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Absolutely Continuous Invariant Measures for Multidimensional Expanding Maps

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Abstract

We investigate the existence and statistical properties of absolutely continuous invariant measures for multidimensional expanding maps with singularities. The key point is the establishment of a spectral gap in the spectrum of the transfer operator. Our assumptions appears quite naturally for maps with singularities. We allow maps that are discontinuous on some extremely wild sets, the shape of the discontinuities being completely ignored with our approach.

Key-Words: Expanding maps with singularities, Invariant measure, Decay of correlations, Perron-Frobenius operator.

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Contents

1 Introduction

We consider piecewise invertible expanding maps T on some compact subset Ω of $M = \mathbb{R}^N$. The transformation T satisfies uniform expansiveness, contrary to non uniformly hyperbolic maps considered in other papers, as [LSV2, Yu]. However, due to the presence of persistent singularities, they may not satisfy some nice combinatorial behavior (like Markov partition, finite range structure, etc.). Therefore abstract dynamical coding like symbolic dynamic will not be considered.

The main focus of this paper is to prove the existence of a T-invariant probability measure absolutely continuous with respect to the Lebesgue measure m (in short ACIM). One way to find such a measure (and its properties) is to study the spectrum of the Perron-Frobenius (PF) or transfer operator, defined by the dynamic.

This approach has been successfully carried out in the one dimensional case. It is now proven that any piecewise monotonic map of the interval which is eventually expanding[1] —provided that T has some smoothness— posses an ACIM and the dynamic present a very simple decomposition into "chaotic" elements.

Despite several attempts, the same question is still open in the multidimensional case. The unexpected difficulties are essentially due to the following facts :

[1]A map T is called eventually expanding when the inverse of the Jacobian matrix of some iterate T^n of the map has a norm strictly bounded by one. For maps of the interval, this property reduces to the fact that for some $n > 0$, DT^n is bounded from below by some constant strictly larger than one.

(1) One has to find a functional space rich enough to contain the density (which may be discontinuous) but not too wide (to give PF a nice spectral decomposition).

(2) In more than one dimension, the geometry of the dynamical partition becomes a crucial ingredient.

Point (1) has been extensively solved with functions of bounded variation[2] and some adaptation of them. This method although highly powerful in dimension one led to difficult problem in our case, essentially coming from point (2).

The analysis of PF requires, in the non-Markovian case, a combination of an extension and a trace theorem. This is one of the main problem in the multidimensional case[3]. The main difficulty comes from the fact that the variation does not control the supremum of a function. This point is the source of most of the problems; the need to control the integral of the function (actually, its trace on the surface) along a codimension one smooth surface appears to be unavoidable. There is the point where assumptions on the shape of the partition arise (in particular, codimension one, piecewise $\mathcal{C}^{(2)}$ smoothness of the boundary and absence of cusps). We have to mention [Za], where the author avoid this problem, essentially using the fact that from outside, cusps can be simply forgotten (see fig. 1). But the difficulty remains if "the outside" does not exist, like in the example by [BGP], where the persistence of the problem is clearly identified.[4]

We propose here for point (1) to work with a functional space introduced in this context by G. Keller [Ke] for one dimensional maps. This space indeed is still relevant in the multidimensional case, at least for absolutely continuous measures. In [Bl] M. Blank worked with this space and gave similar results with an ad hoc hypotheses (unfortunately, this hypotheses appears to be unsatisfied even in some simple cases) on the dynamical partition.

Using a different approach, we are able to state a theorem with highly relaxed hypotheses on the dynamical partition. This allows us to deal with maps that are discontinuous on some *very irregular sets*, even on fractal sets. Moreover, *general conditions* are given for a map with piecewise $\mathcal{C}^{(1)}$ domains to enjoy the required properties. Finally, in a *concrete example* (a map discontinuous on a von Koch's fractal set) we show how to compute our

[2]In more than one dimension, the notion of bounded variation does not rely on any ordered structure. Bounded variation functions are functions for which the derivative, in the sense of distributions, is a measure whose total variation is finite.

[3]Also in dimension one if the Darboux property does not holds (a map satisfies the Darboux property whenever the image of an interval is again an interval).

[4]The title of the paper, "Inadequacy of the bounded variation technique..." does not leave any doubt !

conditions.

Spectral properties shown for the PF operator ensure immediately the well known decomposition into a finite number of ergodic components (of positive Lebesgue measure), decomposable into mixing components for some iterates of the map.

As a byproduct, we obtain that on these components, Hölder observable are *mixed exponentially* fast by the map.

A great property of the function space involved in this paper gives, without much effort, a *constructive upper bound* on the number of positive Lebesgue measure *ergodic components* (see Theorem 4.2).

We also investigate the question of *computing an upper bound for the rate of decay* of the correlations function [5], for Hölder observable. A method is presented where such a bound is computed ; the strategy, already involved in [L2] for expanding maps of the interval, in [LSV1] for one dimensional equilibrium states and in [L1] for multidimensional symplectic maps, is to define a projective metric space, on which PF is a contraction, the contracting factor being computable.

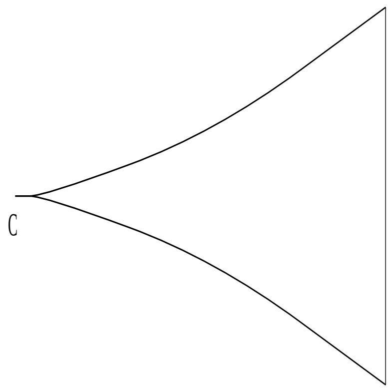

Figure 1 : An example of a domain containing a cusp in C.

Some notations

Given a Borel subset S of M, we denote respectively its closure, its interior and its boundary by $\mathrm{clos}(S), \mathrm{int}(S)$ and $\partial S = \mathrm{clos}(S) \backslash \mathrm{int}(S)$.
For a function $f \in L^\infty$, $\underset{S}{\mathrm{Esup}}\, f$ and $\underset{S}{\mathrm{Einf}}\, f$ will represent the essential supremum and infimum of f on B with respect to the Lebesgue measure. We denote by $B_\varepsilon(x)$ the open ball of radius ε and center x, in the Euclidean

[5] We compute explicitly the speed of convergence of an initial density towards the invariant one in some Hölder like norm, for covering maps (see section 5 for the definition of covering).

metric d.
Moreover, given a real number $a > 0$, we write $B_a(S) = \{x \in M | d(x,S) \leq a\}$.
In addition, we write indifferently $m(f) = \int f(x)dx = \int f dm$ for all $f \in L^1_m$.
$\gamma_N = \frac{\pi^{N/2}}{(N/2)!}$ denotes the N-volume of the N-dimensional unit ball of $\mathbb{R}^N$.

We now introduce the maps under consideration in this paper

Hypotheses

Let Ω be a compact subset of M, with $\mathrm{clos}(\mathrm{int}(\Omega)) = \Omega$, and $T : \Omega \to \Omega$. Let us give our definition of a piecewise expanding map. We assume that there exists a finite[6] family of open sets $U_i \subset \Omega$ and maps $T_i = T_{|U_i}$ satisfying:

(PE1) for all $i \neq j$, $U_i \cap U_j = \emptyset$,

(PE2) for all i, $T_i \in \mathcal{C}^{(1+\alpha)}(\overline{U}_i)$, T_i injective and $T_i^{-1} \in \mathcal{C}^{(1+\alpha)}(\overline{TU_i})$,

(PE3) $m(\Omega \backslash \bigcup_i U_i) = 0$,

(PE4) there exists ε_0 and $s < 1$ such that for all $u, v \in TU_i$ such that $d(u,v) \leq \varepsilon_0$ we have $d(T_i^{-1}u, T_i^{-1}v) \leq s d(u,v)$,

(PE5) For all ε_0 sufficiently small $s^\alpha + \sup_{\varepsilon \leq \varepsilon_0} \frac{G(\varepsilon)}{\varepsilon^\alpha} \varepsilon_0^\alpha < 1$, where $G(\varepsilon)$ is the maximal sum over the domains of the proportion of points in an arbitrary small ball (of radius $(1-s)\varepsilon_0$) to be extremely close to the boundary (in an $s\varepsilon$-neighborhood) of the domain.

$$G(\varepsilon) = \sup_{x \in M} \sum_i \frac{m\left(B_{s\varepsilon}(\partial U_i) \cap B_{(1-s)\varepsilon_0}(x)\right)}{m\left(B_{(1-s)\varepsilon_0}(x)\right)}.$$

Remark 1.1 *1. We emphasize the fact that Ω nor the U_i's do not have to be connected.*

2. (PE5) may look very hard to obtain. Basically, it implies that typical trajectories will not feel the discontinuities, so that the dynamic is essentially a smooth one. In section 6 we will compute this quantity for

[6]Clearly there are some cases for which we can allow countable family, whereas there is not yet any satisfactory sufficient condition. See the discussion at the end of section 6 for more details about that.

some examples of maps discontinuous on a fractal set. Remark that this condition imply that the boundaries are of co(box)dimension bigger than α. In Corollary 4.1, we will give explicit conditions to get (PE5) in the case of piecewise $\mathcal{C}^{(1)}$ boundaries. Fig. 2 gives an example of some domains U_i with piecewise smooth boundaries ($G(\varepsilon)$ is the sum of the three areas between the dotted lines, divided by the area of the disk $B(x)$).

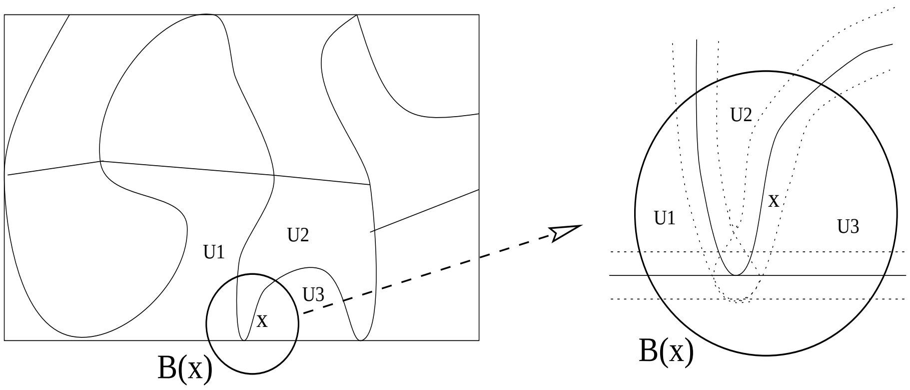

Figure 2 : Typical domains U_i with neighborhood of boundaries.

With these assumptions, it is easy to see that the Perron-Frobenius operator is well defined on L^1_m, and reads like :

$$Ph = \sum_i (gh) \circ T_i^{-1} \chi_{TU_i},$$

where the weight g is given by $g = \dfrac{1}{|\det DT|}$. We recall that P verify for all $h \in L^1, f \in L^\infty$,

$$\int_\Omega f \circ T h dm = \int_\Omega f Ph\, dm.$$

Moreover, $h \in L^1$ is an invariant density of an ACIM if and only if h is a positive eigenvector of P with eigenvalue 1.

2 Quasi-Hölder space

Let $0 < \alpha < 1$ and $\varepsilon_0 > 0$ be real numbers, and $f \in L^1_m(M, \mathbb{R})$. For a Borel subset S of M, we define the oscillation of f on S by

$$\operatorname{osc}(f, S) = \operatorname*{Esup}_S f - \operatorname*{Einf}_S f.$$

By definition of the oscillation, it is easy to prove :

Proposition 2.1 *For each $f \in L^1_m(M, \mathbb{R})$, the function $x \to \operatorname{osc}(f, B_\varepsilon(x))$ is lower semi-continuous, hence measurable.*

Remark 2.1 *Although functions in L^1 are only defined almost everywhere, the oscillation is a real positive function defined everywhere on M. In addition,* $\operatorname{supp} \operatorname{osc}(f, B_\varepsilon(\cdot)) \subset B_\varepsilon(\operatorname{supp} f)$.

By proposition 2.1, we can define

$$|f|_\alpha = \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha} \int_M \operatorname{osc}(f, B_\varepsilon(x))dx.$$

Remark 2.2 *Although it is not explicitly written, it is important to remember that $|f|_\alpha$ depends on ε_0.*

We define now

$$V_\alpha = \left\{ f \in L^1_m(M, \mathbb{R}) \;\middle|\; |f|_\alpha < \infty \right\}.$$

It is clear that any compact supported α-Hölder continuous function belongs to V_α, but this space is bigger, since functions in V_α may have discontinuities[7].

Proposition 2.2 *Let $f, f_i, g \in L^\infty(M, \mathbb{R})$ be positive functions, $0 < a, b, c$ and S a Borel subset of M. The oscillation has the following properties :*

(i) $\operatorname{osc}(\sum_i f_i, B_a(\cdot)) \leq \sum_i \operatorname{osc}(f_i, B_a(\cdot))$,

(ii) $\operatorname{osc}(f\chi_S, B_a(\cdot)) \leq \operatorname{osc}(f, S \cap B_a(\cdot))\chi_S(\cdot) + \operatorname*{Esup}_{B_a(\cdot)\cap S}(f)\chi_{B_a(\partial S)}(\cdot)$,

(iii) the oscillation of the product can be controlled by

$$\operatorname{osc}(fg, S) \leq \operatorname{osc}(f, S) \operatorname*{Esup}_S g + \operatorname{osc}(g, S) \operatorname*{Einf}_S f,$$

(iv) if $a + b \leq c$, then for all $x \in M$ holds

$$\operatorname*{Esup}_{B_a(x)} f \leq \frac{1}{m(B_b(x))} \int_{B_b(x)} [f(y) + \operatorname{osc}(f, B_c(y))]\, dy.$$

PROOF. *Straight forward.* $\diamond$

Using basic functional analysis, the following can be shown (see [Ke] for more properties).

[7] We recall that we cannot avoid discontinuities, since we do not ask any Markovian property on T.

Proposition 2.3 *If we put the norm $\|\cdot\|_\alpha = \|\cdot\|_{L^1_m} + |\cdot|_\alpha$ then V_α becomes a Banach space. Moreover, it is continuously injected in L^∞_m, and V_α is an algebra with the usual sum and product of functions in L^∞_m. In addition, for Ω is compact, the intersection of the unit ball of V_α with the set of functions supported on Ω is compact in $L^1(M)$.*

Lemma 2.1 *for every $h \in V_\alpha$ there exists a ball on which the infimum of h is strictly positive. The radius ε of the ball can be taken as*

$$\varepsilon = \min\left(\varepsilon_0, \left(\frac{\int hdm}{|h|_\alpha}\right)^{\frac{1}{\alpha}}\right).$$

Proof. *Let $h \in V_\alpha$ and ε as in the Lemma. We suppose that h is not constant (otherwise the lemma is proven) We claim that the infimum of h is strictly positive on some ball of radius ε. Otherwise, the infimum of h on every balls would be null, this would imply (the first inequality being strict for h is not constant)*

$$\int h(x)dx < \int \operatorname*{Esup}_{B_\varepsilon(x)} hdx = \int \operatorname{osc}(h, B_\varepsilon(x))dx \leq |h|_\alpha \varepsilon^\alpha,$$

which is contradictory by our choice of ε. $\diamond$

3 A Lasota-Yorke type Inequality

In this section we will establish the key inequality, which says that not only P acts continuously on V_α but also P is a quasi-compact operator, hence the spectral results follows. Notice that we consider functions defined on all M, despite the fact that a density is always supported on Ω. Whereas we could have done the same computations on Ω, we would have found very bad estimates with respect to the dimension or the shape of Ω.[8]

Lemma 3.1 *Let us assume (PE1)–(PE5). Then there exists $\eta < 1$ and $D < \infty$ such that, for $0 \leq f \in V_\alpha$, $0 \leq Pf \in V_\alpha$ with the following*

$$|Pf|_\alpha \leq \eta |f|_\alpha + D\int_M f.$$

Proof. *By (PE2) there exists for each i a neighborhood of $\operatorname{clos}(TU_i)$ where T_i^{-1} can be extended. We assume without loss*

[8] It is just because on M, the Lebesgue measure of a ball depends only on its radius, while the volume of $B_\varepsilon(x) \cap \Omega$ may vary a lot when x is near the boundary of Ω.

of generality that this neighborhood can be taken as $B_{\varepsilon_0}(TU_i)$. *From now on* T_i *and* T_i^{-1} *will denote the extended maps. We assume further that* $B_{s\varepsilon_0}(U_i) \supset T_i^{-1}B_{\varepsilon_0}(TU_i)$, *and given* $\varepsilon < \varepsilon_0$ *and* x, y *in* $B_{s\varepsilon}(z)$, *we have*

$$|\det D_xT_i - \det D_yT_i| \leq c|\det D_zT_i|\,\varepsilon^\alpha. \tag{1}$$

Remark that all of this can be achieved by taking ε_0 *small enough. Let* $f \in V_\alpha$, *positive and* $\varepsilon < \varepsilon_0$. *We have for all* $x \in M$

$$\begin{aligned}
\operatorname{osc}(Pf, B_\varepsilon(x)) &\leq \sum_i \operatorname{osc}((fg)\circ T_i^{-1}\chi_{TU_i}, B_\varepsilon(x)) \\
&\leq \sum_i \operatorname{osc}((fg)\circ T_i^{-1}, TS_i \cap B_\varepsilon(x))\chi_{TU_i}(x) + \\
&\qquad + \operatorname*{Esup}_{TU_i\cap B_\varepsilon(x)} (fg)\circ T_i^{-1}\cdot \chi_{B_\varepsilon(\partial TU_i)}(x) \\
&\leq \sum_i \operatorname{osc}(fg, U_i \cap T^{-1}B_\varepsilon(x))\chi_{TU_i}(x) + \\
&\qquad + \operatorname*{Esup}_{U_i\cap T^{-1}B_\varepsilon(x)} (fg)\cdot \chi_{B_\varepsilon(\partial TU_i)}(x).
\end{aligned}$$

Our task is to show that this function has an integral bounded by a constant $\eta < 1$ *times* $|f|_\alpha\varepsilon^\alpha$, *plus an arbitrary large constant times the* L^1 *norm of* f *times* ε^α. *We now look at the first term involved in this inequality; for* $x \in TU_i$, *by expansiveness of* T, *if* $y = T_i^{-1}x$,

$$R_i^{(1)}(x) := \operatorname{osc}(fg, U_i \cap T_i^{-1}B_\varepsilon(x)) \leq \operatorname{osc}(fg, U_i \cap B_{s\varepsilon}(y)),$$

which is, by proposition (2.2.iii), almost everywhere $x \in TU_i$ *less or equal than*

$$\begin{aligned}
R_i^{(1)}(x) &\leq \operatorname{osc}(f, B_{s\varepsilon}(y)) \operatorname*{Esup}_{U_i\cap B_{s\varepsilon}(y)} g + \operatorname{osc}(g, U_i\cap B_{s\varepsilon}(y)) \operatorname*{Einf}_{B_{s\varepsilon}(y)} f \\
&\leq (1 + cs^\alpha\varepsilon^\alpha)\operatorname{osc}(f, B_{s\varepsilon}(y))g(y) + f(y)g(y)cs^\alpha\varepsilon^\alpha.
\end{aligned}$$

This gives an estimate of the first term,

$$\sum_i R_i^{(1)}\chi_{TU_i} \leq (1 + cs^\alpha\varepsilon^\alpha)P\Big(\operatorname{osc}(f, B_{s\varepsilon}(\cdot))\Big) + cs^\alpha\varepsilon^\alpha Pf. \tag{2}$$

Then the integration gives

$$\begin{aligned}
\int_M \sum_i R_i^{(1)}\chi_{TU_i} &\leq (1 + cs^\alpha\varepsilon^\alpha)\int \operatorname{osc}(f, B_{s\varepsilon}) + cs^\alpha\varepsilon^\alpha \int_M f \\
&\leq (1 + cs^\alpha\varepsilon^\alpha)|f|_\alpha s^\alpha\varepsilon^\alpha + cs^\alpha\varepsilon^\alpha \int_M f.
\end{aligned}$$

The estimates for the second term involves more geometrical structure, and all the assumptions about the partition arises from this point.

$$R_i^{(2)}(x) := \operatorname*{Esup}_{U_i \cap T_i^{-1} B_\varepsilon(x)} (fg) \cdot \chi_{B_\varepsilon(\partial T U_i)}(x).$$

For $x \in B_{\varepsilon_0}(TU_i)$ *we obtain* (y *is still* $T_i^{-1}x$)

$$R_i^{(2)}(x) \leq \operatorname*{Esup}_{B_\varepsilon(y)} f \cdot |\det D_x T_i^{-1}|(1 + cs^\alpha \varepsilon^\alpha) \cdot \chi_{T_i(B_{s\varepsilon}(\partial U_i))}(x).$$

A first integration over M*, followed by a change of variable yields*

$$\frac{1}{(1 + cs^\alpha \varepsilon^\alpha)} \int_M \sum_i R_i^{(2)}(x) dx \leq \sum_i \int_M \chi_{B_{s\varepsilon}(\partial U_i)}(y) \operatorname*{Esup}_{B_{s\varepsilon}(y)} f dy. \quad (3)$$

By proposition (2.2.iv) the supremum of f *can be bounded by its oscillation plus its average. This gives (3) less or equal than*

$$\int_M dy \sum_i \frac{\chi_{B_{s\varepsilon}(\partial U_i)}(y)}{m(B_{(1-s)\varepsilon_0}(y))} \int_{B_{(1-s)\varepsilon_0}(y)} [f(z) + \operatorname{osc}(f, B_{\varepsilon_0}(z))]\, dz,$$

which after having exchanged the order of integration is equal to

$$\int dz\, [f(z) + \operatorname{osc}(f, B_{\varepsilon_0}(z))] \int dy \sum_i \frac{\chi_{B_{s\varepsilon}(\partial U_i)}(y) \chi_{B_{(1-s)\varepsilon_0}(y)}}{m(B_{(1-s)\varepsilon_0}(y))}.$$

Finally, for the measure of a ball depends only on its radius, (PE5) allows the following estimate

$$\begin{aligned}\int_M \sum_i R_i^{(2)}(x) dx &\leq (1 + cs^\alpha \varepsilon^\alpha) G(\varepsilon) \left(\int_M \operatorname{osc}(f, B_{\varepsilon_0}(\cdot)) dm + \int_M f dm \right) \\ &\leq (1 + cs^\alpha \varepsilon^\alpha) G(\varepsilon) |f|_\alpha \varepsilon_0^\alpha + (1 + cs^\alpha \varepsilon^\alpha) G(\varepsilon) \int_M f dm.\end{aligned}$$

To conclude, putting together estimates (1) and (2) yields

$$\begin{aligned}\int_M \operatorname{osc}(Pf, B_\varepsilon(x)) dx &\leq (1 + c(s\varepsilon)^\alpha) [s^\alpha \varepsilon^\alpha + G(\varepsilon) \varepsilon_0^\alpha] |f|_\alpha + \\ &\quad + [c(s\varepsilon)^\alpha + (1 + c(s\varepsilon)^\alpha) G(\varepsilon)] \int_M f dm \\ &\leq \left(\eta |f|_\alpha + D \int_M f dm \right) \varepsilon^\alpha.\end{aligned}$$

Where

$$\eta = (1 + cs^\alpha \varepsilon_0^\alpha) \left(s^\alpha + \sup_{\varepsilon \leq \varepsilon_O} \frac{G(\varepsilon) \varepsilon_0^\alpha}{\varepsilon^\alpha} \right)$$

and

$$D = cs^{\alpha} + (1 + cs^{\alpha}\varepsilon_0^{\alpha}) \sup_{\varepsilon \leq \varepsilon_O} G(\varepsilon)\varepsilon^{-\alpha}.$$

The lemma is proven by definition of $|\cdot|_\alpha$, *provided* ε_0 *is chosen small enough to get* $\eta < 1$. $\diamond$

4 Spectral results

In this section we will present the main result, which is the spectral decomposition of PF for piecewise expanding maps satisfying our hypotheses. This theorem follows from Lemma 3.1 and an ergodic theorem of Ionescu-Tulcea and Marinescu (see [IM]). Its statement, adapted here for our special case, is entirely borrowed from [Ke].

Theorem 4.1 *Under assumptions (PE1)–(PE5) holds :*

(i) $P : L^1_m \to L^1_m$ *has a finite number of eigenvalues* $\lambda_1, \dots, \lambda_r$ *of modulus one.*

(ii) $E_i = \{f \in L^1_m | Pf = \lambda_i f\} \subset V_\alpha$ *and* $\dim(E_i) < \infty$ *for* $i = 1, \dots, r$.

(iii) $P = \sum_{i=1}^{r} \lambda_i P_i + Q$, *where the* P_i *are projections onto the eigenspaces* E_i, $\|P_i\|_1 \leq 1$, *and* Q *is a linear operator on* L^1_m *with* $Q(V_\alpha) \subset V_\alpha$, $\sup_{n \in I\!N} \|Q^n\|_1 < \infty$, *and* $\|Q^n\|_\alpha = O(q^n)$ *for some* $0 < q < 1$. *Furthermore* $P_i P_j = 0$ *if* $i \neq j$ *and* $P_i Q = Q P_i = 0$ *for all* i. *(This means that* P *is quasi-compact as an operator on* $(V_\alpha, \|\cdot\|_\alpha)$*).*

(iv) 1 *is an eigenvalue of* P, *and assuming* $\lambda_1 = 1$ *and* $h_* = P_1 1$, $\mu = h_* m$ *is the greatest* ACIM *for* T, *i.e. if* $\tilde{\mu}$ *is* T*-invariant and* $\tilde{\mu} \ll m$ *then* $\tilde{\mu} \ll \mu$.

PROOF. *Since* $\left\{ f \in V_\alpha \,\middle|\, \operatorname{supp} f \subset \Omega,\ \|f\|_\alpha \leq 1 \right\}$ *is compact in* $L^1(M)$ *by proposition 2.3, by Ionescu-Tulcea and Marinescu Theorem and Lemma 2.1 we get the usual spectral decomposition of* P. $\diamond$

Remark 4.1 *It is sufficient to have (PE4)–(PE5) for some iterate of the map. The contraction* $s(T^n)$ *of the map* T^n *is bounded by* s^n, *while the quantity into* $G(\varepsilon)$ *has "usually" no reasons to grow much (but* ε_0 *must be chosen small enough for that).*

Theorem 4.2 *Let T be a map which satisfy (PE1)–(PE5). Then the number $E(T)$ of ergodic components of positive Lebesgue measure is bounded by*

$$E(T) \leq \frac{m(\Omega)}{\gamma_N}\left(\frac{D}{1-\eta}\right)^{N/\alpha}.$$

PROOF. *For all ergodic component, there exist a positive eigenfunction h associated to the eigenvalue 1 on L^1_m.*

¿From Theorem 4.1.(ii) we know that $h \in V_\alpha$, so Lemma 3.1 yields $|Ph|_\alpha \leq \eta|h|_\alpha + D$. Hence $|h|_\alpha \leq \frac{D}{1-\eta}$. By Lemma 2.1 the infimum of h on some ball of radius $\varepsilon = \left(\frac{1-\eta}{D}\right)^{\frac{1}{\alpha}} < \varepsilon_0$ is strictly positive.

Consequently, each ergodic component must contains a ball of radius ε. It follows immediately that the number of ergodic component is limited by the maximal number of balls of radius ε contained in Ω, which is roughly[9] bounded by $\dfrac{m(\Omega)}{\gamma_N \varepsilon^N}$. $\diamond$

This corollary states a sufficient condition for a map to satisfy the hypotheses (PE5). It links the expanding constant with the local behavior of discontinuities.

Corollary 4.1 *Let T be a map satisfying (PE1)–(PE4). Suppose that the boundaries of the U_i are included in piecewise $\mathcal{C}^{(1)}$ codimension one imbedded submanifolds, and let Y be the maximal number of these smooth components that can meet in one point, i.e.*

$$Y = \sup_{x\in\Omega} \sum_i \#\{\textit{smooth pieces containing } \partial U_i \textit{ containing } x\}.$$

Setting $\eta_0 := s^\alpha + \dfrac{2s}{1-s} Y \dfrac{\gamma_{N-1}}{\gamma_N}$, the following holds :

If $\eta_0 < 1$ then the map satisfy (PE5).

Remark 4.2 *We want to stress that there is no need to control the angles between smooth elements of the partition, contrary to other methods. This condition is very interesting when iterating the map, because $s(T^n)$ decreases exponentially fast, while, generically $Y(T^n)$ grows subexponentially (see [Bu]). This ensure that generically any eventually expanding map posses an* ACIM. *In the case of an affine mod $\mathbb{Z}^N$ map of the unit cube, $Y(T^n)$ grows only polynomially with n, ensuring that (PE5) is automatically true for some iterate.*

[9] Actually, if we do not put any restrictions on Ω, this bound is optimal. Sphere packing is a very hard problem, and it is an open question to calculate the real bound.

PROOF. *In a first step, we consider each smooth component separately. Then, for each point x in M, we will compute a bound for the quantity which come into $G(\varepsilon)$. To do this, we bring the problem to the one when the boundary is a piece of hyperplane.*

Let $\delta > 0$ which will be fixed later and $\nu < \delta$. Let us consider some smooth piece Γ of the boundary of U.

Let x in M. The aim is to compute the following quantity :

$$m(B_\nu(\Gamma) \cap B_\delta(x)). \tag{4}$$

If x does not belong to the ν-neighborhood of Γ then nothing has to be done. Else, let us consider the preferred local coordinate map Φ of the imbedded submanifold Γ. We can suppose $\Phi : B_\delta(x) \to I\!R^N$ for some δ independent of x (remember that Ω is compact). We recall that $\Phi(\Gamma)$ is contained in a hyperplane H of $I\!R^N$. To simplify, let us choose Φ such that $D_x\Phi = I$.

The following estimates holds for $y \in B_\delta(x)$ ($|\cdot|$ represent the Euclidean norm in $I\!R^N$) :

$$|\Phi(y) - \Phi(x)| \leq |D_x\Phi(y-x)| + o(\delta) \leq \delta(1+o(1)),$$

hence ($o(1)$ stands for a function which vanishes with δ)

$$\Phi(B_\delta(x)) \subset B_{\delta(1+o(1))}(\Phi(x)).$$

Moreover, given $z \in \Gamma$ and $y \in B_\nu(z) \cap B_\delta(x)$ we have

$$|\Phi(y) - \Phi(z)| \leq |D_z\Phi(y-z)| + o(\nu) \leq (1+o(\delta))\nu + o(\nu),$$

which imply

$$\Phi(B_\nu(\Gamma)) \subset B_{\nu(1+o(\delta))+o(\nu)}(H) \subset B_{\nu(1+o(1))}(H).$$

For the coordinate map is a $\mathcal{C}^{(1)}$ perturbation of a translation, it changes volumes by a factor $1+o(1)$. This mean that (4) is bounded by $1+o(1)$ times the measure of $B_{\delta(1+o(1))}(\Phi(x)) \cap B_{\nu(1+o(1))}(H)$. It is obviously maximal when the hyperplane H cross the center of the ball. In this case, the quantity can be easily estimated by $2\nu(1+o(1))$ times the $(N-1)$ volume of the $(N-1)$-dimensional ball of radius $\delta(1+o(1))$. Hence

$$(4) \leq 2\nu\gamma_{N-1}\delta^{N-1}(1+o(1)).$$

By compactness of Ω, it is possible to get all the $o(1)$ uniform in x. Moreover, if δ is chosen small enough, then the number of smooth pieces of the boundary does not exceed Y. Consequently,

$$G(\varepsilon) \leq Y 2 \frac{\gamma_{N-1}}{\gamma_N} \frac{s\varepsilon}{(1-s)\varepsilon_0}(1+o(1)).$$

Finally, (PE5) will holds provided $\eta_0 < 1$ and ε_0 is small enough.
$\diamond$

Remark 4.3 *The fact that the pieces of the boundaries are imbedded submanifold is not essential to get (PE5), although it is unclear how to give a general statement which allow singular points (like edge of cones, etc...).*

5 Decay of correlations

In section 4, existence of a spectral gap was shown for the PF operator. As already said, exponential decay of correlations follows immediately. However, Ionescu-Tulcea and Marinescu Theorem does not give any information about the rate of decay, which is precisely the absolute value of the second eigenvalue, if it exists (the first one being 1). In this section we propose to compute an upper bound for the rate of decay. One cannot expect to derive just from the preceding any information about the second eigenvalue, for this some knowledge about the speed of mixing is needed.

Under an additional assumption (Covering), one can compute explicitly an upper bound for the rate of decay (each step of the paper [L2, LSV1] can be followed in the same way).

We emphasize the fact that in what follows we do not refer to section 4, only the Lasota-Yorke inequality of Lemma 3.1 and the existence of the density are used. For example, one do not need the compactness of Ω to obtain Theorems 5.1 and 5.2 if Lemma 3.1 and existence of $h_* \in V_\alpha$ are assumed.

Definition 5.1 *A map T is called covering when holds :*

(C1) For any $\delta > 0$ there exists an integer $K = K(\delta)$ such that for any ball $B \subset \Omega$ of radius δ, $T^K B$ covers Ω but a zero measure set.

Remark 5.1 *This assumption imply that the map is topologically mixing, hence that there exists an unique* ACIM, *with full topological support on Ω. In the case of an affine modulo $\mathbb{Z}^N$ map of the unit cube, it is known that the covering holds provided the expansion is big enough ($s(1+\sqrt{N}) < 1$).*

For simplicity, we assume that $m(\Omega) = 1$.

Theorem 5.1 *Let T be a map satisfying (PE1)–(PE5) and (C1). Let μ be the unique* ACIM *given by theorem 4.1 (the uniqueness is an immediate consequence of covering). Then the decay of correlation is exponential on α-Hölder observable, and the rate of mixing is given, for $f \in L^1$ such that $\mu(f) = 0$, by*

$$\int_\Omega f \circ T^n h d\mu \leq C\|f\|_{L^1}\|h\|_\alpha \Lambda^n.$$

where $\Lambda < 1$ is given by Lemma 5.5.

Actually, we will prove a bit more. The convergence of any density occurs not only in the weak sense (correlation goes to zero) but also in the following strong sense :

Theorem 5.2 *The convergence of $P^n f$ for $f \in V_\alpha$ to the density h_* of the unique* ACIM *is exponential in both L^∞ and $|\cdot|_\alpha$. The rate is given by Λ^n, where $\Lambda < 1$ is given by Lemma 5.5.*

> PROOF.*[Theorem 5.1] The decay of the correlations is given by the rate of convergence in L^∞ of α-Hölder functions (which belongs to V_α) which is by Theorem 5.2 at an exponential rate Λ^n.*
> $\diamond$

To prove Theorem 5.2, we will establish some preparatories Lemmas, which shows step by step that PF is a contraction for a suitable metric (Hilbert-Birkhoff projective metric on a convex cone) defined on V_α.
Before entering into the proof, let us see some preliminary definitions and properties of this metric. For a more detailed review of these properties, see for example [LSV1].

Definition 5.2 *We define a partial order relation on a convex cone $\mathcal{C}$ by*

$$f \preceq h \Leftrightarrow h - f \in \mathcal{C}.$$

The distance $\Theta(f,h)$ between two points $f,h \in \mathcal{C}$ is given by

$$\begin{aligned} \alpha(f,h) &= \sup\{\lambda > 0 | \lambda f \preceq h\}, \\ \beta(f,h) &= \inf\{\mu > 0 | h \preceq \mu f\}, \\ \Theta(f,h) &= \log \frac{\beta(f,h)}{\alpha(f,h)}. \end{aligned}$$

Where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

The distance Θ is a pseudo-metric, because two elements can be at an infinite distance from each others, and it is a projective metric because any two proportional elements have a null distance.
The next Theorem, due to G. Birkhoff [Bi] show that every positive linear operator is a contraction, provided that the diameter of the image is finite.

Theorem 5.3 *Let $\mathcal{V}$ be a vector space, $\mathcal{C} \subset \mathcal{V}$ a convex cone*[10] *and $L : \mathcal{V} \to \mathcal{V}$ a positive linear operator (which means $L(\mathcal{C}) \subset \mathcal{C}$). Let Θ be the Hilbert metric associated to the cone $\mathcal{C}$. If we denote*

$$\Delta = \sup_{f,h \in L(\mathcal{C})} \Theta(f,h) \quad ,$$

then

$$\Theta(Lf, Lh) \leq \tanh\left(\frac{\Delta}{4}\right)\Theta(f,h) \ \ \forall f,h \in \mathcal{C},$$

$(\tanh(\infty) = 1)$.

Now the strategy is clear. We want to show that the cone $\mathcal{C}_a$, where

$$\mathcal{C}_a := \left\{ f \in L^1_m(M, I\!R) \mid 0 \leq f, \text{supp } f \subset \Omega, |f|_\alpha \leq a \int f \right\} \backslash \{0\},$$

and $a > 0$ has to be fixed later, is strictly contracted by PF, that is to say that the diameter of its image is finite. Moreover, the contraction rate will be given by this diameter, which is explicitly computable. The last part will be to show that the Hilbert metric is stronger than the uniform and the $\|\cdot\|_\alpha$ norms.
First we show that the cone $\mathcal{C}_a$ is contracted by PF.

Lemma 5.1 *The cone $\mathcal{C}_a$ is left invariant by the operator PF provided a is big enough. Moreover, $P\mathcal{C}_a \subset \mathcal{C}_{\sigma a}$ for some $\sigma < 1$.*

PROOF. *It is an immediate consequence of Lemma 3.1, since $m(Pf) = m(f)$ and*

$$|Pf|_\alpha \leq \eta |f|_\alpha + Dm(f) \leq (\eta a + D)m(f) \leq \sigma a m(f)$$

provided $\eta a + D \leq \sigma a$, which can be achieved by taking a big enough whenever $\sigma > \eta$. $\diamond$

Second, we compute the distance between two points in $\mathcal{C}_{\sigma a}$.

[10] To be completely honest, $\mathcal{C}$ cannot be any convex cone, weak properties have to be satisfied : (i) $\mathcal{C} \cap -\mathcal{C} = \emptyset$, (ii) $\forall \lambda > 0, \lambda\mathcal{C} = \mathcal{C}$, (iii) $\mathcal{C}$ is convex and (iv) $\forall f,h \in \mathcal{C}, \forall t_n \in I\!R$ such that $t_n \to t$, $h - t_n f \in \mathcal{C}$ imply $h - tf \in \mathcal{C} \cup \{0\}$.

Lemma 5.2 *The distance from two elements of $\mathcal{C}_{\sigma a}$ in $\mathcal{C}_a$ can be estimated as follows*

$$d(f,h) \le \log \frac{\max(\tau \frac{m(h)}{m(f)}, \operatorname{Esup} \frac{h}{f})}{\min(\tau^{-1} \frac{m(h)}{m(f)}, \operatorname{Einf} \frac{h}{f})},$$

where $\tau = \dfrac{1+\sigma}{1-\sigma}$.

PROOF. *Let f, h in $\mathcal{C}_{\sigma a}$. The distance between f and g is given by $\log \frac{\mu}{\lambda}$ where μ and λ are such that $\lambda f \preceq h \preceq \mu f$.*

$$\begin{aligned}
\lambda f \preceq h \quad &\Leftrightarrow \quad \begin{cases} h - \lambda f \ge 0 \\ |h - \lambda f|_\alpha \le am(h - \lambda f) \end{cases} \\
&\Leftarrow \quad \begin{cases} \lambda f \le h \\ |h|_\alpha + \lambda |f|_\alpha \le \ am(h - \lambda f) \end{cases} \\
&\Leftarrow \quad \begin{cases} \lambda \le \operatorname{Einf}_\Omega \frac{h}{f} \\ \sigma am(h) + \lambda \sigma am(f) \le am(h) - \lambda am(f) \end{cases} \\
&\Leftarrow \quad \begin{cases} \lambda \le \operatorname{Einf}_\Omega \frac{h}{f} \\ \lambda \le \frac{1-\sigma}{1+\sigma} \frac{m(h)}{m(f)} \end{cases}
\end{aligned}$$

The same computation with μ yields

$$d(f,h) \le \log \frac{\max(\tau \frac{m(h)}{m(f)}, \operatorname{Esup} \frac{h}{f})}{\min(\tau^{-1} \frac{m(h)}{m(f)}, \operatorname{Einf} \frac{h}{f})}.$$

$\diamond$

Then, this lemma shows that functions in V_α cannot be small too often

Lemma 5.3 *Let $h \in V_\alpha$, then there exists a point $x \in \Omega$ such that*

$$\operatorname*{Einf}_{B_{\varepsilon_1}(x)} h \ge \frac{1}{2} \int h dm, \tag{5}$$

when $\varepsilon_1 := \min(\varepsilon_0, (2a)^{\frac{1}{\alpha}})$.

PROOF. *Let $h \in V_\alpha$. Suppose that (5) is wrong for all x, then, m-almost everywhere $x \in \Omega$ holds*

$$\operatorname{osc}(h, B_\varepsilon(x)) > \operatorname*{Esup}_{B_\varepsilon(x)} h - \frac{1}{2} \int h dm \ge h(x) - \frac{1}{2} \int h dm.$$

An integration on Ω *yields*

$$|h|_\alpha \varepsilon^\alpha \geq \int_\Omega \operatorname{osc}(h, B_\varepsilon(x))dx > \int_\Omega h - \frac{1}{2}\int_\Omega h.$$

Accordingly, since $h \in \mathcal{C}_a$,

$$a\varepsilon^\alpha > \frac{1}{2},$$

which is contradictory with $\varepsilon = \varepsilon_1$. $\diamond$

The above Lemmas give enough control on the metric to see that the diameter of the image of $\mathcal{C}_a$ is finite.

Lemma 5.4 *The diameter of* $P^K\mathcal{C}_a$ *in* $\mathcal{C}_a$ *is bounded by* $\Delta < \infty$ *for some* K*, where*

$$\Delta = 2\log\max\left(\tau, \frac{2(\sigma a\varepsilon_0^\alpha + 1)\|\det DT^K\|_\infty}{\gamma_N \varepsilon_0^N}\right).$$

PROOF. *Let* $f, h \in \mathcal{C}_a$. *We propose to bound the distance between* $P^K f$ *and* $P^K h$. *Since the distance is projective, we may assume that* $m(f) = m(h) = 1$. *By Lemma (5.2), it is sufficient to bound uniformly the extrema of* $P^K f$ *and* $P^K h$. *Proposition (2.2.iv) tells that*

$$\begin{aligned}\operatorname{Esup} P^K h &\leq \frac{1}{\gamma_N\varepsilon_0^N}[\int \operatorname{osc}(P^K h, B_{\varepsilon_0}(x))dx + m(P^K h)]\\ &\leq \frac{1}{\gamma_N\varepsilon_0^N}[|P^K h|_\alpha \varepsilon_0^\alpha + 1] \leq \frac{\sigma a \varepsilon_0^\alpha + 1}{\gamma_N \varepsilon_0^N}.\end{aligned}$$

Lemma (5.3) ensure that there exists a ball $B(f) \subset \Omega$ *of radius* ε_1 *such that* $\operatorname{Einf}_{B(f)} f \geq \frac{1}{2}$. *Clearly,* $B(f) \subset \Omega$. *Moreover, for the map is covering (C1), there exists an integer* $K = K(\varepsilon_1)$ *such that* $B(f)$ *is mapped over all* Ω *after* K *iterations of* T. *Hence,*

$$P^K f \geq P^K(\frac{1}{2}\chi_{B(f)}) \geq \frac{1}{2\|\det DT^K\|_\infty}.$$

Finally, we obtain

$$\operatorname{Esup}\frac{P^K h}{P^K f} \leq \frac{2(\sigma a \varepsilon_0^\alpha + 1)\|\det DT^K\|_\infty}{\gamma_N \varepsilon_0^N}.$$

The Lemma is concluded by remarking that the role of f *and* h *can be exchanged.* $\diamond$

We are now able to prove that PF is a contraction :

Lemma 5.5 *The convergence of $P^n f$ for $f \in \mathcal{C}_a$ to the density h_* of the unique* ACIM *is exponential in the Hilbert metric of $\mathcal{C}_a$, the rate being given by Λ^n, where $\Lambda = \tanh(\Delta/4)^{1/K} < 1$.*

PROOF. *For the diameter of the image $P^K \mathcal{C}_a$ is finite (Lemma 5.4), the contraction of P^K given by Theorem (5.3) is equal to $\Lambda^K < 1$. Writing $n = K(m+1)+p$ with $0 \le p < K$ we obtain*

$$\begin{aligned}
\Theta(P^n f, h_*) &\le \Theta(P^n f, P^n h_*) \\
&\le \Theta((P^K)^m P^{K+p} f, (P^K)^m P^{K+p} h_*) \\
&\le \Lambda^{Km} \Theta(P^K P^p f, P^K P^p h_*) \\
&\le \Delta \Lambda^{Km}.
\end{aligned}$$

Hence the distance between $P^n f$ and h_* is bounded by $\dfrac{\Delta}{\Lambda^{2K}} \Lambda^n$.
$\diamond$

The above facts imply the strong convergence announced before.

PROOF.*[Theorem 5.2] Let $n \ge K$ be an integer. Let us start with $f \in \mathcal{C}_a$. We assume for simplicity that $m(f) = 1$. We already prove that $\Theta(P^n f, h_*) < \infty$. For all $0 < \lambda \le \mu$ such that $\lambda h_* \preceq P^n f \preceq \mu h_*$ in the sense of the cone $\mathcal{C}_a$, we have*[11]

$$\|P^n f - h_*\|_\infty \le \max(1-\lambda, \mu - 1)\|h_*\|_\infty.$$

¿From which follows, by definition of Θ

$$\|P^n f - h_*\|_\infty \le \Theta(P^n f, h_*)\|h_*\|_\infty \le c_1 \Lambda^n.$$

This conclude the first part of the theorem. The second one is obtained via the same argument, since

$$|P^n f - h_*|_\alpha \le |P^n f - \lambda h_*|_\alpha + (1-\lambda)|h_*|_\alpha.$$

Accordingly,

$$|P^n f - h_*|_\alpha \le 2a \max(1-\lambda, \mu - 1),$$

which again imply

$$|P^n f - h_*|_\alpha \le 2a\Theta(P^n f, h_*) \le c_2 \Lambda^n.$$

To conclude we just need to remark that any function f in V_α belongs to $\mathcal{C}_a + \{th_*, t \in I\!\!R\}$ because $h_* \in \mathcal{C}_{\sigma a}$. $\diamond$

[11]Observe that $\lambda \le 1 \le \mu$ since $m(f) = m(h_*) = 1$.

6 Some examples

Let us consider a map T on $\Omega = [0,1]^2$ which is discontinuous on a von Koch's like fractal set Γ. The construction of Γ is done as follows : Start from the diagonal D of the unit square. Then cut the diagonal in three segments of equal length, and replace the one in the middle by two segments of the same length, as shown in Fig. 3. repeat this procedure with each small segment, and so on... Call Γ the set so obtained.

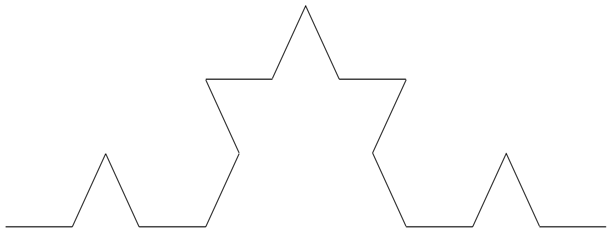

Figure 3 : Third step of the construction of Γ.

Proposition 6.1 *Any map defined on Ω which satisfy (PE1)–(PE4) and discontinuous on Γ verify also (PE5) provided s is small enough.*

PROOF. *Let $\beta = \frac{\log 4}{\log 3}$ (it is the fractal dimension of Γ). Let $\delta > 0$ be small enough, $\nu < \delta$ and $\alpha \leq 2 - \beta$. We want to compute the quantity (4) (see Lemma 4.1). Let $n = [\frac{\log \nu^{-1}}{\log 3}] + 1$. Since our accuracy is ν, it suffices to consider the n-step approximation of the fractal Γ. At the order n, each segment is of length $(1/3)^n$, and there is at most $4^n (2\delta)^\beta$ segments in a ball of size δ. It follows that*

$$(4) \leq 2^\beta [(1/3)^n 2\nu + \pi\nu^2] 4^n \delta^\beta \leq 2^\beta (2+\pi) \nu^{2-\beta} \delta^\beta.$$

So the measure of the intersection divided by the measure of the ball of radius δ is bounded by

$$2^\beta (1 + 2/\pi) \delta^{\beta-2} \nu^{2-\beta}.$$

Hence the contribution to $G(\varepsilon)$ from the boundary Γ can be bounded with by

$$2^\beta (1 + 2/\pi) (\frac{s}{1-s})^{2-\beta} (\frac{\varepsilon}{\varepsilon_0})^{2-\beta}. \tag{6}$$

Hence (PE5) holds provided the contribution from the other boundaries is such that

$$s^\alpha + \eta(\Gamma) + \sup_{\varepsilon \leq \varepsilon_0} \frac{G'(\varepsilon)}{\varepsilon^\alpha} \varepsilon_0^\alpha < 1,$$

for some $\alpha \leq \beta$, where

$$\eta(\Gamma) := 2^{\beta+1}(1 + 2/\pi)(\frac{s}{1-s})^{2-\beta},$$

and $G'(\varepsilon)$ is $G(\varepsilon)$ minus the contribution of the boundary Γ. ($\eta(\Gamma)$ is twice (6) because Γ divide the square into two pieces). $\diamond$

We claimed in footnote ([6]) that there are some cases of maps with a countable number of smooth components that we can handle. Despite the lack of simple condition which ensure that the assumptions are satisfied, we propose to give some examples of such maps.

Definition 6.1 *Let us consider a map T such that there is a countable number of open sets U_i. It T satisfies (PE1),(PE2$_\infty$),(PE3),(PE4) and (PE5$_\infty$) then we call T a σ-piecewise expanding map*

(PE2$_\infty$) we require (PE2), and the neighborhood on which T_i^{-1} extends can be taken of size ε_0 for all i. Moreover, the Hölder estimate on the determinant (inequality (1) in the proof of Lemma 3.1) holds.

(PE5$_\infty$) provided ε_0 is small enough the following holds

$$s^\alpha + \sup_{\varepsilon \leq \varepsilon_0} \frac{G(\varepsilon)}{\varepsilon^\alpha} \varepsilon_0^\alpha < 1,$$

where

$$G(\varepsilon) := \sup_x \sum_i \frac{m(T_i^{-1} B_\varepsilon(\partial T U_i) \cap B_{(1-s)\varepsilon_0}(x))}{m(B_{(1-s)\varepsilon_0}(x))}.$$

Proposition 6.2 *Theorem 4.1 and Theorem 4.2 remain true for a σ-piecewise expanding map.*

PROOF. *We just give a brief sketch of what has to be changed. Essentially, it suffices to treat the last part of Lemma 3.1 (when $R_i^{(2)}$ is computed) more precisely. If we look at the computation, then it is clear that the condition (PE5$_\infty$) suffices.* $\diamond$

Now we state a condition which can replace the covering, in the case of a map with a countable number of smooth components. It is the definition given in [LSV1].

Definition 6.2 *We call the σ-piecewise expanding map covering if holds :*

(C1$_\infty$) For any $\delta > 0$ there exists an integer $K = K(\delta)$ and $c = c(\delta) > 0$ such that for any ball $B \subset \Omega$ of radius δ we have $\operatorname{Einf} P^K \chi_B > c$.

This condition implies that any ball is mapped onto Ω, hence topological mixing for T.

Proposition 6.3 *Theorem 5.1 and Theorem 5.2 remain true for a covering σ-piecewise expanding map.*

> Proof. *The only difference occurs while estimating the diameter in Lemma 5.4. Because of (PE5$_\infty$), it is impossible to hope that $\|\det DT^K\|_\infty < \infty$. The condition (C1$_\infty$) is enough to get the diameter finite.* $\diamond$

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