open. Notably, the "gap problem", as described below, is still an impediment to constructing transition chains of $O(1)$ length in the action space.

Another remark will be that real analyticity is indeed one of the main sources of difficulty. For in the C^∞ case, by using suitable perturbations with compact support, it's reasonably well understood how one can construct transition chains, due to Douady [1988]. Apparently, in this case the Nekhoroshev theorem does not apply.

The obvious strategy to attack the Arnold diffusion problem has been to attempt to adapt to it or slightly modify certain things, known about a-priori unstable systems. Since the geometric picture behind it still remains the same, construction of a transition chain would in principle suffice to vindicate "Arnold's diffusion", as far as the existence is concerned. This is undoubtedly the most important direction of research in this area today.

In this connection, the main difficulty, as is mostly regarded, is the splitting size versus the gaps between the surviving whiskered tori caused by the presence of resonances of higher multiplicity for the progenitor a-priori stable system (or, resonances one order lower multiplicity, if the system is viewed as a-priori unstable). The latter have a maximum size of the square root of the perturbation parameter, whereas the former are either exponentially small for a-priori stable systems, or of the order of the perturbation parameter for a-priori unstable systems. In both cases, in principle, for one wants the splitting size to exceed the size of the gaps, one can try to restrict the analysis by considering non-resonant regions in the action space (in particular, with respect to the perturbation spectrum), where the gaps are considerably smaller, and go fairly far with a-priori unstable systems, as did Chierchia and Gallavotti [1994]. However, such a domain will become uninterestingly small (of the characteristic size, given by some power of the perturbation parameter, larger than $\frac{1}{2}$) in the a-priori stable case. Practically, it can be achieved by fulfilling a number of normalization steps, as it is done in the analytic part of the proof of the Nekhoroshev theorem. For a-priori unstable systems it would require a finite fixed number of such steps (one or two) to make sure that in the new coordinates the splitting size exceeds the remaining gaps that have not been cut out in the process of construction of the Normal form. Whereas in the a-priori stable case, it would depend on the perturbation parameter, being its certain inverse power, which is another incarnation of the singular nature of the problem. Hence, as a price to pay, the characteristic size of the domain in the action space where such a Normal form is valid will never exceed the radius of confinement in the Nekhoroshev theorem.

Therefore, Arnold's mechanism for the instability is really interesting if one can establish the existence of transition chains which are longer than the Nekhoroshev confinement radius, and at least longer than $\sqrt{\varepsilon}$, with ε being a perturbation parameter for an a-priori stable system. The fact that the values of the actions can "oscillate" rather rapidly, but with a small amplitude of the order $\sqrt{\varepsilon}$ near resonances is quite well known. Furthermore, the exhaustive treatment of Arnold's mechanism, even for a-priori unstable systems, will not be complete until one attempts to understand the dynamics involving the resonances of higher multiplicity, or the resonance junctions, and how they can be incorporated into the overall picture. This part is clearly missing in the original recipe due to Arnold.

On the other hand, both technically and from the point of view of optimality of the estimates, construction of a real orbit shadowing the transition chain has so far been very little attended and is likely to turn out to be very hard just as well. Traditionally, this has been considered an easier aspect of the matter. Moeckel [1996] (considering an a-priori unstable system and using the first order of perturbation theory to construct a very short transition chain) says that the diffusion rate has the same order as the perturbation parameter, which is a very optimistic statement (Bernard [1996] believes that for a-priori unstable systems, perturbed by order μ the fastest possible drift rate will not exceed $O(\mu\log^{-1}\mu^{-1})$, whereas he himself ends up getting $O(\mu^2)$). It's not clear how it can be neatly done in the general setting for a-priori unstable systems, and whether there is any principal difference between a-priori stable and unstable systems, provided that the transition chain has been constructed. So far, the constructive estimates, obtained by Chierchia and Gallavotti [1994] and Marco [1996] seem to be anything but optimal. As for the variational methods used by Bessi [1996], and after him Bernard [1996], which produced much better results, they seem to have heavily needed the fact that the perturbation vanished on the tori, thus being free to choose arbitrary frequencies for the transition tori.

Eventually, there is another source of severe difficulties which pertain in our opinion exclusively to a-priori stable systems, and so far have displayed themselves in our attempts to study the splitting, but will almost surely be around on any further step of the program above outlined. These problems come from the necessity to fulfill certain uniform (with respect to the tori connected into a transition chain) linear approximations of the frequency vectors on these tori, with the error going to zero as the perturbation parameter goes to zero. This, in particular, implies a certain non-uniformity in the constructions, for as the perturbation size goes to zero the approximation properties of the frequency vectors in consideration may change. The last chapter in Rudnev and Wiggins [1997b] (see also Rudnev and Wiggins [1997c]) deals with such, when the tori have dimension 2, in which case the well developed tools of Continued fractions theory can be employed. Even then the obstacles are numerous, and the problem solved is, perhaps, the simplest of this nature. In higher dimension, though, the situation is more dour, for no approximation tools that compare in efficiency with the theory of Continued fractions are available. The rest of this note demonstrates one such problem for which we can make some progress.

3 The Generalized Arnold Model

We will illustrate the main issues concerning the use of the Melnikov vector as as measure of the splitting distance in a-priori stable systems by considering a generalization of Arnold's model given in (1) that we described earlier. We will refer to this as the *generalized Arnold model*, which is given by the following "two-parameter" system:

$$H(y,x,\vec{I},\vec{\varphi}) = \sum_{j=1}^{n-1} \frac{I_j^2}{2} + \frac{y^2}{2} + \varepsilon(\cos x - 1) + \mu F(x,\vec{\varphi}), \tag{9}$$

where

$$(y,x) \in \mathbb{R} \times T, \quad (\vec{I},\vec{\varphi}) \in \mathbb{R}^{n-1} \times T^{n-1}, \quad T = \mathbb{R}/\mathbb{Z},$$

and ε and μ are (small) parameters. Later, we will take μ as a specific algebraic function of ε so that (9) is indeed an a-priori stable system.

> *Since the purpose of this note is to point out the difficulties with justifying the Melnikov integral as the dominant term in the perturbation series for the distance between the stable and unstable manifolds of a given torus with diophantine frequency, we will refer the reader to Rudnev and Wiggins [1997b] for proofs of many of the technical estimates, as well as discussion and proofs of the analyticity properties of the functions and clarification of the sizes of certain expressions with respect to various norms.*

We refer to the Hamiltonian (9) with $\mu = 0$ as the unperturbed Hamiltonian, and we will discuss the assumptions on the perturbation $F(x,\vec{\varphi})$ in the next subsection.

If $\mu = 0$, the Hamiltonian (9) describes the uncoupled motion of a pendulum in the $y - x$ plane (these variables we will further call *hyperbolic*) and $n-1$ rotors, rapidly (compared to the time-scale of the pendulum) gyrating with constant frequencies.

The Hamiltonian vector field generated by (9) is given by:

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \varepsilon \sin x - \mu g(x,\vec{\varphi}), \\
\dot{\vec{\varphi}} &= \vec{I}, \\
\dot{\vec{I}} &= -\mu \vec{f}(x,\vec{\varphi}).
\end{aligned} \tag{10}$$

Here $g(x,\vec{\varphi}) = \partial_x F(x,\vec{\varphi})$, $\vec{f}(x,\vec{\varphi}) = \partial_{\vec{\varphi}} F(x,\vec{\varphi})$. This system was originally considered by Lochak [1990].

For $\mu = 0$ the motion, described by the Hamiltonian (9), is very simple. There is a $2n-2$ dimensional invariant manifold

$$\mathcal{M}_0 \equiv \{(x, y, \vec{\varphi}, \vec{I}) : (x, y) = (0, 0)\},$$

which is foliated by $(n-1)$-tori with frequencies $\vec{\omega}(\vec{I}) = \vec{I}$. The manifold $\mathcal{M}_0$ is normally hyperbolic. Its $2n-1$ dimensional stable and unstable manifolds coincide and form an invariant manifold $\mathcal{M}_0^h$ which is foliated by the coinciding stable and unstable manifolds (whiskers) of each individual torus. The separatrix of the pendulum is defined by the relation

$$\frac{y^2}{2} + \varepsilon(\cos x - 1) = 0. \tag{11}$$

The unperturbed trajectories in $\mathcal{M}_0^h$ are given by

$$\begin{aligned} x_0(t) &= 4\arctan\left(e^{-\sqrt{\varepsilon}t}\right), \\ y_0(t) &= \tfrac{2\sqrt{\varepsilon}}{\cosh(\sqrt{\varepsilon}t)}, \\ \vec{\varphi}_0(\vec{\omega}, t, \vec{\alpha}) &= \vec{\alpha} + \vec{\omega}t, \\ \vec{I}_0(\vec{\omega}) &= \vec{\omega}. \end{aligned} \tag{12}$$

where $\vec{\omega} \in D \subseteq \mathbb{R}^n$, and $\vec{\alpha}$ corresponds to the initial condition at $t = 0$.

Our analysis will simultaneously apply to any of the tori of the normally hyperbolic manifold $\mathcal{M}_0$, whose frequencies $\vec{\omega} \equiv \partial_{\vec{I}} H|_{\mu=0}$ enjoy the classical diophantine condition, formulated in terms of two parameters γ, τ, $0 < \gamma \leq 1$, $\tau \geq n-2$, the latter will be considered fixed. In other words, we require that $\vec{\omega} \in \Omega_\gamma$ for

$$\Omega_\gamma \equiv \{\vec{\omega} : |\vec{\omega} \cdot \vec{k}| \geq \gamma |\vec{k}|^{-\tau}, \ \forall \vec{k} \neq \vec{0}\}. \tag{13}$$

3.1 The Perturbation

There are a number of key features that we require the perturbation to possess.

1. **Parity.** We choose the perturbation to be time-angle even, or equivalently, even in the variables $(x, \vec{\varphi})$. This is, as we already mentioned, necessary to be able to ascertain the existence of zeroes of the splitting distance function to all orders.

2. **Analyticity.** We require that the perturbation be a trigonometric polynomial of finite order in the x variable. The (technical) reason why it is necessary is as follows. To develop canonical perturbation theory for the whiskers, the derivatives of the perturbations shall be evaluated on the unperturbed homoclinic orbit of the pendulum, which has a singularity away from the real axis. We need to have quantitative control over the perturbation in the whole complex domain for the unperturbed homoclinic manifold, and at the aforementioned singularity of the unperturbed solution, the perturbation will have a pole of the order, proportional to its degree in the x variable. Thus, if this degree is not bounded, we will lose the desired control.

3. **Full spectrum.** In order to guarantee the dominance of the Melnikov vector for the splitting distance, expressed as a convergent Fourier series in $\vec{\alpha}$, the former shall possess all the harmonics. The reason for it is the appearance of the expressions $\frac{\vec{k}\cdot\vec{\omega}}{\sqrt{\varepsilon}}$ in the exponents further in (20) in the first, and generally in all, orders of perturbation theory. Therefore, depending upon the approximation properties of $\vec{\omega}$, a certain (depending on ε and changing as the latter goes to zero) mode is likely to dominate the rest, and we must be sure, it is present in the Melnikov function. The precise analysis of the way it happens can be done for three degrees of freedom. In higher dimensions, one has to be content with most elementary estimates.

4. **Control over Fourier coefficients.** Since we are not going to compute the higher orders of perturbation theory, all we will know about the remainder to the splitting distance function will be its norm estimate and analyticity properties, in particular in the angle variables, which will bound from above its Fourier coefficients. Therefore, to insure that the Melnikov function dominates, we will need to know that its Fourier coefficients do not vanish too fast in the ultraviolet region, but stick in their behavior to the upper bound. That's why (14) below has an exponential factor $\exp(-|\vec{k}|\sigma_0)$. So, not only shall the perturbation have a full spectrum, but all its Fourier coefficients must be in a way maximum possible (allowed by the norm). One can compare this with the previous passage; one can also see that in (20) there are two competing quantities in the exponents: $|\vec{k}|\sigma_0$ which grows as $|\vec{k}|$ grows, and $\frac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}$, which can become smaller. Where the balance is achieved depends on the approximation properties of $\vec{\omega}$, and appears to be fairly tractable when the number of degrees of freedom equals 3 (see Rudnev and Wiggins [1997b], [1997c]).

Remark 1. We remark that the last two requirements above can be interpreted in terms of analyticity properties of the perturbation in the $\vec{\varphi}$ variable. Delshams et al. [1996] for a similar purpose considered a meromorphic perturbation with a known location of the nearest pole in the complex extension of a torus. Indeed, our Fourier series for the perturbation (14) will diverge if $|\Im\vec{\varphi}| \geq \sigma_0$. Besides, the presence of a singularity in the $\vec{\varphi}$ variables certainly makes the spectrum infinite, and is, perhaps, enough to insure that the Fourier coefficients vanish not too fast as we require it.

Remark 2. We also remark that if the perturbation does not have a certain Fourier mode, it can be made nonzero in a certain order of canonical perturbation theory by either computing the Linstedt series or doing the averaging. If the perturbation has "gaps" in its spectrum, one such step will probably suffice. If it is a finite trigonometric polynomial in $\vec{\varphi}$, the number of such steps will depend on ε as its some inverse power. Clearly, this renders the task to estimate the norm of the splitting distance function in this case much more difficult.

We choose the perturbation as follows:

$$F(x,\vec{\varphi}) = \sum_{\vec{k}\neq\vec{0}} P_{\vec{k}}(x)\exp(-|\vec{k}|\sigma_0)\cos(\vec{k}\cdot\vec{\varphi}), \tag{14}$$

where $\sigma_0 > 0$ is a parameter (which is unimportant for the purposes of this discussion, but which actually describes the size of the extension of the domain of the perturbation to a complex neighborhood of the torus with respect to the angle variables) and $P_{\vec{k}}(x) = P_{-\vec{k}}(x)$ is given explicitly by

$$P_{\vec{k}}(x) = 1 + \sum_{j=1}^{\nu_{\vec{k}}} 2^{-j}(2j-1)!A_{j\vec{k}}(1-\cos x)^j, \tag{15}$$

where the integers $\nu_{\vec{k}}$ are such that $0 \leq \nu_{\vec{k}} \leq \nu_0$, with ν_0 a finite, nonnegative integer, and $0 \leq A_{j,-\vec{k}} = A_{j\vec{k}}$ (this latter *parity property* is important for certain technical matters, besides, it allows to ascertain the existence of homoclinic points to all orders of perturbation theory; this parity property will further not play a visible role in our discussion). The factors $2^{-j}(2j-1)!$ merely play a normalizing role that serves to simplify certain expressions.

The following assumptions are crucial for the Melnikov integral to "work" for this example.

Assumption 3.1 (Property of the Perturbation) *We assume that*

$$\forall\vec{k}\neq\vec{0}, \quad and \quad \forall j\in\{1,..,\nu_{\vec{k}}\} \quad A_{j\vec{k}} \geq 0 \quad and \quad \exists j_{\vec{k}}\in\{1,..,\nu_{\vec{k}}\}: \ A_{j_{\vec{k}}\vec{k}} \geq C_1 > 0 \tag{16}$$

for some positive constant C_1.

The significance of this assumption is that it is one of the ways to insure that all the Fourier modes in the Melnikov function (21) are nonzero.

We will be measuring the splitting between the stable and unstable manifolds of a torus labeled by the frequency ω, for which we will need the following property to hold.

Assumption 3.2 (Assumption on the Frequencies of the Torus) *For a given torus, labeled by the frequency ω, we assume that any l-dimensional subvector of ω, $l = 2, \dots, n-1$, satisfies the diophantine property (13) with $\tau = l - 2\delta$, for some $0 < \delta << \frac{1}{2}$.*

4 The Splitting of Separatrices: Justification of the Melnikov Vector as the Dominant Term for the Splitting Distance

For a whiskered torus with the diophantine frequency $\vec{\omega}$ we will define the homoclinic splitting distance function as follows:

$$\begin{aligned} \Delta(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= (\Lambda,\vec{\Delta})(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\ &\text{where} \\ \Lambda(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= y^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon) - y^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\ \vec{\Delta}(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= \vec{I}^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon) - \vec{I}^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \end{aligned} \tag{17}$$

where $\vec{\alpha} \in T^{n-1}$ is an angular parameter, whose choice up to a near identity transformation depends upon the way one chooses to parameterize the whiskers.

The measure of transversality of the splitting will be the quantity

$$\Upsilon = |\det \partial_{\vec{\alpha}} \vec{\Delta}|, \tag{18}$$

evaluated at a zero of $\vec{\Delta}$, which is sufficient if one recalls that Λ and $\vec{\Delta}$ are dependent via energy conservation.

This is the measure of transversality, or the size of the splitting which is meant to be exponentially small when we say that splitting is exponentially small and in which one is interested in the Arnold diffusion problem.

Using standard perturbation arguments (see, e.g., Wiggins [1988]), one can express the splitting distance as a perturbation series in μ of the following form (μ_0 will be discussed shortly):

$$\vec{\Delta}(\vec{\omega},\vec{\alpha},\mu,\varepsilon) = \mu\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon)) + \left(\frac{\mu}{\mu_0}\right)^2 \vec{N}(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \tag{19}$$

where the leading order term

$$\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon) = -\int_{-\infty}^{+\infty} \vec{f}(x_0(t),\vec{\psi})dt \tag{20}$$

is the Melnikov (vector) integral with $\vec{\psi} = \vec{\alpha} + \vec{\omega}t$, and $\vec{N}(\vec{\omega},\vec{\alpha},\mu,\varepsilon)$ denotes the remainder.

Generally, $\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon)$ will be exponentially small with respect to ε, i.e.,

$$|\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon)| = \mathcal{O}(\exp(-\varepsilon^{-a})), \quad a > 0.$$

However, if μ is independent of ε (with $\vec{N}$ bounded), then a simple implicit function theorem argument allows one to conclude that simple zeros of $\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon)$ (with respect to $\vec{\alpha}$) imply the existence of simple zeros of $\vec{\Delta}(\vec{\omega},\vec{\alpha},\mu,\varepsilon)$.

A very difficult situation arises if $\mu = \mu(\varepsilon)$ (i.e., the a-priori stable case). In this case if we do not have detailed knowledge of the size of $\vec{N}(\vec{\omega},\vec{\alpha},\mu,\varepsilon)$ then we can only assume that the Melnikov integral "dominates" the distance measurement provided we take

$$\mu = \mathcal{O}(\exp(-\varepsilon^{-b})), \quad b > a > 0.$$

This was clearly pointed out by Arnold [1964] in his original paper.

In practical applications these issues are important for determining the size of the region in parameter space where Arnold diffusion exists. Recently, Xia [1993], [1994] has published two papers concerned with Arnold diffusion in versions of the three body problem. Each of these papers also uses the "two-parameter stratagem" that we described above. In Xia [1994] his e (corresponding to our ε) denotes eccentricity and his m_3 (corresponding to our μ) denotes the mass of a "third" body, and he requires $m_3 << e$. The two parameters in Xia [1993] are a reduced mass $\mu < 1$ (corresponding to our ε) and the eccentricity e (corresponding to our μ), where $e << \mu$.

We emphasize that choosing μ exponentially small with respect to ε is the most specific feature of Arnold's example, this is this assumption that allowed to treat the Hamiltonian (1) as a-priori unstable, for from the point of view of canonical perturbation theory in ε, μ is simply zero, for it is smaller than any power of ε; conversely, from the point of view of canonical perturbation theory in μ, ε is order one, for $\varepsilon^{-1}\mu^{1/p} \to 0$ for any finite p.

4.1 Splitting Distance in the Generalized Arnold Model

This Melnikov function can be explicitly computed, and is found to be:

$$\vec{M}(\vec{\omega},\vec{\alpha},\varepsilon) = \sum_{\vec{k}\neq\vec{0}} \vec{k}\sin(\vec{k}\cdot\vec{\alpha})\exp(-|\vec{k}|\sigma_0)\frac{\pi(\vec{k}\cdot\vec{\omega})}{\varepsilon\sinh\left(\frac{\pi}{2}\frac{\vec{k}\cdot\vec{\omega}}{\sqrt{\varepsilon}}\right)}\sum_{j=1}^{\nu_{\vec{k}}} A_{j\vec{k}}\prod_{l=1}^{j-1}\left(\frac{(\vec{k}\cdot\vec{\omega})^2}{\varepsilon}+4l^2\right), \tag{21}$$

and is clearly exponentially small in ε.

The remainder $\vec{N}$ can be shown to be exponentially small in ε also (Rudnev and Wiggins [1997b]). We fix an ultraviolet cutoff parameter $K(\varepsilon) = [\varepsilon^{-\frac{1}{2}+\delta}]$ (where $[\cdot]$ denotes the integer part of a number). It is shown in Rudnev and Wiggins [1997b] that the Fourier coefficients of $\vec{N}$ obey the following bounds:

$$\begin{aligned} |\vec{N}_{\vec{k}}|\varepsilon^{\frac{\pi}{2}} &\leq \exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right), \quad |\vec{k}| < K(\varepsilon), \\ \textstyle\sum_{|\vec{k}|\geq K(\varepsilon)} |\vec{N}_{\vec{k}}| &\leq \exp\left(-\varepsilon^{-\frac{1}{2}+2\delta}\right), \end{aligned} \tag{22}$$

where $0 < \delta << \frac{1}{2}$. It would then appear that from (19), by comparing (21) with (22), that the contribution of the terms of the second and higher orders in μ in components of the splitting distance are negligible, and hence that the Melnikov function provides the "answer" for the splitting distance. However, this is not immediately true, for in (21) note that $\vec{k}$ multiplies each term in the sum over $\vec{k}$, and we cannot rule out a-priori that some components of $\vec{k}$ are zero. Moreover, it is easy to see that this multiplication by $\vec{k}$ is a typical property of the Melnikov integral since it is the integral of the $\vec{\varphi}$ derivative of a scalar valued function. Hence, if the function is represented as a Fourier series in φ (with exponents $\exp(\vec{k}\cdot\vec{\varphi})$), then differentiation by $\vec{\varphi}$ will always "bring down" the factor $\vec{k}$. So in order to show the dominance of the Melnikov function we must somehow handle the Fourier harmonics which have some component of $\vec{k}$ zero, since the latter terms do not contribute to the same component of $\vec{M}$. Clearly, this will involve some additional assumptions on the frequency $\vec{\omega}$, which is the motivation behind assumptions 3.1 and 3.2.

For $|\vec{k}| \leq K(\varepsilon)$ we can write in (21) with an exponentially small relative error

$$|\vec{k}\cdot\vec{\omega}|\exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{k}\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right)$$

$$\text{instead of} \tag{23}$$

$$\exp(-|\vec{k}|\sigma_0)\frac{\vec{k}\cdot\vec{\omega}}{\sinh\left(\frac{\pi}{2}\frac{\vec{k}\cdot\vec{\omega}}{\sqrt{\varepsilon}}\right)},$$

if ε is small enough. Now we are seeking both lower and upper bounds on the splitting distance. The classical Dirichlet theorem provides the key in our strategy for seeking a lower bound:

Theorem 4.1 (Dirichlet) *Suppose $\vec{\beta}$ is a real vector with components $\beta_1,..,\beta_l$, and $M > 1$ is an integer. Then there exists an integer l-vector $\vec{q}$ and an integer p such that*

$$1 \leq \max(|q_1|,..,|q_l|) \leq M^{\frac{1}{l}} \textit{ and } |\vec{\beta}\cdot\vec{q}-p| \leq \frac{1}{M}. \tag{24}$$

¿From the statement of the theorem it should be clear how the Dirichlet theorem can be used to find a *lower bound* for the splitting distance function. It allows us to estimate the dot products $\vec{k}\cdot\vec{\omega}$ in the exponents from above. Hence, one can argue that there exists an integer vector $\vec{k}^*$ with its norm bounded as $\mathcal{O}(M^{\frac{1}{n-2}})$, such that $|\vec{k}^*\cdot\vec{\omega}| \leq M^{-1}$, and then use a simple fact that the Fourier norm, which is the sum of the absolute values of all the Fourier components, will be greater than the absolute value of the particular Fourier coefficient in the Fourier series for the splitting distance function, corresponding to $\vec{k}^*$.

Unfortunately, for an arbitrary $\vec{\omega}$ it is not clear that all the $q's$ and p in the Dirichlet Theorem, or equivalently all the components of the aforementioned vector $\vec{k}^*$, are going to be nonzero, which one must necessarily have to be able to perform the estimate from below following this scheme because of the presence of $\vec{k}$ in the formula (21) for the Melnikov function.

One cannot totally avoid this difficulty, and given $\vec{\omega}$, the strategy for getting the lower bound may vary. The approach that we follow can be applied to all the frequency vectors for which the ratio of the absolute value of the maximum modulus component and the absolute value of the second maximum modulus component is independent of ε.

We denote

$$\omega_+ = \max_{j=1,..,n-1} |\omega_j| = |\vec{\omega}|,$$

and without loss of generality we can assume that the maximum is achieved on the first component: $\omega_+ = |\omega_1|$. Then for the dot products $\vec{k}\cdot\vec{\omega}$ we can write

$$|\vec{k}\cdot\vec{\omega}| = \omega_+|p+\vec{q}\cdot\vec{\beta}|, \tag{25}$$

where $p = \pm k_1$ - the first component of the integer $(n-1)$-vector $\vec{k}$, which is multiplied by -1 if ω_1 is negative (the latter cannot be zero by the diophantine condition), $\vec{q} = (k_2,..,k_{n-1}) \in \mathbb{Z}^{n-2}$, and $\vec{\beta} = (\frac{\omega_2}{\omega_+},..,\frac{\omega_{n-1}}{\omega_+}) \in \mathbb{R}^{n-2}$. We will also denote

$$\omega_- = \max_{j=2,..,n-1} |\omega_j|,$$

namely the "second maximum", then again, without loss of generality we can assume that $\omega_- = |\omega_2|$. (In the standard notation $[\cdot]$ will stand for an integer part and $\{\cdot\}$ for a fractional part of a real number.)

Then it is easy to establish the following Proposition, which is proved in Rudnev and Wiggins [1997b] by using the standard box argument due to Dirichlet.

Proposition 4.1 *Given an $(n-1)$-vector $(\pm 1,\vec{\beta})$, where $\vec{\beta}\in\mathbb{R}^{n-2}$ has rationally independent components, with $|\vec{\beta}| = \max_{j=1,..,n-2}|\beta_j|$, and a real number $M > 2$ such that also*

$$M > \left(\frac{(n-3)(n-2)}{2}\right)^{n-2}, \tag{26}$$

there exists an $(n-1)$-integer vector $\vec{k}^*$, whose all components, except for possibly the first one, are nonzero, and such that

$$|\vec{k}^*| < 2(n-1)M^{\frac{1}{n-2}}\max(1,|\vec{\beta}|), \quad |\vec{k}^*\cdot\vec{\omega}| < \frac{1}{M}. \tag{27}$$

Using this Proposition we can argue that given $M > 2$ and satisfying (26) there exists an integer vector $\vec{k}^*$ with no component, except for possibly the first one, equal to zero, such that the corresponding exponential term in (23) will be bounded from below as follows:

$$\begin{aligned}\exp\left(-|\vec{k}^*|\sigma_0-\frac{\pi}{2}\frac{|\vec{k}^*\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0-\frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right)\\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-}-\frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right).\end{aligned} \tag{28}$$

Unfortunately, we don't know whether k_1^* is nonzero. If it is, then we repeat what we have just done, representing in the same notation $\vec{k}\cdot\vec{\omega}=\omega_-\vec{k}\cdot(\pm 1,\vec{\beta})$, where now $\vec{\beta}=(\frac{\omega_1}{\omega_-},\frac{\omega_3}{\omega_-},..,\frac{\omega_{n-1}}{\omega_-})$, and again apply Proposition 4.1 to establish the existence of an integer vector $\vec{k}'$ which satisfies (27) and has all the nonzero components except for possibly the second one, yielding the following estimate for the corresponding exponential term:

$$\begin{aligned}\exp\left(-|\vec{k}'|\sigma_0-\frac{\pi}{2}\frac{|\vec{k}'\cdot\vec{\omega}|}{\sqrt{\varepsilon}}\right) &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-}-\frac{\pi}{2}\frac{\omega_-}{M\sqrt{\varepsilon}}\right)\\ &\geq \exp\left(-2(n-1)M^{\frac{1}{n-2}}\sigma_0\frac{\omega_+}{\omega_-}-\frac{\pi}{2}\frac{\omega_+}{M\sqrt{\varepsilon}}\right).\end{aligned} \tag{29}$$

The absolute value of the exponent in the coinciding rightmost sides of (28) and (29) as a function of M has a minimum when

$$M=M^*=\left(\frac{\pi\omega_-(n-2)}{4(n-1)\sigma_0\sqrt{\varepsilon}}\right)^{\frac{n-2}{n-1}}, \tag{30}$$

which for ε small enough is certainly bigger than 2 and satisfies (26), provided that the quantity ω_- is ε-independent.

Substituting (30) into the lower bound at (28) or (29), we can rewrite it as:

$$\exp\left(-\omega_+(n-1)\left(\frac{2\sigma_0(n-1)}{\omega_-(n-2)}\right)^{\frac{n-2}{n-1}}\left(\frac{\pi}{2\sqrt{\varepsilon}}\right)^{\frac{1}{n-1}}\right). \tag{31}$$

So in the Fourier series for (19) there are at least two terms below the ultraviolet cutoff (labeled by $\vec{k}^*$ and $\vec{k}'$), whose exponentially small part is estimated from below by (31). That's why we can give an estimate *from below* for the Fourier norm of the splitting distance function. Moreover, it is in applying Proposition 4.1 to construct $\vec{k}^*$ and $\vec{k}'$ where we use Assumption 3.1. This is because we have no real control over the construction of $\vec{k}^*$ and $\vec{k}'$ (Proposition 4.1 is an existence proof), and once constructed, we need to know that the corresponding Fourier amplitudes are nonzero. This is guaranteed by Assumption 3.1.

Therefore, the Fourier norm (see Rudnev and Wiggins [1997a], [1997c] for definition and properties) for the ith component of the homoclinic splitting-distance function ($i=1,..,n-1$) will be bounded from below by the following expression (we neglect the exponentially small ultraviolet part of the remainder from (22), thinking that ε is small enough, and the exponentially small error has also been taken care of in the quite rough estimates for the norm of $\vec{k}^*$ or $\vec{k}'$, given by Proposition 4.1. We use the notation

$$\vec{k}^{\delta_{i1}}=\left[\begin{array}{l}\vec{k}' \text{ for } i=1\\ \vec{k}^* \text{ for } i\neq 1,\end{array}\right.$$

which means that for $i=1$ we take the mode, indexed by $\vec{k}'$, since we know for sure that $k_1'\neq 0$, and for the rest of the components of the splitting distance function we take the mode indexed by $\vec{k}^*$, because we also know that $k_2^*,..,k_{n-1}^*\neq 0$. This can be used to obtain the following estimate for a lower bound on the sup-norm of each component of the splitting distance:

$$
\begin{aligned}
|\Delta_i| \quad &\geq \quad \left(|\mu| \varepsilon^{-1+\frac{\tau}{2}} \sum_{j=1}^{\nu_{\vec{k}^{\delta_{i1}}}} A_{j\vec{k}^{\delta_{i1}}} \prod_{l=1}^{j-1} \left(\frac{(\vec{k}^{\delta_{i1}} \cdot \vec{\omega})^2}{\varepsilon} + 4l^2 \right) - \left| \frac{\mu}{\mu_0} \right|^2 \varepsilon^{1-\frac{\tau}{2}} \right) \\
&\times \quad \exp \left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right) \qquad (32) \\
&\geq \quad \frac{|\mu|}{\mu_0} \exp \left(-\omega_+ (n-1) \left(\frac{2\sigma_0 (n-1)}{\omega_- (n-2)} \right)^{\frac{n-2}{n-1}} \left(\frac{\pi}{2\omega_- \sqrt{\varepsilon}} \right)^{\frac{1}{n-1}} \right).
\end{aligned}
$$

The last thing to show is that the Melnikov function really dominates over the remainder. This basically follows from (22) if we manage to compare the lower bound for the norm of the Melnikov function with the upper bound for the sum of all the terms in N_i such that $k_i = 0$ for $|\vec{k}| < K(\varepsilon)$ and $i = 1, .., n-1$. But by assumption 3.2 on the frequency vector if at least one component of $\vec{k}$ is zero, then $|\vec{k} \cdot \vec{\omega}| \geq \gamma_0 |k|^{n-2-2\delta}$, therefore

$$
\min \left(|\vec{k}| \sigma_0 + \frac{\pi}{2} \frac{|\vec{k} \cdot \vec{\omega}|}{\sqrt{\varepsilon}} \right) \geq \sigma_0 \left(\frac{\pi \gamma_0}{\sigma_0 \sqrt{\varepsilon}} \right)^{\frac{1}{n-1-2\delta}}.
$$

To obtain this estimate we have simply used the diophantine condition with $\gamma = \gamma_0$ and $\tau = n - 2 - 2\delta$, where δ can be chosen much smaller than $\frac{1}{2}$, to find the minimum of the exponent. Therefore, if ε is small enough one ends up with the following estimate:

$$
|\Delta_i| = \mu |M_i| \left\{ 1 + O\left(\frac{\mu}{\mu_0} \right) + O\left(e^{-\varepsilon^{-\frac{1}{2(n-1)}}} \right) \right\}. \qquad (33)
$$

We point out, that knowing the exact zeroes of the splitting distance function (due to parity, they correspond to $\vec{\alpha}$ such, that its every component is either 0 or π) a-priori is important, for we cannot generally argue that the quantity Υ at a zero of $\vec{M}$ will be dominated by the first order of perturbation theory, unless we have a specific frequency vector to deal with or the number of degrees of freedom equals 3, when a lengthy analysis using Continued fractions for linear approximation succeeds in Rudnev and Wiggins [1997b], [1997c] in establishing this for an asymptotically full measure set of frequencies.

4.2 The Role of μ

Recall the form of the splitting distance given in (19):

$$
\vec{\Delta}(\vec{\omega}, \vec{\alpha}, \mu, \varepsilon) = \mu \vec{M}(\vec{\omega}, \vec{\alpha}, \varepsilon)) + \left(\frac{\mu}{\mu_0} \right)^2 \vec{N}(\vec{\omega}, \vec{\alpha}, \mu, \varepsilon).
$$

In our arguments above we showed that $\vec{M}$ was exponentially small with respect to ε, as well as $\vec{N}$. Hence, it would appear that μ could be an arbitrary small parameter. This is not true. One point that was neglected is that the expression (19) must be given mathematical meaning. In Rudnev and Wiggins [1997b] it is shown that (19) is an analytic function in a certain domain, and it is these analyticity properties that allow one to obtain the exponentially small estimates on $\vec{M}$ and $\vec{N}$. In the process of deriving the analyticity properties it is shown that for

$$
\mu_0 = \varepsilon^{\max\left(\frac{15}{4} + \frac{n}{2} + \tau + \delta,\ 2\nu_0 + 1 + \delta \right)}
$$

the splitting distance estimates hold for all $\mu \in \mathbb{C}$ such that $|\mu| \leq \mu_0$.

Apparently, if μ were exponentially small in ε, none of the above difficulties would appear, even if ϵ we let go to zero, due to the ultraviolet cutoff and estimates (22), or equivalently the fact that the splitting distance function is analytic in $\vec{\alpha}$ in a complex neighborhood of a torus, arbitrarily close to the initial analyticity domain of the perturbation in the angle variables.

4.3 The Splitting Size: Transversality

As one can see now that under certain assumptions on the frequency one can vindicate the Melnikov function as the leading-order answer for the splitting distance. In fact, all we need is that the perturbation possess an infinite number of harmonics.

It is much harder to follow through a similar argument for the transversality measure Υ, unless one is dealing with some fixed frequencies whose approximation properties are nice and uniform, and well known. Otherwise, it's very hard to deal with this quantity, for being a determinant, it becomes much more sensitive to any errors in its entries, at least if the problem is ill-conditioned, which seems to be the situation.

If we roughly rewrite (32) as

$$|\Delta_i| \geq \mu e^{-K\varepsilon^{-\frac{1}{2(n-1)}}}(1+O(\mu)),$$

for some $\mathcal{O}(1)$ constant K, we can estimate $\Upsilon \sim \mu^{n-1} e^{-K(n-1)\varepsilon^{-\frac{1}{2(n-1)}}}$. Thus, this quantity becomes smaller than the error in the preceding estimate, therefore, in general the leading order cannot be proved to dominate for the determinant unless there is an estimate on the condition number of the leading order matrix.

In other words, if we want to deal with the determinant and suggest some uniform estimates for exponentially small quantities, we have to worry not only about the power of ε in the exponents, but also about the values of certain constants in the exponents, which is hardly manageable in a more or less general setting, when the number of degrees of freedom $n > 3$. Indeed, for ε small enough and positive constants K_1, K_2, such that $K_1 > K_2$ and exponents a, p such that $0 < a \leq \frac{1}{2}$ and $p > 0$, we always have

$$\varepsilon^p \exp\left(-K_2 \frac{1}{\varepsilon^a}\right) \gg \exp\left(-K_1 \frac{1}{\varepsilon^a}\right),$$

which means that if μ is a finite power of ε, to prove the fact that the first order of perturbation theory dominates in the quantity Υ we shall have to compare the constants K_1 in the first order and K_2 in the remainder, which is practically almost impossible in a broad setting when $n > 3$. Nevertheless, when $n = 3$, namely when there are two rotators in the Generalized Arnold Model, it's fairly easy, see Rudnev and Wiggins [1997b], [1997c].

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