Effective Perturbation Methods for One-Dimensional Schrödinger Operators

Alexander Kiselev[1,2], Christian Remling[3,4], and Barry Simon[3,5]

1 Introduction

In this paper, we want to discuss a new set of equations that let us relate solutions of

$$-u'' + (V + V_0)u = Eu \tag{1}$$

to solutions of

$$-\varphi'' + V_0\varphi = E\varphi. \tag{2}$$

These equations will be particularly useful in regions where all solutions of (2) remain bounded as $x \to \infty$. We will also discuss the discrete analogs,

$$u(n+1) + u(n-1) + (V + V_0)(n)u(n) = Eu(n) \tag{3}$$

$$\varphi(n+1) + \varphi(n-1) + V_0(n)\varphi(n) = E\varphi(n). \tag{4}$$

If V is in L^1, virtually any perturbation technique allows one to control u (and, in fact, to show that all solutions of (1) are bounded as $x \to \infty$). We are interested in cases where V is not L^1 but is small at infinity in some sense. We want to generalize what has turned out to be a powerful set of tools in case $V_0 \equiv 0$, namely, the use of modified Prüfer equations and their discrete analogs, which were dubbed EFGP equations in [6] on account of contributions of [4, 5, 11].

Explicitly, in the continuum case when $V_0 \equiv 0$ and $E = k^2 > 0$, one defines $R(x), \theta(x)$ by

$$u(x) = R(x)\sin(\theta(x)) \tag{5}$$

$$u'(x) = kR(x)\cos(\theta(x)) \tag{6}$$

and finds that R, θ obey

$$\frac{d}{dx}\ln R(x) = \frac{V(x)}{2k}\sin(2\theta(x)) \tag{7}$$

$$\frac{d\theta}{dx} = k - \frac{V(x)}{k}\sin^2(\theta(x)). \tag{8}$$

[1]Mathematical Sciences Research Institute, 1000 Centennial Drive, Berkeley, CA 94720.

[2]Permanent address: Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago IL 60637-1546. E-mail: kiselev@math.uchicago.edu

[3]Division of Physics, Mathematics, and Astronomy, 253-37, California Institute of Technology, Pasadena, CA 91125.

[4]Permanent address: Fachbereich Mathematik/Informatik, Universität Osnabrück, D-49069 Osnabrück, Germany. E-mail: cremling@mathematik.uni-osnabrueck.de

[5]E-mail: bsimon@caltech.edu. This material is based upon work supported by the National Science Foundation under Grant No. DMS-9401491. The Government has certain rights in this material.

In the discrete case when $V_0 \equiv 0$ and $E = 2\cos(k) \in (-2,2)$, we define $R(n), \theta(n)$ by

$$u(n-1) = R(n)\frac{\sin(\theta(n))}{\sin(k)} \tag{9}$$

$$u(n) = R(n)\frac{\sin(\widetilde{\theta}(n))}{\sin(k)} \tag{10}$$

$$\widetilde{\theta}(n) \equiv \theta(n) + k \tag{11}$$

(in earlier references, the equivalent formula with (10), (11) replaced by $R(n)\cos(\theta(n)) = u(n) - \cos(k)u(n-1)$ is used; also what we denote $\tilde{\theta}$ is called $\bar{\theta}$, but since there will be complex conjugates below, we use $\tilde{}$ in place of $\bar{}$ to avoid confusion) and find that

$$R(n+1)^2 = R(n)^2[1 + U(n)\sin(2\tilde{\theta}(n)) + U(n)^2\sin^2(\tilde{\theta}(n))] \tag{12}$$

$$\cot(\theta(n+1)) = \cot(\tilde{\theta}(n)) + U(n) \tag{13}$$

$$U(n) \equiv -\frac{V(n)}{\sin(k)}. \tag{14}$$

These equations are useful in studying spectral properties [6] and tunneling [8]. One of our main goals is to extend them to situations with $V_0 \not\equiv 0$.

An important ingredient in our extension is the realization that R and θ should be viewed as pieces of a single complex valued function. As a bonus of this point of view, we have a rewriting of (7), (8) and (12), (13) that makes the fact that they are analogs totally transparent and, moreover, is a more tractable version of (13) or its equivalent form noted by Figotin-Pastur:

$$e^{2i\theta(n+1)} = e^{2i\widetilde{\theta}(n)} + \frac{iU(n)}{2}\left(\frac{(e^{2i\widetilde{\theta}(n)}-1)^2}{1-\frac{iU(n)}{2}(e^{2i\widetilde{\theta}(n)}-1)}\right).$$

Namely, in the continuum case, define

$$\rho(x) = R(x)e^{i(\theta(x)-kx)}.$$

Then (7), (8) are equivalent to

$$\frac{d\rho}{dx} = \rho(x)\frac{V(x)}{k}\sin(\theta(x))e^{-i\theta(x)}. \tag{15}$$

In the discrete case, define

$$\rho(n) = R(n)e^{i(\tilde{\theta}(n)-kn)}. \tag{16}$$

Then (12), (13) are equivalent to

$$\rho(n+1) - \rho(n) = \rho(n)U(n)\sin(\tilde{\theta}(n))e^{-i\tilde{\theta}(n)}. \tag{17}$$

It is clear that (17) is the discrete analog of (15); this is not so clear from the form (7), (8) and (12), (13).

In applications, the critical feature of the (R, θ) variables is that $R(x) \sim (u'(x)^2 + u(x)^2)^{1/2}$ (resp. $(u(n)^2 + u(n-1)^2)^{1/2}$) in the sense that for some C independent of V (but dependent on k),

$$C^{-1} R(x)^2 \leq u'(x)^2 + u(x)^2 \leq C R(x)^2.$$

Our variables when $V_0 \not\equiv 0$ will not be quite as simple as (5), (6) and (9), (10), (11) when expressed in that format, but will yield an R, θ with $R \sim (u^2 + (u')^2)^{1/2}$ and will obey

$$\frac{d}{dx} \ln R(x) = \frac{V(x)}{2\gamma'(x)} \sin(2\theta(x))$$
$$\frac{d\theta}{dx} = \gamma'(x) - \frac{V(x)}{\gamma'(x)} \sin^2(\theta(x))$$

where γ' is no longer a constant k and now obeys $0 < \alpha \leq \gamma'(x) \leq \beta < \infty$ for suitable α, β. In the discrete case, (12), (13) will hold but $U(n)$ will no longer be a constant multiple of $V(n)$; rather for suitable α, β: $0 < \alpha \leq -\frac{V(x)}{U(x)} \leq \beta < \infty$.

We will discuss two different applications of these equations in this paper. First, we'll study embedded eigenvalues. We will show that it's possible to generate bound states by perturbations of order $V(x) = O(1/x)$. Then we will generalize Naboko's construction [9] to the case of a periodic background potential.

As our second application, we will show that sufficiently regular ac spectrum can be turned into sc spectrum by a perturbation that tends to zero. This V will be a sparse potential of Pearson type (cf. [12]). We will also prove an auxiliary result on the asymptotic distribution of the function $\gamma(x, E)$ from above which seems to be of independent interest.

As for the applications, there are few differences between the continuous and discrete case. We'll discuss embedded eigenvalues in the continuous case and Pearson potentials in the discrete case, but we might as well have done it the other way around.

The tools we develop here were also used in [14] to generalize a result on stability of ac spectrum to situations with general background potentials. For a different (in fact, earlier) approach to this problem, see [2].

Research of A.K. done at MSRI was supported in part by NSF grant DMS 9022140. C.R. would like to thank for the hospitality of Caltech, where most of this work was done. He would also like to thank the Deutsche Forschungsgemeinschaft for financial support.

2 Basic Variables and Equations in the Continuum Case

Let V_0 be a real valued $L^1_{\rm loc}$ function on $[0,\infty)$. Then, as usual for any E, we define a transfer matrix $T_0(x,E)$ by

$$T_0(x,E)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varphi'(x) \\ \varphi(x) \end{pmatrix}$$

where φ is the unique solution of (2) with $\varphi'(0)=a$, $\varphi(0)=b$. We will suppose $E\in\mathbb{R}$ and that

$$K \equiv \sup_{x>0} \|T_0(x,E)\| < \infty.$$

As usual, define the Wronskian $W(f,g)$ of the C^1 functions f,g by

$$W(f,g)(x) = f(x)g'(x) - g(x)f'(x).$$

If $f''(x) = A(x)f(x)$ and $g''(x) = B(x)g(x)$, then

$$W'(f,g)(x) = (B(x)-A(x))f(x)g(x). \tag{18}$$

In particular, if f,g both solve (2), then W is constant.

Fix $E\in\mathbb{R}$ and φ, a complex solution of (2). Then $W(\bar\varphi,\varphi)$ is constant and

$$W(\bar\varphi,\varphi) = 2i\,\mathrm{Im}(\,\overline{\varphi(x)}\varphi'(x)).$$

If φ is essentially complex, then $W(\bar\varphi,\varphi)\neq 0$. By interchanging φ and $\bar\varphi$, we can suppose $\mathrm{Im}\,W(\bar\varphi,\varphi)>0$. Thus,

$$W(\bar\varphi,\varphi)(x) = i\omega; \qquad \omega = \text{constant} > 0.$$

We could normalize φ so that $\omega=1$ but do not do so to allow the standard choice $\varphi(x)=e^{ikx}$ (corresponding to $\omega=2k$) in case $V_0\equiv 0$. For definiteness, you can think about the solution with $\binom{\varphi'(0)}{\varphi(0)} = \binom{i}{1}$; but in the periodic case, it will be useful to take a Floquet solution instead.

We'll define two phases $\gamma(x),\delta(x)$ by

$$\varphi(x) = |\varphi(x)|e^{i\gamma(x)} \tag{19}$$

$$\varphi'(x) = i|\varphi'(x)|e^{i\delta(x)}. \tag{20}$$

γ will play a central role; δ will not.

Proposition 2.1 *(a)* $2|\varphi(x)|\,|\varphi'(x)|\cos(\gamma(x)-\delta(x)) = \omega$
(b) $\gamma'(x) = \frac{\omega}{2|\varphi(x)|^2} > 0$
(c) Let $\varphi_0 \equiv (|\varphi(0)|^2 + |\varphi'(0)|^2)^{1/2}$. *Then*

$$\frac{\omega}{2K\varphi_0} \leq |\varphi(x)| \leq K\varphi_0$$

$$\frac{\omega}{2K\varphi_0} \leq |\varphi'(x)| \leq K\varphi_0$$

(d) $\frac{\omega}{2K^2\varphi_0^2} \leq \gamma'(x) \leq \frac{K^2\varphi_0^2}{2\omega}$

Proof. (a) This is just evaluating the Wronskian.
(b) $\gamma' = \mathrm{Im}(\frac{\varphi'}{\varphi}) = |\varphi|^{-2}\mathrm{Im}(\bar{\varphi}\varphi') = \frac{\omega}{2|\varphi|^2}$
(c) By definition of K, $|\varphi(x)| \leq K\varphi_0$ and $|\varphi'(x)| \leq K\varphi_0$. On the other hand, by (a), $|\varphi(x)|\,|\varphi'(x)| \geq \frac{\omega}{2}$. Thus, the upper bounds imply the lower bounds.
(d) follows from (b), (c). $\square$

Remark. δ' will not be non-negative in general; indeed,

$$\delta'(x) = \mathrm{Im}\left(\frac{\varphi''}{\varphi'}\right) = (V_0(x) - E)\,\mathrm{Im}\left(\frac{\varphi}{\varphi'}\right)$$

so if $V_0(x) - E \geq 0$, then $\delta' \leq 0$; and if $V_0(x) - E < 0$, then $\delta' > 0$. In case $V_0 \equiv 0$, $\delta' > 0$ but if there are regions with $V_0(x) - E > 0$, then there are regions with $\delta' < 0$. But $\gamma(x) \geq \gamma(0) + \frac{\omega x}{2K^2\varphi_0^2}$, and $|\delta - \gamma|$ is bounded because of Proposition 2.1(a), so δ' is "mainly" positive.

Given a reference complex solution φ to (2) and a real valued solution u of (1), we define $\rho(x) \in \mathbb{C}$ by

$$\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix} = \frac{1}{2i}\left[\rho(x)\begin{pmatrix} \varphi'(x) \\ \varphi(x) \end{pmatrix} - \bar{\rho}(x)\begin{pmatrix} \bar{\varphi}'(x) \\ \bar{\varphi}(x) \end{pmatrix}\right] \tag{21}$$

$$= \mathrm{Im}\left[\rho(x)\begin{pmatrix} \varphi'(x) \\ \varphi(x) \end{pmatrix}\right]. \tag{22}$$

$\binom{\varphi'}{\varphi}$ and $\binom{\bar{\varphi}'}{\bar{\varphi}}$ are linearly independent since $\omega \neq 0$ and so $\binom{u'}{u} = \alpha\binom{\varphi'}{\varphi} + \beta\binom{\bar{\varphi}'}{\bar{\varphi}}$. The reality of u implies that $\beta = \bar{\alpha}$.

That ρ is a reasonable perturbation parameter follows from the fact that if $V = 0$, then ρ is a constant.

We now define $R(x), \eta(x)$ and $\theta(x)$ by

$$R(x) = |\rho(x)| \tag{23}$$

$$\eta(x) = \mathrm{Arg}(\rho(x)) \tag{24}$$

$$\theta(x) = \gamma(x) + \eta(x). \tag{25}$$

η can be normalized by $\eta(0) \in (-\pi, \pi]$ and η continuous. By (19), (20), (22)–(25), we have

$$u(x) = R(x)|\varphi(x)|\sin(\theta(x)) \tag{26}$$

$$u'(x) = R(x)|\varphi'(x)|\cos(\theta(x) + \delta(x) - \gamma(x)). \tag{27}$$

Example. If $V_0 \equiv 0$ and $\varphi(x) = e^{ikx}$ with $E = k^2$, then $|\varphi(x)| \equiv 1$, $|\varphi'(x)| = k$, and $\gamma(x) = \delta(x) = kx$ so $\gamma - \delta = 0$. Thus, (26), (27) become (5), (6) and our R, θ reduce to the standard ones.

One can invert (21) by using Wronskians. Take the Wronskian of both sides with $\bar{\varphi}$ using $W(\bar{\varphi}, \bar{\varphi}) = 0$ to see that $\frac{\rho}{2i} = \frac{W(u,\bar{\varphi})}{W(\varphi,\bar{\varphi})}$ or

$$\rho = \frac{2}{\omega} W(\bar{\varphi}, u). \tag{28}$$

Thus,

$$R^2 = \frac{4}{\omega^2} W(\bar{\varphi}, u) W(\varphi, u)$$
$$\theta = \operatorname{Arg}(W(\bar{\varphi}, u)).$$

By (22) and (28), $R(x)^2$ is comparable to $|u(x)|^2 + |u'(x)|^2$:

Proposition 2.2

$$\frac{(|u(x)|^2 + |(u'(x)|^2)^{1/2}}{K\varphi_0} \leq R(x) \leq \frac{2}{\omega} K\varphi_0[|u(x)|^2 + |u'(x)|^2]^{1/2}$$

Proof. By (22), $(|u'(x)|^2 + |u'(x)|^2)^{1/2} \leq |\rho(x)| \, (|\varphi(x)|^2 + |\varphi'(x)|^2)^{1/2} \leq K_0\varphi_0 R(x)$, yielding the lower bound. By (28),

$$R \leq \frac{2}{\omega} |W(\bar{\varphi}, u)| \leq \frac{2}{\omega} (|\varphi(x)|^2 + |\varphi'(x)|^2)^{1/2} (|u(x)|^2 + |u'(x)|^2)^{1/2}.$$

□

Theorem 2.3

$$\text{(a)} \qquad \rho'(x) = \rho(x) \frac{2|\varphi|^2}{\omega} V \sin(\theta) e^{-i\theta} \tag{29}$$

$$\text{(b)} \qquad [\ln R(x)]' = \frac{V(x)}{2\gamma'(x)} \sin(2\theta(x)) \tag{30}$$

$$\text{(c)} \qquad \theta(x)' = \gamma'(x) - \frac{V(x)}{\gamma'(x)} \sin^2(\theta(x)). \tag{31}$$

Proof. By (18) and (28),

$$\rho' = \frac{2}{\omega} V(x) \, \overline{\varphi(x)} \, u(x).$$

Now by (26), $u(x) = \rho(x) e^{-i\eta(x)} |\varphi(x)| \sin(\theta(x))$ and $\overline{\varphi(x)} = |\varphi(x)| e^{-i\gamma(x)}$ so (a) follows from $\eta + \gamma = \theta$.

$\rho' \rho^{-1} = (\ln R + i\eta)'$ so (b) is just the real part of (29) and $\frac{2|\varphi|^2}{\omega} = (\gamma')^{-1}$ (by Proposition 2.1(b)), and (c) is just the imaginary part. □

3 The Discrete Case

The approach is similar to the continuum case. The transfer matrix $T_0(n,E)$ is defined so that

$$T_0(n,E)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varphi(n+1) \\ \varphi(n) \end{pmatrix}$$

if φ obeys (4) with $\varphi(1)=a$, $\varphi(0)=b$. We will suppose that $E\in\mathbb{R}$ and

$$K \equiv \sup_{n>0} \|T_0(n,E)\| < \infty.$$

The Wronskian of two functions f,g on $\mathbb{Z}^+$ is defined by

$$W(f,g)(n) = f(n)g(n+1) - f(n+1)g(n).$$

If $f(n+1)+f(n-1) = A(n)f(n)$, $g(n+1)+g(n-1) = B(n)g(n)$, then

$$W(f,g)(n) - W(f,g)(n-1) = (B-A)(n)f(n)g(n). \tag{32}$$

Fix φ, a complex valued solution of (4), so that

$$W(\bar{\varphi},\varphi)(n) = 2i\,\mathrm{Im}(\varphi(n+1)\,\overline{\varphi(n)}\,) = i\omega$$

with $\omega=$ constant >0. The free case ($V_0\equiv 0$) is $E=2\cos(k)$, $\varphi(n)=e^{ikn}$, $\omega = 2\sin(k)$. We define $\gamma(n)$ by

$$\varphi(n) = |\varphi(n)|e^{i\gamma(n)}. \tag{33}$$

By constancy of the Wronskian,

$$2|\varphi(n)|\,|\varphi(n+1)|\sin(\gamma(n+1)-\gamma(n)) = \omega \tag{34}$$

so we can fix non-uniqueness in γ by requiring $\gamma(0)\in[0,2\pi)$, $\gamma(n)-\gamma(n-1)\in(0,\pi)$. With this choice, $\gamma(n)=kn$ in the free case.

Proposition 3.1 *(a)* $\gamma(n+1)-\gamma(n) = \begin{cases} \mathrm{Arcsin}\big(\frac{\omega}{2|\varphi(n)||\varphi(n+1)|}\big), & \textit{or} \\ \pi - \mathrm{Arcsin}\big(\frac{\omega}{2|\varphi(n)||\varphi(n+1)|}\big) \end{cases}$

(b) Let $\varphi_0 = (|\varphi(0)|^2+|\varphi(1)|^2)^{1/2}$. *Then*

$$\frac{\omega}{2K\varphi_0} \le |\varphi(n)| \le K\varphi_0.$$

(c) $\mathrm{Arcsin}(\frac{\omega}{2K^2\varphi_0^2}) \le \gamma(n+1)-\gamma(n) \le \pi - \mathrm{Arcsin}(\frac{\omega}{2K^2\varphi_0^2})$.

Proof. (a) follows from (34) if we note that if $\theta\in(0,\pi)$, then either $\theta = \mathrm{Arcsin}(\sin(\theta))$ or $\theta = \pi - \mathrm{Arcsin}(\sin(\theta))$. To prove (b), note that $|\varphi(n)|\le K\varphi_0$ is trivial and then (34), which implies $2|\varphi(n)||\varphi(n+1)|\ge\omega$, yields the lower bound. (c) follows from (a), (b) and monotonicity of Arcsin. □

Given a complex reference solution φ to (4) and a real valued solution u to (3), define $\rho(n) \in \mathbb{C}$ by

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = \frac{1}{2i}\left[\rho(n)\begin{pmatrix} \varphi(n) \\ \varphi(n-1) \end{pmatrix} - \overline{\rho(n)}\begin{pmatrix} \overline{\varphi}(n) \\ \overline{\varphi}(n-1) \end{pmatrix}\right] \tag{35}$$

$$= \operatorname{Im}\left[\rho(n)\begin{pmatrix} \varphi(n) \\ \varphi(n-1) \end{pmatrix}\right] \tag{36}$$

and then $R(n)$, $\eta(n)$, $\theta(n)$, and $\tilde{\theta}(n)$ by

$$\rho(n) = R(n)e^{i\eta(n)} \tag{37}$$

$$\theta(n) = \eta(n) + \gamma(n-1) \tag{38}$$

$$\tilde{\theta}(n) = \eta(n) + \gamma(n) \tag{39}$$

with η normalized by $\eta(0) \in (-\pi, \pi]$ and $\eta(n+1) - \eta(n) \in (-\pi, \pi]$. (We will soon see if $|V(n)| \to 0$, then $|\eta(n+1) - \eta(n)| \to 0$ also.)

By (33), (36), and (38), (39),

$$\begin{aligned} u(n) &= R(n)|\varphi(n)| \sin(\tilde{\theta}(n)) \\ u(n-1) &= R(n)|\varphi(n-1)| \sin(\theta(n)) \end{aligned} \tag{40}$$

which in the free case, where $|\varphi(n)| = 1$ and $\gamma(n) = kn$, is essentially (9), (10). Our $\theta, \tilde{\theta}$ are the same as the EFGP θ's. Our R differs from theirs by a constant $\sin(k)$. We could have made the definitions agree (by using $\rho_{\text{New}} = \frac{\omega\rho}{2}$) but chose to make the normalization in the continuum and discrete cases identical. Since a constant factor $(\frac{\omega}{2})$ is involved, no difference equations change and Proposition 3.1 changes in an elementary way.

We can use Wronskians to invert (35) to get

$$\rho(n) = \frac{2}{\omega}\, W(\overline{\varphi}, u)(n-1). \tag{41}$$

The strange fact that n appears on one side of (41) and $(n-1)$ on the other is a consequence of our reconciling the standard definition of W (involving $n, n+1$) and the EFGP definition (involving $n, n-1$). As in the continuum case, (36), (41) immediately imply:

Proposition 3.2

$$\frac{(u(n)^2 + u(n-1)^2)^{1/2}}{K\varphi_0} \leq R(n) \leq \frac{2}{\omega}\, K\varphi_0 (u(n)^2 + u(n-1)^2)^{1/2}$$

Now we can apply (32) to get the evolution equations for ρ, θ, R.

Theorem 3.3 *Let*

$$U(n) = -\frac{2V(n)}{\omega}\, |\varphi(n)|^2. \tag{42}$$

Then,

$$\text{(a)} \qquad \rho(n+1) - \rho(n) = U(n)\rho(n)\sin(\tilde{\theta}(n))e^{-i\tilde{\theta}(n)} \tag{43}$$

$$\text{(b)} \qquad \textit{If } |U(n)| \le 1, \textit{ then } |\eta(n+1) - \eta(n)| = |\theta(n+1) - \tilde{\theta}(n)| \le \frac{\pi}{2}\,|U(n)| \tag{44}$$

$$\text{(c)} \qquad R(n+1)^2 = R(n)^2[1 + U(n)\sin(2\tilde{\theta}(n)) + U(n)^2\sin^2(\tilde{\theta}(n))] \tag{45}$$

$$\text{(d)} \qquad \cot(\theta(n+1)) = \cot(\tilde{\theta}(n)) + U(n). \tag{46}$$

Remarks 1. (45), (46) are, of course, just (12), (13), so this generalizes the free case.

2. As noted in the introduction, from the usual form of the free equations in the continuum and discrete cases ((7), (8) vs. (12), (13)), the analogy appears vaguer, but the complex form of the equations (29) vs. (43) are clearly analogs!

3. (b) strengthens slightly the result in [6] that if $|U(n)| \le \frac{1}{2}$, then (44) holds with $\frac{\pi}{2}$ replaced by π.

Proof. (a) (3), (4), (32), and (41) imply that

$$\begin{aligned}\rho(n+1) - \rho(n) &= -\frac{2}{\omega}\,V(n)u(n)\overline{\varphi(n)} \\ &= -\frac{2}{\omega}\,V(n)R(n)\sin(\tilde{\theta}(n))|\varphi(n)|^2 e^{-i\gamma(n)}\end{aligned}$$

by (40). Given (42), (37), and (39), this is precisely (43).

(b) The distance of $z_0 = 1$ from the line $\mathrm{Arg}(z) = \theta$ or $\theta + \pi$ is $|\sin(\theta)|$. Thus, if $|z - 1| \le 1$, we have that

$$|\mathrm{Arg}(z)| \le \frac{\pi}{2}\,|\sin(\mathrm{Arg}(z)| \le \frac{\pi}{2}\,|z-1|. \tag{47}$$

But (43) implies

$$\left|\frac{\rho(n+1)}{\rho(n)} - 1\right| \le |U(n)|$$

and $\mathrm{Arg}(\frac{\rho(n+1)}{\rho(n)}) = \eta(n+1) - \eta(n)$, so (47) is just (44).

(c) By (43),

$$R(n+1)^2 = R(n)^2\big|1 + U(n)\sin(\tilde{\theta}(n))e^{-i\tilde{\theta}(n)}\big|^2.$$

Since $|1+\alpha|^2 = 1 + |\alpha|^2 + 2\,\mathrm{Re}\,\alpha$, we get (45).

(d) Multiply (43) by $e^{i\gamma(n)}$ to see that

$$R(n+1)e^{i\theta(n+1)} = R(n)[e^{i\tilde{\theta}(n)} + U(n)\sin(\tilde{\theta}(n))]. \tag{48}$$

The real part of (48) divided by its imaginary part is precisely (46). □

4 Embedded eigenvalues

As a warm-up, we show how to use the basic equations (30), (31) to construct a potential of Wigner-von Neumann type.

Theorem 4.1 *Suppose that $K = \sup_{x\geq 0} \|T_0(x,E)\| < \infty$. Then for every boundary condition at $x=0$, we can find a potential V of order $|V(x)| \leq C(1+x)^{-1}$, such that the equation* (1) *has an L_2 solution that satisfies the prescribed boundary condition.*

Proof. Fix a reference solution φ and consider the differential equation

$$\psi'(x) = \gamma'(x) + \frac{C}{\gamma'(x)(1+x)} \sin 2\psi \sin^2 \psi, \tag{49}$$

where $C>0$ will be chosen later. By Proposition 2.1(d), there are positive constants C_1, C_2 so that $C_1 \leq \gamma'(x) \leq C_2$. Hence the right-hand side of (49) satisfies a global Lipschitz condition with respect to ψ, and thus (49) has a unique global solution satisfying the initial condition $\psi(0)=\psi_0$ (see, e.g., [3]). Now set

$$V(x) = -\frac{C}{1+x} \sin 2\psi(x).$$

Then, by uniqueness, the generalized Prüfer angle θ with the initial value $\theta(0) = \psi_0$ is just $\psi(x)$. Thus the equation (30) for R becomes

$$(\ln R)' = -\frac{C}{2(1+x)\gamma'(x)} \sin^2 2\psi(x). \tag{50}$$

By (49), $\psi'(x)$ is also bounded away from zero and infinity for large enough x, so (50) implies

$$\ln R(x) \leq -AC \ln x$$

(x sufficiently large) with some constant $A>0$ that depends on C_1, C_2. If we now take C big enough, then R is in L_2. By Proposition 2.2, this also holds for the solution u corresponding to the Prüfer variables R, θ. By adjusting ψ_0, we can achieve that u satisfies any given boundary condition. □

Next, we study embedded point spectrum for perturbed periodic operators. So, let $V_0(x)$ be a periodic function of period 1 (say). Then the spectrum of $H_0 = -\Delta + V_0$ (on the whole axis) is purely absolutely continuous and has band structure

$$\sigma(H_0) = \sigma_{ac}(H_0) = \bigcup_{n=1}^{\infty} [a_n, b_n].$$

Theorem 4.2 *Let $F(x)$ be a positive, increasing function with $\lim_{x\to\infty} F(x) = \infty$. Then there is a potential V satisfying $|V(x)| \leq F(|x|)/(1+|x|)$ so that $\sigma_{pp}(H_0+V) \supset \sigma(H_0)$.*

Proof. Clearly it suffices to consider the case where $F(x) \le x^\beta$ for some $\beta > 0$ (otherwise let $\tilde{F}(x) = \min(F(x), x^\beta)$ and construct a potential V for $\tilde{F}$, it will also work for the original F). We will use the generalized Prüfer equations (30), (31), where we take a Bloch solution as reference solution φ. So

$$\varphi(x,E) = p(x,E)e^{ik(E)x}, \tag{51}$$

where p is periodic with period 1. Since the quasimomentum $k(E)$ is monotone in every band $[a_n, b_n]$, we can find a countable set of energies $E_n \in \sigma^{int}(H_0)$ so that $\{E_n\}$ is dense in $\sigma(H_0)$ and the numbers $\{\pi, k(E_n)\}$ are rationally independent.

It suffices to consider the half-line problem. Namely, we will prove that given F as above and a set of boundary conditions $\{\alpha_n\}$, there is a potential V on $(0,\infty)$ satisfying $|V(x)| \le F(x)/(1+x)$, such that for every n, (1) with $E = E_n$ has an $L_2(0,\infty)$ solution, and this solution satisfies the boundary condition α_n.

On intervals with $V \equiv 0$, (31) says that $\theta' = \gamma'$. Since p is periodic, (51) implies that

$$\theta(x+1,E) - \theta(x,E) = \gamma(x+1,E) - \gamma(x,E) = k(E)(\mathrm{mod} 2\pi).$$

By construction, the $\{k(E_n)\}$ are rationally independent. Given this observation, the argument proceeds similarly to the original Naboko paper [9] (see also [10]). For the sake of completeness we provide a sketch of the argument. Fix a sector Γ_ϵ,

$$\Gamma_\epsilon = \{\alpha | |\alpha + \frac{\pi}{4}| < \epsilon\}.$$

The value of ϵ needs to be chosen sufficiently small; we will assume $\epsilon < \frac{\pi}{12}$. Let $n_0 = 1$ and consider the first N_0 values of energies from our set: $\{E_j\}_{j=1}^{N_0}$. We suppose that the ordering in our sequence is fixed once and for all in some arbitrary way. The choice of N_0 is also arbitrary. We define V to be zero on the interval (n_0, n_1), where n_1 is chosen to be such that $\theta(n_1, E_j)$, $j = 1, \dots N_0$ all lie in Γ_ϵ. This is possible because the rotation on the torus given by

$$(\theta_1, \dots, \theta_N) \mapsto (\theta_1 + k(E_1), \dots, \theta_N + k(E_N)) \pmod \pi$$

is an ergodic map. Moreover, there is an a priori estimate on $n_1 - n_0$ which is independent of the initial values $\theta(n_0, E_n)$. We denote the maximal possible value of $n_1 - n_0$ by $D(N(n_0), \epsilon)$. Similarly, we will denote $D(M, \epsilon)$ the maximal distance needed to bring all values $\theta(x, E_j)$, $j = 1, \dots M$ into Γ_ϵ (for any initial data). Set

$$h_M = \frac{12}{\pi} \sup_{x \in R^+, j=1,\dots M} \{|\gamma'(x, E_j)|, 1\}. \tag{52}$$

Let $\chi(x)$ be a characteristic function of the interval $(0,1)$. On the interval (n_1, n_1+1) we define V as follows:

$$V(x) = \frac{F(n_1)}{2+n_1} \chi\left(h_{N_0}(x - n_1)\right).$$

We continue the construction inductively. V is set to be zero on the intervals (n_l+1, n_{l+1}) and is defined by

$$V(x) = \frac{F(n_l)}{2+n_l}\chi\left(h_{N_{l-1}}(x-n_l)\right). \tag{53}$$

on the intervals (n_l, n_l+1). The formula (53) is devised in a way that the angles $\theta(x,E_j)$, $j=1,\dots N_{l-1}$ do not change much in the region where $V \neq 0$, staying close to the phase which guarantees the fastest decay of $R(x,E_j)$. It may seem that we forget the second term in (31) influencing the change of $\theta(x,E_j)$, but this term becomes arbitrary small at large distances and we can safely ignore it.

We need to gradually add solutions at every energy E_j to our consideration, but to do it carefully and slowly enough so that we can control the L^2 norm of every solution $u(x,E_j)$ and make sure that they are square integrable. The following simple algorithm works well. Suppose that we have constructed V up to and on the interval (n_l, n_l+1). We check whether the following two conditions hold true:

$$F(n_l)h^{-2}_{N_{l-1}+1} > F^{\frac{1}{2}}(n_l) \tag{54}$$

$$D(N_{l-1}+1,\epsilon)+1 < F^{\frac{1}{4}}(n_l). \tag{55}$$

Here N_{l-1} is the number of solutions that we took into account on the interval $(n_{l-1}+1, n_l)$. If both conditions are verified, we add one more solution to our consideration in the interval (n_l+1, n_{l+1}), so that $N_l = N_{l-1}+1$. If any of the conditions fails, we do not add any new solutions in the next step so that $N_l = N_{l-1}$.

It is clear that the potential we construct satisfies the decay condition. It remains to show two things: that N_l eventually goes to infinity , so that we take into account every E_j, and that it yields L^2 solutions. The first is immediate from (54), (55) since the function F tends to infinity as n_l grows. We now indicate how to verify the second claim. Consider any of the solutions $u(x,E_j)$ (satisfying the right boundary condition at zero). Then we can find n_l such that the following estimate holds by (30), (31) (52), (53), (54):

$$\|u(x,E_j)\|^2_{L^2(n_l+1,\infty)} \leq CR^2(n_l,E_j)\sum_{m=l}^{\infty}\exp\left(-C_1\sum_{i=l}^{m}\frac{F^{\frac{1}{2}}(n_i)}{n_i+2}\right)(D(N(n_m),\epsilon)+1).$$

Employing (55), we find

$$\|u(x,E_j)\|^2_{L^2(n_l+1,\infty)} \leq C\sum_{m=l}^{\infty}\exp\left(-C_1\sum_{i=l}^{m}F^{\frac{1}{4}}(n_i)\log\left(\frac{2+n_{i+1}}{2+n_i}\right)\right)F^{\frac{1}{4}}(n_m).$$

This sum is obviously finite for any growing function F bounded by some power.
□

5 Asymptotic Distribution of $\gamma(n, E)$

In these final two sections, we will work in the discrete setting. So, we consider the operators H_λ acting on $l_2(\mathbb{N})$ as

$$(H_\lambda y)(n) = \begin{cases} y(2) + (V_0(1) + \lambda)y(1) & n = 1 \\ y(n-1) + y(n+1) + V_0(n)y(n) & n \geq 2 \end{cases}.$$

The parameter λ plays the role of a boundary condition. If $\lambda = 0$, then the corresponding index will usually be dropped.

We need some notation and some elementary facts. Let $u(n, E), v(n, E)$ be the solutions of (4) with the initial values $u(0) = v(1) = 1, u(1) = v(0) = 0$. Write

$$m_\lambda(z) \equiv \langle \delta_1, (H_\lambda - z)^{-1}\delta_1 \rangle,$$

where $\delta_1(n) = \delta_{1n}$. Although $m_\lambda(z)$ is defined originally only off the spectrum of H_λ, the limit $m_\lambda(E) \equiv \lim_{\epsilon \to 0+} m_\lambda(E + i\epsilon)$ exists almost everywhere. In regions where $m(E)$ does exist and, moreover, Im $m(E) > 0$, a natural choice for the complex solution φ from Section 3 is

$$\varphi(n, E) = u(n, E) - \overline{m(E)}v(n, E) \tag{56}$$

(the complex conjugation being necessary to have $\omega > 0$). Note that for non-real E, $\overline{\varphi}$ would be the l_2 solution of (4). Our goal in this section is to show that the $\gamma(n, E)$ gotten from (56) is approximately uniformly distributed as a function of E for large n.

We need some more preliminaries. Denote by H_λ^N the operator restricted to $l_2(\{1, \ldots, N\}) \equiv \mathbb{C}^N$ with Dirichlet boundary condition at N. More precisely, define H_λ^N by

$$(H_\lambda^N y)(n) = \begin{cases} (H_\lambda y)(n) & n \leq N-1 \\ y(N-1) + V_0(N)y(N) & n = N \end{cases}.$$

Correspondingly set $m_\lambda^N(z) = \langle \delta_1, (H_\lambda^N - z)^{-1}\delta_1 \rangle$. The functions $m_\lambda^N(z)$ are meromorphic with precisely N simple poles on the real axis, and if E is not one of these poles, then Im $m_\lambda^N(E) = 0$. Also, it's not hard to see that the solution $f_N(n, z) \equiv u(n, z) - m^N(z)v(n, z)$ satisfies $f_N(N+1, z) = 0$, and thus

$$m^N(z) = \frac{u(N+1, z)}{v(N+1, z)}. \tag{57}$$

So, if we write $m(z) = a(z) + ib(z)$ (similar notations will be used for the other m-functions introduced above), then (56), (57) yield

$$\cot \gamma(N+1, E) = \frac{m^N(E)}{b(E)} - \frac{a(E)}{b(E)} \tag{58}$$

The m-functions obey the following well-known transformation formula (see, e.g., [15]): $m_\lambda = m/(1+\lambda m)$. In particular,

$$b_\lambda(z) = \frac{b(z)}{(1+\lambda a(z))^2 + \lambda^2 b^2(z)}. \tag{59}$$

Of course, analogous formulae hold for m^N, m^N_λ.

Finally, denote by $d\rho_\lambda, d\rho^N_\lambda$ the corresponding spectral measures, i.e. $d\rho(t) = d\|E(t)\delta_1\|^2$ etc., where $E(t)$ is the spectral resolution of H. Recall that $d\rho^N_\lambda$ converges weakly (i.e., when integrated against continuous functions of compact support) to $d\rho_\lambda$ as $N\to\infty$. Moreover, $d\rho_\lambda$ can also be obtained as the weak limit $\pi d\rho_\lambda(E) = \lim_{\epsilon\to 0+} b_\lambda(E+i\epsilon)\, dE$.

Theorem 5.1 *Suppose that $m(E) = a(E) + ib(E)$ exists on $I = [E_1, E_2]$, is continuous, and $b(E) > 0$. Then*

$$\lim_{N\to\infty} \frac{1}{|J|} \left|\{E \in J : \gamma(N,E) \in S \,(\mathrm{mod}\ \pi\}\right| = \frac{|S|}{\pi},$$

where J, S are subintervals of I and $\mathbb{T}^1 = [0,\pi)$, respectively.

Remark. Some elements of the following proof are related to the spectral averaging formula from the general theory of rank one perturbations (cf. [15]). Results of a flavor similar to our Theorem 5.1 have been obtained in [13].

The proof uses the following elementary fact.

Lemma 5.2 *Suppose that $A(\epsilon) = A_0 + A_1\epsilon + O(\epsilon^2), B(\epsilon) = B_1\epsilon + O(\epsilon^2)$ with $B_1 > 0$. Then*

$$\frac{1}{\pi}\lim_{\epsilon\to 0+}\int_c^d dt\, \frac{B(\epsilon)}{(t-A(\epsilon))^2 + B^2(\epsilon)} = \begin{cases} 0 & A_0 \notin [c,d] \\ \in (0,1) & A_0 \in \{c,d\} (\text{the limit exists!}) \\ 1 & A_0 \in (c,d) \end{cases}.$$

Proof. Evaluate the integral. □

Proof of Theorem 5.1. We first consider the case when $S = [\gamma_1, \gamma_2]$ with $\gamma_i \not\equiv 0 \,(\mathrm{mod}\ \pi)$. By (58), the Lemma, and the properties of $m^N(z)$, we get

$$\pi|\{E\in J : \gamma(N+1,E) \in S\}| = \int_J dE \lim_{\delta\to 0+}\lim_{\epsilon\to 0+}\int_{c_2(E)-\delta}^{c_1(E)+\delta} dt\, \frac{b^N(E+i\epsilon)}{(t-a^N(E+i\epsilon))^2 + (b^N(E+i\epsilon))^2},$$

where $c_i(E) = a(E) + b(E)\cot\gamma_i$. The basic idea is to change the order of integration, then let $N\to\infty$, and finally go back to the original order. In practice, things are a little messy, unfortunately; here are the details.

$$\begin{aligned}
\pi|\{E \in J : &\gamma(N+1,E) \in S\}| \\
&\geq \int_J dE \lim_{\epsilon\to 0+} \int_{c_2(E)}^{c_1(E)} dt\, \frac{b^N(E+i\epsilon)}{(t-a^N(E+i\epsilon))^2 + (b^N(E+i\epsilon))^2} \\
&= \lim_{\epsilon\to 0+} \int_J dE \int_{c_2(E)}^{c_1(E)} dt\, \frac{b^N(E+i\epsilon)}{(t-a^N(E+i\epsilon))^2 + (b^N(E+i\epsilon))^2} \\
&= \lim_{\epsilon\to 0+} \int_J dE \int_{c_2(E)}^{c_1(E)} \frac{dt}{t^2}\, b^N_{-t^{-1}}(E+i\epsilon) \\
&\equiv \lim_{\epsilon\to 0+} \int_M d(E,t)\, \frac{b^N_{-t^{-1}}(E+i\epsilon)}{t^2}.
\end{aligned}$$

We used dominated convergence in the second line and (59) in the third line. The region of integration $M \subset \mathbb{R}^2$ is given by

$$M = \{(E,t) : E \in J, c_2(E) \leq t \leq c_1(E)\}.$$

Since $c_{1/2}(E)$ are continuous, we can find an increasing sequence of sets M_n, such that every M_n is a finite, disjoint union of open rectangles, and $\chi_{M_n}(E,t) \to \chi_M(E,t)$ as $n \to \infty$ for almost every pair $(E,t) \in \mathbb{R}^2$.

Now, applying Fatou's Lemma and using the properties of the sets M_n, we get

$$\begin{aligned}
\pi|\{E \in J : \gamma(N+1,E) \in S\}| &\geq \liminf_{\epsilon\to 0+} \int_{M_n} d(E,t)\, \frac{b^N_{-t^{-1}}(E+i\epsilon)}{t^2} \\
&= \liminf_{\epsilon\to 0+} \int \frac{dt}{t^2} \int_{I_n(t)} dE\, b^N_{-t^{-1}}(E+i\epsilon) \\
&\geq \int \frac{dt}{t^2} \liminf_{\epsilon\to 0+} \int_{I_n(t)} dE\, b^N_{-t^{-1}}(E+i\epsilon) \\
&\geq \pi \int \frac{dt}{t^2}\, \rho^N_{-t^{-1}}(I_n(t)).
\end{aligned}$$

Here, the sets $I_n(t) \subset J$ are finite, disjoint unions of open intervals (this follows from the construction of the sets M_n). We can let $N \to \infty$ to obtain

$$\begin{aligned}
\pi \liminf_{N\to\infty} |\{E \in J : \gamma(N,E) \in S\}| &\geq \pi \int \frac{dt}{t^2} \liminf_{N\to\infty} \rho^N_{-t^{-1}}(I_n(t)) \\
&= \pi \int \frac{dt}{t^2}\, \rho_{-t^{-1}}(I_n(t)) \\
&= \int \frac{dt}{t^2} \int_{I_n(t)} dE\, b_{-t^{-1}}(E) \\
&= \int_{M_n} d(E,t) \frac{b_{-t^{-1}}(E)}{t^2}.
\end{aligned}$$

Finally, let also $n \to \infty$ (using monotone convergence) and reverse the steps from above:

$$\begin{aligned}\pi \liminf_{N\to\infty} |\{E \in J : \gamma(N,E) \in S\}| &\geq \int_M d(E,t) \frac{b_{-t^{-1}}(E)}{t^2} \\ &= \int_J dE \int_{c_2(E)}^{c_1(E)} dt\, \frac{b(E)}{(t-a(E))^2 + b^2(E)} \\ &= \int_J dE \int_{\cot\gamma_2}^{\cot\gamma_1} du\, \frac{1}{u^2+1} = |J|(\gamma_2 - \gamma_1).\end{aligned}$$

Now it's easy to see that the proven statement on $\liminf |\{\gamma \in S\}|$ actually implies the full claim. Namely, assume that on the contrary

$$\lim_{k\to\infty} \frac{1}{|J|} |\{E \in J : \gamma(N_k, E) \in S\}| = \frac{|S|}{\pi} + \delta$$

for some $S, N_k \to \infty$ and $\delta > 0$. Pick a closed interval $S' \subset \mathbb{T}^1 \setminus S$ with length $\geq \pi - |S| - (\pi\delta)/2$. Then, by what has already been shown,

$$\liminf_{k\to\infty} \frac{1}{|J|} |\{E \in J : \gamma(N_k, E) \in S \cup S'\}| \geq \frac{|S|}{\pi} + \delta + 1 - \frac{|S|}{\pi} - \frac{\delta}{2} = 1 + \frac{\delta}{2},$$

an obvious contradiction. □

In the next section, we will use the following easy consequence of Theorem 6.1.

Corollary 5.3 *Suppose that $g : \mathbb{T}^1 \to \mathbb{C}$ is continuous. Then, under the assumptions of Theorem 5.1, we also have that*

$$\lim_{N\to\infty} \frac{1}{E_2 - E_1} \int_{E_1}^{E_2} g(\gamma(N,E))\, dE = \frac{1}{\pi} \int_0^{\pi} g(\gamma)\, d\gamma.$$

Proof. Let $\epsilon > 0$ be given. Pick $\delta > 0$ so that $|g(\gamma) - g(\gamma')| < \epsilon/3$ if $|\gamma - \gamma'| \leq \delta$. We may also assume that $\pi/\delta \in \mathbb{N}$. Let

$$I_n(N) = \{E \in [E_1, E_2] : \gamma(N,E) \in [(n-1)\delta, n\delta)\}.$$

By Theorem 5.1, we can find an N_0 so that for $n = 1, \ldots, \pi/\delta$

$$\left| |I_n(N)| - \frac{\delta}{\pi}(E_2 - E_1) \right| < \frac{\delta\epsilon(E_2 - E_1)}{3\pi \max|g|} \tag{60}$$

if $N \geq N_0$. Now for $N \geq N_0$, we get (with error terms $|\eta_i| < \epsilon/3$)

$$
\begin{aligned}
\frac{1}{E_2 - E_1} \int_{E_1}^{E_2} g(\gamma(N,E))\, dE &= \frac{1}{E_2 - E_1} \sum_{n=1}^{\pi/\delta} \int_{I_n(N)} g(\gamma(N,E))\, dE \\
&= \frac{1}{E_2 - E_1} \sum_{n=1}^{\pi/\delta} g(n\delta)|I_n(N)| + \eta_1 \\
&= \frac{\delta}{\pi} \sum_{n=1}^{\pi/\delta} g(n\delta) + \eta_1 + \eta_2 \\
&= \frac{1}{\pi} \int_0^{\pi} g(\gamma)\, d\gamma + \eta_1 + \eta_2 + \eta_3.
\end{aligned}
$$

□

6 Sparse Perturbations

In this section, we want to show that absolutely continuous spectrum can be transformed to singular continuous spectrum by a perturbation that tends to zero. We need a number of technical assumptions. First of all, we assume that $m(E)$ satisfies the hypotheses of Theorem 5.1 on some interval $[E_1, E_2]$. Then, as usual, we suppose that

$$\sup_{n \in \mathbb{N}} \max_{E_1 \leq E \leq E_2} \|T_0(n,E)\| < \infty.$$

As in the preceding section, let $\varphi(n,E) = u(n,E) - \overline{m(E)}v(n,E)$. We further assume that $|\varphi|$ is equicontinuous, i.e., for every $\epsilon > 0$ there is a $\delta > 0$ so that for all $n \in \mathbb{N}$ we have that

$$\big| |\varphi(n,E)| - |\varphi(n,E')| \big| < \epsilon \quad \text{ if } |E - E'| < \delta.$$

Note that φ has these properties if $V_0 \equiv 0$ or if V_0 is periodic.

Theorem 6.1 *Under the above assumptions, there is a perturbation $V(n) \to 0$, so that $H_0 + V$ has purely singular continuous spectrum on (E_1, E_2).*

Sketch of the proof. V will be a Pearson type potential (this name refers to [12], of course). We can take, say,

$$V(n) = \sum_{k=1}^{\infty} k^{-1/2} \delta_{n x_k},$$

with x_k to be chosen later. The critical features are that the weights $k^{-1/2}$ are not square summable and that the x_k increase sufficiently rapidly.

The argument follows closely [6, Proof of Theorem 1.6(2)], with Corollary 5.3 as important additional ingredient. We use the generalized Prüfer equations

(45), (46) with reference solution φ as above (so $\omega(E) = 2b(E)$). Clearly, R is constant on every interval $\{x_{k-1}+1, \ldots, x_k\}$ and

$$\widetilde{\theta}_k(E) \equiv \widetilde{\theta}(x_k, E) = \gamma(x_k, E) + \psi_{k-1}(E) \qquad (61)$$

where $\psi_{k-1}(E) = \widetilde{\theta}(x_{k-1}+1, E) - \gamma(x_{k-1}+1, E)$. Write $R_k(E) = R(x_k, E)$,

$$U_k(E) = U(x_k, E) = -\frac{|\varphi(x_k, E)|^2}{k^{1/2} b(E)}.$$

A Taylor expansion shows

$$\ln R_{k+1}^2(E) = \sum_{m=1}^{k} (X_m(E) + Y_m(E)) + \frac{1}{4} \sum_{m=1}^{k} U_m^2(E) + O(1) \quad (k \to \infty),$$

where

$$\begin{aligned} X_m(E) &= U_m(E) \sin 2\widetilde{\theta}_m(E) \\ Y_m(E) &= \frac{1}{2} U_m^2(E) \left(\frac{1}{2} \cos 4\widetilde{\theta}_m(E) - \cos 2\widetilde{\theta}_m(E) \right). \end{aligned}$$

The remainder $O(1)$ is uniformly bounded. This follows from the definition of U, the usual bound on $|\varphi|$ (see Proposition 3.1(b)), and the fact that $\min b(E) > 0$.

By [7, Theorem 1.2], in order to show $\sigma_{ac} \cap (E_1, E_2) = \emptyset$, it suffices to find a subsequence $y_n \to \infty$ so that $\lim_{n\to\infty} R(y_n, E) = \infty$ for almost every $E \in (E_1, E_2)$. In the case at hand, we already have a diverging term: Obviously, $\sum_{m=1}^{k} U_m^2(E) \geq c(E) \ln k$. So it's sufficient to prove that $\sum_{m=1}^{k} X_m(E), Y_m(E)$ are of order $o(\ln k)$ (at least, on a suitable subsequence) for almost every $E \in (E_1, E_2)$.

Write $S_k(E) = \sum_{m=1}^{k} X_m(E)$. An elementary probabilistic argument (compare [6, Sect. 6]) shows that if

$$\int_{E_1}^{E_2} S_k^2(E)\, dE = o(\ln^2 k) \quad (k \to \infty), \qquad (62)$$

then, as desired, $S_{k_n}(E) = o(\ln k_n)$ on a certain subsequence for almost all $E \in (E_1, E_2)$, and similarly for $\sum Y_m$.

In order to prove (62), we note that

$$\int_{E_1}^{E_2} S_k^2(E)\, dE \leq \int_{E_1}^{E_2} S_{k-1}^2(E)\, dE + \int_{E_1}^{E_2} X_k^2(E)\, dE + 2 \left| \int_{E_1}^{E_2} S_{k-1}(E) X_k(E)\, dE \right|. \qquad (63)$$

By (61) and the complex representation of the sine, the last term of (63) is a sum of four contributions of the form

$$\int_{E_1}^{E_2} \sum_{m=1}^{k-1} U_m(E) U_k(E) e^{2i(\pm\psi_{m-1}(E) \pm \psi_{k-1}(E) \pm \gamma(x_m, E))} e^{\pm 2i\gamma(x_k, E)}\, dE$$

$$\equiv \int_{E_1}^{E_2} f(E, x_k) e^{\pm 2i\gamma(x_k, E)}\, dE.$$

For fixed $x_1, \ldots, x_{k-1}$, the family $\{f(\cdot, x_k) : x_k \in \mathbb{N}\}$ is equicontinuous and uniformly bounded. This is easily inferred from the corresponding properties of $|\varphi|$ and the continuity of m, ψ, γ.

Now Corollary 5.3 implies that

$$\lim_{x_k \to \infty} \int_{E_1}^{E_2} f(E, x_k) e^{\pm 2i\gamma(x_k, E)} \, dE = 0.$$

This is shown as follows. Given any $\epsilon > 0$, pick $\delta > 0$ so that

$$\sup_{x \in \mathbb{N}} \sup_{|e-e'|<\delta} |f(e,x) - f(e',x)| < \frac{\epsilon}{2(E_2 - E_1)}.$$

Then (assuming $N \equiv (E_2 - E_1)\delta^{-1} \in \mathbb{N}$)

$$\int_{E_1}^{E_2} f(E, x_k) e^{\pm 2i\gamma(x_k, E)} \, dE = \sum_{n=1}^{N} f(E_1 + n\delta, x_k) \int_{E_1+(n-1)\delta}^{E_1+n\delta} e^{\pm 2i\gamma(x_k, E)} \, dE + \eta,$$

where $|\eta| < \epsilon/2$. By Corollary 5.3, the integrals $\int e^{\pm 2i\gamma(x_k, E)} \, dE$ tend to zero as $x_k \to \infty$, and $|f(E, x)| \leq C$ for all x, E, so the claim follows.

So the last term of (63) can be made arbitrarily small by taking x_k large enough, and the second one can obviously be estimated by Ck^{-1}, so (62) indeed holds (in fact, $\int S_k^2 = O(\ln k)$). The proof for $\sum Y_m$ is similar.

Finally, non-existence of l_2 solutions follows easily by also taking x_k sufficiently large. $\square$

References

[1] R. Carmona, *One-dimensional Schrödinger operators with random or deterministic potentials: New spectral types*, J. Funct. Anal. **51** (1983), 229–258.

[2] M. Christ and A. Kiselev, *Absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials: some optimal results*, preprint.

[3] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[4] T. Eggarter, *Some exact results on electron energy levels in certain one-dimensional random potentials*, Phys. Rev. **B5** (1972), 3863–3865.

[5] S.A. Gredeskul and L.A. Pastur, *Behavior of the density of states in one-dimensional disordered systems near the edges of the spectrum*, Theor. Math. Phys. **23** (1975), 132–139.

[6] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, to appear in Commun. Math. Phys.

[7] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, preprint.

[8] Y. Last and B. Simon, *Modified Prüfer and EFGP transforms and deterministic models with dense point spectrum*, to appear in J. Funct. Anal.

[9] S.A. Naboko, *Dense point spectra of Schrödinger and Dirac operators*, Theor. Math. Phys. **68** (1986), 18–28.

[10] S.N. Naboko and S.I. Yakovlev, *On the point spectrum of discrete Schrödinger operators,* Funct. Anal. **26(2)**(1992), 145–147.

[11] L. Pastur and A. Figotin, *Spectra of Random and Almost Periodic Operators*, Springer-Verlag, Berlin, 1992.

[12] D.B. Pearson, *Singular continuous measures in scattering theory*, Commun. Math. Phys. **60** (1978), 13–36.

[13] D.B. Pearson, *Value distribution and spectral analysis of differential operators*, J. Phys. A **26** (1993), 4067–4080.

[14] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials*, to appear in Commun. Math. Phys.

[15] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Lecture Notes Vol. 8 (J. Feldman, R. Froese, L. Rosen, eds.), 109–149, Amer. Math. Soc., Providence, RI, 1995.