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One-dimensional quantum scattering for potentials defined as measures

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Abstract

We generalize the basic one-dimensional scattering formalism to potentials defined as measures and retrieve the classical results that hold for smooth potentials. We introduce a set of generalized eigenfunctions for the corresponding Schrödinger operator and study their analytical properties. This allows a characterization of the spectrum and an eigenfunction expansion. We also prove the existence and completeness of the wave operators and give explicit formulae for these latter.

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1 Introduction

From the very beginning of quantum scattering theory, the one-dimensional case has always received much attention, as a first step towards more complicated multi-dimensional problems. There are two typical situations in the one-dimensional approach. The first one is the so-called half-line scattering, motivated by its connection to the three-dimensional radial problem, in which case the S-matrix reduces to a unimodular complex number. The second one is the full-line scattering, which is formally much richer since the S-matrix becomes a two-two matrix, owing to the two-fold degeneracy of the spectrum. At present, the spectral and scattering theory for the one-dimensional Schrödinger equation have been well worked-out and have given rise to a huge literature; for signifiant contributions, we refer to [New80b],[DAR65] for the half-line, to [Fad64],[DT79],[New80a] for the full-line, and to [BS91] for both. The characteristic results can be roughly summarized as follows. If the potential satisfies some short-range condition (it is usually required that the potential and its first two moments be absolutely integrable), then the wave operators exist and are asymptotically complete, the scattering matrix is analytic (with respect to the spectral parameter) in the upper-half plane with a finite number of simple poles corresponding to the bound states, the positive spectrum is purely absolutely continuous, and the eigenfunctions form a complete set. In the derivation of these results, many efforts have been made to optimize the short-range condition, but few consideration has been taken into the local regularity of the potential. A natural question, however, might be whether small scales properties of the potential can influence the scattering process, even under some severe short-range condition. A fairly complete treatment of this question has been done in [Sch81] for measurable potentials with local non-integrable singularities. For more singular interactions such as potentials defined as measures or distributions, a scattering theory has been developped for point-interactions (see e.g [PV85],[Zor80],[Dol78]), but only partial results have been established in the general case. The existence and completeness of the wave operators have been proved in an abstract n-dimensional sheme ([For92],[Her89],[GP84]) and the nature of the spectrum has been investigated ([BEKS94]). Further results such as the characterization of the domain of Scrödinger operators perturbed by measures and unicity properties of eigenfunctions have been stated ([Tip90] in one dimension. However, to our knowledge, there are no explicit formulae at our disposal for the wave operators and the eigenfunction expansion associated to such potentials.

In this paper, we want to study the scattering problem on the line when the potential is an arbitrary finite Borel measure. The idea we have in mind (for subsequent work) while taking measures is to set a frame for the study of the time evolution generated by fractal potentials. Therefore, we would like to derive an explicit eigenfunctions expansion. To this end, we will extend (partially) the one-dimensional formalism to the case of measure-potentials. The paper is organized as follows. We start in section 2 with some necessary mathematical preliminaries. In section 3, we briefly recall how to define the hamiltonian properly for our type of potential. In section 4 and 5, we prove the existence of scattering wave functions and study their basic properties in section 6. In section 7, we study the spectrum and give an expression for the sprectral projectors. In the last section, we prove the existence and completeness of the wave operators and derive the eigenfunction expansion, thereby retrieving formally the formulae wich are valid for smooth potentials.

2 Preliminaries

Before entering the core of this paper, we will introduce some tools and notations that are necessary for the following. We refer e.g to [Rud74] for all classical results on measure theory and to e.g [Yos80] for the functional analysis. We denote by $C_0^k(\mathbb{R})$ the space of complex-valued k times continuously differentiable functions with compact support on $\mathbb{R}$. $\mathcal{D} = C_0^\infty(\mathbb{R})$ is the space of test functions and $\mathcal{D}'$, its dual, is the space of distributions on $\mathbb{R}$. We write (η, f) for the action of a distribution η on a function f. We use the notation $\eta_n \xrightarrow{\mathcal{D}'} \eta$ to say a sequence of distributions (η_n) converges in the weak topology of $\mathcal{D}'$ towards some other distribution η (that is $(\eta_n, s) \overset{n\to\infty}{\longrightarrow} (\eta, s)$ for any test function s). We denote by $C_b^0(\mathbb{R})$ the Banach space of continuous bounded functions on $\mathbb{R}$, equipped with the supremum norm $\|\cdot\|_\infty$. We call $\mathcal{M}$ the dual of $C_0^0(\mathbb{R})$, that is the space of Radon measures on $\mathbb{R}$, and $\mathcal{M}_\text{o}$ the dual of $C_b^0(\mathbb{R})$, which is the space of finite Radon measures on $\mathbb{R}$. The action of a measure μ in $\mathcal{M}$ or $\mathcal{M}_\text{o}$ on a continuous function f in $C_0^0(\mathbb{R})$ or $C_b^0(\mathbb{R})$ is given by

$$(\mu, f) = \int f(x)\mu(dx),$$

where the right-hand side is the Lebuesgue integral of f with respect to μ. We use the notation $\mu_n \xrightarrow{\mathcal{M}} \mu$, respectively $\mu_n \xrightarrow{\mathcal{M}_\text{o}} \mu$, to say a sequence of measures (μ_n) converges in the weak topology of $\mathcal{M}$, respectively $\mathcal{M}_\text{o}$, towards some other measure μ (that is $(\mu_n, f) \overset{n\to\infty}{\longrightarrow} (\mu, f)$ for any f in $C_0^0(\mathbb{R})$, respectively $C_b^0(\mathbb{R})$). A strong topology on $\mathcal{M}_\text{o}$ can also be defined via the norm

$$\|\mu\| = \sup_{\substack{f \in C_b^0(\mathbb{R}),\\ \|f\|_\infty = 1}} \left| \int f\mu(dx) \right| = \int |\mu|(dx) = \text{total variation of the measure } \mu.$$

If μ is a Borel measure, we use the canonical notation μ_{ac}, μ_{sc} and μ_{pp} for its absolutely continuous part, singular continuous part and purely ponctual part, respectively. In the following, we will always implicitely assume that the measures are $\sigma-$finite Borel measures. When $\mu(\mathbb{R}^-) < \infty$, one may define a "primitive" of μ by

$$L_\mu(x) = \mu(] - \infty, x[)$$

L_μ is a locally bounded measurable function, which is continuous to the left and admits at most a countable set of discontinuities located at the atomic points of μ (i.e $\mu(\{x\}) \neq 0$). Moreover, L_μ is almost everywhere (a.e) differentiable in the classical sense with

$$L'_\mu(x) = f_\mu(x) \ a.e,$$

where $f_\mu \in L^1_{loc}(\mathbb{R})$ is the Radon-Nykodim derivative of μ_{ac}, that is $\mu_{ac}(dx) = f_\mu(x)dx$. Unless explicitely stated, the word "almost everywhere" will always refer to the Lebesgue measure.

A function f is said to be absolutely continuous on a set K (possibly unbounded) if for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |b_i - a_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for any finite collection of disjoints intervals $[a_1, b_1], ...[a_n, b_n]$ of K. Absolutely continous functions can be thought as primitive of absolutely continuous measures: a function is absolutely continuous on K if and only if it is the indefinite integral of some locally integrable function on K. To distinguish between the classical derivatives and the distributional derivatives of a function, say f, we will write $f', f", ..f^{(n)}$ for the former and $Df, \Delta f, ..D^{(n)}f$ for the later. This distinction is especially important for functions such as L_μ, for which $L'_\mu = f_\mu$ a.e but $DL_\mu = \mu$, as can be seen from the following

Proposition 2.1 *If μ is a Borel measure such that $\mu(\mathbb{R}^-) < \infty$ and $s \in C^1_0(\mathbb{R})$, then we have the following "integration by part" formula*

$$\int dx \ s'(x) L_\mu(x) = - \int s(x)\mu(dx). \tag{2.1}$$

Hence, μ is the distributional derivative of L_μ in $\mathcal{D}$.

Proof. First suppose μ is finite. The formula is certainly true if μ can be identified with a smooth function. Otherwise, take some positive test function $h \in C_0^\infty(\mathbb{R}^-)$ such that $\int h = 1$ and set $h_n(x) = nh(nx)$ and $\mu_n = \mu * h_n \in C^\infty(\mathbb{R})$. The sequence h_n is an approximate identity, so $\mu_n \xrightarrow{\mathcal{M}_q} \mu$. Now the formula (2.1) holds for μ_n and thus

$$\int dx \ s'(x) L_{\mu_n}(x) = - \int dx \ s(x)\mu_n(x) \to - \int s(x)\mu(dx).$$

On the other hand we have, by Fubini's theorem,

$$\begin{aligned} L_{\mu_n}(x) &= \int_{-\infty}^{x} dy \int h_n(y-t)\mu(dt) \\ &= \int \mu(dt) \int_{-\infty}^{x} dy \ h_n(y-t) \\ &= \int \mu(dt)\rho_n(x-t), \end{aligned}$$

where

$$\rho_n(u) = \int_{-\infty}^{u} dy \ h_n(y).$$

Since we have choosen the function h to be supported by the negative axis only, we have

$$\rho_n(u) \stackrel{n\to+\infty}{\longrightarrow} \left\{ \begin{array}{l} 0 \text{ if } u < 0 \\ 1 \text{ if } u \geq 0 \end{array} \right. .$$

Thus, if we introduce the Heavyside function centered at x

$$\theta_x(t) = \left\{ \begin{array}{l} 0 \text{ if } t < x \\ 1 \text{ if } t \geq x \end{array} \right. ,$$

we have $\rho_n(x-t) \to 1 - \theta_x(t)$ for any fixed x and t. Since $|\rho_n(x-t)| \leq \int h_n = 1$ (which is μ-integrable), the Lebesgue dominated convergence theorem implies

$$L_{\mu_n}(x) \to \int (1 - \theta_x(t))\mu(dt) = L_\mu(x)$$

for every x. Furthermore, μ_n is weakly convergent and therefore strongly bounded, so $\|L_{\mu_n}\|_\infty < \|\mu_n\| < M$ for some constant M. It follows from Lebuesgue dominated convergence theorem again that

$$\int dx\ s'(x)L_\mu^n(x) \to \int dx\ s'(x)L_\mu(x).$$

Thus the formula (2.1) holds for finite measures. If μ is not finite but $\mu(\mathbb{R}^-) < \infty$, it suffices to troncate μ to the right in such a way that the troncation $\tilde{\mu}$ coincide with μ on the support of s. The conclusion follows by writing down the formula for $\tilde{\mu}$. $\diamondsuit$

The two notions of derivation for L_μ coincide if μ is absolutely continuous. This suggest the following simple but convenient criterium for absolute continuity.

Proposition 2.2 *Let $\eta \in \mathcal{D}'$. If $D\eta = f$, where $f \in L^1_{loc}(\mathbb{R})$ (locally integrable functions), then η is absolutely continuous on every compact.*

Proof. Take some compact K and a cut-off function $\chi \in C_0^\infty$ such that $\chi \equiv 1$ on K, and define the function

$$G(x) = \int_{-\infty}^x dx\ f(x)\chi(x).$$

G is absolutely continuous and $DG = f\chi$. Moreover, for all $s \in C_0^\infty(K)$ we have $(DG, s) = (D\eta, s)$ and thus η and G coincide on K up to a constant. It follows that η is absolutely continuous on K. $\diamondsuit$

The differentiation rule for the product has to be adapted to the functions of type L_μ . If f and g are absolutely continuous functions, it is well-known that fg is also absolutely continuous with $(fg)' = f'g + fg'$ a.e. An easy generalization of this is

Proposition 2.3 *Let μ be a Borel measures such that $\mu(\mathbb{R}^-) < \infty$ and g be an absolutely continuous function. Then the function $G = gL_\mu$ is a.e differentiable and*

$$G'(x) = g'(x)L_\mu(x) + g(x)f_\mu(x) \ a.e, \tag{2.2}$$

where $\mu_{ac}(dx) = f_\mu(x)dx$.

Proof. For all x we have

$$\begin{aligned}\frac{G(x+h)-G(x)}{h} &= \frac{g(x+h)-g(x)}{h}L_\mu(x) + \frac{L_\mu(x+h)-L_\mu(x)}{h}g(x+h)\\ &= \mathrm{Term}_1 + \mathrm{Term}_2\end{aligned}$$

g and L_μ are differentiable a.e with $L'_\mu = f_\mu$ a.e. Hence $\mathrm{Term}_1 \to L_\mu(x)g'(x)$ and $\mathrm{Term}_2 \to g(x)f_\mu(x)$ for a.e x as $h \to 0$. $\diamondsuit$

If μ is a Borel measure such that $\mu(\mathbb{R}^+) < \infty$ one can define a primitive by integrating to the right :

$$R_\mu(x) = \mu([x, +\infty[).$$

All the results stated for L_μ can be translated in an obvious way to R_μ. Here and everywhere, we will adopt the the notational convention

$$\int_{-\infty}^{x} \mu(dt) := \mu(]-\infty, x[), \ \int_{x}^{+\infty} \mu(dt) := \mu([x, +\infty[), \ \int_{a}^{b} \mu(dt) := \mu([a, b[), \ a < b.$$

Note that this notation preserves the chain rule for integrals.

We end this section with a useful identity. If $h \in L^1(\mathbb{R})$ it is well-known that

$$\int_{x}^{+\infty} dx_1 h(x_1) \int_{x_1}^{+\infty} dx_2 ... \int_{x_{p-1}}^{+\infty} dx_p h(x_p) = \frac{1}{p!}\left(\int_{x}^{+\infty} dy \ h(y)\right)^p,$$

and if furthermore $\int dx \ \ |xh(x)| < \infty$, then

$$\int_{x}^{+\infty} (x_1 - x) |h(x_1)| \, dx_1 \int_{x_1}^{+\infty} dx_2 ... \int_{x_{p-1}}^{+\infty} (x_p - x_{p-1}) dx_p \, |h(x_p)| \le \frac{1}{p!}\left(\int_{x}^{+\infty} dy \ (y-x) \, |h(y)|\right)^p.$$

The last inequality can be extended to measures as shows the next

Proposition 2.4 *If μ is a positive Borel measure such that $\int (1+|x|)\mu(dx) < \infty$, then*

$$\int_{x}^{+\infty} (x_1 - x)\mu(dx_1) ... \int_{x_{p-1}}^{\infty} (x_p - x)\mu(dx_p) \le \frac{1}{p!}\left(\int_{x}^{+\infty} dy \ (y-x)\mu(dy)\right)^p. \tag{2.3}$$

Proof. First suppose the measure μ is of compact support. Let us introduce the function

$$K(t) = \begin{cases} 0 \text{ if } t > 0 \\ -t \text{ if } t \leq 0 \end{cases}.$$

Then

$$G_\mu^1(x) = K * \mu(x) = \int_x^{+\infty} (y-x)\mu(dy)$$

is a continuous bounded function. Now take $h \in C_0^\infty(\mathbb{R})$ and $h_n(x) = nh(nx)$ and define $\mu_n = \mu * h_n$ and $G_{\mu_n}^1 = K * \mu_n = G_\mu^1 * h_n$. Then we have $\mu_n \xrightarrow{\mathcal{M}_a} \mu$ and $G_{\mu_n}^1(x) \to G_\mu^1(x)$ for all x. What is more, we claim that

$$\mu_n G_{\mu_n}^1 \xrightarrow{\mathcal{M}_a} \mu G_\mu^1.$$

Indeed, for all $f \in C_b^0(\mathbb{R})$ we have

$$\left|(\mu_n G_{\mu_n}^1, f) - (\mu G_\mu^1, f)\right| \leq \|\mu_n\| \sup_{x\in\Omega} \left|f(x)(G_{\mu_n}^1(x) - G_\mu^1(x))\right|_\infty + \left|(\mu_n - \mu, G_\mu^1 s)\right|,$$

where Ω is some compact including all the supports of μ_n. The second term on the right-hand side converges to zero. On the other hand, the function $G_{\mu_n}^1$ is easily seen to be lipschitz with a liptschitz constant $2\|\mu_n\|$. Since the sequence μ_n is strongly bounded for it is weakly convergent, there is a common lipschitz constant (precisely $\sup_n \|\mu_n\|$) for all the $G_{\mu_n}^1$ and thus this family of functions is equicontinuous. It follows that $G_{\mu_n}^1 \to G_\mu^1$ uniformly on every compact and thus the first term converges to zero. Iterating this procedure, one can show that

$$\mu_n G_{\mu_n}^p \xrightarrow{\mathcal{M}_a} \mu G_\mu^p,$$

where G_μ^p and $G_{\mu_n}^p$ are defined recursively by

$$G_\mu^p(x) = K * (\mu G_\mu^{p-1})(x),\ G_{\mu_n}^p(x) = K * (\mu_n G_{\mu_n}^{p-1})(x).$$

It follows that

$$\begin{aligned} \int_x^{+\infty} (x_1 - x)\mu_n(dx_1) ... \int_{x_{p-1}}^\infty (x_p - x)\mu_n(dx_p) &= K * (\mu_n G_{\mu_n}^p)(x) \\ &\to K * (\mu G_\mu^p)(x) \\ &= \int_x^{+\infty} (x_1 - x)\mu(dx_1) ... \int_{x_{p-1}}^\infty (x_p - x)\mu(dx_p) \end{aligned}$$

for all x. Since the formula (2.3) holds for μ_n, this proves the proposition for μ. It μ is not compactly supported, it suffices to apply the formula to truncations $\mu_\lambda(dx) = \theta_\lambda(x)\mu(dx)$ and to verify that

$$\int_x^{+\infty} (x_1-x)\mu_\lambda(dx_1) ... \int_{x_{p-1}}^\infty (x_p-x)\mu_\lambda(dx_p) \xrightarrow{\lambda\to+\infty} \int_x^{+\infty} (x_1-x)\mu(dx_1) ... \int_{x_{p-1}}^\infty (x_p-x)\mu(dx_p).$$

$\diamondsuit$

Remark: Surprisingly, the analog formula

$$\int_x^{+\infty} \mu(dx_1) \ldots \int_{x_{p-1}}^{\infty} \mu(dx_p) = \frac{1}{p!} \left(\int_x^{+\infty} \mu(dy) \right)^p$$

is false, as can be easily seen on the conter-example of the Dirac function. The difference is essentially due to the fact that G_μ is more regular than R_μ.

3 Definition of the Hamiltonian

The first task for our scattering problem is to define properly the hamiltonian. Let $L^2(\mathbb{R})$ be the Hilbert space of complex square integrable functions on $\mathbb{R}$, equipped with the usual scalar product $< f, g >= \int \overline{f} g$. One wishes to define a self-adjoint operator on $L^2(\mathbb{R})$ corresponding to the formal hamiltonian $H = -\Delta + V$, where V is the multiplication operator by some singular measure V. In one dimension, there are several ways of constructing such operator. The first one is to consider self-adjoint extensions of some densely defined symmetric operator. Among all possible extensions, there is exactly one corresponding in some sense to the formal hamiltonian $H = -\Delta + V$. This method has been mainly developped in the case of point interactions (see [Aa88] for a survey). However, besides some elementary school-cases as the delta potential, it is in general difficult to decide which extension is the "good one". A more handy method, that we shall adopt, consists in defining the hamiltonian as a quadratic form. It has the advantage of providing without ambiguity the physical hamiltonian and moreover it applies to a wider class of measure (at least in the one-dimensional case). There exists also other methods based on Dirichlet forms or norm-resolvent convergence of operators, but they seem to us more complicated to deal with.

To every quadratic form q with domain $\mathcal{Q}(q)$ on some Hilbert space $\mathcal{H}$ is uniquely associated a symmetric operator H by

$$< Hf, g >= q(f, g), \; g \in \mathcal{Q}(q), \; f \in D(H),$$

where $D(H) = \{f \in \mathcal{Q}(q)$ such that $h \mapsto q(f, h)$is continuous in the norm of $\mathcal{H}\}$. The representation theorem (see e.g [Far75]) establishes a one-to-one correspondance between densely defined closed positive quadratic forms and positive self-adjoint operators. The theorem holds in fact not only for positive form but more generally lower semi-bounded forms, since one can apply it to $q(f, g) - \lambda < f, g >$ instead of $q(f, g)$. Now if V is a finite real measure on $\mathbb{R}$, one may define the form

$$q(f, g) = \int \overline{f'} g' + \int \overline{f} g dV = q_0(f, g) + q_V(f, g),$$

on the domain $W^1(\mathbb{R})$, where $W^p(\mathbb{R})$ is the Sobolev space of L^2 functions whosedistributional derivatives up to order p are also square integrable. The form q_0 is known to be closed on $W^1(\mathbb{R})$ for it corresponds to the self-adjoint Laplacian. The form q_V is semi-bounded but is not closed in general. However, the sum $q = q_0 + q_V$ happens to be closed if q_V is in some sense a "small" perturbation of q_0. This is expressed by the so-called KLMN theorem (see [RS75], Th. X.17), which asserts that q_V needs only be relatively form bounded (with relative bound smaller than one) with respect to q_0 for the sum to be closed. When V is a finite measure, this can easily be verified by use of Sobolev's inequality (for details see [Bra85]). Hence, we may define the self-adjoint operator $H = -\Delta + V$ on $L^2(\mathbb{R})$ as the operator uniquely associated to the quadratic form $q_0 + q_V$. In fact, the operator H can be identified with its formal expression $-\Delta + V$ in the following way. For all f in $D(H)$ and g in $\mathcal{D}$, we have

$$< g, Hf >= ((-\Delta + V), \overline{g}), \tag{3.1}$$

where $(-\Delta + V)f \in \mathcal{D}'(\mathbb{R})$. It is useful to explicit a little more the domain $D(H)$. By definition,

$$D(H) = \{f \in W^1(\mathbb{R}) \mid |q(f,g)| \leq C \|g\|_2 \text{ for all } g \in L^2(\mathbb{R})\}$$

for some constant $C > 0$. In fact, it suffices to verify $|q(f,g)| \leq C \|g\|_2$ for g in some dense domain of $L^2(\mathbb{R})$, for instance $\mathcal{D}$. Thus,

$$\begin{aligned} D(H) &= \{f \in W^1(\mathbb{R}) \mid |((-\Delta + V)f, \overline{g})| \leq C \|g\|_2 \text{ for all } g \in \mathcal{D}\}, \\ &= \{f \in W^1(\mathbb{R}) \mid (-\Delta + V)f \in L^2(\mathbb{R})\}. \end{aligned}$$

Hence, it has a sense to write $Hf = (-\Delta + V)f$ for all $f \in D(H)$. This implies in particular a convenient characterization of the eigenvalues. $\zeta \in \mathbb{R}$ is an eigenvalue of H if and only if $(-\Delta + V - \zeta)f = 0$ for some $f \in L^2(\mathbb{R})$, where the equation is to be understood a priori in the sense of distributions. A classical example for computing domains is

The delta potential. Let $\delta_0(x)$ be the Dirac distribution and $\theta_0(x)$ the Heavyside function centered at 0, $\theta_0(x) = 1$ if $x \geq 0$, $\theta_0(x) = 0$ else. Define $H = -\Delta + \lambda\delta_0$, with $\lambda \in \mathbb{R}$, in the sense of quadratic forms. Then

$$\begin{aligned} f \in D(H) &\Leftrightarrow f \in W^1(\mathbb{R}) \text{ and } f'' = g'' + \lambda f(0)\delta_0 \text{ for some } g \in W^2(\mathbb{R}), \\ &\Leftrightarrow f \in W^1(\mathbb{R}) \text{ and } f' = g' + \lambda f(0)\theta_0 \text{ for some } g \in W^1(\mathbb{R}), \\ &\Leftrightarrow f \in W^1(\mathbb{R}) \cap W^2(\mathbb{R}\backslash 0) \text{ and } f'(0^+) - f'(0^-) = \lambda f(0) \end{aligned}$$

4 Scattering wave functions for smooth potentials

Let us now attack the scattering problem. We begin by introducing the usual definitions for smooth potentials. Let H_0 be the self-adjoint realization of the negative Laplacian on $L^2(\mathbb{R})$,

$$H_0 = -\Delta, \ D(H_0) = W^2(\mathbb{R}),$$

let $V(x)$ be some continuous real function with compact support and V be the associated multiplication operator. Then the hamiltonian $H = H_0 + V$ is a self-adjoint operator on $D(H) = D(H_0)$. The scattering wave functions associated to this operator are solutions of the stationary Schödinger equation or, what amounts to the same, the Helmholtz equation,

$$(\Delta + z^2)\varphi = V\varphi, \tag{4.1}$$

where $z > 0$ and φ satisfy an appropriate asymptotic condition (radiation condition). For the spectral analysis of H, it is useful to let the parameter z run in the complex upper half-plane. We define the family of functions $\varphi_z^\pm$, with $\Im z \geq 0$, as the unique solutions of the Helmholtz equation (4.1) with asymptotic behaviour

$$\varphi_z^\pm(x) \to \begin{cases} t_z^\pm e^{\pm izx}, & x \to \pm\infty, \\ e^{\pm izx} + r_z^\pm e^{\mp izx}, & x \to \mp\infty. \end{cases} \tag{4.2}$$

The coefficients t_z^+ and t_z^- are called transmission coefficients from the left and from the right, respectively, and the coefficients r_z^+ and r_z^- are the reflection coefficients from the left and from the right, respectively. The functions $\varphi_z^\pm(x)$ are generalized eigenfunctions in the sense that $H\varphi_z^\pm(x) = z^2\varphi_z^\pm(x)$, formally -because $\varphi_z^\pm \notin L^2(\mathbb{R})$. For $z \neq 0$, the Helmholtz equation (4.1) together with asymptotic condition (4.2) is equivalent to the integral equation

$$\varphi_z^\pm(x) = e^{\pm izx} + \frac{1}{2iz}\int dy\; e^{iz|x-y|}V(y)\varphi_z^\pm(y). \tag{4.3}$$

Indeed, the above function verifies the Helmholtz equation and has the same asymptotic as $\varphi_z^\pm$, so by the unicity of the solution it is $\varphi_z^\pm$. This integral equation may be used to derive an expression of the transmission and relection coefficients. Letting $x \to \pm\infty$ in (4.3) and comparing with (4.2) gives

$$t_z^\pm = 1 + \frac{1}{2iz}\int dy\; e^{\mp izy}V(y)\varphi_z^\pm(y), \tag{4.4}$$

$$r_z^\pm = \frac{1}{2iz}\int dy\; e^{\pm izy}V(y)\varphi_z^\pm(y). \tag{4.5}$$

5 Scattering wave functions for measure-potentials

When the potential is a measure, we cannot treat the Helmholtz equation as an ordinary differential equation anymore. However, the integral equation (4.3) can still makes sense if we understand $V(y)$ as a distribution. So, the idea is to seek a solution of (4.3) and verify a posteriori that it has the same properties than the usual generalized eigenfunctions. Along this paper, we will assume that V is a real Borel measure satisfying the typical short-range condition

$$\int (1 + |x|^2)\,|V|\,(dx) < \infty. \tag{5.1}$$

We define the generalized eigenfunctions of our problem as the continuous bounded solutions of

$$\varphi_z^{\pm}(x) = e^{\pm izx} + \frac{1}{2iz}\int e^{iz|x-y|}V(dy)\varphi_z^{\pm}(y),\ \Im z \geq 0. \tag{5.2}$$

Only the case $\Im z \geq 0$ is of interest because it corresponds to the physical sheet. The aim of this section is to prove that such functions actually exist and are unique. To remove the trivial oscillatory part, it is convenient to introduce the functions $h_z^{\pm}(x) = e^{\mp izx}\varphi_z^{\pm}(x)$. These functions satisfy the modified integral equations

$$h_z^{+}(x) = 1 + \frac{1}{2iz}\int h_z^{\pm}(y)V(dy) + \int_x^{+\infty} e^{iz(y-x)}\frac{\sin(z(y-x))}{z}h_z^{+}(y)V(dy), \tag{5.3}$$

$$h_z^{-}(x) = 1 + \frac{1}{2iz}\int h_z^{\pm}(y)V(dy) + \int_{-\infty}^{x} e^{iz(x-y)}\frac{\sin(z(x-y))}{z}h_z^{-}(y)V(dy). \tag{5.4}$$

In order to solve these integral equations, we introduce the formal operators

$$G_z^{+}f(x) = \int_x^{+\infty} e^{iz(y-x)}\frac{\sin(z(y-x))}{z}f(y)V(dy), \tag{5.5}$$

$$G_z^{-}f(x) = \int_{-\infty}^{x} e^{iz(x-y)}\frac{\sin(z(x-y))}{z}f(y)V(dy). \tag{5.6}$$

Note that $G_z^{\pm}$ can also be defined in $z = 0$. The properties of these operators are summarized in

Theorem 5.1 *Let V be a real Borel measure satisfying condition (5.1). Then,*

a) For $\Im z \geq 0$, $G_z^{\pm}$ and $(1-G_z^{\pm})^{-1}$ are well-defined operators in $\mathcal{B}(C_b^0(\mathbb{R}))$, the space of bounded operators on $C_b^0(\mathbb{R})$. They are uniformly bounded in the upper half plane, that is

$$\sup_{\Im z \geq 0}\left\|G_z^{\pm}\right\| < \infty,\ \sup_{\Im z \geq 0}\left\|(1-G_z^{\pm})^{-1}\right\| < \infty. \tag{5.7}$$

b) The operator-valued functions $z \mapsto G_z^{\pm}$ and $z \mapsto (1-G_z^{\pm})^{-1}$ are analytic in $\Im z > 0$, continuous in $\Im z \geq 0$ and continuously differentiable on $\mathbb{R}\backslash 0$. Their derivatives are uniformly bounded in the upper-half plane.

Proof.

a) Let us introduce the kernels

$$g_z^{\pm}(x) = \mp\theta_0(\mp x)e^{\mp izx}\frac{\sin(zx)}{z}, \tag{5.8}$$

where as before θ_0 is the Heavyside function centered at 0. These functions satisfy the following estimates, where the upper dot indicates the derivative with respect to the variable z.

Lemma 5.2 *For $\Im z \geq 0$ and for all x we have*

$$\begin{aligned}
\left|g_z^{\pm}(x)\right| &\leq C\theta_0(\mp x)\frac{|x|}{1+|zx|} \leq C\theta_0(\mp x)\,|x|\,, && (5.9)\\
\left|\dot{g}_z^{\pm}(x)\right| &\leq C\theta_0(\mp x)\left(|x| + \frac{|x|^2}{1+|zx|}\right) \leq C\theta_0(\mp x)(1+|x|^2), && (5.10)\\
\left|\ddot{g}_z^{\pm}(x)\right| &\leq C\theta_0(\mp x)\left(|x| + |x|^2 + \frac{|x|^3}{1+|zx|}\right) \leq C\theta_0(\mp x)\left(1+|x|^2+\frac{|x|^2}{|z|}\right). && (5.11)
\end{aligned}$$

for some constant $C > 0$.

Proof. It suffices to write $g_z^{\pm}(x) = \mp\theta_0(\mp x)e^{\mp izx}x\ \mathrm{sc}(zx)$, where $\mathrm{sc}(x) = \sin x/x$, and to use the following estimates, that can be obtained by elementary but tedious computations :

$$\begin{aligned}
|\mathrm{sc}(zx)| &\leq \mathcal{O}(1)e^{|\Im z||x|}\frac{1}{1+|zx|},\\
|\dot{\mathrm{sc}}(zx)| &\leq \mathcal{O}(1)e^{|\Im z||x|}\frac{|zx|}{1+|zx|},\\
|\ddot{\mathrm{sc}}(zx)| &\leq \mathcal{O}(1)e^{|\Im z||x|}\frac{1}{1+|zx|^3}.
\end{aligned}$$

$\diamondsuit$

Now, $G_z^{\pm}f$ can be rewritten as the convolution of a bounded continuous function with a finite measure, $G_z^{\pm}f = g_z^{\pm} * (Vf)$, and so $G_z^{\pm}f$ is itself continuous and bounded. Let us show that it is in fact uniformly bounded. For, recall that

$$\sup_{x\in\mathbb{R}}\int_x^{+\infty}(y-x)\,|V|\,(dy) < \infty,\ \sup_{x\in\mathbb{R}}\int_x^{+\infty}(y-x)^2\,|V|\,(dy) < \infty \tag{5.12}$$

as soon as condition (5.1) is verified. Using the estimations (5.9) and (5.12) we obtain

$$\left|G_z^{\pm}f(x)\right| \leq C_1\,\|f\|_\infty\int\theta_0(\pm(x-y))\,|x-y|\,|V|\,(dy) < C_2\,\|f\|_\infty$$

for some constants $C_1, C_2 > 0$. Thus, $G_z^{\pm}$ is uniformly bounded on $\Im z \geq 0$. To prove the existence of $(1-G_z^{\pm})^{-1}$, we will show that its Neumann series is absolutely convergent. We detail the proof for $(1-G_z^{+})^{-1}$, the proof for $(1-G_z^{-})^{-1}$ being similar. Let us form the n^{th} iterate

$$(G_z^{+})^n f(x) = \int_x^{+\infty} g_z^{+}(x_1-x)V(dx_1)\int_{x_1}^{+\infty}\ldots\int_{x_{n-1}}^{+\infty}g_z^{+}(x_n-x_{n-1})f(x_n)V(dx_n).$$

Since $|g_z^{\pm}(x)| \leq c\,|x|$ for some constant $c>0$, we have

$$\left|(G_z^+)^n f(x)\right| \leq c^n \left\|f\right\|_\infty \int_x^{+\infty} (x_1-x)\,|V|\,(dx_1) \int_{x_1}^{+\infty} \dots \int_{x_{n-1}}^{+\infty} (x_n - x_{n-1})\,|V|\,(dx_n)$$

and it follows from proposition (2.4) that

$$\left|(G_z^+)^n f(x)\right| \leq \left\|f\right\|_\infty \frac{c^n}{n!} \left(\int_x^{+\infty} (y-x)\,|V|\,(dy)\right)^n \leq \frac{C^n}{n!}$$

for some constant $C>c>0$. Thus,

$$\left\|(G_z^+)^n\right\| \leq \frac{C^n}{n!}$$

in the operator norm of $\mathcal{B}(C_b^o(\mathbb{R}))$. It follows that the Neumann series $\sum (G_z^+)^n$ is uniformly absolutely convergent and so the inverse $(1-G_z^+)^{-1}$ exists and its norm is uniformly bounded.

b) Let us define the operator $\dot{G}_z^{\pm}$, $\Im z \geq 0$, by

$$\dot{G}_z^{\pm} f(x) = \int \dot{g}_z^{\pm}(x-y) f(y) V(dy)$$

Using (5.10) and (5.12), one can easily check that $\dot{G}_z^{\pm}$ is a well-defined operator on $\mathcal{B}(C_b^o(\mathbb{R}))$, uniformly bounded on $\Im z \geq 0$. We omit the details. For $\Im z > 0$, one can show by direct computation that $\dot{G}_z^{\pm}$ is the norm derivative of $G_z^{\pm}$ and $(1-G_z^{\pm})^{-1}\dot{G}_z^{\pm}(1-G_z^{\pm})^{-1}$ is the norm derivative of $(1-G_z^{\pm})^{-1}$. So $G_z^{\pm}$ and $(1-G_z^{\pm})^{-1}$ are norm-differentiable on $\Im z > 0$, hence analytic there. Moreover, by virtue of (5.9) and (5.12) we have, for any $\varepsilon > 0$,

$$\begin{aligned}
\left|(G_{k+i\varepsilon}^{\pm} - G_k^{\pm}) f(x)\right| &\leq \left\|f\right\|_\infty \int \left|g_{k+i\varepsilon}^{\pm}(x-y) - g_k^{\pm}(x-y)\right|\,|V|\,(dy) \\
&\leq \varepsilon \left\|f\right\|_\infty \int \sup_{\Im z \geq 0} \left|\dot{g}_z^{\pm}(x-y)\right|\,|V|\,(dy) \\
&\leq \varepsilon C \left\|f\right\|_\infty \int \theta_0(\mp(x-y))(1+|x-y|^2)\,|V|\,(dy) \\
&\leq \varepsilon C' \left\|f\right\|_\infty
\end{aligned}$$

for some constants $C', C > 0$, and thus $G_z^{\pm}$ is continuous on $\Im z \geq 0$. That $(1-G_z^{\pm})^{-1}$ is also continuous on $\Im z \geq 0$ follows from the second resolvent formula

$$\begin{aligned}
\left\|(1-G_{k+i\varepsilon})^{-1} - (1-G_k)^{-1}\right\| &= \left\|(1-G_{k+i\varepsilon})^{-1}(G_{k+i\varepsilon} - G_k)(1-G_k)^{-1}\right\| \\
&\leq \left\|(1-G_{k+i\varepsilon})^{-1}\right\| \left\|G_{k+i\varepsilon} - G_k\right\| \left\|(1-G_k)^{-1}\right\|.
\end{aligned}$$

In the same manner, one can prove using (5.11) and (5.12) that $\dot{G}_z^{\pm}$ is continuous on $\Im z \geq 0,\ z \neq 0$, as well as $(1-G_z^{\pm})^{-1}\dot{G}_z^{\pm}(1-G_z^{\pm})^{-1}$. This concludes the proof of theorem (5.1). $\diamondsuit$

The integral equations (5.3) and (5.4) can now be rewritten in an operatorial form,

$$h_z^\pm = \lambda_z^\pm + G_z^\pm h_z^\pm, \tag{5.13}$$

with

$$\lambda_z^\pm = 1 + \frac{1}{2iz}\int h_z^\pm(y)V(dy).$$

In the general case, it seems not possible to invert equation (5.13) directly, unless $\lambda_z^\pm$ is independent of $h_z^\pm$, that is $\lambda_z^\pm = 1$. Therefore, we first consider the more simple equation

$$j_z^\pm = 1 + G_z^\pm j_z^\pm, \tag{5.14}$$

For $\Im z \geq 0$, the solution of this last equation exists in $C_b^0(\mathbb{R})$ and is unique. It is given by

$$j_z^\pm = (1 - G_z^\pm)^{-1} I,$$

where $I \equiv 1$ is the constant function one. Now set

$$\begin{aligned} \alpha_z^\pm &= 1 - \frac{1}{2iz}\int j_z^\pm(y)V(dy), & (5.15)\\ h_z^\pm &= j_z^\pm/\alpha_z^\pm \text{ whenever } \alpha_z \neq 0. & (5.16)\end{aligned}$$

Then $h_z^\pm$ is a solution of (5.13) for it verifies

$$h_z^\pm = 1/\alpha_z^\pm + G_z^\pm h_z^\pm,$$

with

$$\alpha_z^\pm = 1 - \frac{1}{2iz}\int \alpha_z^\pm h_z^\pm(y)V(dy) = \left(1 + \frac{1}{2iz}\int h_z^\pm(y)V(dy)\right)^{-1} = 1/\lambda_z^\pm.$$

Conversely, if $h_z^\pm$ satisfies (5.13), then $h_z^\pm/\lambda_z^\pm$ is solution of (5.14) and therefore coincides with $j_z^\pm$, by the unicity of the solution. Thus, the functions $h_z^\pm$ exist and are unique in $C_b^0(\mathbb{R})$ as soon as $\Im z \geq 0$ and $\alpha_z^\pm \neq 0$. Hence, for $\Im z \geq 0$, the integral equation (5.2) admits a unique solution $\varphi_z^\pm(x) = e^{\mp izx} j_z^\pm(x)/\alpha_z^\pm$ in $C_0^b(\mathbb{R})$ as soon as $\alpha_z^\pm \neq 0$. We will see in the next section that the condition $\alpha_z^\pm \neq 0$ is verified in some strip $0 \leq \Im z < \varepsilon$ around the real axis.

The terminology "generalized eigenfunctions" for $\varphi_z^\pm$ is justified by the following fact.

Proposition 5.1 *For $\Im z \geq 0$, the functions $\varphi_z^\pm$, whenever they exist, satisfy the weak Helmholtz equation*

$$(\Delta + z^2)\varphi_z^\pm = V\varphi_z^\pm \text{ in } \mathcal{D}'. \tag{5.17}$$

Proof. Let us define the kernel $K_z(x) = (2iz)^{-1}e^{iz|x|}$. When $\alpha_z^\pm \neq 0$, the integral equation (5.2) satisfied by $\varphi_z^\pm$ can be rewritten as

$$\varphi_z^\pm(x) = e^{\pm izx} + K_z * (V\varphi_z^\pm)(x),$$

where the convolution is well defined since

$$\left|e^{\mp izx}K_z * (V\varphi_z^{\pm})(x)\right| = \left|\frac{1}{2iz}\int e^{iz|x-y|}e^{\pm iz(y-x)}h_z^{\pm}(y)V(dy)\right| \leq \frac{1}{2\,|z|}\,\|V\|\,\left\|h_z^{\pm}\right\|_{\infty}.$$

The function K_z is well-known to be a fundamental solution in $\mathcal{D}'$ of

$$(\Delta + z^2)K_z = \delta_0.$$

Now take some test function $s \in \mathcal{D}$. Then

$$\begin{aligned}((\Delta + z^2)\varphi_z^{\pm}, s) &= ((\Delta + z^2)(K_z * (V\varphi_z^{\pm})), s) \\ &= (K_z * (V\varphi_z^{\pm}), s" + z^2 s).\end{aligned}$$

The formula $(f * g, h) = (g, h * \tilde{f})$, where $\tilde{f}(x) = f(-x)$, holds as soon as both sides are defined. Since $\tilde{K_z} = K_z$, it follows that

$$\begin{aligned}((\Delta + z^2)\varphi_z^{\pm}, s) &= (V\varphi_z^{\pm}, K_z * (s" + z^2 s)) \\ &= (V\varphi_z^{\pm}, s).\end{aligned}$$

This proves the proposition. $\diamondsuit$

On the real z-axis, one can state a unicity theorem for the Helmholtz equation.

Proposition 5.2 *Let $f \in C_b^0(\mathbb{R})$ be a weak solution of*

$$(\Delta + k^2)f = Vf,\ k \in \mathbb{R}.$$

If φ_k^+ exists and $\lim_{|x|\to+\infty}\left|\varphi_k^+(x) - f(x)\right| = 0$, then $f = \varphi_k^+$. The analogous result holds for φ_k^- and any linear combination of φ_k^+ and φ_k^-.

Proof. The function $K_k * (Vf)$ is well-defined in $C_b^0(\mathbb{R})$ because it is the convolution of a continuous bounded function K_k with a finite measure Vf. Now $(\Delta+k^2)(f-K_k*(Vf)) = 0$, and thus the general form of f is

$$f(x) = K_k * (Vf)(x) + ae^{ikx} + be^{-ikx}.$$

Hence,

$$\varphi_k^+(x) - f(x) = K_k * V(\varphi_k^+ - f) + (1-a)e^{ikx} - be^{-ikx}.$$

The condition on the asymptotic behaviour imposes $a = 1$ and $b = 0$. Thus,

$$f(x) = K_k * (Vf)(x) + e^{+ikx}.$$

Since the solution of the above equation is unique when it exists (in $C_b^0(\mathbb{R})$), we have in fact $f = \varphi_k^+$. This concludes the proof. $\diamondsuit$

6 Properties of the generalized eigenfunctions

The aim of this section is to derive the few regularity properties that are necessary for the eigenfunctions expansion. Vaguely speaking, we will show that the functions $\varphi_z^\pm$ exist and are well-behaved in some neighbourhood of the real z-axis. The main resuls we want to show in this section can be summarized in

Theorem 6.1 *For $\Im z \geq 0$ and $\alpha_z^\pm \neq 0$, the functions $\varphi_z^\pm(x)$ have the following asymptotics,*

$$\varphi_z^\pm(x) = \begin{cases} t_z^\pm e^{\pm izx} + O\left(x^{-2} e^{\mp \Im z x}\right), \ x \to \pm\infty, \\ e^{\pm izx} + r_z^\pm e^{\mp izx} + O\left(x^{-2} e^{\pm \Im z x}\right), \ x \to \mp\infty, \end{cases} \tag{6.1}$$

for some complex numbers $t_z^\pm$ and $r_z^\pm$ that we refer to as the transmission and reflexion coefficients, respectively. For fixed x, the functions $\varphi_z^\pm(x)$ are meromorph in $\Im z > 0$ with a finite number of poles located on the imaginary axis, corresponding to the zeros of $\alpha_z^\pm$. Moreover, the functions $\varphi_z^\pm(x)$ are continuous on the closed upper half plane $\Im z \geq 0$ and continuously differentiable on $\mathbb{R}\backslash 0$. The same result holds for the transmission and reflection coefficients. For $\Im z \geq 0$ we have

$$t_z^+ = t_z^- =: t_z. \tag{6.2}$$

On the real k-axis, the functions $\varphi_k^\pm(x)$ and $(1+|x|)^{-1}\dot{\varphi}_k^\pm(x)$ are uniformly bounded and we have the additional properties :

$$\overline{\varphi_k^\pm} = \varphi_{-k}^\pm, \ \overline{t_k} = t_{-k}, \ \overline{r_k^\pm} = r_{-k}^\pm, \tag{6.3}$$

and

$$r_k^+ \overline{t_k} - t_k \overline{r_k^-} = 0, \ |t_k|^2 + \left|r_k^\pm\right|^2 = 1 \tag{6.4}$$

Proof. For the proof, it is convenient to introduce an other family of functions, namely

$$J_z^\pm(x) = e^{\pm izx} j_z^\pm(x) = \varphi_z^\pm / \alpha_z^\pm. \tag{6.5}$$

These functions also satisfy the weak Helmholtz equation (5.17) and have nice properties that can be carried over the functions $\varphi_z^\pm$, so we will prove everything for $J_z^\pm$, the correspondance to $\varphi_z^\pm$ being trivial. We begin with the easy statements. The integral equations for $J_z^\pm$ write

$$J_z^+(x) = e^{izx} + \int_x^{+\infty} e^{izy} \sin(z(y-x)) j_z^+(y) V(dy), \tag{6.6}$$

$$J_z^-(x) = e^{-izy} + \int_{-\infty}^{x} e^{-izy} \sin(z(x-y)) j_z^-(y) V(dy). \tag{6.7}$$

For regular potentials (i.e $V(dy) = V(y)dy$ with $V(y) \in L^1_{loc}(\mathbb{R})$), these functions are known in the literature as the Jost functions. Letting $x \to \pm\infty$ and using the short-range condition (5.1) together with (5.9), a straightforward computation shows that

$$J_z^\pm(x) = \begin{cases} e^{\pm izx} + O\left(x^{-2}e^{\mp\Im zx}\right), \ x \to \pm\infty, \\ \alpha_z^\pm e^{\pm izx} + \beta_z^\pm e^{\mp izx} + O\left(x^{-2}e^{\pm\Im zx}\right), \ x \to \mp\infty, \end{cases} \tag{6.8}$$

where

$$\alpha_z^\pm = 1 - \frac{1}{2iz}\int j_z^\pm(y)V(dy), \ \beta_z^\pm = \int \frac{e^{\pm 2izy}}{2iz} j_z^\pm(y)V(dy), . \tag{6.9}$$

This proves (6.1) with $t_z^\pm = 1/\alpha_z^\pm$ and $r_z^\pm = \beta_z^\pm/\alpha_z^\pm$. The functions $J_z^\pm$ are given by $e^{\pm izx}j_z^\pm(x)$, where $j_z^\pm = (1-G_z^\pm)^{-1}I$. Hence the analytic properties of $J_z^\pm$ follow from theorem (5.1), and carry easily over $\alpha_z^\pm$ and $\beta_z^\pm$. It follows also from theorem (5.1) that $j_z^\pm$ and $\dot{j}_z^\pm$ are uniformly bounded on $\Im z \geq 0$. This implies the boundedness properties of $\varphi_k^\pm$ and $\dot{\varphi}_k^\pm$ for real k. Now let us look at the symmetry properties for real k. Since V is real, we have $\overline{G_k^\pm I} = G_{-k}^\pm I$, and also $\overline{(G_k^\pm)^n I} = (G_{-k}^\pm)^n \overline{I}$ for all n. This property carries on the Neumann serie, leading to

$$\overline{j_k^\pm} = \overline{(1-G_k^\pm)^{-1}I} = (1-G_{-k}^\pm)^{-1} = j_{-k}^\pm.$$

This symmetry property carries directly over $\alpha_k^\pm$, $\beta_k^\pm$ and $J_k^\pm$, that is

$$\overline{J_k^\pm} = J_{-k}^\pm, \ \overline{\alpha_k^\pm} = \alpha_{-k}^\pm, \ \overline{\beta_k^\pm} = \beta_{-k}^\pm, \ k \in \mathbb{R}. \tag{6.10}$$

Now let us attack the difficult part of the theorem. We will prove

Proposition 6.1 *For $\Im z \geq 0$, the coefficients $\alpha_z^\pm$ verify*

$$\alpha_z^+ = \alpha_z^- =: \alpha_z, \tag{6.11}$$

and for real k we have the additional relations

$$\beta_k^+ = -\overline{\beta_k^-}, \ |\alpha_k|^2 - \left|\beta_k^\pm\right|^2 = 1 \tag{6.12}$$

These last relations are equivalent to the relations (6.4). They express the unitarity of the scattering matrix, as shall be seen later.

Proof. The classical method (see e.g [MS96]) to prove (6.12) relies upon the properties of the wronskian associated to the functions $J_z^\pm$. In our case the manipulations with the wronskian become touchy because of the poor regularity of the involved functions. Therefore, we have to proceed more carefully. We start with

Lemma 6.2 *For $\Im z \geq 0$, the functions $J_z^\pm$ are absolutely continuous on every compact.*

Proof. The functions $J_z^\pm$ satisfy the following estimates as $x \to \pm\infty$:

$$\left|J_z^\pm(x) - e^{\pm izx}\right| \leq \mathcal{O}(1/x^2), \ \left|J_z^\pm(x)\right| \leq \mathcal{O}(1).$$

So we may define the functions

$$\begin{aligned} F_z^-(x) &= \int_{-\infty}^x J_z^-(t)V(dt) - z^2\int_{-\infty}^x dt\ (J_z^-(t) - e^{-izt}) - ize^{-izx}, \\ F_z^+ = (x) &= -\int_x^{+\infty} J_z^+(t)V(dt) + z^2\int_x^{+\infty} dt\ (J_z^+(t) - e^{+izt}) + ize^{+izx}. \end{aligned}$$

Using propostion (2.1) we see that $DF_z^\pm = (V - z^2)J_z^\pm = \Delta J_z^\pm$. Hence $DJ_z^\pm = F_z^\pm + C^\pm$ for some constant $C^\pm$. To estimate the constants, take some sequence of test functions $s_n^\pm(x) = s(x \mp n)$, with $s \in C_0^\infty$. Then

$$(DJ_z^\pm, s_n^\pm) = -(J_z^\pm, s_n^{'\pm}) \overset{n\to\pm\infty}{\longrightarrow} -(e^{\pm izx}, s_n^{'\pm}) = \pm iz(e^{\pm izx}, s_n^\pm).$$

On the other hand,

$$(F_z^\pm, s_n^\pm) \overset{n\to\pm\infty}{\longrightarrow} \pm iz(e^{\pm izx}, s_n^\pm).$$

Hence $C^\pm = 0$ and thus

$$DJ_z^\pm = F_z^\pm. \tag{6.13}$$

Now $F_z^\pm$ is locally bounded and measurable, hence locally integrable. It follows from proposition (2.2) that $J_z^\pm$ is absolutely continuous. This concludes the proof of lemma (6.2). $\diamondsuit$

The function $F_z^\pm(x)$ are defined for all x and so we may define everywhere the "wronskian"

$$\mathcal{W}[J_z^+, J_z^-](x) = J_z^+(x)F_z^-(x) - J_z^-(x)F_z^+(x).$$

A priori, $\mathcal{W}[J_z^+, J_z^-]$ is only measurable with a countable number of discontinuities. However, a cancellation of the singularities makes the wronskian more regular that expexted.

Lemma 6.3 *$\mathcal{W}[J_z^+, J_z^-](x)$ is absolutely continuous on every compact.*

Proof. We keep the notations of lemma (6.2). For any x we can rewrite

$$\mathcal{W}[J_z^+, J_z^-](x) = w_1(x) + w_2(x),$$

where

$$w_1(x) = J_z^+(x)\int_{-\infty}^x J_z^-(t)V(dt) + J_z^-(x)\int_x^{+\infty} J_z^+(t)V(dt)$$

and w_2 is an absolutely continuous function. So it suffices to prove the absolute continuity of w_1. After some splittings and rearrangements of integrals, we obtain, for any $b > a$,

$$\begin{aligned} w(b) - w(a) &= [J_z^+(b) - J_z^+(a)]\int_{-\infty}^a J_z^-(t)V(dt) \\ &\quad + \int_a^b [J_z^+(b)J_z^-(t) - J_z^+(t)J_z^-(a)]V(dt) + [J_z^-(b) - J_z^-(a)]\int_b^{+\infty} J_z^+(t)V(dt) \\ &= \mathrm{Term}_1 + \mathrm{Term}_2 + \mathrm{Term}_3. \end{aligned}$$

We then estimate each term independtly,

$$\begin{aligned}
|\mathrm{Term}_1| &\le \|V\| e^{\Im z b} \left\| j_z^- \right\|_\infty \left| J_z^+(b) - J_z^+(a) \right|, \\
|\mathrm{Term}_2| &\le \left\| j_z^+ \right\|_\infty e^{-\Im z a} \int_a^b \left| J_z^-(t) - J_z^-(a) \right| |V|(dt) \\
&\quad + \left\| j_z^- \right\|_\infty e^{\Im z b} \int_a^b \left| J_z^+(t) - J_z^+(b) \right| |V|(dt), \\
|\mathrm{Term}_3| &\le \|V\| \left\| j_z^+ \right\|_\infty e^{-\Im z a} \left| J_z^-(b) - J_z^-(a) \right|.
\end{aligned}$$

Now the functions $J_z^{\pm}$ are themselves absolutely continuous on every bounded interval $[a,b]$. So given $\varepsilon > 0$, there exists some $\delta > 0$ such that for any finite collection of disjoint segments $[a_1,b_1],[a_2,b_2],...[a_n,b_n]$ of $[a,b]$,

$$\sum |b_i - a_i| < \delta \Rightarrow \sum \left| J_z^{\pm}(b_i) - J_z^{\pm}(a_i) \right| < \varepsilon.$$

For such a_i and b_i we have, in view of the previous estimations,

$$\sum |w(b_i) - w(a_i)| \le 2\|V\| (e^{\Im z b} \left\| j_z^- \right\|_\infty + e^{-\Im z a} \left\| j_z^+ \right\|_\infty)\varepsilon,$$

which proves the absolute continuity of the function w_1 on the interval $[a,b]$. This concludes the proof of lemma (6.3). $\diamondsuit$

Corrolary 6.4 *The wronskian is constant.*

Proof. $J_z^{\pm}$ is absolutely continuous and $F_z^{\pm}$ is the sum of a smooth function and a function of type L_μ, so we may apply the differentiation rule of proposition (2.3) to the wronskian.

$$\mathcal{W}[J_z^+, J_z^-]'(x) = J_z^+(x) F_z^{-'}(x) + J_z^{+'}(x) F_z^-(x) - J_z^-(x) F_z^{+'}(x) + J_z^{-'}(x) F_z^+(x).$$

Now $F_z^{\pm'} = J_z^{\pm} f_V - z^2 J_z^{\pm}$ a.e (where f_V is the density of V_{ac}) and $J_z^{\pm'} = F_z^{\pm}$ a.e. Using these identities, we obtain $\mathcal{W}[J_z^+, J_z^-]'(x) = 0$ a.e. Since $\mathcal{W}[J_z^+, J_z^-](x)$ is absolutely continuous on every compact, it follows that $\mathcal{W}[J_z^+, J_z^-](x)$ is constant. $\diamondsuit$

The value of the wronskian can be computed by letting $x \to \pm\infty$. This yields

$$\mathcal{W}[J_z^+, J_z^-](x) = -2iz\alpha_z^+ = -2iz\alpha_z^-, \tag{6.14}$$

proving $\alpha_z^+ = \alpha_z^-$. For real z, one can derive other relations. Since the measure V is real, the functions $J_k^{\pm}$ and $\overline{J_k^{\pm}}$ satisfy for real k the same equation (5.17), so we may apply the same method (that is the wronskian method) to the pairs of solutions $\mathcal{W}[J_k^{\pm}, \overline{J_k^{\pm}}]$ and $\mathcal{W}[J_k^{\pm}, \overline{J_k^{\mp}}]$. This yields (6.12) and completes the proof of proposition (6.1). $\diamondsuit$

We conclude the proof of theorem (6.1) with

Proposition 6.2 *α_z has a finite number of zeros in the upper-half plane. They lie on the positive imaginary axis, away from zero. If z_0 is a zero of α_z, then z_0^2 is an eigenvalue of the hamiltonian H.*

Proof. By the mere definition, we see that $\alpha_z = 0$ if and only if $\int j_z^\pm(t)V(dt) = 2iz$. Now, $j_z^\pm$ is uniformly bounded in the upper half-plane and thus $|z| = \left|\int j_z^\pm(t)V(dt)\right|/2 \leq M$ for some constant $M > 0$. It follows that the zeros of α_z lie in some semi-circle $|z| \leq M$, $\Im z \geq 0$. Moreover, the condition $|\alpha_k|^2 - \left|\beta_k^\pm\right|^2 = 1$ when $k \in \mathbb{R}$ forces $|\alpha_k| \geq 1$. Since α_z is continuous on $\Im z \geq 0$, this means that the possible zeros are relagated in some compact region $|z| \leq M$, $\Im z \geq \varepsilon > 0$, in which α_z is analytic. Hence, there is only a finite number of zeros in the upper half-plane. Now suppose z_0 is a zero of α_z. Then $J_{z_0}^\pm$ is in $L^2(\mathbb{R})$. Since $J_z^\pm$ is also solution of the weak Helmholtz equation (5.17), this implies that z_0^2 is an eigenvalue of H and $J_{z_0}^\pm$ an associated eigenvector. Since H is self-adjoint, this implies $z_0^2 \in \mathbb{R}$. This concludes the proof of proposition (6.2) and theorem (6.1). $\diamondsuit$

7 Spectral properties

We are now in a position to study the resolvent and the spectrum of the hamiltonian. Let $R(\zeta) = (H - \zeta)^{-1}$ be the resolvent of H. Let us look for a kernel of $R(\zeta)$, that is a function $\mathcal{G}(x, y, \zeta)$ such that $\mathcal{G}(x, \cdot, \zeta) \in L^2(\mathbb{R})$ and for all $f \in L^2(\mathbb{R})$,

$$R(\zeta)f = \int dy\ \mathcal{G}(\cdot, y, \zeta)f(y). \tag{7.1}$$

For continuous potentials V, the Green's function $\mathcal{G}(x, y, \zeta)$ may be easily computed ny the standard method,

$$\mathcal{G}(x, y, \zeta) = \frac{J_z^+(\max(x, y))J_z^-(\min(x, y))}{\mathcal{W}(J_z^+, J_z^-)}, \tag{7.2}$$

where z is the complex square root of ζ in the upper half-plane

$$z^2 = \zeta,\ \Im z > 0.$$

In fact, this expression of the kernel remains valid if V is a finite measure, as shows the next

Theorem 7.1 *Let V be a finite real Borel measure with compact support and $H = -\Delta + V$ be defined in the sense of quadratic forms. Then the resolvent $R(\zeta) = (H - \zeta)^{-1}$ is given by (7.1) and (7.2).*

Proof. We begin the proof with the following

Lemma 7.2 *For all $y \in \mathbb{R}$, we have the distributional equality*

$$(-\Delta + V - \zeta)\mathcal{G}(\cdot, y, \zeta) = \delta_y, \tag{7.3}$$

where δ_y is the Dirac distribution centered at y.

Proof of lemma (7.2). The function $\mathcal{G}(x, y, \zeta)$ can be rewritten as

$$\mathcal{G}(x, y, \zeta) = \frac{J_z^+(y)J_z^-(x)}{\mathcal{W}(J_z^+, J_z^-)} + \theta_y(x)\tilde{\mathcal{G}}_y(x) \tag{7.4}$$

where

$$\tilde{\mathcal{G}}_y(x) = \frac{J_z^+(y)J_z^-(x) - J_z^+(x)J_z^-(y)}{\mathcal{W}(J_z^+, J_z^-)},$$

and, as usual, θ_y is the Heavyside function centered at y. Since $J_z^\pm$ satisfy the weak Helmholtz equation (5.17) we have

$$(-\Delta + V - \zeta)\mathcal{G}(\cdot, y, \zeta) = (-\mathit{\Delta} + V - \zeta)\theta_y\tilde{\mathcal{G}}_y.$$

Let us first compute the term $\Delta(\theta_y\tilde{\mathcal{G}}_y)$. Take some positive function $h \in C_0^\infty$ such that $\int dx h(x) = 1$ and set $h_n(x) = nh(nx)$ and $\theta_{y,n}(x) = \theta_y * h_n(x)$. The sequence h_n is an approximate identity and so $\theta_{y,n} \xrightarrow{\mathcal{D}'} \theta_y$. The same holds for the derivatives, that is $D^{(k)}\theta_{y,n} \xrightarrow{\mathcal{D}'} D^{(k)}\theta_y$. Now the function $\theta_{y,n}$ is C^∞ and so we may apply the usual differentiation rule (see e.g [Hör90]) for the distribution $\theta_{y,n}\tilde{\mathcal{G}}_y$, that is

$$\begin{aligned}
\Delta(\theta_{y,n}\tilde{\mathcal{G}}_y) &= D(\theta'_{y,n}\tilde{\mathcal{G}}_y + \theta_{y,n}D\tilde{\mathcal{G}}_y) \\
&= D(\theta'_{y,n}\tilde{\mathcal{G}}_y) + \theta'_{y,n}D\tilde{\mathcal{G}}_y + \theta_{y,n}\Delta\tilde{\mathcal{G}}_y \\
&= D(\theta'_{y,n}\tilde{\mathcal{G}}_y) + \theta'_{y,n}D\tilde{\mathcal{G}}_y - \theta_{y,n}(V - \zeta)\tilde{\mathcal{G}}_y.
\end{aligned}$$

Now

$$\begin{aligned}
\Delta(\theta_{y,n}\tilde{\mathcal{G}}_y) &\xrightarrow{\mathcal{D}'} \Delta(\theta_y\tilde{\mathcal{G}}_y), \\
\theta'_{y,n}\tilde{\mathcal{G}}_y &\xrightarrow{\mathcal{D}'} \delta_y\tilde{\mathcal{G}}_y(y) = 0, \\
\theta_{y,n}(V - \zeta)\tilde{\mathcal{G}}_y &\xrightarrow{\mathcal{D}'} \theta_y(V - \zeta)\tilde{\mathcal{G}}_y.
\end{aligned}$$

It remains the term $\theta'_{y,n}D\tilde{\mathcal{G}}_y$ which must be manipulated more carefully. We have the distributional equality

$$D\tilde{\mathcal{G}}_y(x) = \frac{J_z^+(y)F_z^-(x) - F_z^+(x)J_z^-(y)}{\mathcal{W}(J_z^+, J_z^-)},$$

where we have kept the notations of lemma (6.2). The right-hand side is defined everywhere. Let us show that it is continuous at y. We may write

$$J_z^+(y)F_z^-(x) - F_z^+(x)J_z^-(y) = \rho_1(x) + \rho_2(x),$$

where

$$\rho_1(x) = J_z^+(y)\int_{-\infty}^{x} J_z^-(t)V(dt) + J_z^-(y)\int_{x}^{+\infty} J_z^+(t)V(dt),$$

and ρ_2 is some continuous function. Now for any $h>0$ we have

$$\begin{aligned}
|\rho_1(y)-\rho_1(y-h)| &= \left|\int_{y-h}^{y} [J_z^+(y)J_z^-(t) - J_z^-(y)J_z^+(t)]V(dt)\right| \\
&\le \|V\| \sup_{y-h\le t\le y} \left|J_z^+(y)J_z^-(t) - J_z^-(y)J_z^+(t)\right| \\
&\to 0 \quad, h\to 0^+.
\end{aligned}$$

Thus, ρ_1 is right-continuous at y. Analogously, ρ_1 is shown to be left-continuous. Hence, $D\tilde{\mathcal{G}}_y$ coincide with a function which is continuous at y, and takes the value 1 there. It follows that $\theta'_{y,n}D\tilde{\mathcal{G}}_y \xrightarrow{\mathcal{D}'} \delta_y$. Putting altogether, we obtain (7.3). $\diamondsuit$

Now set

$$R^\sharp(\zeta)f(x) = \int dy\ \mathcal{G}(x,y,\zeta)f(y).$$

For $\zeta = z^2$, $\Im z > 0$, $R^\sharp(\zeta)$ is a bounded operator (in fact Hilbert-Schmidt) on $L^2(\mathrm{I\!R})$. Moreover, $R^\sharp(\zeta)f$ is easily seen to be a continuous bounded function for all $f\in\mathcal{D}$, so we may consider the distribution $(-\Delta+V-\zeta)R^\sharp(\zeta)f$. Now, for all $s\in\mathcal{D}$ we have

$$\begin{aligned}
((-\Delta+V-\zeta)R^\sharp(\zeta)f,s) &= (-s" + Vs - \zeta s, R^\sharp(\zeta)f) \\
&= -\int dx(s"(x)+\zeta s(x))\int dy\mathcal{G}(x,y,\zeta)f(y) \\
&\quad + \int V(dx)\int dy\mathcal{G}(x,y,\zeta)f(y) \\
&= \int dyf(y)\left[\int dx(-s"(x)-\zeta s(x))\mathcal{G}(x,y,\zeta) + \int V(dx)\mathcal{G}(x,y,\zeta)\right]
\end{aligned}$$

by Fubini's theorem. On the other hand,

$$\int dx(-s"(x)-\zeta s(x))\mathcal{G}(x,y,\zeta) + \int V(dx)\mathcal{G}(x,y,\zeta) = ((-\varDelta+V-\zeta)\mathcal{G}(\cdot,y,\zeta),s) = s(y).$$

Therefore, we have the distributional equality $(-\Delta+V-\zeta)R^\sharp(\zeta)f = f$. It follows from the remarks of section 3 that $R^\sharp(\zeta)f\in D(H)$ and $(H-\zeta)R^\sharp(\zeta)f = f$. Thus $R^\sharp(\zeta)f = R(\zeta)f$ holds on the dense set $\mathcal{D}$, hence on $L^2(\mathrm{I\!R})$ and so $R^\sharp(\zeta) = R(\zeta)$. This completes the proof of theorem (7.1). $\diamondsuit$

We now turn to the study of the spectrum. Let $\sigma_{ess}(h)$, $\sigma_d(H)$, $\sigma_{sc}(H)$ and $\sigma_{ac}(H)$ stand for the essential, discrete, singular continuous and absolutely continuous spectrum of H, respectively, and let $\sigma_c(H) = \sigma_{ac}(H)+\sigma_{sc}(H)$ be the continuous spectrum of H. The spectrum has the same characteristics as for regular potentials, as shows the next

Theorem 7.3 *The essential spectrum is positive and purely absolutely continuous,*

$$\sigma_{ess}(H) = \sigma_{ac}(H) = [0, +\infty[,\ \sigma_{sc}(H) = \emptyset, \tag{7.5}$$

and the point spectrum is composed of a finite number of negative eigenvalues,

$$\sigma_d(H) = \{-\lambda_1^2, .., -\lambda_N^2\}, \tag{7.6}$$

where $i\lambda_1, ..i\lambda_N$ are the zeros of α_z in the upper-half plane. The spectral projector $E[\alpha, \beta]$ on any interval $[\alpha, \beta] \subset \mathbb{R}^+$ is given by

$$E[\alpha, \beta]f(x) = \int dy\ e_{\alpha,\beta}(x, y)f(y), \tag{7.7}$$

with

$$e_{\alpha,\beta}(x, y) = \frac{1}{\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk\ \Re\left\{\frac{\varphi_k^+(y)\varphi_k^-(x)}{t_k}\right\} \tag{7.8}$$

Proof. In view of proposition (6.1), the kernel of the resolvent $\mathcal{G}(x, y, \zeta)$ is analytic on the negative real axis with only poles produced by the zeros of the wronskian, which are also the zeros of α_k, that is bound states (see proposition (6.2). Hence, there is no continuous spectrum on the negative axis. Let us show that there are no imbedded eigenvalues in the positive spectrum. Suppose $E = k^2$, $k \geq 0$, is an eigenvalue. Then there exists some non-zero $f \in D(H) \subset C_b^0(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$(\Delta + k^2)f = Vf.$$

Now, as was explained in proposition (5.2), any continuous bounded solution of the above equation can be written as $f(x) = K_k * (Vf) + ae^{ikx} + be^{-ikx}$ for some constants a and b. The supplementary condition $f \in L^2(\mathbb{R})$ forces $f(x) \to 0$ as $|x| \to +\infty$. The inspection of the asymptotic behaviour of f then imposes $a = b = 0$ and $f = 0$. Hence E cannot be an eigenvalue. So far, we have proven $\sigma(H) \cap]-\infty, 0[= \sigma_d(H)$, where $\sigma_d(H)$ is given by (7.6). Let us use Stone's formula to derive the spectral projectors on the positive spectrum. If $E(\zeta)$ is the spectral family associated to H, then

$$E(\alpha, \beta) = \operatorname*{s.lim}_{\varepsilon \to 0} \frac{1}{2i\pi}\int_\alpha^\beta d\zeta\ R(\zeta + i\varepsilon) - R(\zeta - i\varepsilon),\ 0 < \alpha < \beta.$$

The above operator is a projector with kernel

$$e_{\alpha,\beta}(x, y) = \frac{1}{2i\pi}\int_\alpha^\beta d\zeta\ \mathcal{G}(x, y, \zeta + i\varepsilon) - \mathcal{G}(x, y, \zeta - i\varepsilon). \tag{7.9}$$

The integrand can be computed more precisely since for $\zeta = k^2$ $(k > 0)$ we have

$$\frac{1}{2i\pi}\left\{\mathcal{G}(x, y, \zeta + io) - \mathcal{G}(x, y, \zeta - io)\right\} = p(x, y, \zeta),$$

with

$$p(x,y,\zeta) = \frac{1}{2\pi}\Re\left\{\frac{J_k^+(y)J_k^-(x)}{k\alpha_k}\right\} = \frac{1}{2\pi}\Re\left\{\frac{\varphi_k^+(y)\varphi_k^-(x)}{kt_k}\right\}.$$

After the change of variables $\zeta = k^2$, we obtain the formulae (7.7) and (7.8) for the spectral projector. This shows in particular that $\sigma_c = [0,+\infty[$. It remains to show that there is no singular continuous spectrum. One possibility is to use the limiting absorption principle (Theo. XIII.20 [RS79]). However, we will proceed more simply. For any state f in $L^2(\mathbb{R})$, let us denote by μ_f the spectral measure associated to any f, i.e $\mu_f(d\zeta) =< f, E(d\zeta)f >$. We will show that μ_f is absolutely continuous on $(0,+\infty)$, whatever the state f we choose. To this end, it suffices to show that the Radon-Nikodym derivative of μ_f exists everywhere on $\mathbb{R}^+$. Take some function f in $\mathcal{D}$, fix some $\alpha > 0$ and look at

$$\begin{aligned} h^{-1}\mu_f([\alpha,\alpha+h]) &= h^{-1}\int dxdy\ e_{\alpha,\alpha+h}(x,y)\overline{f}(y)f(x) \\ &= h^{-1}\int dxdy\ \overline{f}(y)f(x)\int_\alpha^{\alpha+h} d\zeta\ p(x,y,\zeta). \end{aligned}$$

An obvious candidate for $\lim_{h\to 0} h^{-1}\mu_f([\alpha,\alpha+h])$ is

$$m(\alpha) = \int dxdy\ \overline{f}(y)f(x)p(x,y,\alpha).$$

Now,

$$h^{-1}\mu_f([\alpha,\alpha+h]) - m(\alpha) = h^{-1}\int dxdy\ \overline{f}(y)f(x)\int_\alpha^{\alpha+h} d\zeta\ (p(x,y,\zeta) - p(x,y,\alpha))$$

On the other hand, we know from theorem (6.1) that $p(x,y,\cdot)$ is continuously derivable with a derivative $\dot{p}(x,y,\cdot)$ uniformly bounded with respect to x and y. Hence

$$\left|\int_\alpha^{\alpha+h} d\zeta(p(\zeta,x,y) - p(\alpha,x,y))\right| \le \mathcal{O}(h^2)$$

and

$$\left|h^{-1}\mu_f([\alpha,\alpha+h]) - m(\alpha)\right| \le \mathcal{O}(h).$$

Since $m(\alpha)$ is continuous, this proves the absolute continuity of μ_f and completes the proof of theorem (7.3). $\diamondsuit$

8 Eigenfunctions expansion

In this section, we will prove the completeness of the generalized eigenfunctions, whereby we will also show the existence and asymtotic completeness of the wave operators. Let Φ_0 be the unitary Fourier transform on $L^2(\mathbb{R})$ with adjoint Φ_0^*

$$\begin{aligned} \Phi_0 f(k) &= \text{l.i.m}\ \frac{1}{\sqrt{2\pi}}\int dx\ e^{-ikx}f(x), & (8.1)\\ \Phi_0^* g(x) &= \text{l.i.m}\ \frac{1}{\sqrt{2\pi}}\int dk\ e^{+ikx}g(k), & (8.2)\end{aligned}$$

where l.i.m $\int$ stands for the limit in the $L^2(\mathbb{R})$ norm of $\int_{-R}^{+R}$ as $R \to \infty$. To simplify the notation, we will sometimes set $\widehat{f}(k) = \Phi_0 f(k)$. We define the linear maps $\Phi_\pm : L^1 \to L^\infty$ and $\Phi_\pm^* : L^1 \to L^\infty$ by

$$\Phi_+ f(k) = \frac{1}{\sqrt{2\pi}}\int dx\ \overline{\varphi_k^+(x)}f(x), \quad \text{and} \quad \Phi_+^* g(x) = \frac{1}{\sqrt{2\pi}}\int dk\ \varphi_k^+(x)g(k),$$
$$\Phi_- f(k) = \frac{1}{\sqrt{2\pi}}\int dx\ \varphi_k^-(x)f(x), \quad \text{and} \quad \Phi_-^* g(x) = \frac{1}{\sqrt{2\pi}}\int dk\ \overline{\varphi_k^-(x)}g(k).$$

Then

Proposition 8.1 *$\Phi_\pm$ and $\Phi_\pm^*$ can be extended to continuous maps from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. $\Phi_\pm^*$ is the adjoint of $\Phi_\pm$*

Proof. Take some f in $L^2(\mathbb{R})$. Then

$$\begin{aligned} \Phi_- f(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1} dx f(x)(\varphi_k^-(x) - t_k e^{-ikx}) + \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} dx f(x)(\varphi_k^-(x) - e^{-ikx})\\ &\quad + \frac{1}{\sqrt{2\pi}}\int_{1}^{+\infty} dx f(x)(\varphi_k^-(x) - e^{-ikx} - r_k^- e^{ikx}) + \Phi_0 f(k)\\ &\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-1} dx f(x)(1-t_k)e^{-ikx}) + \frac{1}{\sqrt{2\pi}}\int_{1}^{+\infty} dx f(x) r_k^- e^{ikx})\\ &= A_1 f(k) + A_2 f(k) + A_3 f(k) + \Phi_0 f(k) + A_4 f(k) + A_5 f(k),\end{aligned}$$

with obvious notations. For the last three terms on the right-hand side, we have

$$\|\Phi_0 f\|_2 = \|f\|_2\,,\ \|A_4 f\|_2 \le \sqrt{2}\,\|f\|_2\,,\ \|A_5 f\|_2 \le \|f\|_2\,.$$

Now a more careful investigation of the asymptotics of φ_k^- using equations (6.7) shows that

$$\begin{aligned} \left|\varphi_k^-(x) - t_k e^{-ikx}\right| &\le \frac{C}{1+|k|}\frac{1}{x^2},\ x<0, & (8.3)\\ \left|\varphi_k^-(x) - e^{-ikx} - r_k^- e^{ikx}\right| &\le \frac{C}{1+|k|}\frac{1}{x^2},\ x>0, & (8.4)\\ \left|\varphi_k^-(x) - e^{-ikx}\right| &\le \frac{C}{1+|k|},\ x \in \mathbb{R}, \end{aligned}$$

for some constant $C > 0$. These estimations, together with the Schwarz inequality yields to

$$\|A_i f\|_2 \leq C' \|f\|_2 \,, \; i = 1, 2, 3,$$

for some other constant $C' > 0$. Hence Φ_- is a continuous map on $L^2(\mathbb{R})$, and the same can be proven for $\Phi_-^\star$, Φ_+ and $\Phi_+^\star$. It is then straightforward to verify that $\Phi_\pm^*$ is actually the adjoint of $\Phi_\pm$. This concludes the proof. $\diamondsuit$

The operators Φ_+ and Φ_- are usually called generalized Fourier transform. Let us now introduce a family of solutions of physical importance for the scattering problem. For $k \in \mathbb{R}$, we define the functions $\psi_k^\pm$ by

$$\psi_k^+ = \begin{cases} \varphi_k^+ \text{ if } k \geq 0, \\ \\ \overline{\varphi_k^-} \text{ if } k < 0, \end{cases} , \; \psi_k^- = \begin{cases} \varphi_k^- \text{ if } k \geq 0, \\ \\ \overline{\varphi_k^+} \text{ if } k < 0, \end{cases} .$$

They are solutions of the integral equation

$$\psi_k^\pm(x) = e^{\pm ikx} + \int \frac{e^{i|k||x-y|}}{2i\,|k|} V(dy) \psi_k^\pm(y). \tag{8.5}$$

These functions write as the superposition of an ingoing free wave and an outgoing perturbed wave. For regular potentials, (8.5) is the well-known Lippmann-Schwinger equation. In the same way as before, we define the generalized Fourier transforms $\Psi_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $\Psi_\pm^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\Psi_+ f(k) = \frac{1}{\sqrt{2\pi}} \int dx \; \overline{\psi_k^+(x)} f(x), \quad \text{and} \quad \Psi_+^* g(x) = \frac{1}{\sqrt{2\pi}} \int dk \; \psi_k^+(x) g(k),$$
$$\Psi_- f(k) = \frac{1}{\sqrt{2\pi}} \int dx \; \psi_k^-(x) f(x), \quad \text{and} \quad \Psi_-^* g(x) = \frac{1}{\sqrt{2\pi}} \int dk \; \overline{\psi_k^-(x)} g(k),$$

or equivalently,

$$\Psi_\pm = P_+ \Phi_\pm + P_- \Phi_\mp, \; \Psi_\pm^* = \Phi_\pm^* P_+ + \Phi_\mp^* P_-, \tag{8.6}$$

where $P_\pm$ is the projector on positive/negative axis : $P^\pm f(u) = \theta_0(\pm u) f(u)$. The generalized Fourier transforms satisfy the so-called intertwining property, that we state in the following form :

Proposition 8.2 (Intertwining property) *For all bounded measurable function η we have*

$$\eta(H)\Phi_\pm^*\Phi_0 = \Phi_\pm^*\Phi_0\eta(H_0), \tag{8.7}$$
$$\eta(H)\Psi_\pm^*\Phi_0 = \Psi_\pm^*\Phi_0\eta(H_0), \tag{8.8}$$

where $\eta(H)$ and $\eta(H_0)$ defined according to the spectral theorem.

Proof. We will first show that $H\Phi_+^*\Phi_0 f = \Phi_+^*\Phi_0 H_0 f$ holds on $D(H_0)$. For any x, y we have

$$\begin{aligned}
\left|\varphi_k^+(x) - \varphi_k^+(y)\right| &\leq \left|\int \frac{e^{ik|x-t|} - e^{ik|y-t|}}{2ik} \varphi_k^+(t) V(dt)\right| + \left|e^{ikx} - e^{iky}\right| \\
&\leq \|V\| \sup_{k\in\mathbb{R}} \left\|\varphi_k^+\right\|_\infty |x-y| + |k|\,|x-y|\,.
\end{aligned}$$

It follows that $\Phi_+^*\Phi_0 f$ is lipschitz and a fortiori continuous for all $f \in D(H_0)$. So, we may consider the distribution $(-\Delta + V)\Phi_+^*\Phi_0 f$ in $\mathcal{D}'$. Now for any $s \in \mathcal{D}$ we have

$$\begin{aligned}
((-\Delta + V)\Phi_+^*\Phi_0 f, s) &= -(\Phi_+^*\Phi_0 f, s") + (V, \Phi_+^*\Phi_0 f s) \\
&= -\int dx\ s"(x) \int dk\ \varphi_k^+(x)\widehat{f}(k) + \int V(dx) s(x) \int dk\ \varphi_k^+(x)\widehat{f}(k) \\
&= \int dk\ \widehat{f}(k) \left[-\int dx\ s"(x)\varphi_k^+(x) + \int V(dx)\ s(x)\varphi_k^+(x)\right],
\end{aligned}$$

where we have used Fubini's theorem to interchange the integrals. Now,

$$-\int dx\ s"(x)\varphi_k^+(x) + \int V(dx)\ s(x)\varphi_k^+(x) = k^2 \int dx\ s(x)\varphi_k^+(x),$$

owing to the weak Helmholtz equation $(-\Delta + V)\varphi_k^+ = k^2\varphi_k^+$. Hence, using again Fubini's theorem,

$$\begin{aligned}
((-\Delta + V)\Phi_+^*\Phi_0 f, s) &= \int dx\ s(x) \int dk\ k^2 \widehat{f}(k)\varphi_k^+(x) \\
&= (\Phi_+^*\Phi_0 H_0 f, s).
\end{aligned}$$

Now $\Phi_+^*\Phi_0 H_0 f$ is in $L^2(\mathbb{R})$. So, the distribution $(-\Delta + V)\Phi_+^* f$ is in $L^2(\mathbb{R})$, which means that $\Phi_+^* f$ is in $D(H)$ and $H\Phi_+^*\Phi_0 f = \Phi_+^*\Phi_0 H_0 f$. Also we have

$$(H - \zeta)\Phi_+^*\Phi_0 f = \Phi_+^*\Phi_0 (H_0 - \zeta) f$$

for any $\zeta \in \mathbb{C}\backslash\mathbb{R}$ and for all $f \in D(H_0)$. If we introduce the free resolvent $R_0(\zeta) = (H_0 - \zeta)^{-1}$, it follows that

$$R(\zeta)\Phi_+^*\Phi_0 g = \Phi_+^*\Phi_0 R_0(\zeta) g$$

for all $g \in \mathrm{Ran}(H_0 - \zeta)$ (Ran=Range). H_0 being self-adjoint, we have $\mathrm{Ran}(H_0 - \zeta) = L^2(\mathbb{R})$ and thus

$$R(\zeta)\Phi_+^*\Phi_0 = \Phi_+^*\Phi_0 R_0(\zeta)$$

for all $\zeta \notin \mathbb{R}$. This implies

$$E(\Lambda)\Phi_+^*\Phi_0 = \Phi_+^*\Phi_0 E_0(\Lambda)$$

for all Borel set Λ, where $E_0(\cdot)$ is the spectral family associated to H_0 (the last identity can be obtained by means of Stone's formula). The relation (8.7) follows from the spectral theorem, and (8.8) is a consequence of (8.7) and (8.6). $\diamondsuit$

Corrolary 8.1 *Let $\mathcal{H}_{ac}$ be the absolutely continuous subspace of $\mathcal{H}$,*

$$\mathcal{H}_{ac} = \{f \in \mathcal{H} \mid \textit{the spectral measure associated to } f \textit{ is absolutely continuous}\},$$

and P_{ac} is the orthogonal projector on $\mathcal{H}_{ac}$. Then

$$\Phi_{\pm} P_{ac} = \Phi_{\pm}, \;\; \Psi_{\pm} P_{ac} = \Psi_{\pm}. \tag{8.9}$$

Proof. Let $\mathcal{H}_{pp}$ be the purely ponctual subspace, that is the set of bound states. Then $\mathcal{H}_{pp} \oplus \mathcal{H}_{ac} = \mathcal{H}$, so it suffices to prove that $\Phi_{\pm}$ vanishes on $\mathcal{H}_{pp}$. If $-\lambda^2 \in \sigma_d(H)$ is an eigenvalue and f an associated eigenfunction, then $Hf = -\lambda^2 f$. Now by taking the adjoint in equation (8.7), with $\eta(x) = (x-i)^{-1}$, we have

$$\Phi_{\pm}(H+i)^{-1} f = \Phi_0 (H_0 + i)^{-1} \Phi_0^* \Phi f,$$

and thus

$$(-\lambda^2 + i)\Phi_{\pm} f(k) = (k^2 + i)^{-1} \Phi_{\pm} f(k)$$

for all k, which forces $\Phi_{\pm} f(k) = 0$ for all k. The proof for $\Psi_{\pm}$ is similar. $\diamondsuit$

The generalized eigenfunctions of the free Laplacian $e^{\pm ikx}$ form a complete orthonormal set of $L^2(\mathbb{R})$. The orthogonality is expressed by $\Phi_0 \Phi_0^* = 1$ and the completeness by $\Phi_0^* \Phi_0 = 1$. We wish to show that, analogously, the functions $\psi_k^{\pm}$ form a complete orthonormal set of the absolutely continuous subspace $\mathcal{H}^{ac}$, that is we whish to prove $\Psi_{\pm}\Psi_{\pm}^* = 1$ and $\Psi_{\pm}^* \Psi_{\pm} = P_{ac}$. First, we prove

Proposition 8.3 *The generalized Fourier transforms satisfy*

$$\Phi_+^* \Phi_+ + \Phi_-^* \Phi_- = 2P_{ac}, \tag{8.10}$$

$$\Psi_+^* \Psi_+ + \Psi_-^* \Psi_- = 2P_{ac}. \tag{8.11}$$

For the proof, we need the following lemma

Lemma 8.2 *For all $x, y, k \in \mathbb{R}$ we have*

$$2\Re\left\{\frac{\varphi_k^+(y)\varphi_k^-(x)}{t_k}\right\} = \Re\left\{\overline{\varphi_k^+(x)}\varphi_k^+(y)\right\} + \Re\left\{\overline{\varphi_k^-(x)}\varphi_k^-(y)\right\}. \tag{8.12}$$

Proof. We may expand $\varphi^{\pm}(x,k)$ on the basis $\left\{\varphi^{\mp}(x,k), \overline{\varphi^{\pm}(x,k)}\right\}$

$$\varphi_k^+(x) = \frac{1}{\overline{r_k^+}}\overline{\varphi_k^+(x)} + \frac{t_k}{r_k^-}\varphi_k^-(x), \tag{8.13}$$

$$\varphi_k^-(x) = \frac{1}{\overline{r_k^-}}\overline{\varphi_k^-(x)} + \frac{t_k}{r_k^+}\varphi_k^+(x). \tag{8.14}$$

Indeed, the functions on the left- and right hand-side both verify the weak Helmholtz equation and satisfy the same asymptotic condition. They must therefore coincide, by proposition (5.2). Now using the relations (6.4) between the coefficients, (8.12) is easily verified $\diamondsuit$

Proof of theorem (8.3). For f and g in $C_0^\infty(\mathbb{R})$ and $0<\alpha<\beta<\infty$ we have

$2 < E[\alpha,\beta]f, g >= 2\int dx\overline{E[\alpha,\beta]f(x)}g(x)$

$= 2\int dx g(x)\int dy\; e_{\alpha,\beta}(x,y)\overline{f(y)} = \frac{2}{\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk \int dxdy\; \overline{f(y)}g(x)\Re\left\{\frac{\varphi_k^+(y)\varphi_k^-(x)}{t_k}\right\}$

$= \frac{1}{\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk\int dxdy\;\overline{f(y)}g(x)\Re\left\{\overline{\varphi_k^+(x)}\varphi_k^+(y)\right\} + \frac{1}{\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk\int dxdy\;\overline{f(y)}g(x)\Re\left\{\overline{\varphi_k^-(x)}\varphi_k^-(y)\right\}$

$= \frac{1}{2\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk\int dxdy\;\overline{f(y)}g(x)\overline{\varphi_k^+(x)}\varphi_k^+(y) + \frac{1}{2\pi}\int_{-\sqrt{\beta}}^{-\sqrt{\alpha}} dk\int dxdy\;\overline{f(y)}g(x)\overline{\varphi_k^+(x)}\varphi_k^+(y)$

$+\frac{1}{2\pi}\int_{\sqrt{\alpha}}^{\sqrt{\beta}} dk\int dxdy\;\overline{f(y)}g(x)\overline{\varphi_k^-(x)}\varphi_k^-(y) + \frac{1}{2\pi}\int_{-\sqrt{\beta}}^{\sqrt{\alpha}} dk\int dxdy\;\overline{f(y)}g(x)\overline{\varphi_k^-(x)}\varphi_k^-(y).$

Letting $\alpha\to 0$ and $\beta\to+\infty$ yields

$$\begin{aligned} < P_{ac}f, g > &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\int dy\;\overline{f(y)}\varphi_k^+(y)\int dxg(x)\overline{\varphi_k^+(x)} \\ &\quad +\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\int dy\;\overline{f(y)\overline{\varphi_k^-(y)}}\int dxg(x)\varphi_k^-(x) \\ &= <\Phi_+f,\Phi_+g> + <\Phi_-f,\Phi_-g>. \end{aligned}$$

It follows that $\Phi_+^*\Phi_+f + \Phi_-^*\Phi_-f = 2P_{ac}f$ holds on a dense set, hence on $L^2(\mathbb{R})$, and (8.11) is merely a consequence of (8.10) and (8.6). We are now able to prove the main result fom this paper.

Theorem 8.3 *a) The (strong) wave operators $W_\pm$ exist and are complete. They are given by*

$$W_\pm = \mathrm{s.}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0} = \Psi_\mp^*\Phi_0. \tag{8.15}$$

b) The generalized functions $\psi_k^\pm$ form a complete orthonormal set of $\mathcal{H}_{ac}$,

$$\Psi_\pm\Psi_\pm^* = 1,\;\; \Psi_\pm^*\Psi_\pm = P_{ac}. \tag{8.16}$$

c) The scattering operator $\mathcal{S} = W_+^*W_-$ verify

$$P_+\Phi_0\mathcal{S}\mathit{\Phi}_{\rm o}^*P_+ = TP_+,\;\; P_-\mathit{\Phi}_{\rm o}\mathcal{S}\mathit{\Phi}_{\rm o}^*P_- = T^*P_-,$$

$$P_-\Phi_0\mathcal{S}\mathit{\Phi}_{\rm o}^*P_+ = R_+^*P^-,\;\; P_+\mathit{\Phi}_{\rm o}\mathcal{S}\mathit{\Phi}_{\rm o}^*P_- = R_-P_+,$$

where T, T^*, $R_\pm$, $R^*_\pm$ are the operator of multiplication by t_k, $\overline{t_k}$, $r_k^\pm$ and $\overline{r_k^\pm}$, respectively,

$$Tf(k) = t_k f(k),\ \ T^*f(k) = \overline{t_k} f(k),\ \ R_\pm f(k) = r_k^\pm f(k),\ \ R^*_\pm f(k) = \overline{r_k^\pm} f(k).$$

Corrolary 8.4 *The action of the scattering operator on the energy shell k^2 can be represented by the usual scattering matrix*

$$S = \begin{pmatrix} t_k & r_k^- \\ r_k^+ & t_k \end{pmatrix}$$

Thanks to the relations (6.4), S is unitary.

Proof of theorem (8.3).

a) The proof for the existence and completeness of the wave operators is basically the same as for regular potentials (see e.g [BS91] for a sketch of proof)). Let us detail it. The method basically relies upon a stationary phase argument. First note that

$$e^{-itH}\Psi^*_\mp\Phi_0 = \Psi^*_\mp\Phi_0 e^{-itH_0} \tag{8.17}$$

because of the intertwining property. Thus,

$$\begin{aligned} e^{itH}e^{-itH_0} - \Psi^*_\mp\Phi_0 &= e^{itH}(e^{-itH_0} - e^{-itH}\Psi^*_\mp\Phi_0) \\ &= e^{itH}(\Phi^*_0 - \Psi^*_\mp)\Phi_0 e^{-itH_0}. \end{aligned}$$

On the other hand, H_0 is unitarily equivalent to K^2, the operator of multiplication by k^2,

$$H_0 = \Phi^*_0 K^2 \Phi_0.$$

Therefore, proving (8.15) is equivalent to proving

$$\operatorname*{s.\,lim}_{t\to\pm\infty} (\Phi^*_0 - \Psi^*_\mp)e^{-itK^2}\Phi_0 \tag{8.18}$$

Let us detail the proof for $t \to +\infty$. Take some function f with $\widehat{f} \in C_0^\infty(\mathbb{R}\backslash 0)$ (such functions are dense in $L^2(\mathbb{R})$). For every fixed x we have

$$\begin{aligned} (\Psi^*_- - \Phi^*_0)e^{-itK^2}\Phi_0 f(x) &= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} dk(\overline{\varphi_k^-(x)} - e^{ikx})e^{-ik^2t}\widehat{f}(k) \\ &+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 dk(\varphi_k^+(x) - e^{ikx})e^{-ik^2t}\widehat{f}(k) \end{aligned}$$

An integration by parts yields

$$
\begin{aligned}
(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{t}\int_0^{+\infty} dk\ \partial_k \left[\frac{(\overline{\varphi_k^-(x)} - e^{ikx})\widehat{f}(k)}{2ik}\right] e^{-ik^2t} \\
&+ \frac{1}{\sqrt{2\pi}}\frac{1}{t}\int_{-\infty}^{0} dk\ \partial_k \left[\frac{(\varphi_k^+(x) - e^{+ikx})\widehat{f}(k)}{2ik}\right] e^{-ik^2t}
\end{aligned}
$$

It follows from the general properties of $\varphi_k^\pm$ stated in theorem (6.1) that

$$
\left|(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x)\right| \leq \frac{C(x)}{t},
$$

where the function $C(x) \geq 0$ is bounded on every compact interval. Hence, for every compact $[-N, +N]$ we have

$$
\lim_{t\to+\infty}\int_{|x|\leq N} dx\ \left|(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x)\right|^2 = 0.
$$

Now, we have for all $x > N$

$$
\begin{aligned}
(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x) &= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} dk\ e^{-itk^2-ikx}\widehat{f}(k)\overline{r_k^-} \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dk\ e^{-itk^2+ikx}\widehat{f}(k))(t_k - 1) + \rho(x) \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} dk\ \frac{e^{-itk^2-ikx}}{2ikt+ix}\partial_k[\widehat{f}(k)\overline{r_k^-}] \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dk\ \frac{e^{-itk^2+ikx}}{-2ikt+ix}\partial_k[\widehat{f}(k)(t_k - 1)] + \rho(x),
\end{aligned}
$$

where

$$
\rho(x) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} dk\, e^{-itk^2}\widehat{f}(k)(\overline{\varphi_k^-(x)} - e^{ikx} - \overline{r_k^-}e^{-ikx}) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dk\, e^{-itk^2}\widehat{f}(k)(\varphi_k^+(x) - t_k e^{ikx})
$$

The absolute value of the denominators $\pm 2ikt + ix$ is always greater than x for positive t. On the other hand, the estimations (8.3) and (8.4) yields $|\rho(x)| \leq Cx^{-2}$ for the remainder. Thus

$$
\left|(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x)\right| \leq \frac{C}{|x|} + \frac{C}{x^2}.
$$

and

$$
\int_{x>N} dx\ \left|(\Psi_-^* - \Phi_0^*)e^{-itK^2}\Phi_0 f(x)\right|^2 \leq \frac{C}{N}. \tag{8.19}
$$

Upon enlarging N, the above integral can be made arbitrarily small independently of t. The same result can be obtained for the integral running over $-\infty < x < N$. This proves (8.18), which in turn proves the existence of W_+. The proof for the existence of W_- is similar.

Now let us prove the completeness of the wave operators. By theorem (8.3) we have

$$W_+W_+^* + W_-W_-^* = \Psi_-^*\Psi_- + \Psi_+^*\Psi_+ = 2P_{ac} \tag{8.20}$$

Now $\mathrm{Ran}W_\pm W_\pm^* \subset \mathrm{Ran}W_\pm \subset \mathcal{H}_{ac}$ so by (8.20) we have

$$\mathcal{H}_{ac} = \mathrm{Ran}(W_+W_+^* + W_-W_-^*) \subset \mathrm{Ran}W_+W_+^* + \mathrm{Ran}W_-W_-^* \subset \mathcal{H}_{ac}$$

and thus $\mathrm{Ran}W_\pm W_\pm^* = \mathrm{Ran}W_\pm = \mathcal{H}_{ac}$. This proves the completeness of the wave operators.

b) The wave operators are isometric, as strong limit of unitary operators. So

$$W_\pm^* W_\pm = \Psi_\pm\Psi_\pm^* = 1. \tag{8.21}$$

This expresses the 'orthogonality' of the generalized functions $\psi_k^\pm$, for it corresponds formally to

$$< \psi_k^\pm, \psi_{k'}^\pm >= \delta(k-k').$$

Furthermore, $W_\pm W_\pm^* = (W_\pm W_\pm^*)(W_\pm W_\pm^*)$ is a projector onto $\mathcal{H}_{ac}$, hence

$$W_\pm W_\pm^* = \Psi_\pm^*\Psi_\pm = P_{ac}, \tag{8.22}$$

which proves the completeness of the functions $\psi_k^\pm$.

c) The scattering operator is given by $\mathcal{S} = W_+^* W_- = \Phi_0^*\Psi_-\Psi_+^*\Phi_0$. Thus,

$$\Phi_0\mathcal{S}\Phi_0^* = P_+\Phi_-\Phi_+^*P_+ + P_-\Phi_+\Phi_-^*P_- + P_+\Phi_-\Phi_-^*P_- + P_-\Phi_+\Phi_+^*P_+.$$

Now (8.21) and (8.6) imply

$$P_+\Phi_\pm\Phi_\pm^*P_+ = P_+ \text{ and } P_-\Phi_\pm\Phi_\pm^*P_- = P_-. \tag{8.23}$$

On the other hand, we have thanks to (8.12)

$$\Phi_+\Phi_+^* + \Phi_-\Phi_-^* = 2T^{-1}\Phi_-\Phi_+^*,$$

where T^{-1} is the inverse of T. It follows from the last two equations that

$$P_\pm\Phi_-\Phi_+^*P_\pm = TP_\pm, \tag{8.24}$$

and by taking the adjoint

$$P_\pm\Phi_+\Phi_-^*P_\pm = T^*P_\pm. \tag{8.25}$$

Furthermore, (8.13) together with (6.3) and (6.4) provides a cross-relation between the Φ_+ and Φ_-, namely

$$\Phi_- = R_-T^{-1}\Pi\Phi_+ + (T^*)^{-1}\Phi_+, \tag{8.26}$$

where Π is the parity operator $\Pi f(k) = f(-k)$. Using the equations (8.26), (8.24), (8.23) together with (6.3) and (6.4), we obtain easily

$$P_{-}\Phi_{+}\Phi_{+}^{*}P_{+} = R_{+}^{*}P_{-} \tag{8.27}$$

and

$$P_{+}\Phi_{-}\Phi_{-}^{*}P_{-} = R_{-}P_{+}. \tag{8.28}$$

This concludes the proof of theorem (8.3). $\diamondsuit$

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