A LOWER BOUND FOR THE COVARIANCE OF POSITIVE EVENTS

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Abstract. We prove a lower bound on the covariance of two positive events in terms of the probabilities of their pivotal events for probability spaces that verify the conditions of FKG inequality.

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1. Introduction, notation and statement of the results.

FKG inequality, first proved by Harris ([4]) in the contest of product measures, and then extended to more general measures by Fortuin, Kasteleyn and Ginibre ([3]) (see also [2] and [1]), has been proved to be a basic tool in percolation theory, reliability theory, statistical mechanics and other areas of probability theory. It applies to probability spaces endowed with a partially ordered structure and with probability measures satisfying suitable conditions with respect to this structure. We start by giving some necessary notation and definitions.

Let M be a finite set and let $\Omega = \bigotimes_{x\in M} S_x$ where $S_x \subset I\!R$. Given $\omega, \omega' \in \Omega$ we write $\omega \leq \omega'$ when $\omega_x \leq \omega'_x$ $\forall x \in M$. Given $\omega, \omega' \in \Omega$ $\omega'' = \omega \vee \omega'$ is defined by $\omega''_x = \max(\omega_x, \omega'_x)$ and similarly $\omega''' = \omega_x \wedge \omega'_x$ is defined by $\omega'''_x = \min(\omega_x, \omega'_x)$ $\forall x \in M$.

A function $f : \Omega \to I\!R$ is said to be nondecreasing (resp. nonincreasing) if $f(\omega) \leq f(\omega')$ (resp. $f(\omega) \geq f(\omega')$ when $\omega \leq \omega'$. Given an event $E \subset \Omega$ we denote by χ_E its indicator, i. e. $\chi_E(\omega) = 1$ for $\omega \in E$ and $\chi_E(\omega) = 0$ for $\omega \in E^c$. An event is said to be positive (resp. negative) if its indicator is a nondecreasing (resp. nonincreasing) function.

Given an event E, $x \in M$ and $\omega \in \Omega$ we say that x is pivotal for E in ω if there exists $t \in S_x$ such that $\chi_F(\omega') \neq \chi_F(\omega')$ where $\omega'_y = \omega_y$ for $y \neq x$ and $\omega'_x = t$. We denote by $\delta_x E$ the event that x is pivotal for F and we put $\delta_x^{(i)} E = \delta_x E \cap E$ and $\delta_x^{(e)} F = \delta_x E \cap E^c$.

Let

$$\lambda = k \cdot \bigotimes_{x\in M} \nu_x, \tag{1.1}$$

where ν_x is a probability measure and $k : \Omega \to I\!R$ is a nonzero strictly positive function integrable with respect to $\bigotimes_{x\in M} \nu_x$ satisfying $\forall \omega, \omega' \in \Omega$

$$k(\omega \vee \omega') k(\omega \wedge \omega') \geq k(\omega) k(\omega'). \tag{1.2}$$

Let

$$\mathbf{P} = Z^{-1}\lambda \qquad \text{where} \quad Z = \lambda(\Omega). \tag{1.3}$$

FKG inequality ([2]) states that if f and g are nondecreasing measurable functions on Ω such that f, g and fg are integrable with respect to λ, then

$$\mathbf{E}(fg) \geq \mathbf{E}(f)\mathbf{E}(g), \tag{1.4}$$

where $\mathbf{E}$ denotes the expectation with respect $\mathbf{P}$. In particular if E and F are positive events then

$$\mathbf{P}(E \cap F) \geq \mathbf{P}(E)\mathbf{P}(F). \tag{1.5}$$

In many situations it is useful to have a lower bound on the quantity $\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F)$. Our results give estimates of this kind in terms of probabilities of pivotal events and related events. Let us first consider the simpler case when for some $x \in M$ S_x has two elements that for simplicity of notation we take to be 0 and 1.

Theorem 1.1. Let $S_x = \{0, 1\}$ for some $x \in M$ and $\mathbf{P}$ be defined as in (1.1), (1.2) and (1.3); then for every positive events E and F we have

$$\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \max\left(\frac{\mathbf{P}(\omega_x = 0)}{\mathbf{P}(\omega_x = 1)} \mathbf{P}(\delta_x^{(i)} E)\mathbf{P}(\delta_x^{(i)} F), \frac{\mathbf{P}(\omega_x = 1)}{\mathbf{P}(\omega_x = 0)} \mathbf{P}(\delta_x^{(e)} E)\mathbf{P}(\delta_x^{(e)} F) \right). \tag{1.6}$$

In the case that for every $x \in M$ the state space S_x consists of two elements, that for simplicity of notation we take to be 0 and 1, we get the inequality stated in the following corollary.

Corollary 1.2. In the same conditions of theorem 1.1 if $S_x = \{0,1\}$ $\forall x \in M$ we have

$$\begin{aligned} &\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \\ &\max_{x\in M} \max \left(\frac{\mathbf{P}(\omega_x = 0)}{\mathbf{P}(\omega_x = 1)} \mathbf{P}(\delta_x^{(i)} E)\mathbf{P}(\delta_x^{(i)} F), \frac{\mathbf{P}(\omega_x = 1)}{\mathbf{P}(\omega_x = 0)} \mathbf{P}(\delta_x^{(e)} E)\mathbf{P}(\delta_x^{(e)} F) \right). \end{aligned} \tag{1.7}$$

Remark 1.3. We remark that the r. h. s. of (1.7) is 0 if E and F have disjoint supports, i. e. if they depend on the components of ω in two disjoint subsets of M; when k is identically O two events E and F with disjoint supports are independent so that in this case the l. h. s. of (1.4) is actually 0.

Let us now condider the case of a general state space. Given $x \in M$ let P_x denote the marginal of $\mathbf{P}$ on the coordinate x. Given $\sigma \in S_x$ let $\mathbf{P}^{\sigma,x}$ denote conditional measure $\mathbf{P}$ given that the coordinate in x is σ. Given an event E, $x \in M$ and $\sigma \in S_x$, let $E_{\sigma,x}$ be the event defined as the set of ω such that $\omega' \in E$ where $\omega'_y = \omega_y$ for $y \neq x$ and $\omega'_x = \sigma$.

Theorem 1.4. Let $\mathbf{P}$ be as defined in (1.1), (1.2) and (1.3). Then if E and F are positive events we have the two inequalities

$$\begin{aligned} &\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \max_{x\in M} \\ &\left(\int dP_x(\sigma) \int_{\sigma' > \sigma} dP_x(\sigma') \mathbf{P}^{\sigma',x}(E \backslash E_{\sigma,x}) \mathbf{P}^{\sigma',x}(F \backslash F_{\sigma,x}) \right) \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} &\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \max_{x\in M} \\ &\left(\int dP_x(\sigma) \int_{\sigma' > \sigma} dP_x(\sigma') \mathbf{P}^{\sigma,x}(E_{\sigma',x} \backslash E) \mathbf{P}^{\sigma,x}(F_{\sigma',x} \backslash F) \right) \end{aligned} \tag{1.9}$$

Remark 1.4. It is easy to check that (1.8) and (1.9) reduce to (1.7) when $S_x = \{0,1\}$ for every $x \in M$. We remark that as for (1.7) the r. h. s. of (1.8) and (1.9) is 0 if E and F have disjoint supports. Even if (1.8) and (1.9) look more complicate than the bounds given in (1.6) and (1.7), the quantities that appear on their right hand sides are easy to estimate in concrete interesting situations since the two factors inside the integrals depend each only on one of the two events.

2. Proofs of the results. The proof are base on the duplication method of the proof of FKG inequality (see [2]).

Proof of theorem 1.1 and corollary 1.2. By considering two independent copies of Ω we can write

$$2Z^2(\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F)) = \int\int (\chi_E(\omega) - \chi_E(\omega'))(\chi_F(\omega) - \chi_F(\omega'))d\lambda(\omega)d\lambda(\omega'). \tag{2.1}$$

We choose $x \in M$ and with a slight abuse of notation we write $\omega = (\sigma, s)$, $\omega' = (\sigma', t)$, where the first component is the coordinate in x and the other represents the remainder of the coordinates. We denote by ν the probability measure $\bigotimes_{y\in M,\, y\neq x} \nu_y$.

We can then separate the integrals over the two components. Let

$$
\begin{aligned}
A(\sigma) &= \int \chi_E(\sigma, s) k(\sigma, s) d\nu(s) \\
B(\sigma) &= \int \chi_F(\sigma, s) k(\sigma, s) d\nu(s) \\
C(\sigma) &= \int \chi_E(\sigma, s)\chi_F(\sigma, s) k(\sigma, s) d\nu(s) \\
Z(\sigma) &= \int k(\sigma, s) d\nu(s).
\end{aligned}
\tag{2.2}
$$

The expression (2.1) is then equal to

$$
\int\int g(\sigma, \sigma')\, d\nu_x(\sigma) d\nu_x(\sigma'), \tag{2.3}
$$

where

$$
\begin{aligned}
&Z(\sigma)Z(\sigma')g(\sigma,\sigma') = \\
&Z(\sigma)Z(\sigma')(H(\sigma)Z(\sigma') + Z(\sigma)H(\sigma') - A(\sigma)B(\sigma') - B(\sigma)A(\sigma')) = \\
&= Z(\sigma)^2(Z(\sigma')H(\sigma') - A(\sigma')B(\sigma')) + Z(\sigma')^2(Z(\sigma)H(\sigma) - A(\sigma)B(\sigma'))+ \\
&(Z(\sigma)A(\sigma') - Z(\sigma')A(\sigma))(Z(\sigma)B(\sigma') - Z(\sigma')B(\sigma)).
\end{aligned}
\tag{2.4}
$$

By adding and subtracting two terms the r. h. s. of equation (2.4) can be written as

$$
\begin{aligned}
&Z(\sigma)^2(Z(\sigma')C(\sigma') - A(\sigma')B(\sigma')) + Z(\sigma')^2(Z(\sigma)C(\sigma) - A(\sigma)B(\sigma))+ \\
&Z(\sigma)^2A(\sigma')B(\sigma') + Z(\sigma')^2A(\sigma)B(\sigma) - Z(\sigma)Z(\sigma')A(\sigma)B(\sigma') - Z(\sigma)Z(\sigma')A(\sigma')B(\sigma).
\end{aligned}
\tag{2.5}
$$

The last four terms on the r. h. s. of (2.5) can be expressed as a product of two factors so that we have

$$
\begin{aligned}
&Z(\sigma)Z(\sigma')g(\sigma,\sigma') = \\
&Z(\sigma)^2(Z(\sigma')C(\sigma') - A(\sigma')B(\sigma'))+ \\
&Z(\sigma')^2(Z(\sigma)C(\sigma) - A(\sigma)B(\sigma))+ \\
&\left(Z(\sigma)A(\sigma') - +Z(\sigma')A(\sigma)\right)\left(Z(\sigma)B(\sigma') - Z(\sigma')B(\sigma)\right).
\end{aligned}
\tag{2.6}
$$

First we consider the first two terms in the sum on the r. h. s. of equation (2.6).

$$
\begin{aligned}
&Z(\sigma)C(\sigma) - A(\sigma)B(\sigma) = \\
&Z(\sigma)^2\left(\mathbf{P}_\sigma(E\cap F) - \mathbf{P}_\sigma(E)\mathbf{P}_\sigma(F)\right),
\end{aligned}
\tag{2.7}
$$

where $\mathbf{P}_\sigma$ is the measure $\mathbf{P}$ conditioned to fact that the coordinate in x is σ. It is easy to check that $\mathbf{P}_\sigma$ verifies the conditions for FKG inequality so the expression on the l. h. s. of (2.7) is nonnegative. The same holds for the other term.

Let us now consider the last term of the sum on the r. h. s. of (2.6). We observe that it is symmetric with respect to interchange of σ and σ' and that it is 0 when $\sigma = \sigma'$. We can thus consider just the case $\sigma = 0$, $\sigma' = 1$. We have

$$\begin{aligned}
&A(0)Z(1) = \int \chi_E(0,s)k(0,s)\,d\nu(s) \int k(1,s)\,d\nu(s) = \\
&\int \chi_E(0,s)k(0,s)d\nu(s) \int \left(\frac{k(1,s)}{k(0,s)}\right) k(0,s)\,d\nu(s) \leq \\
&\leq \int\int \chi_E(0,s)\left(\frac{k(1,s)}{k(0,s)}\right) k(0,s)d\nu(s)k(0,s)\,d\nu(s) \int k(0,s)d\nu(s) = \\
&= \int \chi_E(0,s)k(1,s)\,d\nu(s) \int k(0,s)d\nu(s),
\end{aligned} \tag{2.8}$$

where we have applied FKG inequality using the fact that the function $h(s) = k(1,s)/k(0,s)$ is nondecreasing by the property (1.2) of the function k.

Therefore

$$\begin{aligned}
&A(0)Z(1) \leq Z(0)A(1) - Z(0)\int (\chi_E(1,s) - \chi_E(0,s))\,k(1,s)\,d\nu(s) = \\
&Z(0)A(1) - \frac{Z}{\nu_x(0)}\mathbf{P}(\delta_x^{(i)}E).
\end{aligned} \tag{2.9}$$

In the same way

$$B(0)Z(1) \leq Z(0)B(1) - \frac{Z}{\nu_x(0)}\mathbf{P}(\delta_x^{(i)}F). \tag{2.10}$$

By putting together equations (2.3), (2.6) and the bounds (2.8), (2.9) we get

$$2Z^2(\mathbf{P}(E\cap F) - \mathbf{P}(E)\mathbf{P}(F)) \geq 2\nu_x(1)\nu_x(0)\frac{Z(0)^2Z^2\mathbf{P}(\delta_x^{(i)}E)\mathbf{P}(\delta_x^{(i)}F)}{\nu_x(1)^2Z(0)Z(1)}. \tag{2.11}$$

We remark that

$$\mathbf{P}(\sigma = i) = \frac{\nu_x(i)Z(i)}{Z} \tag{2.12}$$

so that from (2.10) we get

$$\mathbf{P}(E\cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \frac{\mathbf{P}(\sigma=0)}{\mathbf{P}(\sigma=1)}\mathbf{P}(\delta_x^{(i)}E)\mathbf{P}(\delta_x^{(i)}F) \tag{2.13}$$

If we exchange 0 and 1 FKG property is preserved. We can then apply previous derivation to the events E^c and F^c that become positive if we exchange 0 and 1. This gives

$$\begin{aligned}
&\mathbf{P}(E\cap F) - \mathbf{P}(E)\mathbf{P}(F) = \mathbf{P}(E^c\cap F^c) - \mathbf{P}(E^c)\mathbf{P}(F^c) \geq \\
&\frac{\mathbf{P}(\sigma=1)}{\mathbf{P}(\sigma=0)}\mathbf{P}(\delta_x^{(i)}E^c)\mathbf{P}(\delta_x^{(i)}F^c) = \frac{\mathbf{P}(\sigma=1)}{\mathbf{P}(\sigma=0)}\mathbf{P}(\delta_x^{(e)}E)\mathbf{P}(\delta_x^{(e)}F).
\end{aligned} \tag{2.14}$$

Q. E. D.

Proof of theorem 1.4 We proceed as in the proof of theorem 1.1 up to equation (2.6). The first two terms on the r. h. s. are there shown to be nonnegative by FKG inequality. Let us now consider the last term. It is symmetric and equal to 0 for $\sigma = \sigma'$ so, as in that proof, we can assume that $\sigma < \sigma'$. By proceeding as in equations (2.7) and (2.8) we get

$$
\begin{aligned}
&(Z(\sigma)A(\sigma') - Z(\sigma')A(\sigma))\,(Z(\sigma)B(\sigma') - Z(\sigma')B(\sigma)) \geq \\
&Z(\sigma)^2 \left(\int (\chi_E(\sigma', s) - \chi_E(\sigma, s))\, k(\sigma', s)\, d\nu(s) \right) \\
&\left(\int (\chi_F(\sigma', s) - \chi_F(\sigma, s))\, k(\sigma', s)\, d\nu(s) \right).
\end{aligned}
\tag{2.15}
$$

Therefore from equations (2.1), (2.3), (2.6), (2.15) we obtain

$$
\begin{aligned}
&\mathbf{P}(E \cap F) - \mathbf{P}(E)\mathbf{P}(F) \geq \\
&\int d\nu_x(\sigma) \frac{Z(\sigma)}{Z} \int_{\sigma < \sigma'} d\nu_x(\sigma') \frac{Z(\sigma')}{Z} \\
&\left(\int (\chi_E(\sigma'_x, s) - \chi_E(\sigma_x, s))\, k(\sigma'_x, s)\, \frac{d\nu(s)}{Z(\sigma')} \right) \\
&\left(\int (\chi_F(\sigma', s) - \chi_F(\sigma, s))\, k(\sigma', s)\, \frac{d\nu(s)}{Z(\sigma')} \right).
\end{aligned}
\tag{2.16}
$$

It is easy to check

$$
\begin{aligned}
d\nu_x(\sigma) \frac{Z(\sigma)}{Z} &= d\,P_x(\sigma), \\
k(\sigma, s)\, \frac{d\nu(s)}{Z(\sigma)} &= d\,\mathbf{P}^{\sigma,x}(s).
\end{aligned}
\tag{2.17}
$$

In this way we get (1.8). Equation (1.9) is obtained by the previous derivation after inversion of the order structure in $I\!R$ (which preserves FKG property) and condidering the events E^c and F^c which have the same covariance as E and F. Q. E. D.

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