Separatrix Splittings Near Resonance in Perturbations of Integrable, A-Priori Stable Hamiltonian Systems with Three or More Degrees-of-Freedom

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Abstract

In this note we consider the problem of the splitting of separatrices near resonances in perturbations of a-priori stable, integrable Hamiltonian systems with three or more degrees-of-freedom. For a model problem that is an n degree-of-freedom generalization of Arnold's original model we show that the exponents in the exponentially small (with respect to the perturbation parameter) upper bound for the measure of the transversality of the splitting correspond to the exponents arising in the Nekhoroshev theorem which describes the evolution of the action variables near a resonant torus. These exponents are given by $\frac{1}{2(n-m)}$, where n is the number of degrees-of-freedom and m is the multiplicity of the resonance. In appropriate coordinates, the problem of separatrix splitting near a resonant torus can be viewed as the problem of the splitting of separtrices of a torus with a certain number of "fast" and "slow" frequencies (the number of slow frequencies plus one is the multiplicity of the resonance). We show that a splitting direction corresponding to a slow frequency gives rise to a splitting distance that has an algebraic dependence on the perturbation parameter, a splitting direction corresponding to a fast frequency gives rise to a splitting distance that has an exponentially small dependence on the perturbation parameter, and if there is *at least one* fast frequency, then the measure of transversality of the splitting will be exponentially small with respect to the perturbation parameter.

1 Introduction

A central question in the study of near integrable *analytic* Hamiltonian systems is the stability of the action variables. In particular, consider a Hamiltonian of the form

$$H(\vec{I},\vec{\varphi}) = H_0(\vec{I}) + \varepsilon H_1(\vec{I},\vec{\varphi},\varepsilon), \quad (\vec{I},\vec{\varphi}) \in R^n \times T^n, \tag{1}$$

with associated Hamiltonian vector field

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$$\begin{aligned}\dot{\vec{I}} &= -\varepsilon\frac{\partial H_1}{\partial\vec{\varphi}}(\vec{I},\vec{\varphi},\varepsilon),\\ \dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial\vec{I}}(\vec{I})+\varepsilon\frac{\partial H_1}{\partial\vec{I}}(\vec{I},\vec{\varphi},\varepsilon).\end{aligned} \tag{2}$$

This Hamiltonian system is referred to as "a-priori" stable because the unperturbed system is expressed in action-angle variables and all trajectories are stable. Stated another way, there is no hyperbolicity present in the unperturbed system. We are interested in the behavior of the $\vec{I}$ values for ε small. The KAM theorem gives a partial answer to this question, insuring that generically, the $\vec{I}$ variables are stable for most initial conditions. For those initial conditions not covered by the KAM theorem, Nekhoroshev's theorem provides an upper bound on the evolution of the $\vec{I}$ variables for *finite* times that are exponentially long with respect to ε. We want to state this theorem more carefully since it is central to the results described in this note.

1.1 The Nekhoroshev Theorem

Setting $\varepsilon = 0$ is (2) gives us the following unperturbed system

$$\begin{aligned}\dot{\vec{I}} &= 0,\\ \dot{\vec{\varphi}} &= \frac{\partial H_0}{\partial\vec{I}}(\vec{I}),\end{aligned} \tag{3}$$

whose dynamics are well known. Namely, all trajectories lie on n-dimensional tori, and are given by

$$\begin{aligned}\vec{I} &= \vec{I}_0 = \text{constant},\\ \vec{\varphi} &= \frac{\partial H_0}{\partial\vec{I}}(\vec{I}_0)t+\vec{\varphi}_0.\end{aligned} \tag{4}$$

The n-dimensional tori can be labeled by the action. A particular torus $\vec{I}=\vec{I}^*$ is said to be *resonant of multiplicity* m $(0 \le m \le n-1)$ if there exists m independent integer vectors $\vec{k}_i \in Z^n$, $i = 1,\dots,m$, such that

$$\vec{k}_i \cdot \vec{\omega}(\vec{I}^*) = 0.$$

In the two extreme cases, $m = 0$ corresponds to an n-dimensional *nonresonant* torus and $m = n-1$ corresponds to an n-dimensional torus filled with periodic orbits. Generally, a multiplicity m resonant torus is filled with $n-m$ dimensional *nonresonant* tori.

With this background, we can now state the Nekhoroshev theorem (Nekhoroshev [1975]), which requires analytic Hamiltonians where $H_0(\vec{I})$ is either convex or quasiconvex (i.e., convex when restricted to the energy surface). Typically, the Nekhoroshev theorem is stated for arbitrary initial conditions. However, to make a closer relationship with the topic of this note, we will state it in a form that describes the evolution of initial conditions near resonant surfaces (see Benettin and Gallavotti [1986], Lochak [1992], and Pöschel [1993]).

Suppose $\vec{I} = I^*$ is a resonant torus of multiplicity m $(0 \le m \le n-1)$. Consider a trajectory of the perturbed system (2) which starts $O(\sqrt{\varepsilon})$ near this resonant torus, i.e.,

$$|\vec{I}(0)-\vec{I}^*| \le C_1\sqrt{\varepsilon}.$$

Then

$$|\vec{I}(t)-\vec{I}(0)| \le C_2\,\varepsilon^{\frac{1}{2(n-m)}}, \tag{5}$$

for

$$|t| \le C_3\exp\left(C_4\,\varepsilon^{-\frac{1}{2(n-m)}}\right). \tag{6}$$

The numbers $\frac{1}{2(n-m)}$ are referred to as the *Nekhoroshev exponents*, and will be the subject of later discussions. An important consequence of the Nekhoroshev theorem is that it provides a rigorous *lower bound* on the time required for the action variables to evolve from their initial conditions by an order one amount. Roughly, it will require a time exponentially long with respect to the perturbation parameter ε. Moreover, the Nekhoroshev theorem, as we've stated it, tells us that the more resonant the initial condition (as measured by the notion of multiplicity), the more stable the trajectory. In fact, near a torus of multiplicity $n-1$, the dependence of the Nekhoroshev exponents on the number of degrees-of-freedom disappears entirely. This "stabilization at resonances" property has been emphasized by Benettin and Gallavotti [1986], Lochak [1992], and Pöschel [1993].

1.2 Splitting of Separatrices, Transition Chains, and Arnold Diffusion

In 1964 Arnold published an example of a three degrees of freedom Hamiltonian system containing a mechanism for a global instability of the action variables which has come to be known as *"Arnold diffusion"*. We begin by describing this example.

Arnold [1964] considered a Hamiltonian of the form

$$H = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) + \mu\varepsilon\left(\cos x - 1\right)\left(\sin\varphi_1 + \cos\varphi_2\right), \tag{7}$$

where $x \in T^1$, $y \in R$, $(I_1, I_2) \in R^2$, $(\varphi_1, \varphi_2) \in T^2$, and ε and μ are parameters. The Hamiltonian obtained by setting $\mu = 0$ is referred to as the *unperturbed Hamiltonian*, which one recognizes as the Hamiltonian for three uncoupled systems: a pendulum and two *rotors* described by two action-angle pairs of variables. Hamilton's canonical equations for this system are given by

$$\begin{aligned}
\dot{x} &= \tfrac{\partial H}{\partial y} = y, \\
\dot{y} &= -\tfrac{\partial H}{\partial x} = \varepsilon\sin x + \mu\varepsilon\sin x\left(\sin\varphi_1 + \cos\varphi_2\right), \\
\dot{\varphi}_1 &= \tfrac{\partial H}{\partial I_1} = I_1, \\
\dot{I}_1 &= -\tfrac{\partial H}{\partial \varphi_1} = -\mu\varepsilon\cos\varphi_1\left(\cos x - 1\right), \\
\dot{\varphi}_2 &= \tfrac{\partial H}{\partial I_2} = 1, \\
\dot{I}_2 &= -\tfrac{\partial H}{\partial \varphi_2} = \mu\varepsilon\sin\varphi_2\left(\cos x - 1\right).
\end{aligned} \tag{8}$$

We want to describe the geometry of the phase space for $\mu = 0$. In this case the system has a four dimensional normally hyperbolic invariant manifold given by

$$\mathcal{M}_0 = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, x = y = 0\right\}.$$

whose five dimensional stable and unstable manifolds coincide along

$$W^s\left(\mathcal{M}_0\right) = W^u\left(\mathcal{M}_0\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \,|\, \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0\right\}.$$

$\mathcal{M}_0$ is foliated by a family of two-tori, and the trajectories on the two-tori are given by

$$\begin{aligned}
\varphi_1(t) &= I_1 t + \varphi_{10}, \\
\varphi_2(t) &= t + \varphi_{20}.
\end{aligned}$$

We denote these two-tori by $\mathcal{T}(I_1)$.

Recall that the dynamics is restricted to lie in the five dimensional energy surface given by

$$h = \frac{I_1^2}{2} + I_2 + \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right),$$

and the four dimensional $\mathcal{M}_0$ intersects the five dimensional energy surface in a three dimensional set. This three dimensional set, denoted by $\mathcal{M}_0^h$, is filled out by a family of two tori, and each two-torus has three-dimensional stable and unstable manifolds that coincide and are given by

$$W^s\left(\mathcal{T}(I_1)\right) = W^u\left(\mathcal{T}(I_1)\right) = \left\{(x, y, \varphi_1, I_1, \varphi_2, I_2) \mid \frac{y^2}{2} + \varepsilon\left(\cos x - 1\right) = 0,\ I_1 \text{ fixed}\right\}.$$

The stable and unstable manifolds of a given torus are often referred to as the *whiskers* of the torus. Moreover, $W^s\left(\mathcal{T}(I_1)\right)$ intersects $W^u\left(\mathcal{T}(I_1)\right)$ non-transversely, which is a situation that we do not expect to persist for μ small, but nonzero, and it is to this situation that we now turn.

The particular form of the perturbation in (7) is such that it does not affect $\mathcal{M}_0$ or any of the tori on it. Nevertheless, for any given torus the perturbation may cause the stable and unstable manifolds to intersect transversely. Moreover, because the tori are arbitrarily close on $\mathcal{M}_0^h$ the stable and unstable manifolds of nearby tori may intersect and, consequently, give rise to a mechanism for a global drift in the trajectories. Following Lochak [1995], we refer to this as the *Arnold mechanism* and we now describe it in a bit more detail. The tori can be denoted by the parameter I_1. Arnold showed that one could find a sequence of tori, $\{I_1^i\}$, $i = 1, \ldots, N$, with neighboring tori sufficiently close, i.e. $|I_1^i - I_1^{i+1}| = O(|\mu|^2)$, such that the unstable manifold of $\mathcal{T}_\mu(I_1^i)$ transversely intersects the stable manifold of $\mathcal{T}_\mu(I_1^{i+1})$, $i = 1, \ldots, N-1$. This sequence of whiskered tori is said to form a *transition chain*, see Figs. 1 and 2 for an illustration of the geometry. Arnold has proven the following theorem.

Figure 1: A "cartoon" of the phase space geometry associated with the "whiskered tori" in the unperturbed Arnold model. One must be very careful in interpreting such a cartoon for this three degree-of-freedom system since the relationship between the dimensions cannot be correctly depicted in such a figure. Nevertheless, one could think of the plane as being the normally hyperbolic invariant manifold $\mathcal{M}_0$ (with the angle variables suppressed, and the curve on the plane as being the intersection of the energy surface with $\mathcal{M}_0$.

Theorem 1.1 (Arnold) *For every $\varepsilon, r > 0$ there exists a $\mu_0 > 0$ such that for all $0 < \mu \leq \mu_0$ there are invariant tori $\mathcal{T}_\mu(I_1)$ and $\mathcal{T}_\mu(I_1')$, with $|I_1 - I_1'| > r$, which are connected by a transition chain.*

He then shows that the transition chain is shadowed by a true orbit of the system (see Fig. 3) and in this way one obtains an $O(1)$ drift in the I_1 variable over some finite time interval. Arnold's example is important because it is a model of the behavior near a multiplicity one resonance surface, which we now explain.

For nondegenerate $H_0(\vec{I})$ the resonance condition $\vec{k} \cdot \vec{\omega}(\vec{I}) = 0$ defines a codimension one surface in the action space or, equivalently, a hyperplane in the frequency space. Near this *resonance surface* the Hamiltonian can be transformed to a normal form of the following form

$$H(y, x, \vec{I}, \vec{\varphi}, \sqrt{\varepsilon}) = h_0(\vec{I}, \varepsilon) + P(y, x, \vec{I}, \varepsilon) + \mu F(y, x, \vec{I}, \vec{\varphi}, \sqrt{\varepsilon}), \tag{9}$$

Figure 2: The intersection of the whiskers of neighboring tori and the transition chain.

Figure 3: An illustration of an orbit "shadowing" the transition chain.

where $(y,x) \in R^2$, $(\vec{I},\vec{\varphi}) \in R^{n-1} \times T^{n-1}$, and $|\mu| \leq \mu_0 = O(\varepsilon^d)$, the latter being such that $d \geq \frac{3}{2}$. Precise statements on the size of domains and analyticity conditions can be found in Rudnev and Wiggins [1997a]. We regard μ as the perturbation parameter in (9). Then the important point in relation to Arnold's example is that under certain non-degeneracy conditions, for fixed $\vec{I}$ and ε, $P(y,x,\vec{I},\varepsilon)$ is the Hamiltonian of a one degree-of-freedom Hamiltonian system whose phase portrait is qualitatively that of a simple pendulum. In particular, it has hyperbolic equilibria connected by homoclinic orbits. Thus, we see that (9) has the structure of the Hamiltonian of a pendulum coupled to $n-1$ action-angle pairs, or *rigid rotors*. Therefore, one can view Arnold's example as a model for the global dynamics near a multiplicity one resonance surface, which raises the question "is Arnold's mechanism of transition chains a generic mechanism causing the drift of trajectories along a multiplicity one resonance surface"? To date this question is open.

Higher multiplicity resonances are not so easily treated since the normal forms describing them do not generally have an underlying integrable structure as they do in the case of multiplicity one resonances (see Bennetin and Gallavotti [1986] or Pöschel [1993]). However, in the case where one resonance is dominant in the sense that all other resonances are weak in comparison, one can cast the problem in a setting where there is an underlying integrable system on which one can apply perturbation methods. This can be seen in the local normal forms in Bennetin and Gallavotti [1986] and in the global (near a particular resonance surface) normal forms in Pöschel [1993]. However, we will not pursue this level of generality here. Rather, we will focus on a specific example where this particular structure is evident.

The reason that we are interested in separatrices near resonances is that they are believed by many to be an important mechanism governing transport near resonances. Along these lines, Chirikov [1979] presented an argument that relates the exponent associated with the measure of transversality of the splitting of separatrices to the exponents arising in the Nekhoroshev theorem. This argument was succinctly articulated by Lochak [1995]. The purpose of this note is, for a specific example, to provide a rigorous relation between these exponents. This follows from work of Rudnev and Wiggins [1997b] on exponentially small splitting of separatrices in near-integrable Hamiltonian systems. We emphasize that the results here do not say anything about Arnold diffusion; we are not constructing transition chains.

2 The Generalized Arnold Model

We consider a generalization of Arnold's model given in (7) that we described earlier. We will refer to this as the *generalized Arnold model*, which is given by the following "two-parameter" system:

$$H(y,x,\vec{I},\vec{\varphi})=\sum_{j=1}^{n-1}\frac{I_j^2}{2}+\frac{y^2}{2}+\varepsilon(\cos x-1)+\mu F(x,\vec{\varphi}), \tag{10}$$

where

$$(y,x)\in \mathbb{R}\times T,\quad (\vec{I},\vec{\varphi})\in \mathbb{R}^{n-1}\times T^{n-1},\quad T=\mathbb{R}/\mathbb{Z},$$

and ε and μ are (small) parameters. Later, we will take μ as a specific algebraic function of ε so that (10) is indeed an a-priori stable system.

We refer to the Hamiltonian (10) with $\mu=0$ as the unperturbed Hamiltonian, and we will discuss the assumptions on the perturbation $F(x,\vec{\varphi})$ in the next subsection.

If $\mu=0$, the Hamiltonian (10) describes the uncoupled motion of a pendulum in the $y-x$ plane (these variables we will further call *hyperbolic*) and $n-1$ rotors, rapidly (compared to the time-scale of the pendulum) gyrating with constant frequencies.

The Hamiltonian vector field generated by (10) is given by:

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=\varepsilon\sin x-\mu g(x,\vec{\varphi}),\\ \dot{\vec{\varphi}}&=\vec{I},\\ \dot{\vec{I}}&=-\mu\vec{f}(x,\vec{\varphi}).\end{aligned} \tag{11}$$

Here $g(x,\vec{\varphi})=\partial_x F(x,\vec{\varphi})$, $\vec{f}(x,\vec{\varphi})=\partial_{\vec{\varphi}}F(x,\vec{\varphi})$. This system was originally considered by Lochak [1990].

For $\mu=0$ the motion, described by the Hamiltonian (10), is very simple. There is a $2n-2$ dimensional invariant manifold

$$\mathcal{M}_0\equiv\{(x,y,\vec{\varphi},\vec{I}):(x,y)=(0,0)\},$$

which is foliated by $(n-1)$-tori with frequencies $\vec{\omega}(\vec{I})=\vec{I}$. The manifold $\mathcal{M}_0$ is normally hyperbolic. Its $2n-1$ dimensional stable and unstable manifolds coincide and form an invariant manifold $\mathcal{M}_0^h$ which is foliated by the coinciding stable and unstable manifolds (whiskers) of each individual torus. The separatrix of the pendulum is defined by the relation

$$\frac{y^2}{2}+\varepsilon(\cos x-1)=0. \tag{12}$$

The unperturbed trajectories in $\mathcal{M}_0^h$ are given by

$$
\begin{aligned}
x_0(t) &= 4\arctan\left(e^{-\sqrt{\varepsilon}t}\right),\\
y_0(t) &= \tfrac{2\sqrt{\varepsilon}}{\cosh(\sqrt{\varepsilon}t)},\\
\vec{\varphi}_0(\vec{\omega},t,\vec{\alpha}) &= \vec{\alpha}+\vec{\omega}t,\\
\vec{I}_0(\vec{\omega}) &= \vec{\omega}.
\end{aligned}
\tag{13}
$$

where $\vec{\omega}\in D\subseteq \mathbb{R}^n$, and $\vec{\alpha}$ corresponds to the initial condition at $t=0$.

Our analysis will simultaneously apply to any of the tori of the normally hyperbolic manifold $\mathcal{M}_0$, whose frequencies $\vec{\omega}\equiv\partial_{\vec{I}}H|_{\mu=0}$ enjoy the classical diophantine condition, formulated in terms of two parameters γ,τ, $0<\gamma\leq 1$, $\tau\geq n-2$, the latter will be considered fixed. In other words, we require that $\vec{\omega}\in\Omega_\gamma$ for

$$\Omega_\gamma\equiv\{\vec{\omega}:|\vec{\omega}\cdot\vec{k}|\geq\gamma|\vec{k}|^{-\tau},\ \forall\vec{k}\neq\vec{0}\}. \tag{14}$$

Apropos of the parameter γ, we can let it depend on ε as follows:

$$\gamma=\gamma_0\varepsilon^b,\ \text{with } 0\leq b\leq\frac{1}{2}, \tag{15}$$

where γ_0 is $O(1)$ (and can be assumed to be exactly 1 without loss of generality). Note that if $b>\frac{1}{2}$, then (10) is no longer a model for the near-resonant behavior of an a-priori stable system (see Rudnev and Wiggins [1997a] for details).

2.1 The Perturbation

For the Hamiltonian (10) we assume that $\varepsilon>0$ and $\mu\in C$ are two independent parameters. We refer to the Hamiltonian (10) with $\mu=0$ as the unperturbed Hamiltonian.

The perturbation F in this model will be considered analytic in the $\vec{\varphi}$-variables in a complex strip of the width $\sigma_0>0$ with its sup-norm being $O(1)$ for all real x. If F is a finite trigonometric polynomial in $\vec{\varphi}$, it is certainly analytic in any strip, but the above condition will impose a constraint upon the coefficients of the polynomial.

We consider a *time-angle even* perturbation, namely the one satisfying $F(x,\vec{\varphi})=F(-x,-\vec{\varphi})$, and one way to write it down is

$$F(x,\vec{\varphi})=\sum_{\vec{k}\in Z^{n-1}}F_{\vec{k}}(x)e^{i(\vec{k}\cdot\vec{\varphi})}, \tag{16}$$

where the Fourier coefficients $F_{\vec{k}}(x)$ such that $F_{\vec{k}}(x)=F_{-\vec{k}}(-x)$ are trigonometric polynomials in $x\in T$ satisfying for all $x\in T$:

$$|F_{\vec{k}}(x)|\leq\exp(-|\vec{k}|\sigma_0), \tag{17}$$

where in the standard notation

$$|\vec{k}|=\sum_{i=1}^{n-1}|k_i|.$$

In addition, as we have already mentioned in the Introduction, we assume that

$$F_{\vec{0}}(x)=<F>=0,\ \forall x\in T,$$

where $<\cdot>$ stands for the angle-average.

Once again, we will be dealing with a time-angle even analytic perturbation (16), so the latter can be represented by a convergent Fourier series as follows:

$$F(x,\vec{\varphi})=\sum_{k:\ |k_0|\leq\nu_0,\vec{k}\neq\vec{0}}F_k\exp(-|\vec{k}|\sigma_0)\cos(k_0x+\vec{k}\cdot\vec{\varphi}) \tag{18}$$

with the notation $k=(k_0,\vec{k}),\ k_0\in Z,\ \vec{k}\in Z^{n-1}$, where all the F_k are uniformly bounded from above, and $\nu_0>0$ being an integer constant. We also require (17) to be satisfied. It is to this class of perturbations that our exponentially small upper bound will apply.

3 Separatrix Splitting

For a whiskered torus with the diophantine frequency $\vec{\omega}$ we will define the homoclinic splitting distance function as follows (see Rudnev and Wiggins [1997b]):

$$\begin{aligned} \Delta(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= (\Lambda,\vec{\Delta})(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\ &\text{where} \\ \Lambda(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= y^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon)-y^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon), \\ \vec{\Delta}(\vec{\omega},\vec{\alpha},\mu,\varepsilon) &= \vec{I}^u(\vec{\omega},\vec{\alpha},\mu,\varepsilon)-\vec{I}^s(\vec{\omega},\vec{\alpha},\mu,\varepsilon). \end{aligned} \tag{19}$$

We will sometimes use the notation $\Lambda=\Delta_0$ and $\vec{\Delta}=(\Delta_1,..,\Delta_{n-1})$. We will often omit the $\vec{\omega}$-dependence of the homoclinic splitting, thinking of a fixed torus. We will usually omit $\vec{\omega}$ in the above notation for the homoclinic splitting. We will also often drop μ and ε from the above dependencies, keeping in mind that there is no splitting for $\mu=0$.

The measure of transversality of the splitting will be the quantity

$$\Upsilon=|\det\partial_{\vec{\alpha}}\vec{\Delta}|, \tag{20}$$

evaluated at a zero of $\vec{\Delta}$, which is quite sufficient if one recalls that Λ and $\vec{\Delta}$ are dependent via energy conservation. *This is the measure of transversality, or the size of the splitting which is meant to be exponentially small when we say that splitting is exponentially small and in which one is interested in the Arnold diffusion problem.*

There will be few last pieces of notation that we want to introduce. We will often encounter the quantities

$$\begin{aligned} p_* &= \tfrac{\pi}{2\sqrt{\varepsilon}}-\sqrt{\varepsilon}, \\ \sigma_* &= \sigma_0-\sqrt{\varepsilon}. \end{aligned} \tag{21}$$

The first one is related to the pole of the unperturbed pendulum separatrix (see (13)), whereas the second one indicates that after all the construction we will end up losing very little analyticity in the angle variables.

Apropos of our strategy of making estimates, we do not fancy finding any realistic smallness conditions for ε, always following the convention that ε is small enough to insure for any constants $K,d=O(1)$ and a small positive δ such that

$$\begin{aligned} &K\geq\varepsilon^{\delta}, \\ &1\geq K\varepsilon^{-d}e^{-\varepsilon^{-\delta}}. \end{aligned}$$

This ruse will save us the effort of evaluating many non-significant constants, and we will be utilizing it all the time. Also, once δ is introduced, we endow it with a role of taking care of the constants in the future as well, whenever this is possible.

As we mentioned earlier, the generalized Arnold model is a Hamiltonian system modeling the behavior near a multiplicity one resonance surface. The primary resonance is contained in the pendulum part of the Hamiltonian. We can consider higher multiplicity resonances, which are weaker than the primary resonance, by considering resonances amongst the rotors. The following assumption enables us to quantify this notion more precisely.

Assumption 3.1 (Higher Multiplicity Resonances) *Along with (14) assume that for a fixed number* $s \in \{1, .., n-1\}$*, the frequency* $\vec{\omega}$ *can be decomposed in the following way:*

$$\vec{\omega} = (\sqrt{\varepsilon}\vec{\omega}_0, \vec{\omega}_{\vec{s}}),$$

where $\vec{\omega}_{\vec{s}} = (\omega_{n-s}, .., \omega_{n-1})$ *is such that any* l*-subvector of* $\vec{\omega}_{\vec{s}}$ *for* $l = 2, .., s$ *(so it includes* $\vec{\omega}_{\vec{s}}$ *itself) satisfies the diophantine condition* (14) *with* $\gamma = \gamma_0$ *(see (15)) and* $\tau = l - 2\delta$ *for some small positive* δ *such that* $0 < \delta \ll \frac{1}{2}$*.*

This assumption enables us to consider the splitting of separatrices of tori having s "fast" frequencies and $n-1-s$ "slow" frequencies. This models the behavior close to a whiskered resonant torus, where the resonance is of multiplicity $n-1-s$ (the number of slow frequencies). Let us consider these indices in light of the Nekhoroshev exponents given in (5) and (6), which were $\frac{1}{2(n-m)}$, where n is the number of degrees-of-freedom and m is the multiplicity of the resonance. In our notation, n is still the number of degrees of freedom and the multiplicity of the resonance is $m = n-1-s$. Hence, $n - m = s + 1$. However, it is important that we count the number of frequencies involved in the resonance correctly. Assumption 3.1 is only the weaker higher multiplicity resonances amongst the rotors. We still have the primary resonance described by the pendulum part of the generalized Arnold model. Therefore, taking into account the primary resonance, we have $m = n - s$, and therefore $n - m = s$. So we will be looking for $\frac{1}{2s}$ to appear as the exponent in the exponentially small measure of the transversality of the splitting.

3.1 The Main Theorem

The following theorem is proved in Rudnev and Wiggins [1997b] for the generalized Arnold model.

Theorem 3.1 (Exponential smallness of the splitting size) *Suppose,* $n \geq 3$*. Given a diophantine frequency vector* $\vec{\omega} \in \Omega_\gamma$*, satisfying Assumption 3.1 with* $1 \leq s \leq n-1$*, suppose that in the Hamiltonian (10) the perturbation is given by (18).*

For any small positive number δ*, such that* $0 < \delta \ll \frac{1}{2}$*, there exists a small number* $\varepsilon_0 > 0$*, also depending upon all the other parameters of the problem, excluding* μ *and* ε *only, such that if* $0 < \varepsilon < \varepsilon_0$ *and for*

$$\mu_0 = \varepsilon^{\max\left(\frac{19}{4} + \frac{n}{2} + \tau + \delta,\ 2\nu_0 + 1 + \delta\right)}$$

for all $\mu \in C$ *such that* $|\mu| \leq \mu_0$*, the homoclinic splitting is exponentially small.*

Namely, the splitting distance Δ*, defined in (19) is given by a convergent Fourier series in* $\alpha \in T^{n-1}$ *whose coefficients obey the following upper bound:*

$$|\Delta_{\vec{k}}| \leq \frac{|\mu|}{\mu_0} \frac{\sqrt{\varepsilon}}{\vec{k} \cdot \vec{\omega}} e^{-|\vec{k} \cdot \vec{\omega}| p_* - |k| \sigma_*},$$

with the quantities p_* *and* σ_* *coming from (21).*

Moreover, the measure of transversality of the splitting respects the upper bound

$$\Upsilon_{\vec{k}} \leq \left(\frac{|\mu|}{\mu_0}\right)^{n-1} \exp\left\{-\sigma_0 \left(\frac{\pi\gamma_0}{\sqrt{\varepsilon}}\right)^{\frac{1}{s-2\delta}}\right\}.$$

Besides,

$$\vec{\alpha}: \ \alpha_i = [0 \quad or \quad \pi \quad for \quad i = 1, .., n-1]$$

yield homoclinic connections, i.e., the splitting distance at these points is zero.

Since δ can be taken arbitrarily small, we see that the exponent in the upper bound for the measure of transversailty of the splitting can be made as close as we like to the Nekhoroshev exponent $\frac{1}{2s}$. One can also clearly see that the smaller s, or, equivalently, the higher the multiplicity of a resonance to which $\vec{\omega}$ is close, the smaller the splitting size becomes. This phenomenon is reminiscent of the stabilizing properties of resonances described by the Nekhoroshev theorem.

3.2 Splitting of Separatrices at Resonance

We will now give some idea of the method of proof of this theorem. In Rudnev and Wiggins [1997b] it is shown that the $n-1$ dimensional splitting distance vector is given by the following convergent series expansion:

$$\vec{\Delta} = \frac{|\mu|}{\mu_0}\sqrt{\varepsilon}\sum_{\vec{k}\neq\vec{0}} A_{\vec{k}}(\mu,\varepsilon)\frac{1}{\vec{k}\cdot\vec{\omega}}e^{-|\vec{k}\cdot\vec{\omega}|p_* - |k|\sigma_*}\sin(\vec{k}\cdot\vec{\alpha}), \tag{22}$$

where all $A_{\vec{k}} = A_{-\vec{k}}$ do not exceed 1 in their absolute values.

We fix $K(\varepsilon) = [\varepsilon^{-\frac{1}{2}+\delta}]$, for some small positive δ, where $[\cdot]$ stands for the integer part, in order to make a standard ultraviolet cutoff. It can then be shown that the sum of the absolute values of the coefficients $\Delta_{\vec{k}}$ (we will often call the latter quantity the Fourier norm "with zero width"; it's known to exceed the *sup*-norm), restricted to the ultraviolet region $|\vec{k}| \geq K(\varepsilon)$, can be easily estimated as follows:

$$\sum_{|\vec{k}|\geq K} |\Delta_{\vec{k}}| \leq \frac{|\mu|}{\mu_0}\exp\left(-\varepsilon^{-\frac{1}{2}+2\delta}\right), \tag{23}$$

whereas for $|\vec{k}| \leq K(\varepsilon)$ using (14) the following estimate holds

$$\begin{aligned} |\Delta_{\vec{k}}| &\leq \tfrac{|\mu|}{\mu_0}\varepsilon^{-\frac{\tau}{2}+\delta}\exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{\omega}\cdot\vec{k}|}{\sqrt{\varepsilon}}\right)\exp\left(\varepsilon^{\delta(|\vec{\omega}|+1)}\right) \\ &\leq \tfrac{|\mu|}{\mu_0}\varepsilon^{-\frac{\tau}{2}}\exp\left(-|\vec{k}|\sigma_0 - \tfrac{\pi}{2}\tfrac{|\vec{\omega}\cdot\vec{k}|}{\sqrt{\varepsilon}}\right). \end{aligned}$$

Therefore, we can estimate the determinant only considering the terms under the cutoff; if the estimate turns out to be much larger than the above estimate for the ultraviolet part, then neglecting it will be justified.

Furthermore, under the cutoff we can find the upper bound for the $\vec{k}$th component of the determinant (for the latter will also be given by the Fourier series in $\vec{\alpha}$, which will converge for $|\Im\vec{\alpha}| \leq \sigma_* = \sigma_0 - \sqrt{\varepsilon})$ if we do it for a finite sum of the following terms (recall the rule of computing determinants and the representation (22)):

$$\Upsilon_{\vec{k}} \leq \sum\left(\frac{|\mu|}{\mu_0}\sqrt{\varepsilon}\right)^{n-1}\left|\prod_{i=1}^{n-1} k^i_{j_i}\frac{1}{\vec{k}^i\cdot\vec{\omega}}\exp\left(-|\vec{k}^i|\sigma_0 - \frac{\pi}{2}\frac{|\vec{\omega}\cdot\vec{k}^i|}{\sqrt{\varepsilon}}\right)\right|. \tag{24}$$

In the above sum $\vec{k} = \vec{k}^1 + \vec{k}^2 + .. + \vec{k}^{n-1}$, the array $[j_1, j_2, .., j_{n-1}]$ is a permutation of the set $\{1, 2, .., n-1\}$, the sum is taken over all the permutations and all the ways to represent $\vec{k}$ as a sum of $\vec{k}^i \neq \vec{0}$ for $i = 1, 2, .., n-1$.

¿From the diophantine condition (14) and the cutoff rule, we can get rid of $\frac{1}{\vec{k}^i\cdot\vec{\omega}}$ if we rescale μ_0, having multiplied it by $\varepsilon^{\frac{\tau}{2}}$ (this would also cause $\sqrt{\varepsilon}$ to go away, for γ in the diophantine condition (14) can be $O(\sqrt{\varepsilon})$).

Furthermore, as we have allowed γ to go to zero when ε goes to zero, we can now consider the near-resonant situation modelled by Assumption 3.1. So we assume that some $r < n-1$ first components of $\vec{\omega}$ go to zero when ε goes to zero as $\sqrt{\varepsilon}$ the fastest. We use the notation $\vec{r} = (1, .., r)$ and $\vec{s} = (r+1, .., n-1)$ with $s = n-1-r$ to write

$$\vec{\omega} = (\vec{\omega}_{\vec{r}}, \vec{\omega}_{\vec{s}}) \text{ and accordingly } \vec{k} = (\vec{k}_{\vec{r}}, \vec{k}_{\vec{s}}).$$

One can probably play a lot with arithmetics and obtain many interesting parallels with the Nekhoroshev exponents if one considers different dependencies $\vec{\omega}_{\vec{r}}(\varepsilon)$, getting closer and farther away from the resonances, similar to Lochak's work on the Nekhoroshev theorem (Lochak [1992]). We content ourselves here with several simple conclusions.

Suppose $\vec{\omega}_{\vec{r}} = \sqrt{\varepsilon}\vec{\omega}_0$, where $\vec{\omega}_0$ does not depend on ε and has rationally independent components. Then (24) tells us that if $\vec{k}^i_{\vec{s}} = \vec{0}$, $\forall\, i = 1, 2, .., n-1$, the corresponding term in the sum is zero. We emphasize this fact in our notation by formally introducing the Boolean variable

$$B_* = \begin{cases} 1 & \text{if there exists} \quad i^* \in \{1, .., n-1\} : \; \vec{k}^{i^*}_{\vec{s}} \neq \vec{0}, \\ 0 & \text{otherwise.} \end{cases}$$

and rewriting (24) in the following way:

$$\Upsilon_{\vec{k}} \leq \sum B_* \left(\frac{|\mu|}{\mu_0}\right)^{n-1} \varepsilon^{-\frac{n-1}{2}(\tau+1)} \exp\left(-|\vec{k}_{\vec{s}}^{i^*}|\sigma_0 - \frac{\pi}{2}\frac{|\vec{\omega}_{\vec{k}_{\vec{s}}} \cdot \vec{k}_{\vec{s}}^{i^*}|}{\sqrt{\varepsilon}}\right). \tag{25}$$

This expression clearly shows that Υ is exponentially small. Moreover, there is a well-known fact, in particular reflected by the Dirichlet theorem (see below), that given a vector $\vec{\omega}$ whose components are independent over the rationals the problem of its linear approximation becomes easier as the dimension grows. This is the general motivation behind why the splitting size gets smaller in a resonant situation. Besides, it's already clear that if $r = n-2$, then the exponent in the estimate (24) will be nothing but $\frac{1}{2} - \delta$, where the latter (possibly rescaled) can be arbitrarily small.

We can also derive the following conclusions from (25).

- A splitting direction corresponding to a slow frequency gives rise to a splitting distance that has an algebraic dependence on the perturbation parameter.
- A splitting direction corresponding to a fast frequency gives rise to a splitting distance that has an exponentially small dependence on the perturbation parameter.
- If there is *at least one* fast frequency, then the splitting size will be exponentially small with respect to the perturbation parameter.

The advantage of choosing $\vec{\omega}_{\vec{r}} = \sqrt{\varepsilon}\vec{\omega}_0$ is that under the cutoff one will have $|\vec{\omega}_{\vec{r}} \cdot \vec{k}| \leq |\vec{\omega}_0|\varepsilon^{\delta}$, therefore the estimate (25) will be valid. Otherwise, e.g. if one chooses $\vec{\omega}_{\vec{r}} = \varepsilon^{\frac{1}{4}}\vec{\omega}_0$, then the cutoff parameter should be chosen as $K(\varepsilon) = \varepsilon^{-\frac{1}{4}+\delta}$ and the exponent in the upper bound for the splitting size in the fully resonant situation will be $\frac{1}{4} - \delta$. In fact, the following Proposition is easy to establish:

Proposition 3.1 (Maximum Multiplicity Resonance) *If $\vec{\omega}$ is fully resonant, namely $r = n-2$, and $\vec{\omega}_{\vec{r}} = \varepsilon^b\vec{\omega}_0$, where $\frac{1}{2(n-1)} + 2\delta \leq b \leq \frac{1}{2}$ for any small positive δ and ε small enough, then*

$$|\Upsilon_k| \leq \left(\frac{|\mu|}{\mu_0}\right)^{n-1} e^{-K\varepsilon^{-b+\delta}},$$

where $\mu_0 = \varepsilon^d$ for a finite d depending upon the parameters of the problem and K being $O(1)$.

Choosing b smaller than $\frac{1}{2(n-1)}$ does not make any sense, for if this is the case the resonant upper bound will become comparable with the non-resonant case (see below). Note also that in the case of the maximal multiplicity resonance the dependence of the exponent on the number of degrees-of-freedom disappears, just as for the Nekhoroshev exponents.

The arithmetic properties of the frequency are the major obstruction to making any more or less general conclusions (especially, when $n > 3$), for the small divisors sit in the exponents, and the diophantine condition is not much help at all, for once being introduced to insure convergence of the now well-known KAM iterative scheme and further given a nice measure-theoretical interpretation, it is extremely awkward as far as any approximation business is concerned.

We will use Assumption 3.1 on the frequency vector that will enable us to move an inch further with estimating (25) and to suggest, in particular, a simple upper bound for it. Returning to (25), recall that we assume that

$$\vec{\omega} = (\sqrt{\varepsilon}\vec{\omega}_0, \vec{\omega}_{\vec{s}}),$$

where the vectors $\vec{\omega}_0$ and $\vec{\omega}_{\vec{s}}$ have ε-independent rationally independent components, and the latter satisfies Assumption 3.1 with $\gamma = \gamma_0$ and $\tau \leq s - 2\delta$.

We fix s such that $s \in \{1, .., n-1\}$, being the dimension of a "non-resonant subvector" of $\vec{\omega}$. Then it becomes possible to estimate the exponent in it as follows:

$$|\vec{k}_{\vec{s}}^{i^*}|\sigma_0 + \frac{\pi}{2}\frac{|\vec{\omega}\cdot\vec{k}_{\vec{s}}^{i^*}|}{\sqrt{\varepsilon}} \geq \sigma_0 \left(\frac{\pi\gamma_0}{\sqrt{\varepsilon}}\right)\frac{1}{s-2\delta},$$

just by finding its minimum as a function of $|\vec{k}_{\vec{s}}^{i^*}|$, which occurs when the latter is equal to $\frac{\pi\gamma_0(s-2\delta)}{2\sigma_0\sqrt{\varepsilon}}$. When we were computing the above minimum, we've taken the constant in the exponent actually smaller than it should be to get rid of the inverse powers of ε in the preexponential factor of (25) and to get rid of the summation sign, for the number of summands is clearly proportional to a certain finite (depending on the number of degrees of freedom) power of ε.

3.2.1 Lower Bounds

Our results are a bit unsatisfactory in that we have only been able to obtain upper bounds on the splitting size. In order to find lower bounds one must somehow show that some finite order truncation of the series for the splitting distance dominates. At first order, this is equivalent to showing that the Melnikov function gives the correct answer for the splitting distance. In general, this is not true. Nevertheless, for a special perturbation (in particular, one that results in the Melnikov function function having all Fourier harmonics) Rudnev and Wiggins [1997b], [1997c] show that the Melnikov function does indeed give the answer for the splitting distance and for this case they are able to obtain lower bounds, which approach the upper bounds in the case of *multiplicity one* resonance.

3.2.2 The Exact Answer

In the case of three degrees-of-freedom one is able to determine the "exact answer". By this, we mean that for a given ε (modulo a "small" set of "critical values") and ω, one is able to extract two dominant Fourier modes from the series for the transversality of the splitting distance that dominate the series. This can be done via a constructive procedure and is described in Rudnev and Wiggins [1997b], [1997d]. These results are particular to three degree-of-freedom systems because continued fraction theory is used heavily to deal with the arithmetic issues associated with the frequencies. In the case of multiplicity one resonance one obtains *exactly* the Nekhoroshev exponent $\frac{1}{4}$.

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