# A NOTE ON RELATION BETWEEN QUANTUM MECHANICS AND ALGEBRAIC INVARIANTS

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**Abstract**

We propose a program for construction of nonclassical algebraic structures and quantization by analogy with Klein geometric program. The group of affine canonical transformations is considered in this context in detail.

## 1. INTRODUCTION

The old quantum theory was developing in the direction: The realization of observables of the classical mechanics and their algebra remains without modification and only realization of states to be changed. As the result the theory has been presented a wealthy of material [1]. However, the quantum theory was setting off by the other way: It was constructed in such a way that the theory remains the algebraic structure of classical mechanics but fully renounced idea of phase space and of observables as smooth functions on it [2],[3]. Nevertheless, the idea of phase space appeared in the quantum mechanics at its early period [4],[5] and then took clear form in the work [6], then in the theory of deformational quantization [7],[8] and in a number of works [9],[10],[11]. Besides the Weyl-Wigner-Moyal representation there are others representations of quantum mechanics in phase space including rare representations [12],[13],[14],[15]. However, the Weyl-Wigner-Moyal (WWM) representation occupy a special position among them: In its basis lies a maximal group (more precise definition will be given in what follows). For this reason the WWM representation is most appropriate initial point for algebro-geometric speculations.

The main Klein idea [16] lies in the correspondence to any geometry a group which acts in its space. In general, every group of transformations determines its own geometry. This geometry studies properties of figures which are invariant under the action of a given group transformations. So, by Klein, group is first notion of geometry and it can be interpreted as group of symmetry for geometry which is arising from. All it is well known [16],[17]. The Euclidean geometry is typical example for the group which has the structure of semidirect product of the group of orthogonal matrixes and the additive group of vectors (in an $n$-dimentional space).

In this work we propose an application of the basic idea of geometric Klein program, which is understood in a wide sense, to the problem of construction of nonclassical algebraic structures on the set of classical observables. The starting point are the WWM phase space representation of quantum mechanics and the group of affine canonical transformations of the phase space.

## 2. CLASSICAL AND QUANTUM MECHANICS IN PHASE SPACE

This paper has the primary purpose of first presenting the program in as simple terms as possible, so we deal with the case of systems with one degree of freedom. It is not the exhaustive exposition, of course, it is the occasion to discussion only.

In this section we take as a starting point a brief reminiscence of phase space classical and quantum mechanics notions [18],[1],[10], [4],[5],[6].

### 2.1. Classical Mechanics

Let $x = (q, p)$ denote coordinates of the phase space $\mathcal{M} = \mathbf{R}^2$, identified with canonical variables of a classical mechanical system, and let $\mathcal{A}$ denote the set of smooth functions on $\mathcal{M}$. The observables of a classical system are identified with elements $f, g, ...$ of $\mathcal{A}$. $\mathcal{A}$ is equipped with two algebraic structures: the pointwise multiplication (Jordan product), and the Poisson bracket operation

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \omega^{ij}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \tag{2.1}$$

which makes $\mathcal{A}$ into a Lie algebra. The notion $\omega$ is used for simplectic matrix.

The states of a classical mechanical system are probability distributions on $\mathcal{M}$.

### 2.2. Phase Space Representations of Quantum Mechanics

The main example here is the WWM representation of quantum mechanics in phase space [20],[5],[6].

Let $\mathcal{M}$ and $\mathcal{A}$ are the same as in section 2.1. For each $k \geq 0$ define bilinear partial Moyal bracket of degree $k$

$$\{f, g\}^{(k)} = \omega^{i_1 j_1} \cdots \omega^{i_k j_k} \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}} \frac{\partial^k g}{\partial x^{j_1} \cdots \partial x^{j_k}}. \tag{2.2}$$

For $k = 0$ it is usual pointwise multiplication and for $k = 1$ it is the Poisson bracket (2.1).

It is useful and important to remark that the $\{f, g\}^{(k)}$ has a form of the $k$th *transvection* operator of the invariant theory [19],[20].

The main algebraic structures of the WWM phase space representation of quantum mechanics, the Jordan-Moyal product and the Poisson-Moyal bracket, are of the form

$$f \circ g = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(\hbar}{2}^{2n} \{f, g\}^{(2n)} \tag{2.3}$$

and

$$\{f, g\}_M = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(\hbar}{2}^{2n} \{f, g\}^{(2n+1)}. \tag{2.4}$$

Here $(\cdot \circ \cdot)$ is a genuine Jordan algebra structure and $\{\cdot, \cdot\}_M$ is a genuine structure of Lie algebra on $\mathcal{A}$. $\hbar$ is the Planck constant.

This representation differs from the usual operator formulation of quantum mechanics [2],[3] by form (and only by form) but is of the most close to classical mechanics. Algebraic structures of the classical mechanics is a limited case of the algebraic structures (2.3) , (2.4): the pointwise product and the Poisson bracket may be derived from the Jordan-Moyal product and the Poisson-Moyal bracket by passing to the limit $\hbar \to 0$. The case of quantum states is not so simple with respect to limit $\hbar \to 0$ but it is the case of particular importance which lies outside the paper.

The algebraic structures of standard operator quantum mechanics and the WWM representation are connected by the Weyl-Wigner correspondence rule

$$-\frac{i}{\hbar}[\hat{f}, \hat{g}] \to \{f, g\}_M; \qquad \frac{1}{2}[\hat{f}, \hat{g}]_+ \to f \circ g. \tag{2.5}$$

Here $[\cdot, \cdot]$ is the commutator and $[\hat{f}, \hat{g}]_+ = (\hat{f} + \hat{g})^2 - \hat{f}^2 - \hat{g}^2$.

There is a lot of correspondence rules. And for every correspondence rule there is the associate phase space representation. Rare phase space representations are existing [12],[13],[15],[14]. For example, the representation which is connected with the Blokhintsev bracket

$$\{f, g\}_B = \sum_{n=1}^{\infty} \frac{(-i\hbar)^{n-1}}{n!} \omega^{ij} \frac{\partial^n f}{\partial (x^i)^n} \frac{\partial^n g}{\partial (x^j)^n}. \tag{2.6}$$

It should be noted that in the reminiscences of this section no mention is made of the states of the mechanicses considered, because the algebraic structures are the main object of consideration here, and this is the question of other program [21]. However, in the final section we will consider the connection between the basic group and coherent states notions.

## 3. THE PROBLEM

The main content of this section is the following. In the first place we briefly recall a formulation of the Dirac problem. Then as a preparetion to the solution of this problem we derive the invariance group $\mathcal{G}$ (group of phase space transformations) of the algebraic structures of the WWM representation. Then we to pose a question on the solution of the Dirac problem by means of introducing into consideration of new algebraic structures on $\mathcal{A}$ instead of the algebraic structures of classical mechanics. Exactly, these new algebraic structures to be defined by the required property to be invariant under the action of the main group $\mathcal{G}$ of the phase space transformations. Then these new algebraic structures on the set of classical mechanics observables to be used for the solution of the Dirac problem by means of Weyl-Wigner correspondence.

In such a way, and after obvious generalization, the basic notion of quantization is a group of the affine phase space transformations. The quantization amounts to the construction of relevant nonclassical invariant algebraic structures on the set of classical mechanical observables. It is clear *analog* of the Klein geometric program [16],[17].

### 3.1. The Dirac Problem

The Dirac problem (quantization) can be formulated as the correspondence problem in the following manner [21], [22]: To establish *ab initio* a mapping $Q$, which is defined on the algebra of classical observables $f, g, \ldots$, takes values in the algebra of quantum observables $\hat{f}, \hat{g}, \ldots$ (self-adjoint operators acting in a Hilbert space $\mathcal{H}$ [3], [2]), and has the following properties:

$$
\begin{array}{ll}
(Q1) & Q(\lambda f+\mu g)=\lambda Q(f)+\mu Q(g), \quad \lambda,\mu\in\mathbf{R};\\
(Q2) & Q(\{f,g\})=\frac{1}{i\hbar}[Q(f),Q(g)];\\
(Q3) & Q(f^2)=(Q(f))^2;\\
(Q4) & Q(1)=1_{\mathcal{H}}.
\end{array}
$$

It is known that such the correspondence problem leads to a lot of difficulties [21], [22].

### 3.2. The Basic Group of the WWM Representation

We take as our starting point the set of partial Moyal brackets (2.2). The Jordan-Moyal product (2.3) and the Poisson-Moyal bracket (2.4) are the linear combinations of the partial Moyal brackets. It has been outlined above (section 2.2) that the partial Moyal brackets are connected with transvections of classical invariant theory. Hence, we may ask the question: If the set of partial Moyal brackets is the set of bilinear invariant algebraic structures on the set of classical observables, what is the group of phase space transformations for? The answer on this question is almost obvious: It is the group of affine canonical transformations (general form of affine cononical transformation is: $x\to Cx+\xi$, where $C$ is a linear canonical transformation, and $\xi\in\mathcal{M}$; as group it has the structure of semidirect product of the symplectic group and the group of vectors $\mathcal{M}$: $\mathcal{G}_C=Sp(\mathcal{M})\dot{\times}\mathcal{M}$ ). This answer we can find in classical invariant theory (see [20] or [19]).

### 3.3. The Program

The program for the solution of the Dirac problem can be formulated now in the following manner.

For a given group $\mathcal{G}$ of the phase space $\mathcal{M}$ affine transformations and a system for which smooth functions from $\mathcal{A}$ are the observables, firstly, the set of bilinear maps $\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ , which are invariant under the transformations $\mathcal{G}$, to be constructed. Then, from these invariant maps the multiplication operations of Lie $\{\cdot,\cdot\}_{\mathcal{G}}$ and Jordan $(\cdot\circ\cdot)_{\mathcal{G}}$ to be constructed. If these operations can be defined uniquely in a sense, they are useing in the Dirac problem instead of classical operations of the Poisson bracket and pointwise multiplication:

$$
\begin{array}{ll}
(q1) & q(\lambda f+\mu g)=\lambda q(f)+\mu q(g), \quad \lambda,\mu\in\mathbf{R};\\
(q2) & q(\{f,g\}_{\mathcal{G}})=\frac{1}{i\hbar}[q(f),q(g)];\\
(q3) & q((f\circ f)_{\mathcal{G}})=(q(f))^2;\\
(q4) & q(1)=1_{\mathcal{H}}.
\end{array}
$$

It is known that in the case of algebraic structures of the WWM representation such the mapping $q$ is well defined [23].

In such a way, this program will be indeed the quantization program if, at least, from the set of bilinear maps, which are invariant under the action of the group $\mathcal{G}_C$ of affine canonical transformations, the multiplication operations of Lie $\{\cdot,\cdot\}_{\mathcal{M}}$ (2.4) and Jordan $(\cdot\circ\cdot)$ (2.3) can be constructed uniquely. When this test case is verivied then we may take into consideration others basic affine transformation groups.

## 4. THE BASIC GROUP $\mathcal{G}_C$: NONCLASSICAL ALGEBRAIC STRUCTURES

The purpose of this section is to construct nonclassical Lie and Jordan operations on the basis of the set $\{\{\cdot,\cdot\}^{(k)}, \quad k \geq 0\}$ of $\mathcal{G}_C$-invariant bilinear operations. $\{\cdot,\cdot\}^{(0)}$ and $\{\cdot,\cdot\}^{(1)}$ are classical pointwise multiplication (Jordan product) and Poisson bracket (Lie product) correspondently. It is easy to see that for $k \geq 2$ there is no any $k = m$ such that $\{\cdot,\cdot\}^{(m)}$ is Lie or Jordan product. Hence, for a construction of nonclassical algebraic structures it is necessary to use an infinite linear combination of all $\{\cdot,\cdot\}^{(k)}, \quad k \geq 0$.

### 4.1. Lie Structure

Let us consider an infinite series with coefficients $c_k, \quad k = 0, 1, 2, \ldots$;

$$\{f,g\}_{\mathcal{G}_C} = \sum_{k=0}^{\infty} c_k \{f,g\}^{(k)}. \tag{4.1}$$

The operation $\{f,g\}_{\mathcal{G}_C}$ to be considered as Lie product if

$$\{f,g\}_{\mathcal{G}_C} = -\{f,g\}_{\mathcal{G}_C}, \tag{4.2}$$

and the Jacobi identity

$$\{f,\{g,h\}_{\mathcal{G}_C}\}_{\mathcal{G}_C} + \{h,\{f,g\}_{\mathcal{G}_C}\}_{\mathcal{G}_C} + \{g,\{h,f\}_{\mathcal{G}_C}\}_{\mathcal{G}_C} = 0 \tag{4.3}$$

are satisfied.

We shall test the hypothesis that the conditions (4.2), (4.3) determine the coefficients $c_k$. First of all, it is evident that $\{f,g\}^{(k)} = (-1)^k \{g,f\}^{(k)}$ for all $k$ and it follows

$$\{f,g\}_{\mathcal{G}_C} = \sum_{n=0}^{\infty} c_{2n+1} \{f,g\}^{(2n+1)}. \tag{4.4}$$

Let us now consider the observables from $\mathcal{A}$ with the Fourier representation

$$f(x) \sim \int \tilde{f}(\alpha) \exp(i\alpha(x)) d\alpha, \tag{4.5}$$

where $\alpha(x)$ is 1-form on $\mathcal{M}$. It is easy to see that

$$\{\exp(i\alpha(x)), \exp(i\beta(x))\}_{\mathcal{G}_C} = \sum_{n=0}^{\infty} c_{2n+1} \exp(i\alpha + i\beta)(-\{\alpha,\beta\}^{(1)})^{2n+1}. \tag{4.6}$$

Define function $F(z)$ as

$$F(z) = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}, \qquad F(-z) = -F(z). \tag{4.7}$$

Substituting equations (4.6) and (4.7) in the Jacobi identity (4.3), we derive the functional equation

$$F(z_1 + z_2)F(z_3) + F(-z_2 - z_3)F(z_1) + F(z_3 - z_1)F(-z_2) = 0. \tag{4.8}$$

After formal manipulations we find that

$$F''(z)F'(z) - F'''(z)F(z) = 0 \tag{4.9}$$

since $F(0) = F''(0) = 0$. The equation (4.9) for power series $F$ satisfying condition (4.7) leads immediately to $F''(z) = -d_1^2 F(z)$, and then to

$$F(z) = d_2' \sin(d_1 z) = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (d_1)^{2n} z^{2n+1}, \tag{4.10}$$

where $d_2 = d_2' d_1$. The solution (4.10) define all coefficients $c_{2k+1}$. Hence, $\mathcal{G}_C$-invariant Lie product has the form

$$\{f, g\}_{\mathcal{G}_C} = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (d_1)^{2n} \{f, g\}^{(2n+1)}. \tag{4.11}$$

The coefficient $d_2$ is unessential for $\{\cdot, \cdot\}_{\mathcal{G}_C}$ as Lie product. The operation (4.11) is the same as Moyal bracket (2.4) when $d_2 = 1$ and $d_1 = (\hbar/2)$.

### 4.2. Jordan Structure

Now we consider, once again, an infinite series of the type (4.1) and use the arguments analogous to those presented in section 4.1 for construction of nonclassical Jordan product.

Let

$$(f \circ g)_{\mathcal{G}_C} = \sum_{k=0}^{\infty} c_k \{f, g\}^{(k)}. \tag{4.12}$$

The operation $(f \circ g)_{\mathcal{G}_C}$ to be considered as Jordan product [24] if

$$(f \circ g)_{\mathcal{G}_C} = (g \circ f)_{\mathcal{G}_C}, \tag{4.13}$$

and the Jordan identity

$$((f \circ g)_{\mathcal{G}_C} \circ (f \circ f)_{\mathcal{G}_C})_{\mathcal{G}_C} - (f \circ (g \circ (f \circ f)_{\mathcal{G}_C})_{\mathcal{G}_C})_{\mathcal{G}_C} = 0 \tag{4.14}$$

are satisfied.

We shall test now the hypothesis that the conditions (4.12) and (4.13) determine the coefficients $c_k$ (4.11). Firstly, we see that

$$(f \circ g)_{\mathcal{G}_C} = \sum_{n=0}^{\infty} c_{2n} \{f, g\}^{(2n)}. \tag{4.15}$$

Define function $G(z)$:

$$G(z) = \sum_{n=0}^{\infty} c_{2n} z^{2n}, \qquad G(-z) = G(z). \tag{4.16}$$

Then the repeated application of the section 4.1 procedure gives the folloing equation

$$G(z_1)G(z_2 + z_3) - G(z_1 + z_2)G(z_3) = 0. \tag{4.17}$$

After formal manipulations we find that $G(z)$ satisfy the equation (4.9) : $G''G' - G'''G = 0$ with the condition (4.15) which is another then (4.7). As the result we have

$$G(z) = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (d_2)^{2n} z^{(2n)}. \tag{4.18}$$

Hence, $\mathcal{G}_C$-invariant Jordan product has the form

$$(f \circ g)_{\mathcal{G}_C} = d_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (d_1)^{2n} \{f, g\}^{(2n)}. \tag{4.19}$$

We see that the structure of the Jordan product (4.18) has the same peculiarity as the structure of the Lie product (4.11) but the coefficients $d_1, d_2$ in equatins (4.11) and (4.18) are not connected. To account for compatibility between Lie and Jordan product, we can take into consideration the condition [24]

$$((f \circ g)_{\mathcal{G}_C} \circ h)_{\mathcal{G}_C} - ((f \circ (g \circ h)_{\mathcal{G}_C})_{\mathcal{G}_C} \sim \{g, \{f, h\}_{\mathcal{G}_C}\}_{\mathcal{G}_C}. \tag{4.20}$$

As the *resume* of the sections 4.1 and 4.2, we see that our program give us the result which coinsides with th WWM algebraic structures.

For a further consideration it should be kept in mind that the algebras may be complex.

## 5. AS A CONCLUSION

Further, the following case may be considered with use: the group of affine homothetic transformations. As use we mean Blokhintsev brackets and Klein principle. On the other hand, it is clear that the program is in its beginning stages. This work is the part of a step toward the program's direction. We believe that developments in this area will be forthcoming.

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