Asymptotic of the Density of States for the Schrödinger Operator with Periodic Electric Potential

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December 10, 1996

Abstract

We analyze in this article the spectral properties of the Schrödinger operator with periodic potential on $L^2(I\!R^n)$. It is proven that the integrated density of states $N(\mu)$ has an asymptotic expansion of the form

$$N(\mu) = a_n \mu^{n/2} + a_{n-2}\mu^{\frac{n-2}{2}} + \mathcal{O}(\mu^{(n-3+\epsilon)/2}),\ \forall \epsilon > 0\ .$$

This gives also a proof of the Bethe-Sommerfeld conjecture for $n \leq 4$.

1 Introduction

We are interested in the study of the spectral properties of the Schrödinger operator on $L^2(I\!R^n)$, with $n \geq 2$,

$$P_V = \sum_{j=1}^{n} D^2_{x_j} + V(x); \tag{1.1}$$

where $D_{x_j} = -i\partial_{x_j} = \frac{1}{i}\dfrac{\partial}{\partial x_j}$, for $j = 1, \cdots, n$.
The potential $x \mapsto V(x)$ is real and

$$V(\cdot) \in C^\infty(I\!R^n; I\!R). \tag{1.2}$$

Let Γ be a lattice on $I\!R^n$:

$$\Gamma = \{\sum_{j=1}^{n} k_j e_j;\ k = (k_1, \ldots, k_n) \in Z\!\!Z^n\}\ , \tag{1.3}$$

where $\{e_1, \ldots, e_n\}$ is a basis of $I\!R^n$.
We assume that the potential is periodic:

$$V(x+a) = V(x)\ ,\ \forall a \in \Gamma\ . \tag{1.4}$$

It is well known that P_V is essentially self-adjoint on $L^2(I\!R^n)$, starting from $C_0^\infty(I\!R^n)$, the space of C^∞ functions with compact support. For any function

$f \in C_0^\infty(I\!R)$, the bounded operator on $L^2(I\!R^n)$ $f(P_V)$ has, due to the ellipticity of P_V, a C^∞ kernel $K_{f(P_V)}(x,y)$. The Γ-periodicity of V shows that this kernel satisfies

$$K_{f(P_V)}(x+a,y+a) = K_{f(P_V)}(x,y), \quad \forall a \in \Gamma \ , \ \forall x,y \in I\!R^n \ , \tag{1.5}$$

and in particular the function $x \mapsto K_{f(P_V)}(x,x)$ is Γ-periodic

$$K_{f(P_V)}(x+a,x+a) = K_{f(P_V)}(x,x), \quad \forall a \in \Gamma \ , \ \forall x \in I\!R^n \ , \tag{1.6}$$

see [HeSj] and [Sjö] for details.
Let $\mathcal{K}$ be a periodic cell:

$$\mathcal{K} = \{\sum_{j=1}^n t_j e_j \ ; \ t_j \in [0,1]\}. \tag{1.7}$$

The density of states ([ReSi]) of P_V is the Borel measure ρ_V defined by

$$\rho_V(f) := \int_{I\!R} f(\tau) d\rho_V = |\mathcal{K}|^{-1} \int_{\mathcal{K}} K_{f(P_V)}(x,x) dx. \tag{1.8}$$

As before, the kernel $K_{\chi_I(P_V)}(x,y)$ is, for every characteristic function $f = \chi_I$ of a right semi-bounded interval I, C^∞. Consequently $\rho_V(\chi_I)$ in (1.8) is well defined. The integrated density of state is by definition the real function

$$\mu \mapsto N(\mu; P_V) = \int_{-\infty}^{\mu} d\rho_V = \rho_V(\chi_{]-\infty,\mu[}). \tag{1.9}$$

This increasing function can also be characterized by the two properties

$$\int_{I\!R} f(\tau) d\rho_V = - \int_{I\!R} f'(\tau) N(\tau; P_V) d\tau, \quad \forall f \in C_0^\infty(I\!R), \tag{1.10}$$

and

$$N(\tau; P_V) = 0, \quad \forall \tau \in]-\infty, \inf V[. \tag{1.11}$$

We shall show that, if $n > 1$, the integrated density of states has, as $\mu \to +\infty$, the following asymptotical behavior

$$N(\mu; P_V) = a_n \mu^{\frac{n}{2}} + \mathcal{O}(\mu^{(n-2)/2}) \ , \tag{1.12}$$

with

$$a_n = (2\pi)^{-n} |\mathbf{S}^{n-1}|/n = (2\pi)^{-n} \int_{\{\xi \in I\!R^n, \ |\xi|<1\}} d\xi \ . \tag{1.13}$$

This result has been obtained (by a very simple perturbation argument which is given in section 8) in a more general context by M. Shubin [Sh] (Theorem 4.5) where one can find also a presentation of the main standard results about the Floquet theory.
We recall that the Floquet theory shows that

$$N(\mu; P_V) = |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star} (\sum_{\lambda_j(\theta)<\mu} 1) d\theta, \tag{1.14}$$

where $(\lambda_j(\theta))_j$ is the increasing sequence of eigenvalues of the operator on $L^2(\boldsymbol{T}^n)$,

$$P_V^\theta := \sum_{j=1}^n (D_{x_j} + \theta_j)^2 + V \tag{1.15}$$

Here we have denoted by $\boldsymbol{T}^n$ the torus

$$\boldsymbol{T}^n := I\!R^n / \Gamma \ ,$$

and the basic periodic cell of the dual lattice $\Gamma^\star$ by $\mathcal{K}^\star$

$$\Gamma^\star := \{\gamma \in I\!R^n;\ \gamma \cdot a \in 2\pi \mathbb{Z},\ \forall a \in \Gamma\} = \{\sum_{j=1}^n k_j e_j^\star;\ k_j \in \mathbb{Z}\},$$

where $(e_j^\star)_{1\leq j\leq n}$ is the dual basis of $(e_j)_{1\leq j \leq n}$ in (1.3).
The Fourier integral techniques have permitted to study the counting function of the eigenvalues of elliptic differential operator, see [Hör3]. They show, that, for any θ,

$$n(\mu; P_V^\theta) = a_n \mu^{\frac{n}{2}} + \mathcal{O}_\theta(\mu^{\frac{n-1}{2}}) \ . \tag{1.16}$$

Here $n(\mu; P_V^\theta)$ is the usual counting function of the number of eigenvalues of P_V^θ strictly less than μ.
According to the properties of the flow, one could hope to improve the $\mathcal{O}_\theta(\mu^{\frac{n-1}{2}})$ and get $\circ_\theta(\mu^{\frac{n-1}{2}})$, or use, for example, recent improvements by A. V. Volovoy [Vol1], for showing that, for some strictly positive η,

$$n(\mu; P_V^\theta) = a_n \mu^{\frac{n}{2}} + \mathcal{O}_\theta(\mu^{\frac{n-1-\eta}{2}}) \ . \tag{1.17}$$

One could then hope to get the information for the integrated density of states by integrating over $\theta \in \mathcal{K}^\star$ (See for connected results [CDV], [Bl]).

We will not pursue in this direction, because we do not see how this proof can take a clear account of the cancellations in the remainder appearing when integrating with respect to the θ variable.
The remainder estimate announced in (1.12) is indeed sharper, if we compare it to (1.16) and (1.17).
We shall actually prove the following theorem.

Theorem 1.1 *:*
If V is a C^∞ Γ-periodic potential and if $n \geq 2$, then, for any fixed $\epsilon > 0$, we have the following asymptotic expansion as $\mu \to +\infty$,

$$N(\mu; P_V) = a_n \mu^{\frac{n}{2}} + a_{n-2} \mu^{\frac{n-2}{2}} + \mathcal{O}(1) + \mathcal{O}(\mu^{(n-3+\epsilon)/2}); \tag{1.18}$$

with

$$a_n = (2\pi)^{-n} |\mathbf{S}^{n-1}| / n \ , \tag{1.19}$$

and

$$a_{n-2} = -\frac{n a_n}{2} |\mathcal{K}|^{-1} \int_{\mathcal{K}} V(x) dx \ . \tag{1.20}$$

Our method is related to the one used by [Hör1] for the study of the counting function of the eigenvalues of an elliptic operator. For $\lambda_0 > 0$ such that

$$P_V + \lambda_0 > 0 \ , \tag{1.21}$$

we construct the half-wave operator e^{itQ},

$$Q := [P_V + \lambda_0]^{\frac{1}{2}}, \tag{1.22}$$

as a Fourier integral operator. The calculus is well known (see [Hör1] or [Hör3]) for any t. But we shall need here a fine control with respect to t as $|t| \to +\infty$. This will be done through a control (with respect to t) of the symbol of the (family of) operator(s) $e^{itQ} e^{-itQ_0}$,

$$Q_0 := [P_0 + \lambda_0]^{\frac{1}{2}} = (-\Delta + \lambda_0)^{\frac{1}{2}}, \tag{1.23}$$

which will be seen as a t dependent (family of) pseudo-differential operator(s) (in short p.d.o). The property that these operators $e^{itQ}e^{-itQ_0}$ are pseudo-differential operators can be seen as a consequence of the Fourier-Integral calculus. We shall actually use the Beals characterization in order to get the

control with respect to t.
This operator will be Γ-periodic, that is commuting with the translations over the lattice, and we shall use in this case the Floquet theory. Later on, we will normalize $V(x)$ by adding the condition

$$\int_{\mathcal{K}} V(x)dx = 0\ . \tag{1.24}$$

But this approach can also be adapted to other spectral quantities attached to the Floquet theory which will be recalled in Section 2. As a consequence of our pseudodifferential calculus, we shall in particular recover a proof of the Bethe-Sommerfeld Conjecture for $n \leq 4$.

Acknowledgements:
This article is a first step before the treatment of the more difficult case devoted to the Schrödinger operator with magnetic field which will be analyzed elsewhere [Mo].
A few years ago, it was announced in [GrMo1] and in [BoKo] a much stronger result. It appeared progressively from discussions with Y. Colin de Verdière in 1993 and later with the first author that [GrMo1] contained a serious gap. The present version eliminates this gap but the price to pay is unfortunately a weaker result leading, for the application to a proof of the Bethe-Sommerfeld conjecture, to the condition $n \leq 4$. According to M. Shubin, there is also a gap in [BoKo].
We want to thank A. Grigis for previous collaboration on the same subject and comments on this manuscript, M. Shubin for fruitful correspondence and Y. Colin de Verdière for useful discussions.

2 Main results for the Fourier coefficients of the counting function of the Floquet eigenvalues

This section will recall some of the spectral quantities related through the Floquet theory to the proof of the Bethe-Sommerfeld conjecture.

Let $\lambda_0 > 0$ be fixed such that $(P_V + \lambda_0)^{-1}$ exists. We can take for example

$$\lambda_0 = 1 + \| V \|_{L^\infty(\mathbb{R}^n)} . \tag{2.1}$$

We define the operators Q and Q_0 by (1.22) and (1.23). We recall that an operator $A \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ is Γ-periodic (or simply periodic) if

$$AT_a = T_a A, \quad \forall a \in \Gamma, \tag{2.2}$$

where T_a is the extension to $\mathcal{S}'(\mathbb{R}^n)$, the space of temperated distributions, of the translation defined on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$T_a(f)(x) = f(x+a), \ \forall f \in \mathcal{S}(\mathbb{R}^n) .$$

For example P_V, Q and Q_0 are periodic. More generally any operator of the form $f(Q)$ or $f(Q_0)$ with f borelian is periodic.
If this periodic operator A has a smooth distribution kernel $K_A(x,y)$, the averaged trace of A is denoted by $\widetilde{\mathrm{Tr}}(\mathrm{A})$ and defined by

$$\widetilde{\mathrm{Tr}}(\mathrm{A}) := \frac{1}{|\mathcal{K}|} \int_{\mathcal{K}} \mathrm{K_A(x,x)dx}. \tag{2.3}$$

By this definition, we have

$$\widetilde{\mathrm{Tr}}\ \mathrm{f(Q)} = \rho_{\mathrm{V}}(\mathrm{F}), \quad \forall \mathrm{f} \in \mathrm{C}_0^\infty(\mathbb{R}), \tag{2.4}$$

where F is a borelian function satisfying

$$F(\tau) = f((\tau+\lambda_0)^{\frac{1}{2}}), \text{ if } \tau \in [-\lambda_0 + \inf\{\sigma(P_V)\}, +\infty[.$$

The values of $F(\tau)$, when $\tau < -\lambda_0 + \inf\{\sigma(P_V)\}$, do not matter. More generally, let $b \in \Gamma$ and A be a periodic operator; then (2.2) implies that

$$x \mapsto K_A(x+b, x) \tag{2.5}$$

is Γ-periodic. So we can define its average over the cell $\mathcal{K}$

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}\,(\mathrm{A}) := \widetilde{\mathrm{Tr}}(\mathrm{T_b A}) = |\mathcal{K}|^{-1} \int_{\mathcal{K}} \mathrm{K_A(x+b,x)dx} . \tag{2.6}$$

Then, it can be shown by the Floquet theory that it defines a complex distribution ϱ_V^b by

$$\varrho_V^b(f) = \int_{I\!R} f(\tau) d\varrho_V^b := \widetilde{\mathrm{Tr}}_{\mathrm{b}}\left(\mathrm{f}(\mathrm{P_V})\right), \quad \forall \mathrm{f} \in \mathrm{C}_0^\infty(\mathrm{I\!R}). \tag{2.7}$$

Let us remark that $\varrho_V = \varrho_V^0$.
The Floquet theory gives also

$$\varrho_V^b(f) = -\int_{I\!R} f'(\tau) N_b(\tau, P_V) d\tau, \quad \forall f \in C_0^\infty(I\!R), \tag{2.8}$$

with $N_b(\tau, P_V)$ defined as in (1.14) by

$$N_b(\tau, P_V) := |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star} e^{ib\theta} (\sum_{\lambda_j(\theta)<\tau} 1) d\theta \ . \tag{2.9}$$

We get also the formula

$$\varrho_V^b(f) = |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star} e^{ib\theta} \left(\sum_j f(\lambda_j(\theta)) \right) d\theta \ . \tag{2.10}$$

The formulas (2.8), (2.9) and (2.10) show that ϱ_V^b is actually a Borel measure and that

$$|\varrho_V^b(f)| \leq \varrho_V(|f|), \quad \forall f \in C_0^\infty(I\!R). \tag{2.11}$$

We get also a similar formula to (2.4)

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\mathrm{f}(\mathrm{Q})) = \varrho_{\mathrm{V}}^{\mathrm{b}}(\mathrm{F}), \quad \forall \mathrm{f} \in \mathrm{C}_0^\infty(\mathrm{I\!R}), \tag{2.12}$$

(F is related to f as in (2.4)).
Our main result about this measure is the following theorem.

Theorem 2.1 *:*
Under the assumptions of Theorem 1.1 and the condition (1.24) and if $b \in \Gamma \setminus \{0\}$*, then, for every* $\epsilon > 0$*, there exists a constant* $C_{b,\epsilon}$*, such that*

$$|N_b(\mu; P_V) - N_b(\mu; P_0)| \leq C_{b,\epsilon}(1 + \mu^{(n-3+\epsilon)/2}), \quad \forall \mu > 1 \ . \tag{2.13}$$

This theorem gives the comparison between spectral quantities attached to the free Laplacian and the corresponding spectral quantities attached to the perturbed Laplacian. The study of the free case is rather explicit. It is indeed easy to see that, if $\lambda > 0$, then

$$N_b(\lambda^2; P_0) = N_b(\lambda; (P_0)^{1/2}) = (2\pi)^{-n} \int_{\{\xi \in I\!R^n;\ |\xi|<\lambda\}} e^{ib\cdot\xi} d\xi \tag{2.14}$$

and, using Stokes theorem before to take polar coordinates, we get

$$N_b(\lambda^2; P_0) = -i(2\pi)^{-n} |b|^{-2} \lambda^{n-1} \int_{\mathbf{S}^{n-1}} (b \cdot \hat{\xi})\ e^{i\lambda b\cdot\hat{\xi}} d\hat{\xi}\ . \tag{2.15}$$

We can apply the stationary phase theorem (see [Hör2]) to the last formula. The phase $\hat{\xi} \mapsto b \cdot \hat{\xi}$ has two non degenerate critical points and we then obtain, as $\lambda \to +\infty$,

$$N_b(\lambda^2; P_0) = (2\pi|b|)^{-(n+1)/2} \lambda^{(n-1)/2} \sin(\pi \frac{n-1}{4} - \lambda|b|) + \mathcal{O}(\lambda^{\frac{n-3}{2}})\ . \tag{2.16}$$

So (2.13) is interesting only when $2 \leq n \leq 4$. This is indeed the condition under which the remainder in (2.13) is generically "smaller" than the main term in (2.16): $n-3+\epsilon < \frac{n-1}{2}$. More precisely, we shall use (2.16) in relation with the property that, for $n = 2, 3, 4$, there exists $b \in \Gamma^\star$, $b \neq 0$, $\rho > 0$ and λ_0 such that

$$|\sin(\pi \frac{n-1}{4} - \lambda|b|)| + |\sin(\pi \frac{n-1}{4} - 2\lambda|b|)| \geq \rho\ ,\ \forall \lambda > \lambda_0\ .$$

We note also that if we have only the property for $\epsilon = 1$ then (2.14) is only interesting for $n = 2$.
Let us define now

$$N^+(\mu; P_V) = \sup_\theta \{ \sum_{\lambda_j(\theta)<\mu} 1 \} \text{ and } N^-(\mu; P_V) = \inf_\theta \{ \sum_{\lambda_j(\theta)<\mu} 1 \}. \tag{2.17}$$

Following an idea of Dahlberg and Trubowitz [DaTr], Theorem 2.1 and (2.16) imply the following corollary

Corollary 2.2 *:*
If $2 \leq n \leq 4$, then there exists a non negative constant $\eta_0 > 0$ such that

$$N^+(\mu; P_V) - N^-(\mu; P_V) \geq \eta_0 \mu^{(n-1)/4},\ \forall \mu > 1. \tag{2.18}$$

The estimate (2.18) is known to give the Bethe-Sommerfeld conjecture,

Corollary 2.3 *:*
If $2 \leq n \leq 4$, P_V has at most a finite number of gaps in its spectrum.

The estimate (2.18) implies indeed the existence of μ_0 such that, for all $\mu \geq \mu_0$, we have

$$N^+(\mu; P_V) - N^-(\mu; P_V) \geq 1 \; . \tag{2.19}$$

This implies immediately the non existence of gaps in $[\mu_0, +\infty[$ and, using that, for any μ_1, there is at most a finite number of gaps in $]-\infty, \mu_1]$, Corollary 2.2 (See [DaTr]).
The Bethe-Sommerfeld Conjecture for bounded potential $V(x)$ was proved for $n = 2$ by Skriganov in 1979 (see [Skr1] and also [DaTr]), for $n = 3$ in [Skr2]. On the other hand, a proof is sketched for general $n \geq 2$ in [Vel]. In his book, Skriganov presents also results for large n but under restrictions on the lattice. In any case our estimates (1.18), (2.13) and (2.18) seem to be new, even when $n = 2$.

3 Periodic pseudo-differential calculus

This section is devoted to the presentation of rather standard material concerning periodic pseudodifferential operators in conjonction with the Floquet theory. A possible reference is the book by P. Kuchment [Ku] or M. Shubin [Sh].

3.1 Some notations

If E and F are two Banach spaces, $\mathcal{L}(E; F)$ will denote the space of bounded linear operators from E to F and $\mathcal{L}(E) := \mathcal{L}(E; E)$.
For any $f \in L^2(I\!R^m)$, $\hat{f}$ denotes the Fourier transform of f

$$\hat{f}(\xi) = \int_{I\!R^m} e^{-ix\xi} f(x)\, dx \; ,$$

and for any $x \in I\!R^m$, $< x >= (|x|^2 + 1)^{\frac{1}{2}}$.

We now introduce our classes of symbol. Let $d > 0$ be an integer and

$$g_z = \sum_{j=1}^{d} q_j^2(z) dz_j^2, \quad (\text{with } q_j(z) > 0), \tag{3.1}$$

a Riemannian diagonal metric on $I\!R^d$.

Let $M(z)$ be a non negative continuous function, $M(z) \in C(I\!R^d; I\!R_+)$ and $p(z) \in C^\infty(I\!R^d)$. We say (see [Hör3]) that

$$p(z) \in S(M(z); g_z),$$

if

$$\forall \alpha \in \mathbf{N}^d, \ \exists C_\alpha > 0, \ s.t. \ |\partial_z^\alpha p(z)| \leq C_\alpha M(z) \prod_{j=1}^{d} q_j^{\alpha_j}(z). \tag{3.2}$$

We will denote by g_z^0 the usual Euclidian metric on $I\!R^d$

$$g_z^0 = \sum_{j=1}^{d} dz_j^2, \tag{3.3}$$

and more generally, for $\delta \in [0, 1]$, by g_z^δ the metric on $I\!R^d$

$$g_z^\delta = < z >^{-2\delta} \left(\sum_{j=1}^{d} dz_j^2 \right). \tag{3.4}$$

Finally, for any real m, $H^m(I\!R^d)$ will denote the usual Sobolev space of order m on $I\!R^d$.

3.2 The usual pseudo-differential algebra

In the following, we recall very shortly some well known results about pseudo-differential operators (see [Hör3]).
For $\delta \in [0, 1]$, we consider on $I\!R^{2n}$ the metric

$$g_{x,\xi}^{0,\delta} := g_x^0 + < \xi >^{-2\delta} g_\xi^0, \quad \forall (x, \xi) \in I\!R^{2n} \ . \tag{3.5}$$

For any real m and any symbol $(x,\xi) \mapsto p(x,\xi)$ in $S(<\xi>^m; g^{0,\delta}_{x,\xi})$, the pseudo-differential operator, (p.d.o.),

$$p(x,D) = p(x,D_x) \in \mathcal{L}(\mathcal{S}(I\!R^n); \mathcal{S}'(I\!R^n)),$$

is defined by

$$p(x,D)u(x) = (2\pi)^{-n} \int_{I\!R^n} e^{ix\xi} p(x,\xi)\hat{u}(\xi) d\xi,\ \forall u \in \mathcal{S}(I\!R^n), \tag{3.6}$$

and can be extended to an operator $p(x,D) : \mathcal{S}'(I\!R^n) \longrightarrow \mathcal{S}'(I\!R^n)$.
We shall write

$$A \in OpS(<\xi>^m; g^{0,\delta}_{x,\xi})$$

if there exists a symbol

$$a(x,\xi) \in S(<\xi>^m; g^{0,\delta}_{x,\xi}),\ \text{s.t. } A = a(x,D). \tag{3.7}$$

It is well known, since [CaVa], that

$$OpS(1; g^{0,\delta}_{x,\xi}) \subset \mathcal{L}(L^2(I\!R^n))\ .$$

If a belongs to $S(<\xi>^r; g^{0,\delta}_{x,\xi})$ and b belongs to $S(<\xi>^s; g^{0,\delta}_{x,\xi})$, then

$$a(x,D)\ b(x,D) = c(x,D) \in OpS(<\xi>^{r+s}; g^{0,\delta}_{x,\xi})\ ,$$

with

$$c(x,\xi) = e^{iD_yD_\eta}(a(x,\eta)b(y,\xi))_{|(y,\eta)=(x,\xi)}\ . \tag{3.8}$$

For $z \in I\!R^m$, D_z denotes $(D_{z_1}, \ldots, D_{z_m})$.

3.3 The periodic pseudo-differential operators

A standard pseudodifferential operator which commutes with the translations of the lattice Γ is called Γ-periodic (or more simply periodic). It is easy to see that such an operator has a periodic symbol p in the sense that

$$p(x+a,\xi) = p(x,\xi),\ \forall a \in \Gamma\ . \tag{3.9}$$

In this case, we can expand the symbol in Fourier series (with ξ as parameter) and get

$$p(x,\xi) = \sum_{\gamma\in\Gamma^\star} e^{ix\gamma}\hat{p}_\gamma(\xi)\ .$$

For a symbol in $S(<\xi>^m; g^{0,\delta}_{x,\xi})$, we get, for any $(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n$, the existence of $C_{j,\alpha} > 0$ s.t.

$$|\partial^\alpha_\xi \hat{p}_\gamma(\xi)| \leq C_{j,\alpha} <\gamma>^{-j}<\xi>^{m-\delta|\alpha|}\ , \quad \forall(\gamma,\xi) \in \Gamma^\star \times \mathbb{R}^n. \tag{3.10}$$

For a periodic symbol p in $S(<\xi>^m; g^{0,\delta}_{x,\xi})$, and for any fixed $\theta \in \mathbb{R}^n$, the p.d.o. $p(x,\theta+D)$, defined by the symbol $p(x,\theta+\xi)$, leads to an operator $p^\theta(x,D)$ from $C^\infty(T^n)$ into itself

$$p^\theta(x,D)v(x) = p(x,\theta+D)\widetilde{v}(x)\ ,$$

where $\widetilde{v}(x)$ is the extension of $v(x)$ on $\mathbb{R}^n$ by Γ-periodicity.
If $v(x) = \sum_{\gamma\in\Gamma^\star} \hat{v}_\gamma e^{ix\gamma} \in C^\infty(T^n)$, then

$$p^\theta(x,D)v(x) = \sum_{\gamma\in\Gamma^\star} p(x,\theta+\gamma)\hat{v}_\gamma e^{ix\gamma} = \sum_{\gamma\in\Gamma^\star} e^{ix\gamma} \sum_{\varrho\in\Gamma^\star} \hat{v}_\varrho \hat{p}_{\gamma-\varrho}(\theta+\varrho). \tag{3.11}$$

The operator $p^\theta(x,D)$ can be extended to an operator

$$p^\theta(x,D) : (C^\infty(T^n))' \longrightarrow (C^\infty(T^n))'$$

and $p^\theta(x,D)$ is a p.d.o. of order m on the compact manifold T^n in the sense of [Hör3]. In particular, it is continuous in the standard Sobolev spaces

$$p^\theta(x,D) : H^s(T^n) \longrightarrow H^{s-m}(T^n), \quad \forall s \in \mathbb{R}.$$

If $q(x,\xi)$ is an other periodic symbol,

$$q(x,\xi) = \sum_{\gamma\in\Gamma^\star} \hat{q}_\gamma(\xi)e^{ix\gamma} \in S(<\xi>^l; g^{0,\delta}_{x,\xi}),$$

then the equality $s(x,D) = p(x,D)q(x,D)$ is equivalent to the property that

$$s^\theta(x,D) = p^\theta(x,D)q^\theta(x,D),\ \forall\theta \in \mathcal{K}^\star\ . \tag{3.12}$$

and we have the following identities for its Fourier coefficients symbol

$$s(x,\xi) = \sum_{\gamma\in\Gamma^\star} \hat{s}_\gamma(\xi)e^{ix\gamma},\ \hat{s}_\gamma(\xi) = \sum_{\varrho\in\Gamma^\star} \hat{p}_{\gamma-\varrho}(\xi+\varrho)\hat{q}_\varrho(\xi). \tag{3.13}$$

For the proof of (3.13) , we write $s(x,\xi)$ as an oscillatory integral

$$s(x,\xi) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{-iz\eta} p(x,\xi+\eta)q(x+z,\xi)dzd\eta,$$

and then one gets

$$s(x,\xi) = (2\pi)^{-n} \int_{I\!R^{2n}} \sum_{\varrho \in \Gamma^\star} e^{-iz(\eta-\varrho)} p(x,\xi+\eta) e^{ix\varrho} \hat{q}_\varrho(\xi) dz d\eta,$$

so

$$s(x,\xi) = \sum_{\varrho \in \Gamma^\star} p(x,\xi+\varrho) e^{ix\varrho} \hat{q}_\varrho(\xi)\ .$$

In order to fix some notations, let us recall some elements of the Floquet theory for p.d.o.. The main point is the existence of the Floquet operator,

$$\Phi :\ L^2(I\!R^n) \longmapsto L^2(\mathcal{K}^\star; L^2(\boldsymbol{T}^n))$$

$$\Phi u(x,\theta) = \sum_{c \in \Gamma} e^{-i(x-c)\theta} u(x-c), \quad \forall u \in L^2(I\!R^n), \tag{3.14}$$

whose adjoint $\Phi^\star = \Phi^{-1}$ is, for $v \in L^2(\mathcal{K}^\star; L^2(\boldsymbol{T}^n))$, defined by

$$\Phi^\star v(x) = |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star} e^{ix\theta} \tilde{v}(x,\theta) d\theta\ , \tag{3.15}$$

where $\tilde{v}$ is defined in $L^2(\mathcal{K}^\star; L^2_{loc}(I\!R^n))$ as the Γ-periodic extension (with respect to $x \in I\!R^n$) of v defined on $\mathcal{K} \times \mathcal{K}^\star$. The map Φ sends in particular the Sobolev space $H^k(I\!R^n)$ onto $L^2(\mathcal{K}^\star; H^k(\boldsymbol{T}^n))$.
If T is a Γ-periodic operator, then $\Phi \cdot T \cdot \Phi^\star$ has a nice decomposition as an Hilbertian integral over $\mathcal{K}^\star$ of operators A_θ (not necessarily bounded) on $L^2(\boldsymbol{T}^n)$. In particular, in the pseudodifferential context, we obtain

$$\Phi(\sigma(x,D)u)(x,\theta) = \sigma^\theta(x,D_x)(\Phi u(x,\theta)), \quad \forall u \in \mathbf{S}(I\!R^n)\ . \tag{3.16}$$

This can be rewritten (cf [Ku]) in the statement:

Theorem 3.1 *:*
Let $T = \sigma(x,D)$ be a periodic p.d.o. of order m, then

$$T = |\mathcal{K}^\star|^{-1} \int_{\mathcal{K}^\star}^{\oplus} T^\theta d\theta, \quad \text{with} \quad T^\theta := \sigma^\theta(x,D)\ . \tag{3.17}$$

(See [ReSi] for the definition of direct integral operators).

We shall also need the following result, in order to recover, from the continuity of the operator in L^2, an information on the Fourier coefficients of its symbol,

Proposition 3.2 *:*
If $\sigma(x,D)$ is a pseudodifferential of order 0*, then,* $\forall \xi \in I\!R^n$,

$$\sum_{\gamma\in\Gamma^\star} |\hat{\sigma}_\gamma(\xi)|^2 \leq \| \sigma(x,D) \|^2_{\mathcal{L}(L^2(I\!R^n))} . \tag{3.18}$$

Proof:
Let $\xi = \varrho + \theta \in I\!R^n$, with $\varrho \in \Gamma^\star$ and $\theta \in \mathcal{K}^\star$.
For $\gamma \in \Gamma^\star$, we get by (3.11)

$$\hat{\sigma}_\gamma(\xi) = |\boldsymbol{T}^n|^{-1} < \sigma^\theta(x,D)(e^{ix\varrho})|e^{ix(\gamma+\varrho)} >, \tag{3.19}$$

($<\ .\ |\ .\ >$ is the $L^2(T^n)$ scalar product.)
The Plancherel formula gives

$$\sum_\gamma |\hat{\sigma}_\gamma(\xi)|^2 = \frac{1}{|\boldsymbol{T}^n|}||\sigma^\theta(x,D)(e^{ix\varrho})||^2 \leq ||\sigma^\theta(x,D)||^2_{\mathcal{L}(L^2(\boldsymbol{T}^n))} .$$

Now, one obtains by a simple spectral analysis

$$\sup_\theta \ \| \sigma^\theta(x,D) \|_{\mathcal{L}(L^2(\boldsymbol{T}^n))} = \| \sigma(x,D) \|_{\mathcal{L}(L^2(I\!R^n))} . \tag{3.20}$$

Throughout the following sections, we will denote $g^{0,1}_{x,\xi}$ by $g_{x,\xi}$:

$$g_{x,\xi} := g^{0,1}_{x,\xi}. \tag{3.21}$$

Finally, it is worth to mention that in many cases the considered pseudodifferential operators will depend on other parameters (for example t) and it will be crucial to control the constants involved in the definitions of the symbols with respect to these parameters.

4 The free half wave unitary group

This section is devoted to the analysis of a global version of the so-called Egorov's theorem. Although the results are probably well known, we do not know about a reference where the estimates we will need later are established and we prefer consequently to give a proof of these results. If $c(x,D)$ is a standard pseudodifferential operator, it is well known that the operator

$\sigma(t,x,D) := e^{itQ_0} c(x,D) e^{-itQ_0}$ is for any t a standard pseudo-differential operator, which is periodic if $c(x,D)$ is periodic. Moreover, if c_0 is the principal symbol, it is well known, from the standard Fourier integral calculus, that the principal symbol $(x,\xi) \mapsto \sigma_0(t,x,\xi)$ is given by

$$\sigma_0(t,x,\xi) = c_0(x + t\frac{\xi}{|\xi|}, \xi) \ , \tag{4.1}$$

and this leads to natural estimates on this symbol where the constants are now well controlled with respect to t. The aim of this section is to establish similar results for the total symbol of $\sigma(t,x,D)$. More precisely, we shall establish

Theorem 4.1 *:*
Let $c(x,D)$ be a periodic pseudodifferential operator in $OpS(\langle\xi\rangle^m; g^{0,1}_{x,\xi})$, then the symbol $(t,x,\xi) \mapsto \sigma(t,x,\xi)$ of the pseudodifferential operator $\sigma(t,x,D) := e^{itQ_0} c(x,D) e^{-itQ_0}$ satisfies the conditions

$$|\partial_t^k \partial_\xi^\alpha \partial_x^\beta \sigma(t,x,\xi)| \leq C_{k,\alpha,\beta} \langle t\rangle^{|\alpha|} \ \langle\xi\rangle^{m-|\alpha|} \tag{4.2}$$

Proof of Theorem 4.1:

Let us denote by

$$q^0(\xi) = \sqrt{\xi^2 + \lambda_0} \ ,$$

the symbol of Q_0.
Our starting point is the formula relating the Fourier coefficients of $c(\cdot,\xi)$ and of $\sigma(t,\cdot,\xi)$.

$$\hat{\sigma}_\gamma(t,\xi) = e^{it\phi_\gamma(\xi)} \hat{c}_\gamma(\xi) \ , \tag{4.3}$$

where ϕ_γ is given by

$$\begin{aligned} \phi_\gamma(\xi) \ &= q^0(\xi+\gamma) - q^0(\xi) \\ &= (2\gamma\xi + |\gamma|^2)(q^0(\xi+\gamma) + q^0(\xi))^{-1} \ . \end{aligned} \tag{4.4}$$

It is easy to verify that, for any $\alpha \in I\!N^n$, there exists a constant C_α such that, for any $\gamma \in \Gamma^\star$, we have,

$$|\partial_\xi^\alpha \phi_\gamma(\xi)| \leq C_\alpha \frac{\langle\gamma\rangle^{|\alpha|+1}}{\langle\xi\rangle^{|\alpha|}} \ . \tag{4.5}$$

We recall also that the assumption on c implies, that for all N, we have

$$|\partial_\xi^\alpha \hat{c}_\gamma(t,\xi)| \leq C_{\alpha,N} \langle \xi \rangle^{m-|\alpha|} \langle \gamma \rangle^{-N} \; . \tag{4.6}$$

¿From the three relations (4.3), (4.5) and (4.6), we obtain

$$|\partial_t^k \partial_\xi^\alpha \hat{\sigma}_\gamma(t,\xi)| \leq C_{k,\alpha} \langle t \rangle^{|\alpha|} \frac{\langle \gamma \rangle^{|\alpha|+1}}{\langle \xi \rangle^{|\alpha|}} [\sum_{|\beta| \leq |\alpha|} \langle \xi \rangle^{|\beta|} |\partial_\xi^\beta \hat{c}_\gamma(\xi)|] \; ,$$

and this leads to the estimate, which is satisfied for any $N \in I\!N$,

$$|\partial_t^k \partial_\xi^\alpha \hat{\sigma}_\gamma(t,\xi)| \leq C_{k,\alpha,N} \langle \xi \rangle^{m-|\alpha|} \langle t \rangle^{|\alpha|} \langle \gamma \rangle^{-N} \; . \tag{4.7}$$

Coming back to

$$\sigma(t,x,\xi) = \sum_\gamma e^{i\gamma \cdot x} \hat{\sigma}_\gamma(t,\xi) \; ,$$

we immediately get (4.2) and the theorem is proved.

Remark 4.2 *:*
We get the same conclusion if we start from a t-dependent periodic pseudo-differential operator $c(t,x,D_x)$ whose symbol satisfies the weaker (4.2).

Remark 4.3 *:*
In the next section, we shall analyze the more general situation when we met $W_C(t) := e^{itQ} c(x,D) e^{-itQ_0}$ and we shall get similar but weaker estimates. For this study, it could be helpful to observe that

$$W_C(t) := \left(e^{itQ}\, e^{-itQ_0} \right) \, \left(e^{itQ_0} c(x,D) e^{-itQ_0} \right) \; . \tag{4.8}$$

In particular, it could make easy the study of t-dependent $C = c(t,x,D_x)$ by reducing it, using the preceding remark, immediately to the case of $C = Id$.

5 The half wave unitary group

Let Q the operator defined by (1.21) and (1.22). Then, by [See] (see also [BoCh]), Q is a pseudodifferential operator in $OpS(< \xi >; g_{x,\xi})$. As already observed, Q commutes with the translations of the lattice, and Q is consequently a periodic p.d.o. whose symbol is

$$q(x,\xi) = \sum_{\gamma\in\Gamma^\star} \hat{q}_\gamma(\xi) e^{ix\gamma} \in S(<\xi>; g_{x,\xi}). \tag{5.1}$$

The Seeley calculus [See], (see also [Hör3]), shows that, if

$$q^0(\xi) = (|\xi|^2 + \lambda_0)^{\frac{1}{2}} , \tag{5.2}$$

then there exists an expansion[1] of q

$$q(x,\xi) \sim q^0(\xi) + \sum_{j>0} q_{-j}(x,\xi) , \tag{5.3}$$

in the sense that

$$q_{-j}(x,\xi) \in S(<\xi>^{-j}; g_{x,\xi}),\ \forall j > 0 , \tag{5.4}$$

and that, for any $N > 0$,

$$q(x,\xi) - q^0(\xi) - \sum_{j=1}^{N-1} q_{-j}(x,\xi) \in S(<\xi>^{-N}; g_{x,\xi}); \tag{5.5}$$

The computation can be done rather explicitly and one finds

$$q_{-1}(x,\xi) = \frac{1}{2} V(x)(q^0(\xi))^{-1} , \tag{5.6}$$

$$q_{-2}(x,\xi) = \frac{i}{4} \xi \cdot \partial_x V(x)(q^0(\xi))^{-3} \tag{5.7}$$

and

$$\begin{aligned} q_{-3}(x,\xi) &= \tfrac{1}{8}(q^0(\xi))^{-3}(\Delta V(x) - V^2(x)) \\ &\quad -\tfrac{1}{4}(q^0(\xi))^{-5} \textstyle\sum_{j=1}^n \sum_{k=1}^n \xi_j\xi_k \partial_{x_k}\partial_{x_j} V(x) . \end{aligned} \tag{5.8}$$

[1] This expansion can be done more canonically by introducing artificially a semi-classical parameter h and constructing the functional calculus of [HeRo] for $P(h) = -h^2\Delta + \lambda_0 + h^2 V(x)$. The coefficients of h^j of the semiclassical expansion of $P(h)^{\frac{1}{2}}$ give a more canonical expansion. In particular, it can be observed that the q_j are alternately odd and even with respect to $\xi \mapsto -\xi$.

The main point we will need is that (1.24) implies that

$$\int_{\mathcal{K}} q_{-1}(x,\xi)dx = 0 \;, \tag{5.9}$$

or, if we denote by $\hat{q}_{-j,\gamma}(\xi)$ the γ- Fourier coefficient of $x \mapsto q_{-j}(x,\xi)$,

$$\hat{q}_{-1,0}(\xi) = 0 \;. \tag{5.10}$$

Theorem 5.1 *:*
Under the assumptions of Theorem 2.1, let $c(x,D)$ be a given periodic p.d.o.with symbol

$$c(x,\xi) = \sum_{\gamma\in\Gamma^\star} c_\gamma(\xi)e^{ix\gamma} \;\in S(<\xi>^{-m}; g_{x,\xi}), \quad \text{with } m \geq 0. \tag{5.11}$$

Then for any real t, $W(t) := e^{itQ}c(x,D)e^{-itQ_0}$ is a periodic p.d.o.whose symbol $(t,x,\xi)\mapsto w(t,x,\xi)$ satisfies locally uniformly with respect to t

$$w(t,x,\xi) \in S(<\xi>^{-m}; dt^2 + g_{x,\xi}) \;. \tag{5.12}$$

Proof of Theorem 5.1:
It is well known by the standard Fourier Integral calculus that $W(t)$ is a p.d.o. and, since Q and Q_0 are periodic, the operator $W(t)$ is also periodic. It is also not difficult by using the standard pseudo-differential calculus to analyze the differentiations with respect to t.

Another proof of this theorem could be based on the theorem of characterization of p.d.o. of Beals [Bea] (see also [BoCh]). This approach may be useful in order to control with respect to $|t|$ large the behavior of w as a symbol.
As a preparation, we shall now prove with this spirit a rather weak but useful result in this direction. A more difficult result will be proved in the next section.
As starting point let us recall the basic theorem by R. Beals.

Theorem 5.2 *(R. Beals Theorem):*
Let $R \in \mathcal{L}(\mathcal{S}(I\!R^n), \mathcal{S}'(I\!R^n))$ such that, for any non negative integer j, the operator

$$(-\Delta+1)^{\frac{N(j)}{2}} ad_{L_j} \circ ad_{L_{j-1}} \circ \ldots \circ ad_{L_1}(R) \tag{5.13}$$

with $L_k = D_{x_{l(k)}}$ or $L_k = x_{l(k)}$, can be extended as an operator in $\mathcal{L}(L^2(I\!\!R^n))$. If $N(j)$ in (5.13) is the number of k, $1 \leq k \leq j$, such that $L_k = x_{l(k)}$, then

$$R = r(x,D) \in OpS(1; g_{x,\xi}). \tag{5.14}$$

If in (5.13), $N(j) = 0, \quad \forall j$, then

$$R = r(x,D) \in OpS(1; g^0_{x,\xi}) \ . \tag{5.15}$$

Our weak result is the following

Theorem 5.3 *:*
The operator $W(t)$ is a periodic pseudo-differential operator whose symbol $I\!\!R_t \times I\!\!R^n_x \times I\!\!R^n \ni (t,x,\xi) \mapsto w(t,x,\xi)$ satisfies

$$w \in S(<\xi>^{-m}<t>^{[\frac{n}{2}]+1}; dt^2 + <t>^2 g^0_{x,\xi}) \ . \tag{5.16}$$

Remark 5.4 *:*
This theorem is weak in the sense that the control of the ξ-derivatives is obtained for the metric g^0_ξ.

As a preliminary observation, we recall the immediate property that, for any $t \in I\!\!R$,

$$e^{itQ} \in \mathcal{L}(H^s(I\!\!R^n)) \ , \ e^{itQ_0} \in \mathcal{L}(H^s(I\!\!R^n)) \ , \tag{5.17}$$

and that their norms are bounded independently of t. This is immediate, if we recall that Q^k (or Q_0^k) are isomorphisms from $H^k(I\!\!R^n)$ onto $L^2(I\!\!R^n)$.
For operators L, M and T, let us introduce the two following brackets

$$ad_L(T) = [L,T] = LT - TL \ , \tag{5.18}$$

and

$$ad_{L,M}(T) = LT - TM \ . \tag{5.19}$$

With these notations, we have in particular

$$ad_{Q,Q_0}(Id) = Q - Q_0 \ . \tag{5.20}$$

For $k \in I\!\!N$, we immediately get

$$W^{(k)}(t) = (\frac{d}{dt})^k W(t) = i^k e^{itQ} ad^k_{Q,Q_0}(c(x,D)) e^{-itQ_0} \ . \tag{5.21}$$

We now observe that

$$ad^k_{Q,Q_0}(c(x,D)) \in OpS(<\xi>^{-m}; g_{x,\xi}) \ . \tag{5.22}$$

For any pseudodifferential operators $p(x,D) \in OpS(<\xi>^{-m}; g_{x,\xi})$, we observe indeed that

$$ad_{Q,Q_0}(p(x,D)) = Q\ p(x,D) - p(x,D)\ Q_0 = (Q - Q_0)\ p(x,D) + [Q_0, p(x,D)] \ . \tag{5.23}$$

Observing [2] that $(Q - Q_0) \in OpS(<\xi>^{-1}; g_{x,\xi})$, we immediately obtain, using the standard p.d.o. calculus, the property that

$$ad_{Q,Q_0}(p(x,D)) \in OpS(<\xi>^{-m}; g_{x,\xi})$$

and (5.22) for general k is obtained by recursion.
Coming back to (5.21), we get from (5.22) the existence of a constant $C_k > 0$ such that

$$\| W^{(k)}(t) \|_{\mathcal{L}(L^2(I\!R^n))} \leq C_k, \quad \forall t \in I\!R. \tag{5.24}$$

More generally, for any real s, there exists a constant $C_{k,s} > 0$ such that

$$\| W^{(k)}(t) \|_{\mathcal{L}(H^s(I\!R^n); H^{m+s}(I\!R^n))} \leq C_{k,s}, \quad \forall t \in I\!R. \tag{5.25}$$

This estimate is immediate (according to the ellipticity properties of Q and Q_0) if we observe the relation

$$Q^{m+s} W^{(k)}(t) Q_0^{-s} = i^k e^{itQ} Q^{m+s} ad^k_{Q,Q_0}(c(x,D)) Q_0^{-s} e^{-itQ_0} \ .$$

We can then apply the standard p.d.o calculus to obtain that $Q^{m+s} ad^k_{Q,Q_0}(c(x,D)) Q_0^{-s}$ belongs to $\mathcal{L}(L^2(I\!R^n))$ and that consequently the operator $Q^{m+s} W^{(k)}(t) Q_0^{-s}$ is bounded in $\mathcal{L}(L^2(I\!R^n))$ uniformly with respect to $t \in I\!R$. This is equivalent to (5.25). So we have proved the case $\alpha = \beta = 0$ of the following Lemma:

Lemma 5.5 *:*
If $W^{(k)}(t) = w_t^{(k)}(x, D_x)$, then, for any $(\alpha, \beta) \in I\!N^{2n}$, any $s \in I\!R$, there exists $C_{\alpha,\beta,s} > 0$ such that

$$\| w^{(k),\alpha}_{t,\beta}(x, D_x) \|_{\mathcal{L}(H^s(I\!R^n); H^{m+s}(I\!R^n))} \leq C_{\alpha,\beta,s} \langle t \rangle^{|\alpha|+|\beta|}, \quad \forall t \in I\!R \ , \tag{5.26}$$

where

$$w^{(k),\alpha}_{t,\beta}(x,\xi) = \partial_\xi^\alpha \partial_x^\beta w_t^{(k)}(x,\xi) = \partial_t^k \partial_\xi^\alpha \partial_x^\beta w(t,x,\xi) \ .$$

[2] We actually only need that $(Q - Q_0) \in OpS(1; g_{x,\xi})$

Proof of Lemma 5.5 :
We denote by C the operator $c(x,D)$. Our starting point is that $W = W_C$ is a solution of the differential equation

$$W'(t) = iQW - iWQ_0 = ie^{itQ}\, ad_{Q,Q_0} C\, e^{-itQ_0} \; . \tag{5.27}$$

For general k, we get

$$W^{(k)}(t) = i^k e^{itQ}\, ad^k_{Q,Q_0} C\, e^{-itQ_0} = W_{C^k}(t) \; . \tag{5.28}$$

So $W^{(k)}(t)$ satisfies a similar equation to W with W_C replaced by W_{C^k} where $C^k := i^k ad^k_{Q,Q_0} C$.
Let us now consider the bracket $[L, W_C(t)]$ where L is an operator in $\mathcal{L}(\mathcal{S})$ and let us consider its derivative with respect to t. We get

$$\frac{d}{dt}[L, W_C(t)] = iad_{Q,Q_0}(ad_L(W_C(t)) + i[ad_L(Q)W_C(t) - W_C(t)ad_L(Q_0)] \; . \tag{5.29}$$

Replacing C by $C^{(k)}$, we get

$$\frac{d}{dt}(ad_L(W^{(k)}(t)) = iad_{Q,Q_0}(ad_L(W^{(k)}(t)) + i[ad_L(Q)W^{(k)}(t) - W^{(k)}(t)ad_L(Q_0)]. \tag{5.30}$$

The solution of this differential equation can be written as

$$\begin{aligned} &ad_L(W^{(k)}(t)) \\ &\quad = e^{itQ} ad_L(W^{(k)}(0)) e^{-itQ_0} \\ &\qquad + ie^{itQ}\{\textstyle\int_0^t e^{-i\tau Q}[ad_L(Q)W^{(k)}(\tau) - W^{(k)}(\tau)ad_L(Q_0)]e^{i\tau Q_0} d\tau\} e^{-itQ_0} \; . \end{aligned} \tag{5.31}$$

We now recall that, if $p(x,D)$ is a standard pseudo-differential operator, then

$$\begin{array}{ll} \sigma(ad_{D_{x_j}} p(x,D)) & = -i(\partial_{x_j} p) \\ \sigma(ad_{x_j} p(x,D)) & = i(\partial_{\xi_j} p) \; , \end{array} \tag{5.32}$$

where, for a given pseudo-differential operator $A = a(x,D)$, the symbol of A is denoted by $\sigma(A) := a$.
Then, keeping in mind (5.2) and (5.5), we have

$$ad_{D_{x_j}}(Q_0) = 0$$

and

$$ad_{D_{x_j}}(Q) = ad_{D_{x_j}}(Q - Q_0) \in OpS(< \xi >^{-1}; g_{x,\xi}) \ .$$

In particular this implies that, for all $s \in I\!R$,

$$ad_{D_{x_j}}(Q) \in \mathcal{L}(H^s(I\!R^n), H^{s+1}(I\!R^n)) \ . \tag{5.33}$$

We get from (5.31) that

$$\| \ ad_{D_{x_j}}(W^{(k)}(t)) \ \|_{\mathcal{L}(H^s(I\!R^n);H^{m+1+s}(I\!R^n))} \leq C_{k,s}\langle t\rangle \ , \tag{5.34}$$

for some constant $C_{k,s} > 0$ depending only on k and the real s.
In the same way, we observe that $ad_{x_j}(Q_0)$ and $ad_{x_j}(Q)$ belong to $OpS(1; g_{x,\xi})$.
This gives in particular that for all $s \in I\!R$

$$ad_{x_j}(Q) \in \mathcal{L}(H^s(I\!R^n), H^s(I\!R^n)) \ , ad_{x_j}(Q_0) \in \mathcal{L}(H^s(I\!R^n), H^s(I\!R^n)) \ . \tag{5.35}$$

We consequently get,

$$\| \ ad_{x_j}(W^{(k)}(t)) \ \|_{\mathcal{L}(H^s(I\!R^n);H^{m+s}(I\!R^n))} \leq C_{k,s}\langle t\rangle \ . \tag{5.36}$$

We establish first an abstract commutator formula.
If A is an operator in $\mathcal{L}(\mathcal{S})$ and if $t \mapsto S(t)$ is an operator $\mathcal{L}(\mathcal{S})$ depending smoothly on $t \in I\!R$, then we get [3] the formula

$$\begin{array}{ll} e^{-itQ} \left[ad_A(e^{itQ}S(t)e^{-itQ_0})\right] e^{itQ_0} & \\ = \quad ad_A(S(0)) & \\ \quad +i \left(\int_0^t e^{-i\tau Q}(ad_A Q) \, e^{i\tau Q} S(\tau) \ d\tau\right) & \\ \quad -i \left(\int_0^t S(\tau) \ e^{-i\tau Q_0} \ ad_A(Q_0) \ e^{i\tau Q_0} \ d\tau\right) & \\ \quad + \left(\int_0^t [e^{-i\tau Q} \ ad_A(e^{i\tau Q} \cdot S'(\tau) \cdot e^{-i\tau Q_0}) \ e^{i\tau Q_0}] d\tau\right) \ . & \end{array} \tag{5.37}$$

When

$$ad_A(Q_0) = 0 \ ,$$

[3] An easy way to verify this formula is to differentiate with respect to t

$$e^{-itQ} \left(ad_A(e^{itQ}S(t)e^{-itQ_0})\right) e^{itQ_0} \ .$$

we shall also use this formula in the form

$$\begin{array}{l} e^{-itQ}\left[ad_A(e^{itQ}S(t)e^{-itQ_0})\right] e^{itQ_0} \\ = \quad ad_A(S(0)) \\ \quad +i\int_0^t e^{-i\tau Q}(ad_A(Q-Q_0))\, e^{i\tau Q}S(\tau)\, d\tau \\ \quad + \int_0^t [e^{-i\tau Q}\ ad_A(e^{i\tau Q}\cdot S'(\tau)\cdot e^{-i\tau Q_0})\ e^{i\tau Q_0}]d\tau\ . \end{array} \tag{5.38}$$

We now try to control, the iterated commutator $ad_{L_j}\circ\ldots\circ ad_{L_1}(W^{(k)}(t))$ where $L_1,\ldots L_j$ are as in Beals Theorem 5.2.
Let us assume that by recursion, we have proved that, for any real s, for any $j\leq j_0$ and for any k, there exists $C_{j,s,k}>0$ such that, for all $t\in I\!\!R$

$$\| \, ad_{L_j}\circ\ldots\circ ad_{L_1}(W^{(k)}(t))\, \|_{\mathcal{L}(H^s(I\!\!R^n);H^{m+s}(I\!\!R^n))}\leq C_{j,s,k}\langle t\rangle^j\ , \tag{5.39}$$

where m is the order of the initial C.

Then let us prove that this is true for $j=j_0+1$.

We apply formula (5.37) with

$$S(t)=e^{-itQ}\left[ad_{L_{j_0}}\circ\ldots\circ ad_{L_1}(W^{(k)}(t))\right]\ e^{itQ_0} \tag{5.40}$$

and $A=L_j$ and get

$$\begin{array}{l} e^{-itQ}\left[ad_{L_j}\circ\ldots\circ ad_{L_1}(W^{(k)}(t))\right] e^{itQ_0} \\ \quad = ad_{L_j}\circ\ldots\circ ad_{L_1}(W^{(k)}(0)) \\ \qquad +i\left(\int_0^t e^{-i\tau Q}(ad_{L_j}Q)\ ad_{L_{j_0}}\circ\ldots\circ ad_{L_1}(W^{(k)}(\tau))\, e^{i\tau Q_0}d\tau\right) \\ \qquad -i\left(\int_0^t e^{-i\tau Q}ad_{L_{j_0}}\circ\ldots\circ ad_{L_1}(W^{(k)}(\tau))\ ad_{L_j}(Q_0)\ e^{i\tau Q_0}\ d\tau\right) \\ \quad +\ \left(\int_0^t [e^{-i\tau Q}ad_{L_j}(e^{i\tau Q}S'(\tau)e^{-i\tau Q_0})e^{i\tau Q_0}]d\tau\right)\ . \end{array} \tag{5.41}$$

The two first terms involve only brackets of order j_0 and we recall that $ad_{L_j}Q$ and $ad_{L_j}Q_0$ belong to $\mathcal{L}(H^s(I\!\!R^n))$.
The only difficult term is the last one. Here we observe that:

$$\begin{array}{l} e^{i\tau Q}S'(\tau)e^{-i\tau Q_0} \\ \quad = -i ad_{Q,Q_0}\left(ad_{L_{j_0}}\circ\ldots\circ ad_{L_1}(W^{(k)}(\tau))\right) \\ \qquad +\left(ad_{L_{j_0}}\circ\ldots\circ ad_{L_1}(W^{(k+1)}(\tau))\right). \end{array}$$

But $W^{(k+1)}$ satisfies

$$W^{(k+1)} = iQW^{(k)} - iW^{(k)}Q_0 \; ,$$

and we get

$$\begin{aligned} & e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} \\ & \quad = -iQ \left(ad_{L_{j_0}} \circ \ldots \circ ad_{L_1}(W^{(k)}(\tau)) \right) + i \left(ad_{L_{j_0}} \circ \ldots \circ ad_{L_1}(QW^{(k)}(\tau)) \right) \\ & \qquad + i \left(ad_{L_{j_0}} \circ \ldots \circ ad_{L_1}(W^{(k)}(\tau)) \right) Q_0 - i \left(ad_{L_{j_0}} \circ \ldots \circ ad_{L_1}(W^{(k)}(\tau) Q_0) \right) \; . \end{aligned}$$

Iterating the abstract formula

$$ad_L(UV) = (ad_L U)V + U(ad_L V) \; , \tag{5.42}$$

we immediately see that

$$\begin{aligned} & e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} \\ & \quad = i \textstyle\sum_{|\alpha|+|\beta|=j_0, |\alpha|\neq 0} c_{\alpha,\beta} \left((ad_X)^\alpha Q \right) \left((ad_X)^\beta W^{(k)} \right) \\ & \qquad - i \textstyle\sum_{|\alpha|+|\beta|=j_0, |\beta|\neq 0} c_{\alpha,\beta} \left((ad_X)^\alpha W^{(k)} \right) \left((ad_X)^\beta Q_0 \right) \; . \end{aligned} \tag{5.43}$$

Here $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$, with $k = |\alpha|$ and $\alpha_\ell \in [1, \cdots, 2n]$. $X_\ell = x_\ell$ for $\ell = 1, \cdots, n$ and $X_\ell = D_{x_{\ell-n}}$ for $\ell = n+1, \cdots, 2n$. A similar convention is made for β. Finally $c_{\alpha,\beta}$ is a constant.
What is important here is that the number of brackets acting on $W^{(k)}$ is of order $j_0 - 1$.
We then obtain, using (5.42) and (5.43), that

$$\begin{aligned} ad_{L_j} \left(e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} \right) & = i \textstyle\sum_{|\alpha|+|\beta|=j_0+1, |\alpha|\neq 0} \tilde{c}_{\alpha,\beta} \left((ad_X)^\alpha Q \right) \left((ad_X)^\beta W^{(k)} \right) \\ & \quad - i \textstyle\sum_{|\alpha|+|\beta|=j_0+1, |\beta|\neq 0} \tilde{c}_{\alpha,\beta} \left((ad_X)^\alpha W^{(k)} \right) \left((ad_X)^\beta Q_0 \right) \; . \end{aligned}$$

The last point in order to conclude is to observe that the operators $(ad_X)^\alpha Q$ and $(ad_X)^\alpha Q_0$ are well controlled as operators in $\mathcal{L}(H^s)$ as it is clear from their pseudodifferential character.
The recursion assumption is then satisfied to the rank $j_0 + 1$.
The lemma is proved.

Theorem 5.3 folows taking into account the Beals Theorem, Proposition 3.2 and the Sobolev inbedding

$$H^{[\frac{n}{2}]+1}(\boldsymbol{T}^n) \subset C^0(\boldsymbol{T}^n) \; .$$

6 Sharp estimates with large $|t|$ control

The estimate (5.12) of Theorem 5.1 has to be improved in order to obtain our optimal results. The next theorem will be decisive for the control of the remainders (where the large $|t|$ behavior is important) for example in Theorem 8.10.

Theorem 6.1 *:*
Let $w(t,x,\xi) = \sum_{\gamma \in \Gamma^\star} \hat{w}_\gamma(t,\xi)\ e^{ix\gamma}$ *be the symbol of* $W(t) := e^{itQ} e^{-itQ_0}$. *Then, for every* $\alpha \in I\!N^n$, $|\alpha| > 0$, *there exists a constant* $C_\alpha > 0$ *such that,*

$$(\sum_{\gamma \in \Gamma^\star} |\partial_\xi^\alpha \hat{w}_\gamma(t,\xi)|^2)^{1/2} \le C_\alpha (< t >< \xi >^{-1})^{|\alpha|+1} (1+ < t >< \xi >^{-1})^{|\alpha|-1}, \tag{6.1}$$

for all $(t,\xi) \in I\!R \times I\!R^n$.

<u>Proof</u>

We keep the spirit of Beals and will consequently analyze commutators. This means that we shall control the norms of $ad_{L_j} \circ \ldots \circ ad_{L_1}(W(t))$ (with $L_d = x_{k(d)}$) in $\mathcal{L}(H^s, H^{s_j})$ for suitables s_j and with a uniformly local control with respect to t.
We shall then conclude with the help of Proposition 3.2.
More precisely we shall have to prove that $ad_{L_j} \circ \ldots \circ ad_{L_1}(W(t))$ can be decomposed as a sum of terms $I_{j;k}$, each of them satisfying, for some $q = 0, \cdots, j$, the property, that, for all $s \in I\!R$, there exists C_s such that

$$||I_{j;k}||_{\mathcal{L}(H^s, H^{s+j+q})} \le C_s \cdot \langle t \rangle^{j+q} . \tag{6.2}$$

We need for this two technical lemmas giving a decomposition formula respectively for the commutators with a pseudodifferential operator independent of x and with the differential operator x_k. The first decomposition will permit also to treat the commutator with $M_d = D_{x_{j(d)}}$. The second decomposition will depends heavily on the first one.

Lemma 6.2 *:*
Let $A = s_0(D) \in OpS(1; g_{x,\xi})$. *Let* $j > 0$ *be an integer and, for* $\ell = 0, 1 \ldots, j$,

let $a_\ell(x,D)$ *be periodic p.d.o. with symbol*

$$a_\ell(x,\xi) \in S(<\xi>^{-m_\ell}; g_{x,\xi}), \quad \text{with} \quad m_\ell \geq 0.$$

If

$$\begin{aligned} S(t) \; &= \textstyle\int_0^t e^{-i\tau_j Q} a_j(x,D) e^{i\tau_j Q} \int_0^{\tau_j} e^{-i\tau_{j-1} Q} a_{j-1}(x,D) e^{i\tau_{j-1} Q} \dots \\ &\quad \dots \textstyle\int_0^{\tau_2} e^{-i\tau_1 Q} a_1(x,D) e^{i\tau_1 Q} a_0(x,D)\, d\tau_1 \dots d\tau_j \ , \end{aligned} \tag{6.3}$$

then $e^{-itQ} ad_L(e^{itQ} S(t) e^{-itQ_0}) e^{itQ_0}$ *may be written as a finite sum of terms of two different types 1 and 2 satisfying the following conditions.*

- *Type 1*
  For a term of type 1, generically denoted by I*, there exists* $j+1$ *periodic p.d.o.* $b_\ell(x,D)$ $(\ell = 0,1,\dots,j)$*, with*

  $$b_\ell(x,\xi) \in S(<\xi>^{-p_\ell}; g_{x,\xi}) \ , \ p_\ell \geq 0 \ ,$$

  such that

  $$\begin{aligned} I = \textstyle\int_0^t e^{-i\tau_j Q} b_j(x,D) e^{i\tau_j Q} \int_0^{\tau_j} e^{-i\tau_{j-1} Q} b_{j-1}(x,D) e^{i\tau_{j-1} Q} \dots \\ \dots \textstyle\int_0^{\tau_2} e^{-i\tau_1 Q} b_1(x,D) e^{i\tau_1 Q} b_0(x,D) d\tau_1 \dots d\tau_j \ . \end{aligned} \tag{6.4}$$

  and

  $$1 + \sum_{\ell=0}^{j} m_\ell \leq \sum_{\ell=0}^{j} p_\ell \ . \tag{6.5}$$

- *Type 2.*
  For a term of type 2, generically denoted by I'*, there exist* $j+2$ *periodic p.d.o.* $b_\ell^{'}(x,D)$ $(\ell = 0,1,\dots,j+1)$*, with*

  $$b_\ell^{'}(x,\xi) \in S(<\xi>^{-p_\ell^{'}}; g_{x,\xi}) \ , \ p_\ell^{'} \geq 0 \ ,$$

  such that

  $$\begin{aligned} I' = \textstyle\int_0^t e^{-i\tau_{j+1} Q} b_{j+1}^{'}(x,D) e^{i\tau_{j+1} Q} \int_0^{\tau_{j+1}} e^{-i\tau_j Q} b_j^{'}(x,D) e^{i\tau_j Q} \dots \\ \dots \textstyle\int_0^{\tau_2} e^{-i\tau_1 Q} b_1^{'}(x,D) e^{i\tau_1 Q} b_0^{'}(x,D) d\tau_1 \dots d\tau_{j+1} \ , \end{aligned} \tag{6.6}$$

  and

  $$2 + \sum_{\ell=0}^{j} m_\ell \leq \sum_{\ell=0}^{j+1} p_\ell^{'} \ . \tag{6.7}$$

Proof of Lemma 6.2:

We do the proof by recursion. As previously, we shall use abstract commutation relations.
We recall that
$$Q - Q_0 \ \in OpS(<\xi>^{-1}; g_{x,\xi}).$$
The assumption that $A = s_0(D)$ is in $OpS(1; g_{x,\xi})$ and commutes with Q_0 implies that
$$ad_A(Q) = ad_A(Q - Q_0) \ \in OpS(<\xi>^{-2}; g_{x,\xi}). \tag{6.8}$$
The commutation of A and Q_0 permits to apply (5.38,
$$\begin{aligned} e^{-itQ} ad_A(e^{itQ} T(t) e^{-itQ_0}) e^{itQ_0} \ = \ & \ ad_A(T(0)) \\ & + i \int_0^t e^{-i\tau Q} ad_A(Q - Q_0) e^{i\tau Q} T(\tau) d\tau \\ & + \int_0^t e^{-i\tau Q} ad_A(e^{i\tau Q} T'(\tau) e^{-i\tau Q_0}) e^{i\tau Q_0} d\tau \ , \end{aligned} \tag{6.9}$$
which is valid for any family of operators $t \mapsto T(t)$ depending smoothly on $t \in I\!R$.

We first treat the case $j = 1$.

For $j = 1$, we have to consider
$$S(t) = \int_0^t e^{-i\tau Q} a_1(x, D) e^{i\tau Q} a_0(x, D) d\tau \ , \tag{6.10}$$
and we remark that
$$S'(t) = e^{-itQ} a_1(x, D) e^{itQ} a_0(x, D) \ . \tag{6.11}$$
Let us now apply (6.9) with $T(t) = S(t)$. Because $S(0) = 0$, we get
$$\begin{aligned} e^{-itQ} \ ad_A(e^{itQ} S(t) e^{-itQ_0}) \ e^{itQ_0} \ = \ & i \int_0^t e^{-i\tau Q} ad_A(Q - Q_0) e^{i\tau Q} S(\tau) d\tau \\ & + \int_0^t e^{-i\tau Q} ad_A(e^{i\tau Q} S'(\tau) e^{-i\tau Q_0}) e^{i\tau Q_0} d\tau \ , \end{aligned} \tag{6.12}$$
The first term is of type 2 with
$$b'_2(x, D) = i ad_A(Q - Q_0) \ , \ b'_1(x, D) = a_1(x, D) \ , \ b'_0(x, D) = a_0(x, D) \ . \tag{6.13}$$

This gives also

$$p'_2 = 2 \ , \ p'_1 = m_1 \ , \ p'_0 = m_0 \ , \tag{6.14}$$

corresponding to the condition announced in (6.7).
For the second term, we observe from (6.11) that

$$e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} = a_1(x,D) e^{i\tau Q} a_0(x,D) e^{-i\tau Q_0} \ .$$

We now observe that

$$\begin{aligned} ad_A \left[e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} \right] \ &= (ad_A a_1(x,D)) e^{i\tau Q} a_0(x,D) e^{-i\tau Q_0} \\ &\quad + a_1(x,D) ad_A \left[e^{i\tau Q} a_0(x,D) e^{-i\tau Q_0} \right] \ . \end{aligned} \tag{6.15}$$

This second term can consequently be expressed as

$$\int_0^t e^{-i\tau Q} ad_A \left[e^{i\tau Q} S'(\tau) e^{-i\tau Q_0} \right] e^{i\tau Q_0} d\tau = J^{(1)} + J^{(2)} \ . \tag{6.16}$$

The term $J^{(1)}$ is defined by

$$J^{(1)} := \int_0^t e^{-i\tau Q} (ad_A a_1(x,D)) e^{i\tau Q} a_0(x,D) d\tau \ , \tag{6.17}$$

and is of type 1 with

$$b_1(x,D) = ad_A a_1(x,D) \ , \ b_0(x,D) = a_0(x,D) \ , \tag{6.18}$$

and (using the property of the commutator)

$$p_1 = m_1 + 1 \ , \ p_0 = m_0 \ . \tag{6.19}$$

The term $J^{(2)}$ is defined by

$$J^{(2)} := \int_0^t e^{-i\tau Q} a_1(x,D) ad_A \left[e^{i\tau Q} a_0(x,D) e^{-i\tau Q_0} \right] e^{i\tau Q_0} d\tau \ , \tag{6.20}$$

which can be rewritten as

$$J^{(2)} = \int_0^t e^{-i\tau Q} a_1(x,D) e^{i\tau Q} K^{(2)}(\tau) d\tau \ , \tag{6.21}$$

with

$$K^{(2)}(\tau) := e^{i\tau Q} ad_A \left[e^{i\tau Q} a_0(x,D) e^{-i\tau Q_0} \right] e^{i\tau Q_0} \ .$$

We again apply Formula (6.9) with $T = a_0(x,D)$ which is now independent of t. We consequently get

$$K^{(2)}(\tau) = (ad_A a_0(x,D)) + i\int_0^\tau e^{-i\sigma Q}\, ad_A(Q-Q_0)\, e^{i\sigma Q}\, a_0(x,D)\, d\sigma\ . \quad (6.22)$$

This permits to conclude that $J^{(2)}$ is the sum of

- one term of type 1 with

$$b_1 = a_1\ ,\ b_0 = (ad_A a_0)\ ,\ p_1 = m_1\ ,\ p_0 = m_0 + 1\ , \quad (6.23)$$

- one term of type 2 with

$$b'_2 = a_1\ ,\ b'_1 = i ad_A(Q-Q_0)\ ,\ b'_0 = a_0\ , \quad (6.24)$$

and

$$p'_2 = m_1\ ,\ p'_1 = 2\ ,\ p'_0 = m_0\ . \quad (6.25)$$

We have consequently proved the lemma for $j = 1$.

We have for the moment initiated the recursion and can now start with the recursion argument.

Let $N > 1$. Let us assume that the lemma is valid for any j, such that $j < N$. For $j = N$, let us consider a term of the form

$$\begin{aligned} S_N(t) &= \int_0^t e^{-i\tau_N Q} a_N(x,D) e^{i\tau_N Q} \int_0^{\tau_N} e^{-i\tau_{N-1} Q} a_{N-1}(x,D) e^{i\tau_{N-1} Q} \dots \\ &\qquad \dots \int_0^{\tau_2} e^{-i\tau_1 Q} a_1(x,D) e^{i\tau_1 Q} a_0(x,D)\, d\tau_1 \dots d\tau_N\ , \end{aligned} \quad (6.26)$$

then, applying as in the case $j = 1$ Formula (6.9), we express our analyzed quantity $e^{-itQ}\left(ad_A(e^{itQ} S_N(t) e^{-itQ_0})\right)\ e^{itQ_0}$ as a sum of two terms $J^{(1)}$ and $J^{(2)}$ with

$$J^{(1)} = \int_0^t e^{-i\tau Q} ad_A(Q-Q_0) e^{i\tau Q} S_N(\tau) d\tau\ ,$$

and

$$J^{(2)} = \int_0^t e^{-i\tau Q}\left(ad_A(e^{i\tau Q} S'_N(\tau) e^{-i\tau Q_0})\right) e^{i\tau Q_0} d\tau\ .$$

The first one is immediately seen as having the second type with

$$b'_{N+1} = iad_A(Q - Q_0) \ , \ b'_j = a_j \ \text{ for } \ j = 0, \ldots, N \ ,$$

and

$$p'_{N+1} = 2 \ , \ p'_j = m_j \quad \text{for} \ \ j = 0, \ldots, N \ .$$

For the second one, we first write that

$$e^{i\tau Q} S'_N(\tau) e^{-i\tau Q_0} = a_N(x,D) e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0}$$

with $S_{N-1}(\tau)$ satisfying (6.3) with $j = N-1$. So

$$\begin{aligned} &ad_A(e^{i\tau Q} S'_N(\tau) e^{-i\tau Q_0}) = \\ &\qquad ad_A(a_N(x,D)) e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0} + a_N(x,D) ad_A(e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0}) \ , \end{aligned} \tag{6.27}$$

or

$$\begin{aligned} &e^{-i\tau Q} \left(ad_A(e^{i\tau Q} S'_N(\tau) e^{-i\tau Q_0}) \right) e^{i\tau Q_0} = \\ &\qquad \left(e^{-i\tau Q} ad_A(a_N(x,D)) e^{i\tau Q} \right) S_{N-1}(\tau) \\ &\qquad + \left(e^{-i\tau Q} a_N(x,D) e^{i\tau Q} \right) \left(e^{-i\tau Q} \left(ad_A(e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0}) \right) e^{i\tau Q_0} \right) \ , \end{aligned} \tag{6.28}$$

We may apply the recursion assumption to $S_{N-1}(\tau)$, we get from (6.8), (6.9) and (6.28) that (6.4) and (6.5) are satisfied for $j = N$. We get indeed a finite sum of terms of the two types with for the first type

$$b_N(x,D) = ad_A a_N(x,D) \ , \ b_j = a_j \ , \ \text{ for } \ j = 0, \ldots, N-1 \ ,$$

or

$$b_N = a_N \ , \ b_j = b_j^{(N-1)} \ , \ \text{ for } \ j = 0, \ldots, N-1 \ ,$$

where the $b_j^{(N-1)}$ are attached to the recursion for S_{N-1} to a term of type 1 of $\left(e^{-i\tau Q} \left(ad_A(e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0}) \right) e^{i\tau Q_0} \right)$, and for the second type

$$b'_{N+1} = a_N \ , \ b'_j = b_j^{\prime\,(N-1)} \ , \ \text{ for } \ j = 0, \ldots, N \ ,$$

where the $b_j^{\prime\,(N-1)}$ are attached to the recursion for S_{N-1} to a term of type 2 of $\left(e^{-i\tau Q} \left(ad_A(e^{i\tau Q} S_{N-1}(\tau) e^{-i\tau Q_0}) \right) e^{i\tau Q_0} \right)$.

Lemma 6.2 is proved.

Lemma 6.3 :

Let $S(t)$ be the family of operators satisfying the assumptions of Lemma 6.2. If $L = x_k$, then $e^{-itQ} ad_L(e^{itQ}S(t)e^{-itQ_0})e^{itQ_0}$ may be written as a finite sum of terms of three different types 1, 2 (as introduced in the preceding lemma) and 3.

For a term of type 3, generically denoted by I'', there exist $j+3$ periodic p.d.o. $b''_\ell(x,D)$ ($\ell = 0, 1, \dots, j+2$), with

$$b''_\ell(x,\xi) \in S(<\xi>^{-p''_\ell}; g_{x,\xi}) \ , \ p''_\ell \geq 0 \ ,$$

such that

$$\begin{aligned} I'' = \textstyle\int_0^t e^{-i\tau_{j+2}Q} b''_{j+2}(x,D) e^{i\tau_{j+2}Q} \int_0^{\tau_{j+2}} e^{-i\tau_{j+1}Q} b''_{j+1}(x,D) e^{i\tau_{j+1}Q} \dots \\ \dots \textstyle\int_0^{\tau_2} e^{-i\tau_1 Q} b''_1(x,D) e^{i\tau_1 Q} b''_0(x,D) d\tau_1 \dots d\tau_{j+2} \ , \end{aligned} \tag{6.29}$$

and

$$2 + \sum_{\ell=0}^{j} m_\ell \leq \sum_{\ell=0}^{j+2} p''_\ell \ . \tag{6.30}$$

Proof of Lemma 6.3:

We use again a recursion argument. The starting point is again (5.37) (as in the proof of the previous lemma) but we can no more use the cancellation $ad_L Q_0 = 0$ leading to (5.38). Let us write for convenience (5.37) in a slightly different way,

$$\begin{aligned} & e^{-itQ} \left[ad_L(e^{itQ}S(t)e^{-itQ_0}) \right] e^{itQ_0} \\ & = ad_L(S(0)) \\ & \quad + \left(i \textstyle\int_0^t e^{-i\tau Q} ad_L(Q - Q_0) e^{i\tau Q} S(\tau) d\tau \right) \\ & \quad + \left(i \textstyle\int_0^t e^{-i\tau Q} [ad_{ad_L(Q_0)} (e^{i\tau Q} S(\tau) e^{-i\tau Q_0})] e^{i\tau Q_0} d\tau \right) \\ & \quad + \left(\textstyle\int_0^t e^{-i\tau Q} ad_L(e^{i\tau Q} S'(\tau) e^{-i\tau Q_0}) e^{i\tau Q_0} d\tau \right) \ . \end{aligned} \tag{6.31}$$

The lemma is true for $j = 1$.

We just repeat the proof of the previous lemma for the terms not involving a double commutation and get immediately terms of type 1 or 2. Here we just observe that L does not belong to $OpS(1; g_{x,\xi})$ but behaves like a p.d.o in this class if a commutation is involved (and this is always the case).

The only new term is the term coming from the second term in the right hand side of (6.31) applied to $S(t) = \int_0^t e^{-i\tau Q} a_1 e^{i\tau Q} a_0 d\tau$,

$$\left(\int_0^t e^{-i\tau Q} [ad_{ad_L(Q_0)} (e^{i\tau Q} S(\tau) e^{-i\tau Q_0})] e^{i\tau Q_0} d\tau \, . \right)$$

But, with $A = ad_L Q_0$, the term $T(\tau) := e^{-i\tau Q}[ad_{ad_L(Q_0)} (e^{i\tau Q} S(\tau) e^{-i\tau Q_0})] e^{i\tau Q_0}$ was analyzed in the preceding lemma as a sum of terms of type 1 or 2 and by integration over $[0, t]$ this leads to terms of type 2 or 3 with

- in the case 1
$$b'_2 = 1 \ , \ b'_j = b_j^{(1)} \ , \quad \text{for } j = 0, \, 1 \ ,$$
where $b_j^{(1)}$ corresponds to one term of type 1 in the decomposition of T given in the preceding lemma,
- in the case 2
$$b''_3 = 1 \ , \ b''_j = b_j^{'\,(1)} \ , \quad \text{for } j = 0, \, 1 \ , \ , \quad \text{for } j = 0, 1, 2 \ ,$$
where $b_j^{'\,(1)}$ corresponds to one term of type 2 in this decomposition of T.

The recursion argument.

Let us assume that the lemma is valid for any j, with $j < N$, and let us consider the case $j = N$.
Then we write that

$$e^{itQ} S'_N(t) e^{-itQ_0} = a_N(x, D) e^{itQ} S_{N-1}(t) e^{-itQ_0}$$

with $S_{N-1}(t)$ satisfying (6.3) with $j = N - 1$.
So, for any operator B,

$$\begin{aligned} & e^{-itQ} \left(ad_B(e^{itQ} S'_N(t) e^{-itQ_0}) \right) e^{itQ_0} \\ & \quad = e^{-itQ} ad_B(a_N(x, D)) e^{itQ} S_{N-1}(t) \\ & \qquad + \left(e^{-itQ} a_N(x, D) e^{itQ} \right) \left(e^{-itQ} ad_B(e^{itQ} S_{N-1}(t) e^{-itQ_0}) e^{itQ_0} \right) \, . \end{aligned} \tag{6.32}$$

The remarks given for the case $j = 1$ still apply and we can follow word by word the proof of the preceding lemma. The only new term appearing when applying Formula (6.31) with $S = S_N$ and with $A = ad_L Q_0$, is the term $\int_0^t e^{-i\tau Q}\left(ad_A(e^{i\tau Q}S_N(\tau)e^{-i\tau Q_0})\right)e^{i\tau Q_0}d\tau$, which can be treated directly by the preceding lemma.

So Lemma 6.3 is proved.

<u>End of the proof of the theorem</u> :

The case of one commutor does not enter immediately in the domain of application of the lemmas but we write that, for any $k \in \{1, \ldots, n\}$,

$$\begin{array}{l} e^{-itQ} ad_{x_j}(W(t)) e^{itQ_0} \\ = i \int_0^t e^{-it_1 Q} ad_{x_j}(Q - Q_0) e^{it_1 Q} dt_1 \\ \quad - \int_0^t (\int_0^{t_2} e^{-it_1 Q} ad_{ad_{x_j}(Q_0)}(Q - Q_0) e^{it_1 Q} dt_1) dt_2 \ . \end{array} \tag{6.33}$$

It is then easy to see that this commutator is a sum of two terms and that for each term the estimate (6.2) is satisfied.
This leads us to propose for the higher order commutators the following decomposition, which will be obtained from (6.33) and Lemma 6.3. We get that, for any $j \geq 1$, and with $L_d = x_{k(d)}$ for $d = 1, \ldots, j$,

$$e^{-itQ} ad_{L_j} \circ \ldots \circ ad_{L_1}(W(t)) e^{itQ_0} = \sum_k I_{j,k}^{(\ell(j;k))} \ , \tag{6.34}$$

where, for k belonging to a finite set $\mathcal{K}(j)$, $I_{j,k}^{(\ell(j,k))}$ admits the following decomposition, with $\ell = \ell(j;k) \in \{1, \cdots, 2j\}$,

$$\begin{array}{l} I_{j,k}^{(\ell)} = \\ \quad \int_0^t e^{-i\tau_\ell Q} a_{j,\ell,\ell;k}(x,D) e^{i\tau_\ell Q} \int_0^{\tau_\ell} e^{-i\tau_{\ell_1} Q} a_{j,\ell,\ell-1;k}(x,D) e^{i\tau_{\ell-1} Q} \ldots \\ \qquad \ldots \int_0^{\tau_2} e^{-i\tau_1 Q} a_{j,\ell,1;k}(x,D) e^{i\tau_1 Q} a_{j,\ell,0;k}(x,D) d\tau_1 \ldots d\tau_\ell \ . \end{array} \tag{6.35}$$

Here, for $p = 0, \cdots, \ell$, each of the $a_{j,\ell,p;k}(x,D)$ belongs to $OpS(< \xi >^{-m_{j,\ell,p;k}}; g_{x,\xi})$ for some $m_{j,\ell,p;k} \geq 0$ and the following properties are satisfied

$$\begin{array}{l} \ell(j;k) = \sum_{s=1}^j \epsilon_s \\ \sum_{p=1}^\ell m_{j,\ell,p;k} = \sum_{s=1}^j \eta_s \ , \end{array} \tag{6.36}$$

where we are at each step s in one of the three following cases:

- $\epsilon_s = 0$, $\eta_s = 1$,
- $\epsilon_s = 1$, $\eta_s = 2$,
- $\epsilon_s = 2$, $\eta_s = 2$.

But we have seen also that at the first step, the first case does not occur. We observe consequently that

$$j + 1 \leq \sum_{s=1}^{j} \eta_s \leq 2j \ ,$$

and that we have always

$$\sum_s \epsilon_s \leq \sum_s \eta_s \ .$$

Each of the terms of the decomposition satisfy the estimate (6.2) for some $q = 1, \cdots, j$.
This ends the proof of Theorem 6.1.

Remark 6.4 *:*
Using also the first lemma for the commutators with D_{x_j}, we can obtain more generally the following (non optimal) estimate, for any α, β, such that $|\alpha| + |\beta| \geq 1$,

$$(\sum_{\gamma \in \Gamma^\star} |\gamma^\beta \partial_\xi^\alpha \hat{w}_\gamma(t,\xi)|^2)^{1/2} \leq C_{\alpha,\beta} (< t >< \xi >^{-1})^{|\alpha|} (1+ < t >< \xi >^{-1})^{|\beta|+|\alpha|} \ , \tag{6.37}$$

for all $(t,\xi) \in I\!R \times I\!R^n$. This is interesting to compare with what was obtained for the free case in (4.2). We observe that (6.37) implies the weaker

$$(\sum_{\gamma \in \Gamma^\star} |\gamma^\beta \partial_\xi^\alpha \hat{w}_\gamma(t,\xi)|^2)^{1/2} \leq C_{\alpha,\beta} (< t >< \xi >^{-1})^{|\alpha|+1} < t >^{|\beta|+|\alpha|-1} \ , \tag{6.38}$$

7 An asymptotic decomposition for the half wave unitary group

In the preceding section, we have recognized through the Beals's characterization the operator $W(t) := e^{itQ} e^{-itQ_0}$ as a pseudodifferential operator. Simultaneously, we were able to analyze more general expressions like

$e^{itQ}C(t)e^{-itQ_0}$ where $C(t)$ is a suitable pseudo-differential operator depending on t.
The estimates which were obtained are unfortunately not sufficient for the analysis of the main contribution and we shall have to go more deeply in the construction of an asymptotic expansion of $W(t)$. But the estimates obtained in the preceding sections will be decisive in order to control the remainders in this asymptotic expansion of the integrated density of states.

We have seen already that $W(t)$ is a periodic p.d.o.. We now want to construct a suitable expansion approximating $W(t)$ by using more intensively the symbolic calculus for pseudodifferential operators with Γ-periodic symbols. This was already done in Section 4 for the case of the free case. As before, this approximation will be controlled with respect to t. The main result is

Theorem 7.1 *:*
The operator $t \mapsto W(t)$ admits the following formal decomposition

$$W(t) \sim \sum_{j=0}^{+\infty} W_{-j}(t) \ , \tag{7.1}$$

with

$$\begin{array}{ll} W_0(t) & = I_d \\ W_{-j-1}(t) & = i \left(\int_0^t W_{-j}(\tau) L(\tau) \ d\tau \right) \ , \end{array} \tag{7.2}$$

for any integer j, where

$$L(\tau) = e^{+i\tau Q_0}(Q - Q_0)e^{-i\tau Q_0} \ .$$

Moreover we have, for any $N \in I\!N$,

$$W(t) = \sum_{j=0}^{N} W_{-j}(t) + R^{N+1}(t), \tag{7.3}$$

and W_{-j} and $R^{N+1}(t) = r^{N+1}(t, x, D_x)$, are periodic p.d.o. whose symbols satisfy,

$$w_{-j}(t, x, \xi) \in S(< t >^j < \xi >^{-j}; g_t^0 + g_x^0 + < t >^2 g_\xi^1) \tag{7.4}$$

and

$$\phi(t) r^{N+1}(t, x, \xi) \in S(< \xi >^{-N-1}; g_t^0 + (g_x^0 + g_\xi^1)) \ , \tag{7.5}$$

for any $\phi \in C_0^\infty(I\!R)$.
Moreover, the same property is true for $\frac{1}{t} r^{N+1}(t,x,\xi)$:

$$\frac{\phi(t)}{t}\, r^{N+1}(t,x,\xi) \in S(<\xi>^{-N-1}; g_t^0 + (g_x^0 + g_\xi^1))\ , \tag{7.6}$$

Proof of Theorem 7.1:

We start from $W_0(t) = Id$ and we observe that

$$Q - Q_0 \in OpS(<\xi>^{-1}; g_{x,\xi})\ . \tag{7.7}$$

We observe also that W is a solution of

$$W'(t) = iW(t)L(t)\ ,$$

or

$$W(t) = Id + i\int_0^t W(\tau)L(\tau)d\tau\ .$$

This leads naturally to the recursion given in the proposition. W_{-j-1} is then given for $j \geq 0$ by

$$W_{-j-1}(t) = i\left(\int_0^t W_{-j}(\tau)L(\tau)d\tau\right)\ . \tag{7.8}$$

All the properties of W_{-j} are then clear by recursion using (7.7) and Theorem 4.1 with $C = Q - Q_0$.

But we have now to analyze the remainder. The next formula is obtained by comparison of the equation satisfied by W and the equation satisfied by $\sum_{j=0}^N W_{-j}$. We get

$$R^{N+1}(t) = i\left(\int_0^t W_{-N}(\tau)e^{i\tau Q_0}(Q-Q_0)e^{-i\tau Q}d\tau\right)e^{itQ}e^{-itQ_0}\ , \tag{7.9}$$

for all $N \in I\!N$.
The statement is then a consequence of (7.9), the property established for $W_{-N}(\tau)$ and the a priori knowledge on $e^{i\tau Q_0}(Q-Q_0)e^{-i\tau Q_0}$ and $e^{i\tau Q_0}e^{-i\tau Q}$ as given by Theorem 5.1.

The theorem is proved.

Remark 7.2 *:*
Under the normalization of $V(x)$ given in (1.24), we have

$$w_0(t,x,\xi) = 1 \quad \textit{and} \ \hat{w}_{-1,0}(t,\xi) \in S(\langle\xi\rangle^{-2}; g_t^0 + g_\xi^1) \ , \tag{7.10}$$

where the second inclusion is locally uniform with respect to t.

The first part is immediate.
For the second statement, we observe, using the definition of W_{-1}

$$W_{-1} = \int_0^t L(\tau) d\tau \ .$$

This gives

$$w_{-1}(t,x,\xi) = \int_0^t \sigma(\tau,x,\xi) d\tau \ ,$$

where σ denotes the symbol of L.
But we are actually interested in

$$\hat{w}_{-1,0}(t,\xi) = \int_0^t \hat{\sigma}_0(\tau,\xi) d\tau \ .$$

We recall from (4.3), (4.4), (5.6) and (5.10) (with $c(x,D) = Q - Q_0$) that $\hat{\sigma}_0(\tau,\xi)$ is given by,

$$\hat{\sigma}_0(\tau,\xi) = \hat{q}_{-1}(\xi) = 0 \ ,$$

modulo $\mathcal{O}_t(\langle\xi\rangle^{-2})$ (more precisely modulo an error in $S(\langle\xi\rangle^{-2}; g_\xi^1)$ locally uniform in t).
Similar properties are true for the derivatives with respect to t.
Let us finally recall that by construction

$$\hat{w}_{-1,0}(0,\xi) = 0 \ . \tag{7.11}$$

8 The asymptotic of the density of state

We begin this section by recalling a rather standard result about the pseudodifferential character of $f(Q)$ defined a priori abstractly through the functional calculus.

Theorem 8.1 *:*
Let $f(\tau)$ be a regular real function on $\mathbb{R}$ such that

$$f(\tau) \in S(<\tau>^m ; <\tau>^{-2} g_\tau^0). \tag{8.1}$$

Then the operator $f(Q)$ is a periodic p.d.o. in $OpS(<\xi>^m; g_{x,\xi})$. Moreover, the symbol $\sigma_{f(Q)}(x,\xi)$ has an asymptotic expansion of the form

$$\sigma_{f(Q)}(x,\xi) \sim \sum_{j\geq 0} \sigma_{f(Q),m-j}(x,\xi), \tag{8.2}$$

with

$$\sigma_{f(Q),m-j}(x,\xi) \in S(<\xi>^{m-j}; g_{x,\xi}) \tag{8.3}$$

and for every positive integer N,

$$\sigma_{f(Q)}(x,\xi) - \sum_{j=0}^{N} \sigma_{f(Q),m-j}(x,\xi) \in S(<\xi>^{m-N-1}; g_{x,\xi}). \tag{8.4}$$

Proof:
Since, for an integer k large enough, the symbol $g(\tau) := <\tau>^{-2k} f \in S(<\tau>^{-2}; <\tau>^{-2} g_\tau^0)$ and since we can write $f(Q) = (Q^2+1)^{k+1} g(Q)$, we may assume for the proof that $m < -1$ and then apply the usual pseudodifferential calculus.
In this case we have

$$f(\tau) \in H^s(\mathbb{R}^n), \quad \forall s \in \mathbb{R}. \tag{8.5}$$

Then we can write

$$f(Q) = (2\pi)^{-1} \int_{\mathbb{R}} \hat{f}(t) e^{itQ} dt, \tag{8.6}$$

so

$$\sigma_{f(Q)}(x,\xi) = (2\pi)^{-1} \int_{\mathbb{R}} \hat{f}(t) w(t,x,\xi) e^{itq^0(\xi)} dt. \tag{8.7}$$

Let N be a fixed positive integer. Then by Taylor formula, we may write

$$w(t,x,\xi) = \sum_{j=0}^{N} \frac{t^j}{j!} \partial_t^j w(0,x,\xi) \; + \; t^{N+1} \rho^{N+1}(w)(t,x,\xi)\,, \tag{8.8}$$

so

$$\begin{aligned} \sigma_{f(Q)}(x,\xi) &= \textstyle\sum_{j=0}^{N} \frac{(-i)^j}{j!} f^{(j)}(q^0(\xi)) \partial_t^j w(0,x,\xi) \\ &\quad + (2\pi)^{-1} \textstyle\int_{\mathbb{R}} \hat{f}(t) t^{N+1} \rho^{N+1}(w)(t,x,\xi) e^{itq^0(\xi)} dt\,. \end{aligned} \tag{8.9}$$

Since there exists some constant $\eta_0 > 0$ such that $q^0(\xi) > \eta_0 < \xi >$, we get by integration by parts

$$\begin{array}{l} \int_{I\!R} \hat{f}(t) t^{N+1} \rho^{N+1}(w)(t,x,\xi) e^{itq^0(\xi)} dt \\ \quad = \int_{I\!R} (q^0(\xi))^{-N-1} e^{itq^0(\xi)} (-i\partial_t)^{N+1} \{\hat{f}(t) t^{N+1} \rho^{N+1}(w)(t,x,\xi)\} dt \;. \end{array} \tag{8.10}$$

The property (8.1) with $m < -1$ shows that, for any integers j, k and l satisfying the condition $0 \leq k \leq l$,

$$< t >^j \, \partial_t^k (\hat{f}(t) t^l) \in L^2(I\!R) \;. \tag{8.11}$$

Then Theorem 8.1 follows from Theorem 5.3, (8.1), (8.5) and (8.8)–(8.11). More precisely, let us start from the formula expressing $\rho^{N+1}(w)(t,x,\xi)$,

$$\rho^{N+1}(w)(t,x,\xi) = \frac{1}{N!} \int_0^1 (1-s)^N (\partial_t^{N+1} w)(st,x,\xi) ds \;.$$

Theorem 5.3 gives immediately the following estimates for $\rho^{N+1}(w)$

$$|\partial_x^\alpha \partial_\xi^\beta \rho^{N+1}(w)(t,x,\xi)| \leq C_{\alpha,\beta} \langle t \rangle^{[\frac{n}{2}]+|\alpha|+|\beta|+1} \;.$$

Putting this information in (8.10) and taking account of the control of $\hat{f}$ given by (8.11), we obtain that the remainder

$$r_f^{N+1}(x,\xi) := \int_{I\!R} \hat{f}(t) t^{N+1} \rho^{N+1}(w)(t,x,\xi) e^{itq^0(\xi)} dt \;,$$

satisfies

$$r_f^{N+1} \in S(\langle \xi \rangle^{-N-1}; g^0_{x,\xi}) \;.$$

This is not exactly what is announced in (8.4), but playing as usual in the pseudodifferential theory with the decomposition

$$r_f^{N+1}(x,\xi) = r_f^{N'+1}(x,\xi) + \sum_{j=N+1}^{N'} \sigma_{f(Q),m-j}(x,\xi) \;,$$

valid for any $1 \leq N < N'$, and using (8.3), we obtain effectively (8.4) and the theorem is proved.
We emphasize that only a rather weak estimate on w for large t was needed at this stage. This will be no more the case in the proof of the next theorem.

We are actually able to explicit (8.2) in the following way.

Proposition 8.2 *:*
In the asymptotic expansion (8.2), we may take

$$\sigma_{f(Q),m}(x,\xi) = f(q^0(\xi)), \quad \sigma_{f(Q),m-1}(x,\xi) = 0, \tag{8.12}$$

and, for every $j > 1$,

$$\sigma_{f(Q),m-j}(x,\xi) = \sum_{k=0}^{j} c_{j,k}(x,\xi) f^{(k)}(q^0(\xi)) \ , \tag{8.13}$$

where the $c_{j,k}(x,\xi)$ are periodic symbols,

$$c_{j,k}(x,\xi) \in S(<\xi>^{-j+k}; g_{x,\xi}), \tag{8.14}$$

and independent of f.

The coefficient $c_{jk}(x,\xi)$ is given by

$$c_{jk}(x,\xi) = \frac{(-i)^j}{j!} (\partial_t^k w_{-j+k})(0,x,\xi) \ . \tag{8.15}$$

Formula (8.12) comes easily from the two following properties of the approximation of $W(t)$,

$$w_0(t,x,\xi) = 1 \ \text{ and } \ w_{-d}(0,x,\xi) = 0, \ \text{ if } d > 0. \tag{8.16}$$

Remark 8.3 *:*
Proposition 8.2 shows that, if $\tau \mapsto f(\tau)$ is a smooth function on the real line and if

$$f(\tau) \in S(<\tau>^{-m}; <\tau>^{-2} g_\tau^0), \text{ with } m > n,$$

Then $\widetilde{\mathrm{Tr}}(\mathrm{f(Q)})$ as defined by (2.3)) is given by the integral

$$\widetilde{\mathrm{Tr}}(\mathrm{f(Q)}) = (2\pi)^{-\mathrm{n}} \, |\mathcal{K}|^{-1} \int_{\mathrm{x}\in\mathcal{K}} \int_\xi \sigma_{\mathrm{f(Q)}}(\mathrm{x},\xi) \, \mathrm{dxd}\xi \ . \tag{8.17}$$

Let us also observe that for any θ we have

$$\mathrm{Tr}\ (f(Q_\theta)) = \int_{x\in\mathcal{K}} \sum_\gamma \sigma_{f(Q)}(x,\gamma+\theta)\, dx \ , \tag{8.18}$$

and that (8.17) is obtained by integration over $\mathcal{K}^\star$ of the equality (8.18). The regularizing effects of this last integration will appear through suitable integrations by parts.

The results corresponding to a fixed f are not sufficient for our purpose and in the preceding remark there is no way to see that the remainder is small. We have actually in mind to treat suitable approximations of the characteristic function of the interval $]-\infty,\lambda]$. This will lead us to a much more precise analysis of the functional calculus.

Theorem 8.4 *:*
Let $f \in C_0^\infty(I\!R)$. Then, as $\lambda \to +\infty$, $\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q}-\lambda\mathrm{I_d}))$ has the following asymptotic expansion

$$\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q}-\lambda\mathrm{I_d})) \sim \sum_{\mathrm{j}\geq 0}\int_0^{+\infty} \mathrm{a_{n-1-j}(r)f((r^2+\lambda_0)^{1/2}-\lambda)dr}, \tag{8.19}$$

with

$$\begin{array}{ll} a_{n-1}(r) & = (2\pi)^{-n}|S^{n-1}|r^{n-1} \\ a_{n-2}(r) & = 0 \;, \\ a_{n-3}(r) & \in S(<r>^{n-4}; g_r^1) \;, \end{array} \tag{8.20}$$

and, more generally,

$$a_k(r) \in S(<r>^k; g_r^1) \;. \tag{8.21}$$

More precisely $\sim$ means here that, for any integer $N > n/2$, there exists a constant $C_N > 0$, depending only on N, (and neither on λ nor on f), satisfying

$$\begin{array}{l} |\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q}-\lambda\mathrm{I_d})) \; - \sum_{\mathrm{j}=0}^{\mathrm{N}-1}\int_0^{+\infty} \mathrm{a_{n-1-j}(r)f((r^2+\lambda_0)^{1/2}-\lambda)dr}| \\ \quad \leq C_N\lambda^{n-N}\{\| \, t^N\hat{f}(t) \, \|_{L^1(I\!R)} + \sum_{j=0}^N \| \, \partial_t^j \hat{f}(t) \, \|_{L^1(I\!R)}\} \;. \end{array} \tag{8.22}$$

<u>Proof</u>

Our starting point will be the formula

$$\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q}-\lambda\mathrm{I_d})) = (2\pi)^{-\mathrm{n}-1}\int_{\mathrm{R}\times\mathrm{R}^\mathrm{n}} \mathrm{e}^{\mathrm{it}(\mathrm{q}^0(\xi)-\lambda)}\hat{\mathrm{f}}(\mathrm{t})\hat{\mathrm{w}}_0(\mathrm{t},\xi)\mathrm{dt}\mathrm{d}\xi \;. \tag{8.23}$$

This appears as an oscillatory integral and looking at the critical point of the phase $(t,\xi) \mapsto t(q^0(\xi)-\lambda)$ we find

$$t = 0 \;;\; q^0(\xi) = \lambda \;.$$

This makes natural the idea to eliminate first by non stationary phase arguments the contribution of the region which is far from the critical set. This is what is done in the two first steps of the proof.

Step 1: localization near $q^0 = \lambda$.

Let $\tau \mapsto \chi(\tau)$ be a cut off function satisfying

$$\chi(\tau) \in C_0^\infty(I\!R),\ \chi(\tau) = \chi(-\tau),\ 0 \leq \chi(\tau) \leq 1, \tag{8.24}$$

$$\chi(\tau) = 1,\ \text{if } |\tau| < \frac{1}{2},\ \text{ and }\ \chi(\tau) = 0,\ \text{if } |\tau| > 1.$$

We denote by χ_λ the function

$$\tau \mapsto \chi_\lambda(\tau) = \chi(2\lambda^{-1}(\tau - \lambda)). \tag{8.25}$$

This function is chosen in order to localize near $\tau = \lambda$. Let $w(t, x, \xi)$ and $\hat{w}_0(t, \xi)$ be as in Theorem 6.1. By Formula (6.1), we obtain, for any $(k, \alpha) \in I\!N \times I\!N^n$,

$$< t >^{-2|\alpha|} < \xi >^{|\alpha|}\ \partial_t^k \partial_\xi^\alpha \hat{w}_0(t, \xi) \in L^\infty(I\!R \times I\!R^n). \tag{8.26}$$

The case $|\alpha| = 0$ is analyzed with Proposition 3.2. The treatment of the derivatives with respect to t is not different.
By (2.3) and (8.7),

$$\begin{aligned} &\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0}))) \\ &\quad = (2\pi)^{-n-1} \textstyle\int_{I\!R \times I\!R^n} e^{it(q^0(\xi) - \lambda)} \hat{f}(t) \hat{w}_0(t, \xi)(1 - \chi_\lambda(q^0(\xi))) dt d\xi\ . \end{aligned} \tag{8.27}$$

Since $|q^0(\xi) - \lambda| > \lambda/4$ on the support of $(1 - \chi_\lambda)(\cdot)$, we may integrate by parts in t and get, for any integer M,

$$\begin{aligned} &\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0}))) \\ &= i^M (2\pi)^{-n-1} \textstyle\int_{I\!R \times I\!R^n} e^{it(q^0(\xi) - \lambda)} (q^0(\xi) - \lambda)^{-M} (1 - \chi_\lambda(q^0(\xi))) \times \\ &\qquad \partial_t^M (\hat{f}(t) \hat{w}_0(t, \xi))\, dt\, d\xi\ . \end{aligned}$$

So taking into account (8.26), we get, if $M > n$, the estimate

$$\begin{aligned} &|\widetilde{\mathrm{Tr}}(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0})))| \\ &\leq C_{0,M} \textstyle\sum_{j=0}^M \parallel \partial_t^j \hat{f}(t) \parallel_{L^1(I\!R)} \times \\ &\qquad \times \textstyle\int_{I\!R^n} (q^0(\xi) - \lambda)^{-M} (1 - \chi_\lambda(q^0(\xi)))\, d\xi\ , \end{aligned} \tag{8.28}$$

with $C_{0,M} := \sum_{j=0}^{M} \| \partial_t^j \hat{w}_0(t,\xi) \|_{L^\infty(I\!R\times I\!R^n)}$. We have consequently eliminated the contribution corresponding to the region far from $q^0 = \lambda$ (more precisely the region $|q^0(\xi) - \lambda| > \frac{\lambda}{4}$). More precisely we have obtained the existence, for any $M > n$, of a constant $C_{0,M}$ such that, for any $f \in C_0^\infty$, we have

$$|\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})(\mathrm{I_d} - \chi_\lambda(\mathrm{Q_0})))\,| \leq \mathrm{C_{0,M}} \lambda^{-\mathrm{M}+\mathrm{n}} \left(\sum_{\mathrm{j}=0}^{\mathrm{M}} \| \partial_\mathrm{t}^\mathrm{j} \hat{\mathrm{f}}(\mathrm{t}) \|_{\mathrm{L^1(R)}} \right) . \quad (8.29)$$

Step 2: Elimination of the contribution $|t|$ large.

Let us now consider $\widetilde{\mathrm{Tr}}\,(\mathrm{f}(\mathrm{Q} - \lambda \mathrm{I_d})\chi_\lambda(\mathrm{Q_0}))$ and in order to eliminate the region $|t|$ large, we analyze the integral

$$I := (2\pi)^{-n-1} \int_{I\!R\times I\!R^n} \hat{f}(t)\hat{w}_0(t,\xi)\,(1-\chi(t))\,\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}\,dt\,d\xi \ . \quad (8.30)$$

Since $\xi \cdot \partial_\xi q^0(\xi) = |\xi|^2/q^0(\xi)$ and $|\xi| \neq 0$ on the support of $\chi_\lambda(q^0(\xi))$ for λ large enough, we may integrate by parts in $|\xi|$, using the identity

$$L(\xi, \partial_\xi) e^{itq^0} = it\, e^{itq^0} \ ,$$

with

$$L(\xi,\partial_\xi) = \frac{q^0(\xi)}{|\xi|^2}(\xi\cdot\partial_\xi) \ .$$

We then get, for any integer M,

$$\begin{aligned} &(2\pi)^{-n-1} \textstyle\int_{I\!R\times I\!R^n} (1-\chi(t))\hat{f}(t)\hat{w}_0(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi \\ &= (2\pi)^{-n-1} i^M \textstyle\int_{I\!R\times I\!R^n} (1-\chi(t))t^{-M}\hat{f}(t)e^{it(q^0(\xi)-\lambda)} \times \\ &\qquad \times ({}^tL)^M(\hat{w}_0(t,\xi)\chi_\lambda(q^0(\xi)))\,dt\,d\xi \ . \end{aligned}$$

Using (8.26), we find that there exists a constant C_M, depending only on n and M, such that

$$\begin{aligned} &|(2\pi)^{-n-1} \textstyle\int_{I\!R\times I\!R^n} (1-\chi(t))\hat{f}(t)\hat{w}_0(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi| \\ &\leq C_M(\textstyle\int_{\lambda/2}^{3\lambda/2} r^{n-1-M}dr) \times \int_{I\!R} |t|^M(1-\chi(t))|\hat{f}(t)|dt \ . \end{aligned}$$

This contribution is then controlled by the estimate

$$\begin{aligned} &|(2\pi)^{-n-1} \textstyle\int_{I\!R\times I\!R^n} \hat{f}(t)\hat{w}_0(t,\xi)\,(1-\chi(t))\,\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}\,dt\,d\xi| \\ &\leq C_{1,M}\lambda^{n-M}\ \textstyle\int_{I\!R} |t|^M|\hat{f}(t)|dt \ . \end{aligned} \quad (8.31)$$

Step 3: The use of the approximation of $W(t)$.
After the two first reductions, we have to consider the main contribution, which is defined as

$$J := (2\pi)^{-n-1} \int_{\mathbb{R}\times\mathbb{R}^n} \hat{f}(t)\hat{w}_0(t,\xi)\ \chi(t)\ \chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}\ dt\, d\xi\ . \tag{8.32}$$

We can now use the asymptotic expansion given in Theorem 7.1 to see that (7.6) implies that, for any integer M, there exists C_M, such that, for any $f \in C_0^\infty$,

$$\begin{array}{l} |(2\pi)^{-n-1} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_0(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi \\ -(2\pi)^{-n-1}\sum_{j=0}^{M-1} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_{-j,0}(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi| \\ \leq C_M(\int_{\lambda/2}^{3\lambda/2} r^{n-1-M}dr) \times \int_{\mathbb{R}} \chi(t)|\hat{f}(t)|dt\ . \end{array}$$

We remark that, as announced in the control of the remainder in the expansion, we need only a control of R^{N+1} on the support of χ.
This gives again a remainder controlled by

$$\begin{array}{l} |(2\pi)^{-n-1} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_0(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi \\ -(2\pi)^{-n-1}\sum_{j=0}^{M-1} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_{-j,0}(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi| \\ \leq C_M\lambda^{n-M} \int_{\mathbb{R}} \chi(t)|\hat{f}(t)|dt\ . \end{array} \tag{8.33}$$

Step 4: Taylor's expansion with respect to the t variable.
We are actually proving "à la main" the stationary phase theorem in this particular case. For any integer $j < M$, we use Taylor's formula

$$\hat{w}_{-j,0}(t,\xi) = \sum_{k=0}^{M-j-1} \frac{t^k}{k!}\partial_t^k \hat{w}_{-j,0}(0,\xi)\ +\ t^{M-j}\hat{\rho}^{M-j}(\hat{w}_{-j,0})(t,\xi), \tag{8.34}$$

and the same integration by parts leads to

$$\begin{array}{l} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_{-j,0}(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi \\ = \sum_{k=0}^{M-j-1} \frac{i^k}{k!} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat{w}_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi \\ +\frac{i^{M-j}}{(M-j)!} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^{M-j}\{\hat{\rho}^{M-j}(\hat{w}_{-j,0})(t,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi\ . \end{array} \tag{8.35}$$

So using (7.4) and (8.34), we get the existence of a constant C_M such that, for any $f \in C_0^\infty$,

$$\begin{array}{l} |\int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)\hat{w}_{-j,0}(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi \\ -\sum_{k=0}^{M-j-1} \frac{i^k}{k!} \int_{\mathbb{R}\times\mathbb{R}^n} \chi(t)\hat{f}(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat{w}_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi| \\ \leq C_M(\int_{\lambda/2}^{3\lambda/2} r^{n-1-M}dr) \times \int_{\mathbb{R}} \chi(t)|\hat{f}(t)|dt\ . \end{array}$$

This gives

$$
\begin{aligned}
&|\int_{I\!R\times I\!R^n}\chi(t)\hat f(t)\hat w_{-j,0}(t,\xi)\chi_\lambda(q^0(\xi))e^{it(q^0(\xi)-\lambda)}dtd\xi\\
&-\sum_{k=0}^{M-j-1}\tfrac{i^k}{k!}\int_{I\!R\times I\!R^n}\chi(t)\hat f(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi|\\
&\leq C_M\ \lambda^{n-M}\ \int_{I\!R}|\hat f(t)|dt\ .
\end{aligned}
\tag{8.36}
$$

But, for any integers j and k with $0\leq j<M$ and $0\leq k\leq M-j-1$,

$$
\begin{aligned}
&|\int_{I\!R\times I\!R^n}(1-\chi(t))\hat f(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi|\\
&\quad=|i^{M-j-k}\int_{I\!R\times I\!R^n}(1-\chi(t))\hat f(t)t^{-(M-j-k)}e^{it(q^0(\xi)-\lambda)}\times\\
&\qquad\qquad\times({}^tL)^{M-j}\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi|\\
&\quad\leq C_M(\int_{\lambda/2}^{3\lambda/2}r^{n-1-M}dr)\times\int_{I\!R}(1-\chi(t))|t|^{-M+j+k}|\hat f(t)|dt\ .
\end{aligned}
$$

This is obtained as in step 2 and leads to the estimate

$$
\begin{aligned}
&|\int_{I\!R\times I\!R^n}(1-\chi(t))\hat f(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi|\\
&\quad\leq C_M\lambda^{n-M}\ \int_{I\!R}|\hat f(t)|dt\ .
\end{aligned}
\tag{8.37}
$$

Step 5: Elimination of the cutoff function:
We have now to consider

$$
\begin{aligned}
K_1:=&\sum_{k,j\ ;\ j+k\leq M-1}\\
&\tfrac{i^k}{k!}\int_{I\!R\times I\!R^n}\hat f(t)e^{it(q^0(\xi)-\lambda)}({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}dtd\xi\ .
\end{aligned}
\tag{8.38}
$$

Integrating with respect to t, we obtain

$$
K_1=\sum_{k,j\ ;\ k+j\leq M-1}\frac{i^k}{k!}\int_{I\!R^n}f(q^0(\xi)-\lambda)({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)\chi_\lambda(q^0(\xi))\}d\xi\ .
\tag{8.39}
$$

We now control

$$
K_1^{(j)}=\sum_{k=0}^{M-j-1}\int_{I\!R^n}\frac{i^k}{k!}f(q^0(\xi)-\lambda)({}^tL)^k\{\partial_t^k\hat w_{-j,0}(0,\xi)(1-\chi_\lambda)(q^0(\xi))\}d\xi\ .
\tag{8.40}
$$

Observing that,

$$
|(q^0(\xi)-\lambda)^Mf(q^0(\xi)-\lambda)|\leq\sup_\tau|\tau^Mf(\tau)|\leq C_M\sum_{j=0}^M\parallel\partial_t^j\hat f(t)\parallel_{L^1(I\!R)}\ ,
$$

we conclude essentially as for (8.28) and get

$$|K_1| \leq C_M \lambda^{-M+n} \sum_{j=0}^{M} \| \partial_t^j \hat{f}(t) \|_{L^1(I\!R)} \ . \tag{8.41}$$

Step 6: The main contribution:
We are now left with the main contribution, which is given by

$$K_2 := \sum_{k,j\ ;\ j+k \leq M-1} \int_{I\!R^n} f(q^0(\xi) - \lambda) ({}^tL)^k \{ \partial_t^k \hat{w}_{-j,0}(0,\xi) \} d\xi \ . \tag{8.42}$$

We can now take polar coordinates and we obtain that, for $\ell \leq M$,

$$a_{n-1-\ell}(r) = r^{n-1} \sum_{k \leq \ell} \int_{S^{n-1}} ({}^tL)^k \{ \partial_t^k \hat{w}_{k-\ell,0}(0, r\omega) \} d\omega \ . \tag{8.43}$$

Summing up the different steps, we have finally proved the theorem outside the announced cancellations.

Control of a_{n-3}.

According to (8.43), we have

$$a_{n-3}(r) = r^{n-1} \sum_{k \leq 2} \int_{S^{n-1}} ({}^tL)^k \{ \partial_t^k \hat{w}_{k-2,0}(0, r\omega) \} d\omega \ . \tag{8.44}$$

But from the previous formula (5.10), we get that

$$\hat{w}_{-1,0}(t,\xi) \equiv it \hat{q}_{-1,0}(\xi) = 0 \ , \tag{8.45}$$

modulo $S(< \xi >^{-2}; g_\xi^1)$.
If we observe that by construction

$$\hat{w}_{-2,0}(0,\xi) = 0 \ , \tag{8.46}$$

and that

$$(\partial_t^2 \hat{w}_{0,0})(t,\xi) = 0 \ , \tag{8.47}$$

The smallness of a_{n-3} is proved.

Cancellation of a_{n-2}

We consider the identity

$$a_{n-2}(r) = r^{n-1} \sum_{k \leq 1} \int_{S^{n-1}} ({}^tL)^k \left(\partial_t^k \hat{w}_{k-1,0}(0, r\omega)\right) d\omega \ , \tag{8.48}$$

and observing that $\hat{w}_{-1,0}(t,\xi) = 0$ and that $\partial_t \hat{w}_{0,0}(t,\xi) = 0$, we obtain directly that $a_{n-2}(r) = 0$ and more generally that

$$r^{n-1} \sum_{k \leq 1} \int_{S^{n-1}} e^{irb\cdot\omega} ({}^tL)^k \left(\partial_t^k \hat{w}_{k-1,0}(0, r\omega)\right) d\omega = 0 \ . \tag{8.49}$$

This ends the proof of the theorem.

Remark 8.5 *:*
Let us observe for later use that the same proof gives for any $b \in \Gamma^\star$ the property that

$$r^{n-1} \sum_{k \leq 2} \int_{S^{n-1}} e^{irb\cdot\omega} ({}^tL)^k \left(\partial_t^k \hat{w}_{k-2,0}(0, r\omega)\right) d\omega = \mathcal{O}(< r >^{n-4}) \ . \tag{8.50}$$

Similarly, one has obtained the vanishing of $a_{n-2}(r)$ through a proof which will permit the extension of the result in the form

$$r^{n-1} \sum_{k \leq 1} \int_{S^{n-1}} e^{irb\cdot\omega} ({}^tL)^k \left(\partial_t^k \hat{w}_{k-1,0}(0, r\omega)\right) d\omega = 0 \ . \tag{8.51}$$

Remark 8.6 *:*
This theorem is not sufficient because working with λ-dependent functions f will appear as necessary in order to get a good regularizing approximation of the characteristic function $Y_{]0,\lambda]}$. The fact that we have explicited carefully the semi-norms on f involved in the control of the remainders will play a crucial role in the proof of the next proposition.

Proposition 8.7 *:*
Let $\widetilde{N}(\lambda; Q)$ be the integrated state density of Q

$$\widetilde{N}(\lambda; Q) = \widetilde{\mathrm{Tr}}(\chi_{]-\infty,\lambda[}(\mathrm{Q})). \tag{8.52}$$

Then,

$$\widetilde{N}(\lambda; Q) = \widetilde{N}(\lambda; Q_0) + \mathcal{O}(\lambda^{n-2}) \ , \tag{8.53}$$

as $\lambda \to +\infty$.

Recall that

$$\widetilde{N}(\lambda; Q_0) = (2\pi)^{-n} \int_{q^0(\xi)<\lambda} d\xi = (2\pi)^{-n} \frac{|S^{n-1}|}{n} \lambda^n + \mathcal{O}(\lambda^{n-2}). \tag{8.54}$$

This proposition was established by M. Shubin [Sh] (Theorem 4.5). We observe indeed that, for $C = ||V||_{L^\infty}$,

$$N(\mu - C, P_0) \leq N(\mu, P_V) \leq N(\mu + C, P_0)\ .$$

This can be for example obtained by proving a similar inequality for the counting functions $n(\mu; P_V^\theta)$ and $n(\mu; P_0^\theta)$ and by integrating over θ.
We now use the known asymptotic for $N(\mu, P_0)$

$$N(\mu, P_0) = (2\pi)^{-n} Vol(B_n(0,1)) \mu^{\frac{n}{2}}\ , \tag{8.55}$$

where $B_n(0,1)$ is the ball of radius 1 in $I\!R^n$, and this implies

$$N(\mu - C, P_0) - N(\mu, P_0) = \mathcal{O}(\mu^{\frac{n}{2}-1})\ .$$

Remark 8.8 *:*
For this proof, we have not used the property that the mean value of V vanishes outside the fact that the constant C depends on V. This is coherent with the final result announced in the Theorem 1.1.

Remark 8.9 *:*
Since in the asymptotic expansion (8.19) in Theorem 8.4, $a_{n-2}(r) = 0$ and $a_{n-3}(r) = \mathcal{O}(r^{n-4})$, we may expect that

$$\widetilde{N}(\lambda; Q) = \widetilde{N}(\lambda; Q_0) + \mathcal{O}(\lambda^{n-3})\ ,$$

as $\lambda \to +\infty$.

The two preceding remarks show that it is natural to try to improve the control of the remainder. We can actually prove only the following estimate of the remainder.

Theorem 8.10 *:*
If $n > 2$, then, for any $\epsilon > 0$, we have the following asymptotic expansion

$$\widetilde{N}(\lambda; Q) = \widetilde{N}(\lambda; Q_0) + \mathcal{O}(\lambda^{n-3+\epsilon}) \tag{8.56}$$

as $\lambda \to +\infty$.

Proof of Theorem 8.10:

When regularizing the Heaviside function $Y_{]-\infty,\lambda]}$ or $Y_{[0,\lambda]}$ [4] in order to apply a regular functional calculus, we need to control the error due to this regularization. One important control is given by the following lemma,

Lemma 8.11 *:*
For any fixed $\delta \in]1,2[$ and any $\tau \in]0,1]$, we have the following asymptotic expansion as $\lambda \to +\infty$, when $n > 2$,

$$\widetilde{\mathrm{Tr}}\left(\chi(\lambda^{\delta}(\mathrm{Q}-\lambda\mathrm{I_d}))\right) = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta+\tau}). \tag{8.57}$$

Proof of Lemma 8.11 :

The proof of the lemma is quite similar to the proof of Theorem 8.4.

Step 1:

The first step (See step 1 in Theorem 8.4) corresponds to the elimination of the region which is far from $q^0(x,\xi)=\lambda$. Let us write (cf Formula (8.23))

$$\begin{aligned}&\widetilde{\mathrm{Tr}}\left(\chi(\lambda^{\delta}(\mathrm{Q}-\lambda\mathrm{I_d}))(\mathrm{I_d}-\chi(\lambda^{-\tau}(\mathrm{Q}_0-\lambda\mathrm{I_d})))\right)\\ &= (2\pi)^{-n-1}\lambda^{-\delta}\int_{\mathbb{R}\times\mathbb{R}^n}\hat{\chi}(\lambda^{-\delta}t)e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\left(1-\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))\right)dtd\xi\ .\end{aligned} \tag{8.58}$$

Since $|q^0(\xi)-\lambda| > \lambda^{\tau}/4$ on the support of $\xi \mapsto (1-\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))$, we may integrate by parts in t and get, for any integer M,

$$\begin{aligned}&\widetilde{\mathrm{Tr}}\left(\chi(\lambda^{\delta}(\mathrm{Q}-\lambda\mathrm{I_d}))(\mathrm{I_d}-\chi(\lambda^{-\tau}(\mathrm{Q}_0-\lambda)))\right)\\ &\quad= i^M(2\pi)^{-n-1}\lambda^{-\delta}\int_{\mathbb{R}\times\mathbb{R}^n}(q^0(\xi)-\lambda)^{-M}e^{it(q^0(\xi)-\lambda)}\\ &\qquad\times\partial_t^M\left(\hat{\chi}(\lambda^{-\delta}t)\hat{w}_0(t,\xi)\right)(1-\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))dtd\xi\ .\end{aligned} \tag{8.59}$$

So taking into account (8.26), we get, if $M > n$, the estimate

$$\begin{aligned}&|\widetilde{\mathrm{Tr}}\left(\chi(\lambda^{\delta}(\mathrm{Q}-\lambda\mathrm{I_d}))(\mathrm{I_d}-\chi(\lambda^{-\tau}(\mathrm{Q}_0-\lambda)))\right)|\\ &\leq C_{0,M}\times\int_{\mathbb{R}^n}(q^0(\xi)-\lambda)^{-M}\left((1-\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))\right)d\xi\\ &\leq C_{1,M}\lambda^{-\tau M+n}\ ,\end{aligned} \tag{8.60}$$

---

[4]Let us observe that $\widetilde{\mathrm{Tr}}\left[\mathrm{Y}_{]-\infty,\lambda]}(\mathrm{Q})\right] = \widetilde{\mathrm{Tr}}\left[\mathrm{Y}_{]0,\lambda]}(\mathrm{Q})\right]$ as can be seen from Section 2 and the fact that, for all the Floquet conditions, the operators Q^{θ} are strictly positive according to our choice of λ_0 (cf (1.21)).

We finally obtain that

$$\widetilde{\mathrm{Tr}}\{\chi(\lambda^\delta(\mathrm{Q}-\lambda\mathrm{I_d}))(\mathrm{I_d}-\chi(\lambda^{-\tau}(\mathrm{Q_0}-\lambda\mathrm{I_d})))\} = \mathcal{O}(\lambda^{-\infty}),\ \text{as } \lambda\to+\infty. \quad (8.61)$$

More generally we get also the property with $(1-\chi)$ replaced by any function ψ, whose support is disjoint of 0. We can also consider, for any "good" symbol $\xi\mapsto p(\xi)$ in $S(\langle\xi\rangle^\ell; g^1_\xi)$, and get a similar result:

$$\begin{array}{l}(2\pi)^{-n-1}\lambda^{-\delta}\ \int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)p(\xi)\left(\psi(\lambda^{-\tau}(q^0(\xi)-\lambda))\right)dtd\xi\\ \qquad = \mathcal{O}(\lambda^{-\infty})\ .\end{array} \quad (8.62)$$

Step 2:

We are now localized near $q^0(\xi)=\lambda$ and shall eliminate the contribution corresponding to $|t|$ large. This will obtain through integration by parts involving the operator L introduced in the proof of Theorem 8.4.
For any integer $k>0$, we get, as in step 2 of Theorem 8.4,

$$\begin{array}{l}\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))\,dt\,d\xi\\ =\Big(i^k\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}t^{-k}\times\\ \qquad\qquad [{}^tL]^k(\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda)))dt\,d\xi\Big)\ .\end{array} \quad (8.63)$$

As for the proof of (8.61), we get from (8.63) that, as $\lambda\to+\infty$,

$$\begin{array}{l}\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))dtd\xi\\ = i^k\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}t^{-k}\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))\times\\ \qquad\qquad [{}^tL]^k(\hat{w}_0(t,\xi))dt\,d\xi\ +\ \mathcal{O}(\lambda^{-\infty})\ .\end{array} \quad (8.64)$$

(Here we have used that when the derivation ${}^tL$ meet the cutoff function, then we can use the last result obtained in the preceding step (8.62).)
We now use the estimate of Theorem 6.1 (with $\gamma=0$) concerning $\hat{w}_0(t,\xi)$ which gives

$$|[{}^tL]^k(\hat{w}_0(t,\xi))|\leq C\left((\frac{\langle t\rangle}{\langle\xi\rangle})^{k+1}+(\frac{\langle t\rangle}{\langle\xi\rangle})^{2k}\right)\ , \quad (8.65)$$

Proceeding as in step 2 of Theorem 8.4, we obtain the estimate

$$\begin{array}{l}\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))dtd\xi\\ \qquad =\mathcal{O}(\lambda^{-\infty})+\mathcal{O}(|\lambda|^{n-2-k+\tau+\delta})+\mathcal{O}(|\lambda|^{n-1-2k+\tau+k\delta})\ .\end{array} \quad (8.66)$$

Taking k sufficiently large gives the result. We obtain actually

$$\lambda^{-\delta}\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)(1-\chi(t))e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))dtd\xi=\mathcal{O}(|\lambda|^{-\infty})\ . \tag{8.67}$$

Step 3:

We are now left with the main contribution where we integrate in the t variable over a compact. This leads immediately (using uniquely the boundedness of $|\hat{w}_0(t,\xi)|$ for t in a compact) to the following estimate

$$\begin{aligned}&|\lambda^{-\delta}\textstyle\int_{I\!R\times I\!R^n}\hat{\chi}(\lambda^{-\delta}t)\chi(t)e^{it(q^0(\xi)-\lambda)}\hat{w}_0(t,\xi)\chi(\lambda^{-\tau}(q^0(\xi)-\lambda))\,dt\,d\xi|\\ &\quad\leq C\lambda^{-\delta}\textstyle\int_{\lambda-C\lambda^\tau}^{\lambda+C\lambda^\tau}r^{n-1}dr\\ &\quad\leq C'\lambda^{n-1+\tau-\delta}\ .\end{aligned} \tag{8.68}$$

This ends the proof of the lemma.

<u>Proof of Theorem 8.10</u>:

We now define our approximation of $Y_{]0,\lambda]}$ by introducing the function $\varphi_{\lambda,\delta}$,

$$\varphi_{\lambda,\delta}(\tau)=\begin{cases}1 & \text{if } 0<\tau<\lambda\ ,\\ \chi(\lambda^\delta(\tau-\lambda)) & \text{if } \tau>\lambda\ ,\\ \chi(\tau) & \text{if } \tau<0.\end{cases}\quad , \tag{8.69}$$

where χ was introduced in (8.24).

Then $\varphi_{\lambda,\delta}(\tau)\in C_0^\infty(I\!R)$. Let us compute its Fourier transform. It is easy to see that

$$\varphi'_{\lambda,\delta}(\tau)=\lambda^\delta\chi_{1+}(\lambda^\delta(\tau-\lambda))+\chi_{1-}(\tau)\ ,$$

with $\chi_{1\pm}(\tau)=\partial_\tau\chi(\tau)$, if $\pm\tau\geq 0$ and $\chi_{1\pm}(\tau)=0$ if $\pm\tau\leq 0$. So, by (8.24), $\chi_{1\pm}(\tau)\in C_0^\infty(I\!R)$.
We then obtain

$$\hat{\varphi}_{\lambda,\delta}(t)=-it^{-1}\hat{\chi}_{1-}(t)-it^{-1}\hat{\chi}_{1+}(\lambda^{-\delta}t)e^{-it\lambda}. \tag{8.70}$$

As a consequence, we observe that, there exists C such that,

$$|t\hat{\varphi}_{\lambda,\delta}(t)|\leq C\ ,\ \forall\lambda\geq\lambda_0\ . \tag{8.71}$$

We observe here that we have only

$$|\hat{\varphi}_{\lambda,\delta}(t)| \leq C\lambda \ , \ \forall t.$$

The definitions (8.69) of $\varphi_{\lambda,\delta}$ and (8.52) of $\widetilde{N}(\lambda, Q)$ show that

$$|\widetilde{N}(\lambda, Q) - \tilde{\mathrm{T}}\mathrm{r}(\varphi_{\lambda,\delta}(\mathrm{Q}))| \leq \tilde{\mathrm{T}}\mathrm{r}\left(\chi(\lambda^\delta(\mathrm{Q} - \lambda\,\mathrm{I_d}))\right) \ . \tag{8.72}$$

We have used that, as a measure, the density of state of Q is positive. According to the lemma, this implies

$$|\widetilde{N}(\lambda, Q) - \tilde{\mathrm{T}}\mathrm{r}(\varphi_{\lambda,\delta}(\mathrm{Q}))| = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta+\tau}) \ ,$$

and this will be used by saying that, for any $\epsilon > 0$, there exists $\delta \in]1, 2[$ such that

$$|\widetilde{N}(\lambda, Q) - \tilde{\mathrm{T}}\mathrm{r}(\varphi_{\lambda,\delta}(\mathrm{Q}))| = \mathcal{O}(\lambda^{\mathrm{n}-3+\epsilon}) \ . \tag{8.73}$$

Because we know how to control $|\widetilde{N}(\lambda, Q_0) - \tilde{\mathrm{T}}\mathrm{r}(\varphi_{\lambda,\delta}(\mathrm{Q_0}))|$ and have actually an explicit formula for each of these terms, **it remains consequently to study the expression** $\tilde{\mathrm{T}}\mathrm{r}\,(\varphi_{\lambda,\delta}(\mathrm{Q}))$ and this will be done by following carefully what was done in Theorem 8.4.

We first eliminate the large $|t|$ contribution by considering

$$S := \int_{I\!R \times I\!R^n} (1 - \chi(t))\hat{\varphi}_{\lambda,\delta}(t)\hat{w}_0(t, \xi)e^{it(q^0(\xi))}dtd\xi \ . \tag{8.74}$$

According to the decomposition (8.70), we can write

$$S := S_+ + S_- \ , \tag{8.75}$$

with

$$S_+ := -i \int_{I\!R \times I\!R^n} (1 - \chi(t))t^{-1}\hat{\chi}_{1+}(\lambda^{-\delta}t)\hat{w}_0(t, \xi)e^{it(q^0(\xi)-\lambda)}dt\,d\xi \ . \tag{8.76}$$

Study of S_+.

For estimating this integral, we can first localize in energy (step 1 in the preceding proof) through integration by parts in the t variable. If we proceed like in the proof of Theorem 8.4, we obtain

$$-i \textstyle\int_{I\!R \times I\!R^n} (1 - \chi(t))t^{-1}\hat{\chi}_{1+}(\lambda^{-\delta}t)\hat{w}_0(t, \xi)e^{it(q^0(\xi)-\lambda)}(1 - \chi_\lambda(q^0(\xi))dt\,d\xi = \mathcal{O}(|\lambda|^{-\infty}) \ . \tag{8.77}$$

We are then reduced to the study of

$$\hat{S}_+ = -i \int_{I\!R\times I\!R^n} (1-\chi(t))t^{-1}\hat{\chi}_{1_+}(\lambda^{-\delta}t)\hat{w}_0(t,\xi)e^{it(q^0(\xi)-\lambda)}\chi_\lambda(q^0(\xi))dt\,d\xi\ . \tag{8.78}$$

which can be treated as in step 2 of the preceding lemma (with $\tau = 1$). As a result of this first study we have obtained

$$S_+ = \mathcal{O}(|\lambda|^{-\infty})\ . \tag{8.79}$$

Study of S_-.

We have to analyze

$$S_- := -i \int_{I\!R\times I\!R^n} (1-\chi(t))t^{-1}\hat{\chi}_{1_-}(t)\hat{w}_0(t,\xi)e^{it(q^0(\xi))}dt\,d\xi\ . \tag{8.80}$$

We observe that this term is actually independent of λ. An integration by parts in the ξ variable permits to give a sense to the integral and we immediately obtain

$$S_- = \mathcal{O}(1)\ . \tag{8.81}$$

The main part.

It remains now to analyze the "main" part, that is

$$T := \int_{I\!R\times I\!R^n} \chi(t)\hat{\varphi}_{\lambda,\delta}(t)\hat{w}_0(t,\xi)e^{it(q^0(\xi))}dt\,d\xi\ . \tag{8.82}$$

This is done by following step by step the proof of Theorem 8.4.

A first step is to replace the study of T (cf Step 3 in the proof of Theorem 8.4 by the study of a finite number of $T^{(j)}$ ($j = 0,\cdots,N$),

$$T^{(j)} := \int_{I\!R\times I\!R^n} \chi(t)\hat{\varphi}_{\lambda,\delta}(t)\hat{w}_{0,-j}(t,\xi)e^{it(q^0(\xi))}dt\,d\xi\ . \tag{8.83}$$

The error in this reduction is given by

$$T^{'(N+1)} := \int_{I\!R\times I\!R^n} \chi(t)\hat{\varphi}_{\lambda,\delta}(t)\hat{r}_0^{N+1}(t,\xi)e^{it(q^0(\xi))}dt\,d\xi\ . \tag{8.84}$$

It is then better to write it in the form

$$T'^{(N+1)} = \int_{I\!R\times I\!R^n} \chi(t)\,(t\,\hat{\varphi}_{\lambda,\delta}(t))\left(\frac{1}{t}\,\hat{r}_0^{N+1}(t,\xi)\right) e^{it(q^0(\xi))}dt\,d\xi\ ,$$

and to use (7.6) and (8.71). This leads to the following estimate for the remainder,

$$T'^{(N+1)} = \mathcal{O}(1)\ . \tag{8.85}$$

In a second step, we treat each term $T^{(j)}$ by taking a Taylor expansion at $t=0$. The error is estimated as around (8.35) using integration by parts in ξ based on the relation $ite^{itq^0(\xi)} = Le^{itq^0(\xi)}$. This integration by parts argument is only valid when the support of the amplitude with respect to the ξ-variable avoids the origin. But the contribution when ξ belongs to a compact is directly estimated using (8.71). This error is again in $\mathcal{O}(1)$.
This leads to the analysis of

$$T^{(j,k)} := \int_{I\!R\times I\!R^n} t^k\chi(t)\hat{\varphi}_{\lambda,\delta}(t)(\partial_t^k\hat{w}_{0,-j}(0,\xi))e^{it(q^0(\xi))}dt\,d\xi\ , \tag{8.86}$$

which can be also written as

$$T^{(j,k)} := \int_{I\!R\times I\!R^n} \chi(t)\hat{\varphi}_{\lambda,\delta}(t)(({}^tL^k)\,\partial_t^k\hat{w}_{0,-j}(0,\xi))e^{it(q^0(\xi))}dt\,d\xi\ . \tag{8.87}$$

The elimination of χ is justified in the same way as we have analyzed S, replacing $\hat{w}_0(t,\xi)$ by $(({}^tL^k)\,\partial_t^k\hat{w}_{0,-j}(0,\xi))$.
This leads again to an error in $\mathcal{O}(1)$ and we have consequently

$$T^{(j,k)} = S^{(j,k)} + \mathcal{O}(1)\ , \tag{8.88}$$

with

$$S^{(j,k)} := (2\pi)\int_{I\!R^n} \varphi_{\lambda,\delta}(q^0(\xi))(({}^tL)^k\partial_t^k\hat{w}_{0,-j}(0,\xi))\,d\xi\ . \tag{8.89}$$

We finally get that, for N large enough,

$$\widetilde{\mathrm{Tr}}(\varphi_{\lambda,\delta}(\mathrm{Q})) = \sum_{\mathrm{j}=0}^{\mathrm{N}}\int_0^{+\infty} \mathrm{a}_{\mathrm{n}-1-\mathrm{j}}(\mathrm{r})\varphi_{\lambda,\delta}((\mathrm{r}^2+\lambda_0)^{1/2})\mathrm{dr} + \mathcal{O}(1). \tag{8.90}$$

We can now treat each of these terms separately. We first observe that

$$\begin{aligned}&\textstyle\int_0^{+\infty} a_{n-1}(r)\varphi_{\lambda,\delta}((r^2+\lambda_0)^{1/2})dr\\ &\quad = \textstyle\int_0^{+\infty} a_{n-1}(r)(Y_{[0,\lambda]}(r^2+\lambda_0)^{1/2})dr + \mathcal{O}(|\lambda|^{n-1-\delta})\ .\end{aligned} \tag{8.91}$$

The main term in (8.91) is exactly $\tilde{N}(\lambda, Q_0)$.
Using the property that a_{n-2} is 0 and that a_{n-3} behaves like a_{n-4}, we verify that

$$\begin{aligned}&\sum_{j=1}^{N} \int_0^{+\infty} a_{n-1-j}(r)\varphi_{\lambda,\delta}((r^2+\lambda_0)^{1/2})dr \\ &= \sum_{j=3}^{N} \int_0^{+\infty} \tilde{a}_{n-1-j}(r)\varphi_{\lambda,\delta}((r^2+\lambda_0)^{1/2})dr \\ &= \mathcal{O}(|\lambda|^{n-3}) + \mathcal{O}(1)\ ,\end{aligned} \tag{8.92}$$

with $\tilde{a}_{n-4} = a_{n-3} + a_{n-4}$ and $\tilde{a}_{n-j} = a_{n-j}$ for $j > 4$.

This permits to conclude the proof of the theorem by choosing, for any $\epsilon > 0$, a suitable $\delta \in]1, 2[$ in (8.73).

9 Proof of Theorem 1.1 and Theorem 2.1

Proof of Theorem 1.1 :

Theorem 1.1 follows easily from Theorem 8.10.

If we take $\lambda = (\mu + \lambda_0)^{1/2}$, with $\mu >> 1$, we get that $N(\mu; P_V) = \widetilde{N}(\lambda; Q)$ and $\widetilde{N}(\lambda; Q_0) = a_n \mu^{n/2}$, with the a_n of Theorem 1.1. We can of course reduce the proof to the case when $\int_{\mathcal{K}} V(x)dx = 0$ by changing the potential V by a constant.

Proof of Theorem 2.1:

The formulas (2.6) and (2.7) show that, for every $f \in C_0^\infty(I\!\!R)$,

$$\varrho_V^b(f) = (2\pi)^{-n} |\mathcal{K}|^{-1} \int_{\mathcal{K} \times I\!\!R^n} e^{ib\cdot\xi} \sigma_f(x,\xi) dx d\xi = \widetilde{\mathrm{Tr}}_{\mathrm{b}}(\mathrm{f}(\mathrm{P_V})), \tag{9.1}$$

where $\sigma_f(x, \xi)$ is the symbol of $f(P_V)$: $\sigma_f(x, D) = f(P_V)$.
If $\mu >> 1$, and $\lambda = (\mu + \lambda_0)^{1/2}$, then we get by (2.9)

$$N_b(\mu; P_V) = \varrho_V^b(\chi_{]-\infty,\mu[}(P_V)) = \widetilde{\mathrm{Tr}}_{\mathrm{b}}\left(\chi_{]-\infty,\lambda[}(\mathrm{Q})\right). \tag{9.2}$$

Now, if $\varphi_{\lambda,\delta}(\tau)$ is the function defined by (8.70), then, using (2.11), we get

$$|\widetilde{N}_b(\lambda; Q) - \widetilde{\mathrm{Tr}}_{\mathrm{b}}(\varphi_{\lambda,\delta}(\mathrm{Q}))| \le \widetilde{\mathrm{Tr}}\left(\chi(\lambda^\delta(\mathrm{Q} - \lambda \mathrm{I_d}))\right)\ . \tag{9.3}$$

If δ is as in Lemma 8.11, we get from this lemma and from (9.3) that, for every fixed $\tau > 0$,

$$|\widetilde{N}_b(\lambda; Q) - \widetilde{\mathrm{Tr}}_{\mathrm{b}}(\varphi_{\lambda,\delta}(\mathrm{Q}))| = \mathcal{O}(\lambda^{\mathrm{n}-1-\delta+\tau}) \,, \tag{9.4}$$

as $\lambda \to +\infty$.

We are now reduced to the study of $\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\varphi_{\lambda,\delta}(\mathrm{Q}))$.

By (9.1), we may write

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}(\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}-1} \int_{\mathrm{R}\times\mathrm{R}^{\mathrm{n}}} \hat{\varphi}_{\lambda,\delta}(\mathrm{t})\hat{\mathrm{w}}_0(\mathrm{t},\xi)\mathrm{e}^{\mathrm{i}(\mathrm{tq}^0(\xi)+\mathrm{b}\cdot\xi)}\mathrm{dtd}\xi. \tag{9.5}$$

Now the proof remains essentially the same as for Theorem 8.10. In a first step, we eliminate the contribution for $|t|$ large. The proof is the same (control of S in this proof) if we replace the cut-off function χ by the b-dependent function $t \mapsto \chi_b(t) = \chi(C_b^{-1}t)$, where $C_b > 0$ is a large constant depending only on $|b|$. The integrations by parts in the ξ variable (after elimination of a small neighborhood of 0 in the ξ variable) are now based on the identity

$$(1-\chi_b(t))L_{t,b}e^{i(tq^0(\xi)+b\cdot\xi)} = (1-\chi_b(t))ie^{i(tq^0(\xi)+b\cdot\xi)} \,,$$

where

$$L_{t,b} := [t + q^0(\xi)|\xi|^{-2}b\cdot\xi]^{-1}q^0(\xi)|\xi|^{-2}\xi\partial_\xi \,,$$

(instead of the previous identity $(1-\chi(t))t^{-1}Le^{itq^0(\xi)} = (1-\chi(t))ie^{itq^0(\xi)}$). Here we observe that when $|t|$ is large, then we have

$$t \approx t + q^0(\xi)|\xi|^{-2}b\cdot\xi \,.$$

We have obtained as $\lambda \to +\infty$,

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}-1} \int_{\mathrm{R}\times\mathrm{R}^{\mathrm{n}}} \chi_{\mathrm{b}}(\mathrm{t})\hat{\varphi}_{\lambda,\delta}(\mathrm{t})\hat{\mathrm{w}}_0(\mathrm{t},\xi)\mathrm{e}^{\mathrm{i}(\mathrm{tq}^0(\xi)+\mathrm{b}\xi)}\mathrm{dtd}\xi + \mathcal{O}(1). \tag{9.6}$$

This corresponds, in the proof of Theorem 8.10, to the "main" part (8.82). Using Theorem 7.1 and Remark 7.2, we find that

$$\chi_b(t)[\hat{w}_0(t,\xi) - 1] \ \in S(<\xi>^{-2}; g_t^0 + <\xi>^{-2} g_\xi^0), \tag{9.7}$$

and also

$$\frac{\chi_b(t)}{t}[\hat{w}_0(t,\xi)-1] \ \in S(<\xi>^{-2}; g_t^0+<\xi>^{-2}\, g_\xi^0). \tag{9.8}$$

We let, for $\xi \neq 0$,

$$t_0(\xi) = -q^0(\xi) \mid \xi \mid^{-2} b\xi, \tag{9.9}$$

and it is clear that

$$t_0 \in S(1; g_\xi^1) \ ,$$

after eliminating the small ξ region.
More precisely, this means that, for any compactly supported C^∞ function ψ such that $\psi = 1$ near 0, we have

$$(1-\psi)t_0 \in S(1; g_\xi^1) \ . \tag{9.10}$$

With this notation, we have the following formula

$$Le^{i(tq^0(\xi)+b\cdot\xi)} = i\,(t-t_0(\xi))e^{i(tq^0(\xi)+b\cdot\xi)} \ , \tag{9.11}$$

which will permit us later useful integration by parts.
By Taylor formula, we may write, for $|\xi| \neq 0$, that

$$\hat{w}_0(t,\xi) - 1 - \sum_{j=1}^{n-2} \frac{(t-t_0(\xi))^j}{j!}\partial_t^j \hat{w}_0(t_0(\xi),\xi) = \frac{(t-t_0(\xi))^{n-1}}{(n-1)!}c(t,\xi), \tag{9.12}$$

and by (9.7), we get that

$$(1-\psi(\xi))\chi_b(t)c(t,\xi) \ \in S(<\xi>^{-2}; g_t^0+<\xi>^{-2}\, g_\xi^0). \tag{9.13}$$

This leads first to the following decomposition for $\widetilde{\mathrm{Tr}}_{\mathrm{b}}\,((\varphi_{\lambda,\delta}(\mathrm{Q}))$

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}\,((\varphi_{\lambda,\delta}(\mathrm{Q})) = \mathrm{I}_1 + \mathrm{I}_2 + \mathcal{O}(1) \ , \tag{9.14}$$

with

$$I_1 = (2\pi)^{-n-1}\int_{I\!R\times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(tq^0(\xi)+b\xi)}dtd\xi \ , \tag{9.15}$$

$$I_2 = (2\pi)^{-n-1}\int_{I\!R\times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)[\hat{w}_0(t,\xi)-1]e^{i(tq^0(\xi)+b\xi)}dtd\xi \ . \tag{9.16}$$

We first eliminate the contribution of "$|\xi|$ small" by introducing as above a compactly supported cutoff function $\xi \mapsto \psi(\xi)$ such that $\psi = 1$ near 0. Using a previous argument, we first observe that

$$(2\pi)^{-n-1} \int_{I\!R \times I\!R^n} \psi(\xi)\chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)[\hat{w}_0(t,\xi) - 1]e^{i(tq^0(\xi)+b\xi)}dtd\xi = \mathcal{O}(1) \ . \quad (9.17)$$

We are left with the analysis of

$$I_2' = (2\pi)^{-n-1} \int_{I\!R \times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)(1 - \psi(\xi)) \, [\hat{w}_0(t,\xi) - 1]e^{i(tq^0(\xi)+b\xi)}dtd\xi \ . \quad (9.18)$$

We then decompose I_2', with help of Formula (9.12) multiplied by $(1 - \psi(\xi))$, and integrating by parts (with (9.11)), we get that

$$\begin{aligned} I_2 &= (2\pi)^{-n-1} \int_{I\!R \times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(tq^0(\xi)+b\xi)} \sum_{j=1}^{n-2} \tfrac{i^j}{j!}({}^tL)^j \partial_t^j \hat{w}_0'(t_0(\xi),\xi)dtd\xi \\ &\quad + (2\pi)^{-n-1} \int_{I\!R \times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(tq^0(\xi)+b\xi)} \tfrac{i^{n-1}}{(n-1)!}({}^tL)^{n-1}c'(t,\xi)dtd\xi \\ &:= I_{21} + I_{22} \ . \end{aligned} \quad (9.19)$$

Here we have used the notation $c'(t,\xi) = (1 - \psi(\xi))c(t,\xi)$ and $\hat{w}'(t,\xi) = (1 - \psi(\xi))\hat{w}(t,\xi)$ which will permit to justify the integration by parts using L. We then omit the " ' " but keep in mind that the support of the symbol is disjoint of a neighborhood of 0 with respect to ξ.
We recall that, by (8.70), there exists a constant C such that

$$|t\hat{\varphi}_{\lambda,\delta}(t)| \leq C \ , \ \forall t \in I\!R \ , \ \forall \lambda > \lambda_0 \ . \quad (9.20)$$

Using also (9.13), we obtain that, as $\lambda \to +\infty$,

$$\begin{aligned} I_{22} &= \int_{I\!R \times I\!R^n} \chi_b(t)\hat{\varphi}_{\lambda,\delta}(t)e^{i(tq^0(\xi)+b\xi)} \tfrac{i^{n-1}}{(n-1)!}({}^tL)^{n-1}c(t,\xi)dtd\xi \\ &= \int_{I\!R \times I\!R^n} \hat{\varphi}_{\lambda,\delta}(t)\chi_b(t)e^{i(tq^0(\xi)+b\xi)} \tfrac{i^{n-1}}{(n-1)!}({}^tL)^{n-1}c(0,\xi)dtd\xi + \mathcal{O}(1) \ . \end{aligned}$$

We then write, eliminating the contribution of χ_b as in the beginning of the proof but for each term of the expansion, that

$$\begin{aligned} I_{22} &= \int_{I\!R \times I\!R^n} \hat{\varphi}_{\lambda,\delta}(t)e^{i(tq^0(\xi)+b\xi)} \tfrac{i^{n-1}}{(n-1)!}({}^tL)^{n-1}c(0,\xi)dtd\xi + \mathcal{O}(1) \\ &= 2\pi \int_{I\!R^n} \varphi_{\lambda,\delta}(q^0(\xi))e^{ib\xi} \tfrac{i^{n-1}}{(n-1)!}({}^tL)^{n-1}c(0,\xi)d\xi + \mathcal{O}(1). \end{aligned}$$

So, as $\lambda \to +\infty$,

$$I_{22} = \mathcal{O}(1) \ . \quad (9.21)$$

Let us now analyze I_{21}.

By (9.7), for $j \geq 1$,

$$\partial_t^j \hat{w}_0(t_0(\xi),\xi) \ \in S(<\xi>^{-2}; g_\xi^1), \tag{9.22}$$

then using (9.22) and, eliminating the cut-off function χ_b, we get the existence of c_3 in $S(\langle\xi\rangle^{-3})$ such that, as $\lambda \to +\infty$,

$$I_{21} = \int_{I\!R^n} \varphi_{\lambda,\delta}(q^0(\xi)) e^{ib\xi} c_3(\xi) d\xi + \mathcal{O}(1) \ . \tag{9.23}$$

Using the definition of $\varphi_{\lambda,\delta}$, we finally obtain

$$I_{21} = \int_{q^0(\xi)<\lambda} e^{ib\xi} c_3(\xi) d\xi + \mathcal{O}(\lambda^{n-4-\delta}) + \mathcal{O}(1) \ . \tag{9.24}$$

Similarly, we obtain for the main contribution, after elimination of the cut-off function χ_b and an explicit estimate of $| \int_{q^0(\xi)<\lambda} e^{ib\xi} d\xi - \int \varphi_{\lambda,\delta}(q^0(\xi)) e^{ib\xi} d\xi|$,

$$I_1 = \int_{q^0(\xi)<\lambda} e^{ib\xi} d\xi + \mathcal{O}(\lambda^{n-1-\delta}) + \mathcal{O}(1) \ . \tag{9.25}$$

The control of I_{21} is a simple consequence of the following lemma (which will be applied with $m = 3$)

Lemma 9.1 *:*
If $0 \leq m$, $b \neq 0$ and c belongs to $S(<\xi>^{-m}; g_\xi^1)$, then, as $\tau \to +\infty$,

$$\int_{\xi \in I\!R^n, \ |\xi|<\tau} e^{ib\xi} c(\xi) d\xi = \mathcal{O}(1) + \mathcal{O}(\tau^{\frac{n-1-2m}{2}}). \tag{9.26}$$

Lemma 9.1 comes from the stationary phase Theorem. As done in (2.15) (Section 2), we first use Stokes Theorem and get, if $b \neq 0$,

$$\begin{aligned} \textstyle\int_{\xi \in I\!R^n, \ |\xi|<\tau} e^{ib\xi} c(\xi) d\xi \ &= -i|b|^{-2} \tau^{n-1} \textstyle\int_{S^{n-1}} (b\cdot\omega) c(\tau\omega) e^{i\tau b\cdot\omega} \\ &\quad + \textstyle\int_{\xi \in I\!R^n, \ |\xi|<\tau} e^{ib\xi} \{ i \frac{b}{|b|^2} \partial_\xi \} \ c(\xi) d\xi \ . \end{aligned} \tag{9.27}$$

This can be iterated and we get integrals on the boundary of the type above with different c's in $S(\langle\xi\rangle^{-m}; g_\xi^1)$ (or better), for which the stationary phase Theorem is available, and a remainder of the following type

$$\int_{\xi \in I\!R^n, \ |\xi|<\tau} e^{ib\xi} \{ i \frac{b}{| \ b \ |^2} \partial_\xi \}^k c(\xi) d\xi$$

which is bounded, if $n < m + k$.

Remark 9.2 *:*
In the case when $m > n$, we immediately obtain that the integral is bounded without integration by parts.

Coming back to our control of I_{21}, we get

$$I_{21} = \mathcal{O}(\lambda^{\frac{n-7}{2}}) + \mathcal{O}(1) + \mathcal{O}(\lambda^{n-1-\delta}) \ . \tag{9.28}$$

Combining (9.14), (9.21), (9.25) and (9.28), we have finally obtained

$$\widetilde{\mathrm{Tr}}_{\mathrm{b}}((\varphi_{\lambda,\delta}(\mathrm{Q})) = (2\pi)^{-\mathrm{n}} \int_{\xi \in \mathrm{R}^{\mathrm{n}},\ \mathrm{q}^0(\xi) < \lambda} \mathrm{e}^{\mathrm{ib}\xi} \mathrm{d}\xi + \mathcal{O}(\lambda^{\mathrm{n}-1-\delta}) + \mathcal{O}(\lambda^{\frac{\mathrm{n}-7}{2}}) + \mathcal{O}(1) \ . \tag{9.29}$$

We recall for the next argument that this can be done for any $\delta < 2$.

Theorem 2.1 follows from (9.29) and (9.4). We have indeed obtained, by playing with δ, that, for every fixed $\epsilon > 0$, as $\lambda \to +\infty$,

$$\widetilde{N_b}(\lambda; Q) = \widetilde{N_b}(\lambda; Q_0) + \mathcal{O}(\lambda^{n-3+\epsilon}) \ .$$

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